Exact and converging bounds for SDDP

V. Leclère, P. Carpentier, J-P. Chancelier, A. Lenoir, F. Pacaud

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Introduction

We are interested in multistage stochastic optimization problems of the form

$$\begin{aligned} \min_{\pi} \quad & \mathbb{E}\left(\sum_{t=0}^{T-1} L_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{\xi}_t) + K(\boldsymbol{X}_T) \right) \\ s.t. \quad & \boldsymbol{X}_{t+1} = f_t(\boldsymbol{X}_t, \boldsymbol{U}_t, \boldsymbol{\xi}_t) \\ & \boldsymbol{U}_t = \pi_t(\boldsymbol{X}_t, \boldsymbol{\xi}_t) \end{aligned}$$

where

- $\boldsymbol{x}_t$ is the state of the system,
- $\boldsymbol{u}_t$ is the control applied at time t,
- $\boldsymbol{\xi}_t$ is the noise happening between time t and $t+1$, assumed to be time-independent,
- π is the policy.

Stochastic Dynamic Programming

By the white noise assumption, this problem can be solved by Dynamic Programming, where the Bellman functions satisfy

$$\begin{cases} V_T(x) & = & K(x) \\ \hat{V}_t(x,\xi) & = & \min_{u_t \in \mathbb{U}} L_t(x,u_t,\xi) + V_{t+1} \circ f_t(x,u_t,\xi) \\ V_t(x) & = & \mathbb{E}\left(\hat{V}_t(x,\boldsymbol{\xi}_t)\right) \end{cases}$$

Indeed, π is an optimal policy if

$$\pi_t(x,\xi) \in \arg\min_{u_t \in \mathbb{U}} \left\{ L_t(x,u_t,\xi) + V_{t+1} \circ f_t(x,u_t,\xi) \right\}$$

Bellman operator

For any time t, and any function $R : \mathbb{X} \to \mathbb{R} \cup \{+\infty\}$ we define

$$\hat{\mathcal{T}}_t(R)(x, \xi) := \min_{u_t \in \mathbb{U}} L_t(x, u_t, \xi) + R \circ f_t(x, u_t, \xi)$$

and

$$\mathcal{T}_t(R)(x) := \mathbb{E}\Big[\hat{\mathcal{T}}_t(R)(x, \boldsymbol{\xi})\Big].$$

Incidentally, R induce a policy $\pi_t^R(x, \xi)$ given by a minimizer of the above problem.
Thus the Bellman equation simply reads

$$\begin{cases} V_T = K \\ V_t = \mathcal{T}_t(V_{t+1}) \end{cases}$$

SDDP algorithm

Under linear dynamics, and convex costs, the SDDP algorithm iteratively constructs polyhedral outer approximations of V_t.

More precisely, at iteration k

- We have polyhedral functions $\underline{V}_t^k(\cdot) = \max_{\kappa \le k} \langle \lambda_t^\kappa, \cdot \rangle + \beta_t^\kappa$, such that $\underline{V}_t^k \le V_t$.
- **Forward pass**: We simulate the dynamical system, along one scenario, according to policy $\pi^{\underline{V}^k}$, yielding a trajectory $\{\underline{x}_t^k\}_{t \in [\![0,T]\!]}$.
- **Backward pass**: We compute cuts $x \mapsto \langle \lambda_t^{k+1}, \cdot \rangle + \beta_t^{k+1} \le V_t$ along this trajectory, and update our outer approximations.

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SDDP strengths

- SDDP is a widely used algorithm in the energy community, with multiple applications in
  - mid and long term water storage management problem,
  - long-term investment problems,
  - ...
- Recent works have presented extensions of the algorithm to
  - deal with some non-convexity,
  - treat risk-averse or distributionally robust problems,
  - incorporate integer variables.
- Multiple numerical improvements have been proposed
  - cut selection
  - regularization
  - multi-cut or ε-resolution

SDDP weaknesses

There are still some gaps in our knowledge of this approach:

- there is no convergence speed guaranteed,
- regularization methods are not mature yet, (see next talk for recent developpments)
- there is no good stopping test.

