Stochastic Optimization - 2 hours

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Name: ____________________

1 Linear quadratic control (11 points)

We consider the following problem

$$\min_{\boldsymbol{x}_1,\boldsymbol{x}_2} \quad \mathbb{E}\left[\frac{1}{2}(\boldsymbol{x}_1-\boldsymbol{\xi}_1)^2+\frac{1}{2}(\boldsymbol{x}_2-\boldsymbol{\xi}_2)^2+\frac{1}{2}(\boldsymbol{x}_2-\boldsymbol{x}_1)^2\right] \tag{1a}$$

$$s.t. \quad \sigma(\boldsymbol{x}_1)\subset\sigma(\boldsymbol{\xi}_1) \tag{1b}$$

$$\sigma(\boldsymbol{x}_2)\subset\sigma(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2) \tag{1c}$$

$$\tag{1d}$$

where $\boldsymbol{\xi}_i$ are centered random variable (i.e. with $\mathbb{E}[\boldsymbol{\xi}_i]=0$) with finite covariance matrix Σ. This means that, $var(\boldsymbol{\xi}_i)=\Sigma_{i,i}$ and $cov(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)=\Sigma_{1,2}$. When possible the result are to be given in function of the coefficient of Σ.

1. (1 point) Assuming that $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are deterministic, find the optimal solution to Problem (1) in function of x_0, ξ_1, ξ_2 and k, and show that the optimal value is of the form

$$V^{det}(\xi_1,\xi_2)=\kappa(\xi_2-\xi_1)^2,$$

κ to be determined.

> **Solution:** The problem is convex (0.5), hence optimality is guaranteed by first order condition.
>
> $$\begin{cases} x_1-\xi_1+x_1-x_2=0 \\ x_2-\xi_2+x_2-x_1=0 \end{cases}$$
>
> leading to (1)
>
> $$\begin{cases} x_1=\frac{2\xi_1+\xi_2}{3} \\ x_2=\frac{\xi_1+2\xi_2}{3} \end{cases}$$
>
> pluging the optimal solution yield V^{det} with $\kappa=\frac{1}{6}$(0.5).

2. (2 points) Using Question 1 propose a natural open-loop solution to Problem (1) and compute the associated upper bound. Can you also give a lower bound ?

> **Solution:** Replacing $\boldsymbol{\xi}_1,\boldsymbol{\xi}_2$ by their expectation (0.5) we get as optimal solution (0.5)
>
> $$\begin{cases} x_1^{EF}=0 \\ x_2^{EF}=0 \end{cases}$$
>
> the expected cost of x^{EF} is thus (0.5)
>
> $$v^{EEF}=\frac{\Sigma_{1,1}+\Sigma_{2,2}}{2}$$
>
> Remark : on this problem we can show that this solution is actually the best open-loop solution.
>
> By convexity we know that the value of the expected problem, i.e. 0, is a lower bound (0.5).

3. (1 point) Using Question 1 give a lower bound of Problem (1).

Solution: The anticipative lower (0.5) bound yields (0.5)

$$v^{antipicative} = \frac{1}{6}\mathbb{E}\big[(\boldsymbol{\xi}_2 - \boldsymbol{\xi}_1)^2\big] = \frac{1}{6}(\Sigma_{1,1} - 2\Sigma_{1,2} + \Sigma_{2,2}).$$

4. (4 points) Assume that $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ are independent. Solve Problem (1) giving both the value and optimal strategy.

Solution: By independence of $\boldsymbol{\xi}$ we can use Dynamic Programming (0.5). We have (2.5)

$$\begin{aligned}
\hat{V}_1(x_1, \xi_2) &= \min_{x_2} \frac{1}{2}(x_2 - x_1)^2 + (x_2 - \xi_2)^2 \\
&= \frac{(x_1 - \xi_2)^2}{4} \\
V_1(x_1) &= \frac{x_1^2 + \Sigma_{2,2}}{4} \\
\hat{V}_0(\xi_1) &= \frac{\Sigma_{2,2}}{4} + \min_{x_1} \frac{1}{2}(x_1 - \xi_1)^2 + \frac{1}{4}x_1^2 \\
&= \frac{\Sigma_{2,2}}{4} + \frac{1}{6}\boldsymbol{\xi}_1^2 \\
V_0 &= \frac{\Sigma_{2,2}}{4} + \frac{\Sigma_{1,1}}{6}
\end{aligned}$$

With optimal strategy (1)

$$\begin{aligned}
x_2^\sharp(x_1, \xi_2) &= \frac{x_1 + \xi_2}{2} \\
x_1^\sharp(\xi_1) &= \frac{2}{3}\xi_1
\end{aligned}$$

5. (2 points) For generic $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2$ prove that the policy obtained in the previous question is ε-optimal for Problem (1), where ε is to be given in function of Σ.