SDDP Stopping test

- Exact lower bound of the problem : $\underline{V}_0^k(x_0)$.
- Upper-bound estimated by Monte-Carlo simulation yielding costly statistical stopping tests (Pereira Pinto (1991) or Shapiro (2011))
- Alternative statistical tests have been proposed (see Homem de Mello et al (2011))
- Exact upper-bound computation has been proposed by Philpott et al (2013) but without any proof of convergence, leading to possibly not converging stopping tests.

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Linear Bellman Operator

An operator $\mathcal{B} : F(\mathbb{R}^{n_x}) \to F(\mathbb{R}^{n_x})$ is said to be a *linear Bellman operator* (LBO) if it is defined as follows

$$\mathcal{B}(R) : x \mapsto \inf_{(\boldsymbol{u},\boldsymbol{y})} \mathbb{E}\Big[\boldsymbol{c}^\top \boldsymbol{u} + R(\boldsymbol{y})\Big]$$
$$s.t. \quad Tx + \mathcal{W}_u(\boldsymbol{u}) + \mathcal{W}_y(\boldsymbol{y}) \leq \boldsymbol{h}$$

where $\mathcal{W}_u : \mathcal{L}^0(\mathbb{R}^{n_u}) \to \mathcal{L}^0(\mathbb{R}^{n_c})$ and $\mathcal{W}_y : \mathcal{L}^0(\mathbb{R}^{n_x}) \to \mathcal{L}^0(\mathbb{R}^{n_c})$ are two linear operators. We denote $S(R)(x)$ the set of $\boldsymbol{y}$ that are part of optimal solutions to the above problem.
We also define $\mathcal{G}(x)$

$$\mathcal{G}(x) := \big\{(\boldsymbol{u},\boldsymbol{y}) \mid Tx + \mathcal{W}_u(\boldsymbol{u}) + \mathcal{W}_y(\boldsymbol{y}) \leq \boldsymbol{h}\big\} .$$

Examples

- Linear point-wise operator:

$$\mathcal{W} \quad : \quad \begin{array}{ccc} \mathcal{L}^0(\mathbb{R}^{n_x}) & \to & \mathcal{L}^0(\mathbb{R}^{n_c}) \\ (\omega \mapsto \boldsymbol{y}(\omega)) & \mapsto & (\omega \mapsto A\boldsymbol{y}(\omega)) \end{array}$$

  Such an operator allows to encode almost sure constraints.

- Linear expected operator:

$$\mathcal{W} \quad : \quad \begin{array}{ccc} \mathcal{L}^0(\mathbb{R}^{n_x}) & \to & \mathcal{L}^0(\mathbb{R}^{n_c}) \\ (\omega \mapsto \boldsymbol{y}(\omega)) & \mapsto & (\omega \mapsto A\,\mathbb{E}(\boldsymbol{y})) \end{array}$$

  Such an operator allows to encode constraints in expectation.

Relatively Complete Recourse and cuts

Definition (Relatively Complete Recourse)

We say that the pair $(\mathcal{B}, R)$ satisfy a *relatively complete recourse* (RCR) assumption if for all $x \in \mathrm{dom}(\mathcal{G})$ there exists admissible controls $(\boldsymbol{u}, \boldsymbol{y}) \in \mathcal{G}(x)$ such that $\boldsymbol{y} \in \mathrm{dom}(R)$.

Cut

If R is proper and polyhedral, with RCR assumption, then $\mathcal{B}(R)$ is a proper polyhedral function.
Furthermore, computing $\mathcal{B}(R)(x)$ consists of solving a linear problem which also generates a supporting hyperplane of $\mathcal{B}(R)$, that is, a pair $(\lambda, \beta) \in \mathbb{R}^{n_x} \times \mathbb{R}$ such that

$$\begin{cases} \langle \lambda \,, \cdot \rangle + \beta \leq \mathcal{B}(R)(\cdot) \\ \langle \lambda \,, x \rangle + \beta = \mathcal{B}(R)(x) \,. \end{cases}$$

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Setting

Consider a *compatible* sequence of LBO $\{\mathcal{B}_t\}_{t \in [\![0, T-1]\!]}$, that is, such that all admissible controls of $\mathcal{B}_t$ lead to admissible states of $\mathcal{B}_{t+1}$.

Consider a sequence of functions such that

$$\begin{cases} R_T = K \\ R_t = \mathcal{B}_t(R_{t+1}) \quad \forall t \in [\![0, T-1]\!] \end{cases}$$

Then, the abstract SDDP algorithm generates a sequence of lower polyhedral approximations of R_t. In a forward pass it simulates a trajectory of states, along which the approximation is refined in the backward pass.

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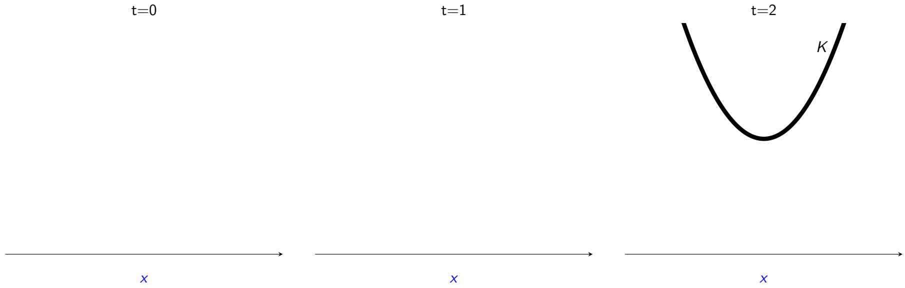
Abstract SDDP



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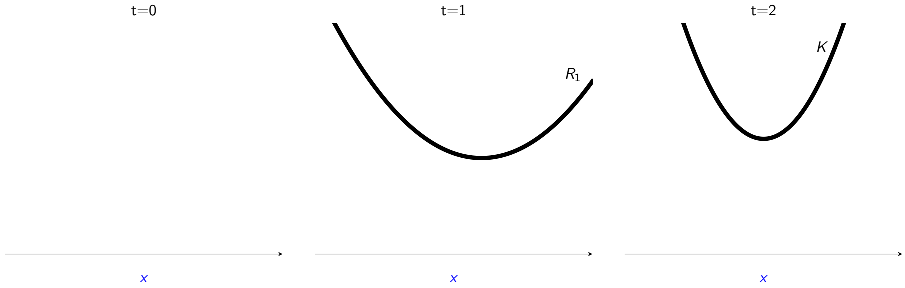

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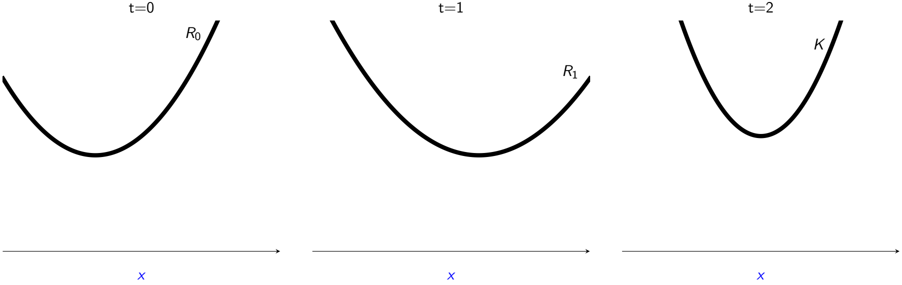