Solution: The expected costs of $x_2^\sharp, x_1^\sharp$ is (1)

$$\begin{aligned}
v^{DP} &= \mathbb{E}\Big[\frac{1}{2}(\frac{2}{3}\boldsymbol{\xi}_1 - \boldsymbol{\xi}_1)^2 + \frac{1}{2}(\frac{x_1^\sharp + \boldsymbol{\xi}_2}{2} - \boldsymbol{\xi}_2)^2 + \frac{1}{2}(\frac{x_1^\sharp + \boldsymbol{\xi}_2}{2} - x_1^\sharp)^2\Big] \\
&= \mathbb{E}\Big[\frac{1}{2}\frac{\Sigma_{11}}{9} + (\frac{\frac{2\boldsymbol{\xi}_1}{3} + \boldsymbol{\xi}_2}{2} - \boldsymbol{\xi}_2)^2\Big] \\
&= \frac{\Sigma_{11}}{6} - \frac{\Sigma_{12}}{3} + \frac{\Sigma_{22}}{4}
\end{aligned}$$

And using the anticipative lower bound we have(1)

$$v^{DP} - v^{anticipative} = \frac{\Sigma_{22}}{12}$$

2 A unit commitment problem (11 points)

We consider an energy production company which has a set of production unit $\mathcal{I}$. Each unit $i \in \mathcal{I}$ is either on ($x^i = 1$) or off ($x^i = 0$) for the day (decided the day before). If it is on, its production u^i_t at time $t \in [\![1, 24]\!]$ should be in $[\underline{u}^i, \bar{u}^i]$. Turning on a unit has a cost c^i (for the day), while the production cost per hour is e^i.

Let's denote $\boldsymbol{z}_t$ the production of the company sold on the market, and $\boldsymbol{\varepsilon}_t \geq 0$ the lost production, i.e. $\sum_{i\in\mathcal{I}} \boldsymbol{u}^i_t = \boldsymbol{z}_t + \boldsymbol{\varepsilon}_t$. For each hour there is a demand $\boldsymbol{d}_t$ such that $\boldsymbol{z}_t \in [0.8\boldsymbol{d}_t, 1.2\boldsymbol{d}_t]$ almost surely. The company is paid $\boldsymbol{p}_t\boldsymbol{z}_t$ at time t.

$\boldsymbol{d}_t$ and $\boldsymbol{p}_t$ are revealed at the beginning of hour t. u^i_t is decided once they are revealed.

The company aims at minimizing the expected cost.

1. (3 points) Write the problem has a multistage stochastic program and give the information structure. Justify the choice of expectation in the objective.

> **Solution:** It's a repeated problem that happens everyday, law of large number justify expectation. (0.5)
> The problem is in hazard decision. (0.5)
>
> $$\begin{aligned} \min \quad & \sum_{i\in\mathcal{I}} c^i x^i + \mathbb{E}\Big[\sum_{t=1}^{24}\Big(\sum_{i\in\mathcal{I}} e^i \boldsymbol{u}^i_t\Big) - \boldsymbol{p}_t\boldsymbol{z}_t\Big] & \\ s.t. \quad & \boldsymbol{z}_t + \boldsymbol{\varepsilon}_t = \sum_{i\in\mathcal{I}} \boldsymbol{u}^i_t & \mathbb{P}-a.s., \forall t \\ & 0.8\boldsymbol{d}_t \leq \boldsymbol{z}_t \leq 1.2\boldsymbol{d}_t & \mathbb{P}-a.s., \forall t \\ & x^i \underline{u}^i \leq \boldsymbol{u}^i_t \leq x^i \overline{u}^i & \mathbb{P}-a.s., \forall i \in \mathcal{I} \\ & \boldsymbol{\varepsilon}_t \geq 0 & \mathbb{P}-a.s., \forall t \in [\![1,24]\!] \\ & x^i \in \{0,1\} & \forall i \in \mathcal{I} \\ & \sigma(u^i_t) \subset \sigma((\boldsymbol{d}_\tau, \boldsymbol{p}_\tau)_{\tau\in[\![1,t]\!]}) & \forall i \in \mathcal{I}, \forall t \in [\![1,24]\!] \end{aligned}$$
>
> (2 points in total : -0.5 by error)

2. (2 points) Justify that this problem can be reduced to a two stage program. Specify the decomposition in first stage and second stage program.

> **Solution:** There is nothing coupling time step t and t', thus all the information needed to take decisions $(u^i_t)_{i\in\mathcal{I}}$ is the information revealed at time t. Consequently all u^i_t can be seen as second stage variable, while x is the first stage decision.
>
> The first stage problem now reads (1)
>
> $$\begin{aligned} \min \quad & \sum_{i\in\mathcal{I}} c^i x^i + \mathbb{E}\big[Q(x, \boldsymbol{\xi})\big] & \\ s.t. \quad & x^i \in \{0,1\} & \forall i \in \mathcal{I} \end{aligned}$$
>
> where $\boldsymbol{\xi} = (\boldsymbol{d}_t, \boldsymbol{p}_t)_{t\in[\![1,24]\!]}$. The second stage problem reads (1)

$$
\begin{aligned}
Q(x,\xi) = \min \quad & \sum_{t=1}^{24} \Big(\sum_{i\in\mathcal{I}} e^i u_t^i \Big) - p_t z_t & \\
s.t. \quad & z_t + \varepsilon_t = \sum_{i\in\mathcal{I}} u_t^i & \forall t \\
& 0.8 d_t \leq z_t \leq 1.2 d_t & \forall t \\
& x^i \underline{u}^i \leq u_t^i \leq x^i \overline{u}^i & \forall i \in \mathcal{I} \\
& \varepsilon_t \geq 0 & \forall t \in [\![1,24]\!]
\end{aligned}
$$

3. (1 point) Give a simple necessary and sufficient condition under which this problem has finite value. Give a necessary and sufficient condition for the decomposition to present relatively complete recourse.

Solution: For the problem to have finite value we need that the minimum demand can always be covered, that is (0.5)

$$
\sum_{i\in\mathcal{I}} \overline{u}^i \geq 0.8 \boldsymbol{d}_t \qquad \mathbb{P}-a.s., \forall t \in [\![1,24]\!].
$$

We are not in relatively complete recourse. To have RCR it is enough to add (0.5)

$$
\sum_{i\in\mathcal{I}} x^i \overline{u}^i \geq 0.8 \boldsymbol{d}_t \qquad \mathbb{P}-a.s., \forall t \in [\![1,24]\!].,
$$

to the first-stage problem.

4. (2 points) Assuming that you have a sample of $S = 1000$ scenarios of $(\boldsymbol{d}_t, \boldsymbol{p}_t)_{t\in[\![1,24]\!]}$. Write the extensive formulation SAA approximation of the above two-stage program as a MILP. Precise the number and type of first stage and second stage variables.

Solution: The problem reads (1)

$$
\begin{aligned}
\min \quad & \sum_{i\in\mathcal{I}} c^i x^i + \frac{1}{S} \sum_{t=1}^{24} \Big(\sum_{i\in\mathcal{I}} e^i u_t^i(s) \Big) - p_t(s) z_t(s) & \\
s.t. \quad & z_t(s) + \varepsilon_t(s) = \sum_{i\in\mathcal{I}} \boldsymbol{u}_t^i(s) & \forall s, \forall t \\
& 0.8 d_t(s) \leq z_t(s) \leq 1.2 d_t(s) & \forall s, \forall t \\
& x^i \underline{u}^i \leq u_t^i(s) \leq x^i \overline{u}^i & \forall i \in \mathcal{I}, \forall s \\
& \varepsilon_t \geq 0 & \forall t \in [\![1,24]\!] \\
& x^i \in \{0,1\} & \forall i \in \mathcal{I}
\end{aligned}
$$

There are $|I|$ first stage binary decision (0.5), and $S \times 24 \times (|\mathcal{I}| + 1)$ continuous recourse decision (0.5).

5. (3 points) Is the SAA problem better addressed by Progressive Hedging or L-Shaped method ? Justify your answer. Write the master and slave problems.

Solution: There are integer first stage decision, thus L-Shaped is adapted to the problem. (1)

Let $\underline{d} = \max_{s,t} d_t(s)$, then the master problem at iteration k reads (1)

$$
\begin{aligned}
\min_{x,\theta} \quad & \sum_{i\in\mathcal{I}} c^i x^i + \frac{1}{S}\sum_{s=1}^{1000}\theta(s) & \\
s.t. \quad & \sum_{i\in\mathcal{I}} x^i \overline{u}^i \geq 0.8\underline{d} & \\
& \theta(s) \geq (\alpha_\kappa(s))^T x + \beta_\kappa(s) & \forall \kappa \leq k
\end{aligned}
$$

while the slave problems reads (1)

$$
\begin{aligned}
\min \quad & \sum_{t=1}^{24}\Big(\sum_{i\in\mathcal{I}} e^i u_t^i\Big) - p_t(s) z_t & \\
s.t. \quad & z_t + \varepsilon_t = \sum_{i\in\mathcal{I}} u_t^i & \forall t \\
& 0.8 d_t(s) \leq z_t \leq 1.2 d_t(s) & \forall t \\
& x_{k+1}^i \underline{u}^i \leq u_t^i \leq x_{k+1}^i \overline{u}^i & \forall i \in \mathcal{I} \\
& \varepsilon_t \geq 0 & \forall t \in [\![1, 24]\!]
\end{aligned}
$$