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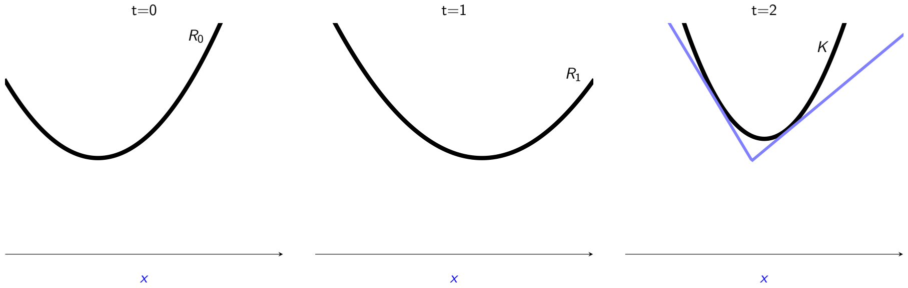

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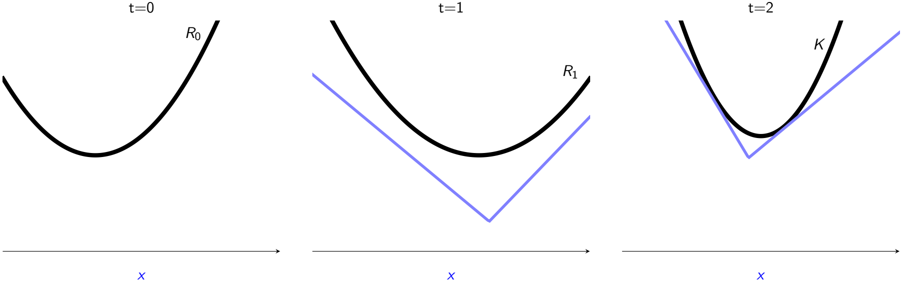

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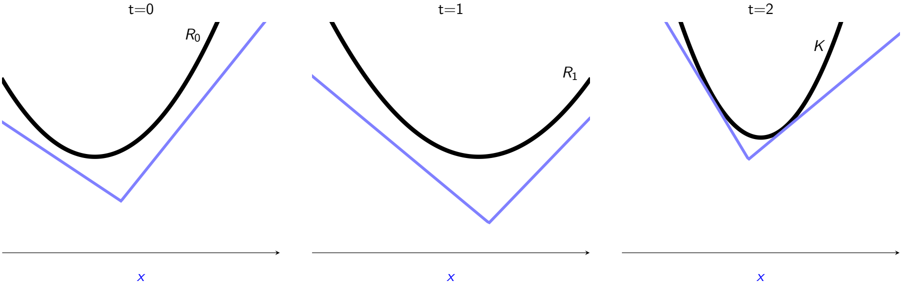

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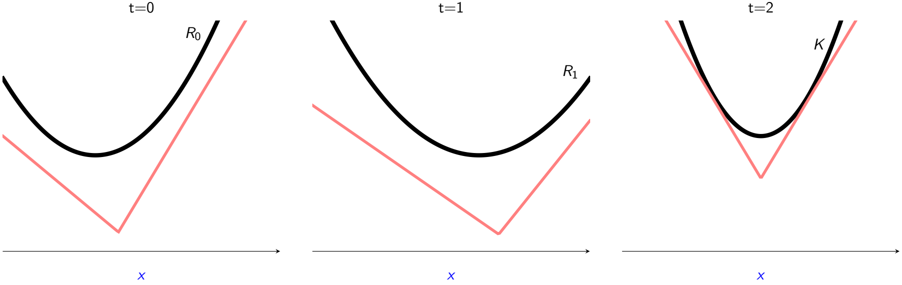

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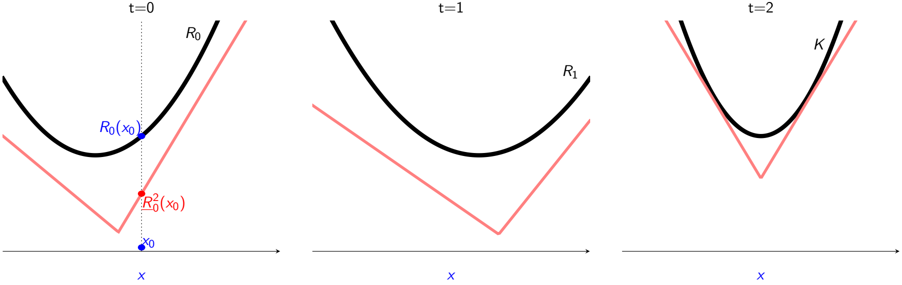

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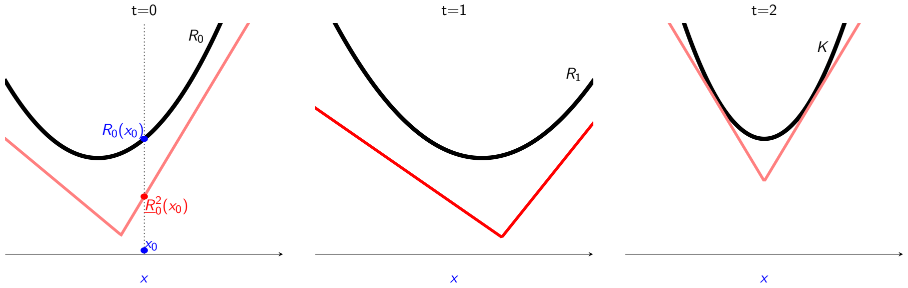

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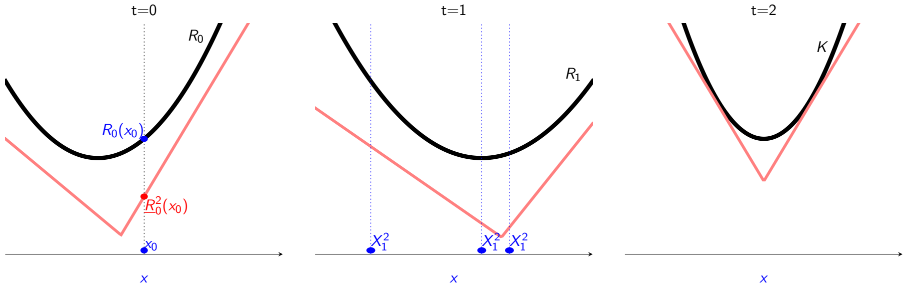

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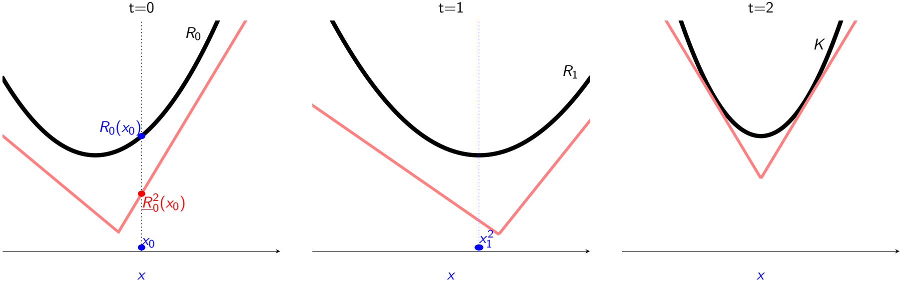

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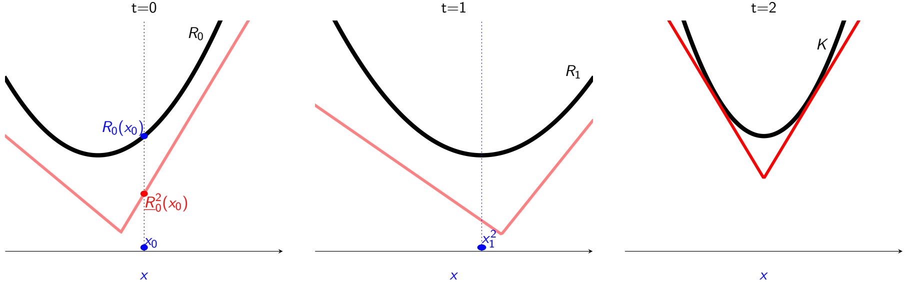

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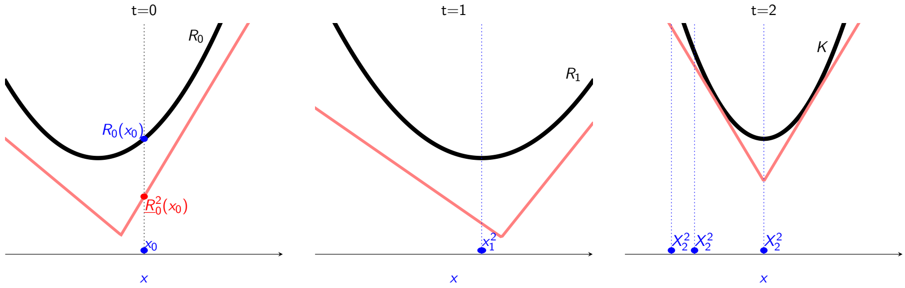

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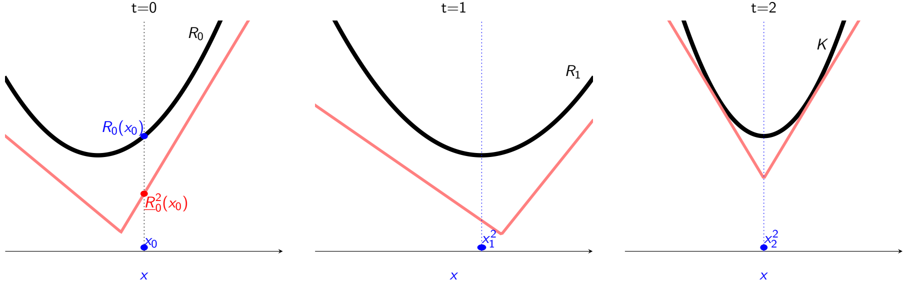

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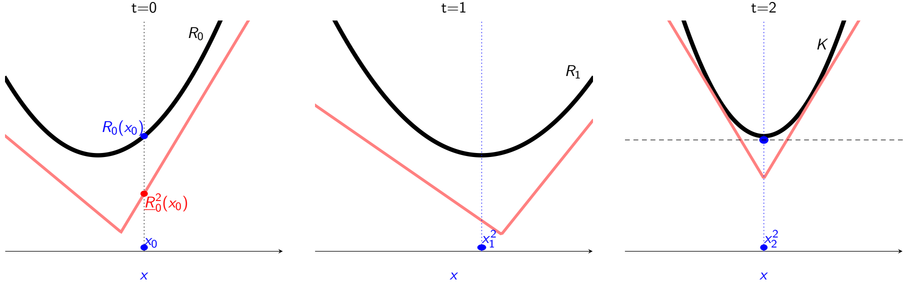

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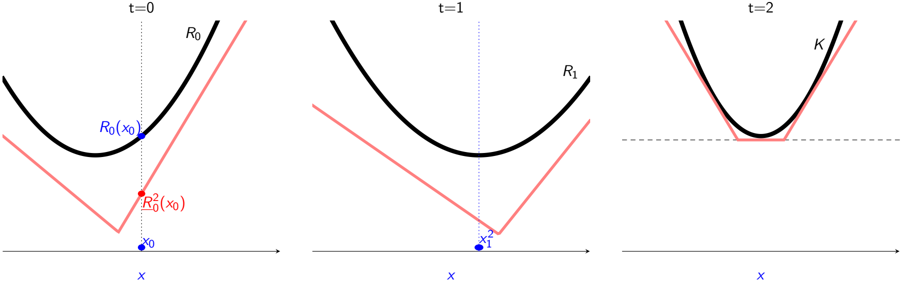

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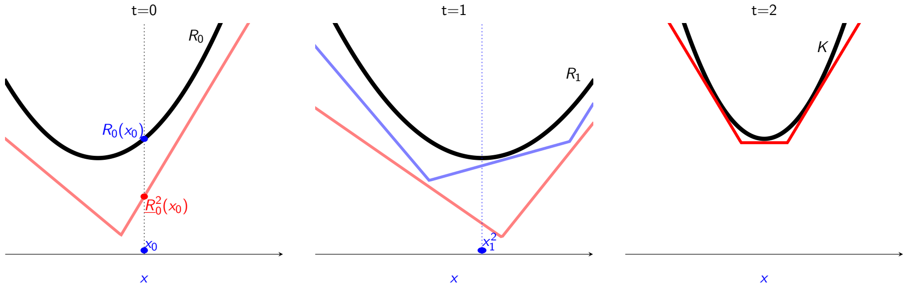

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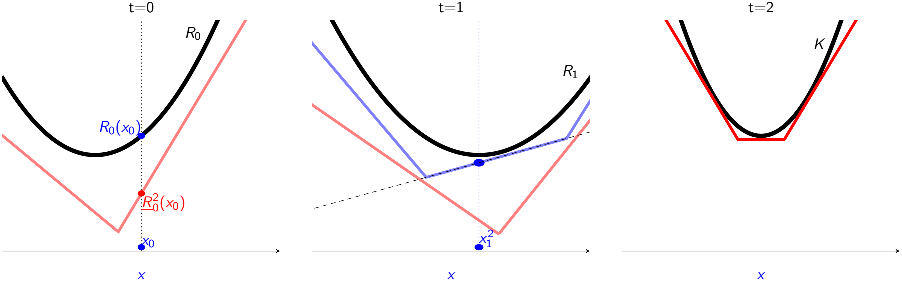

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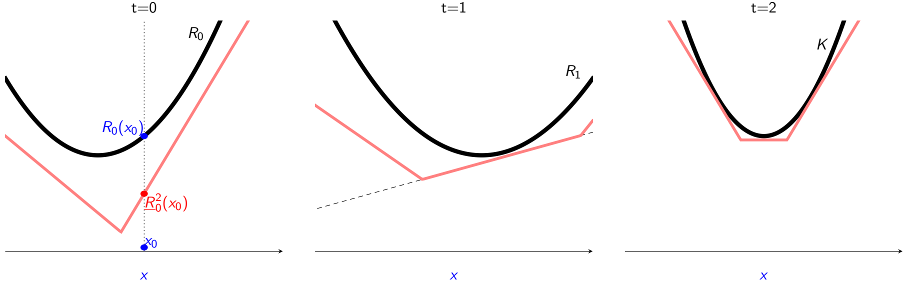

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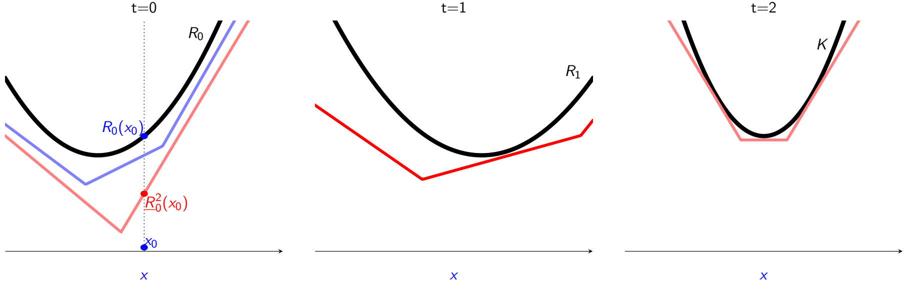

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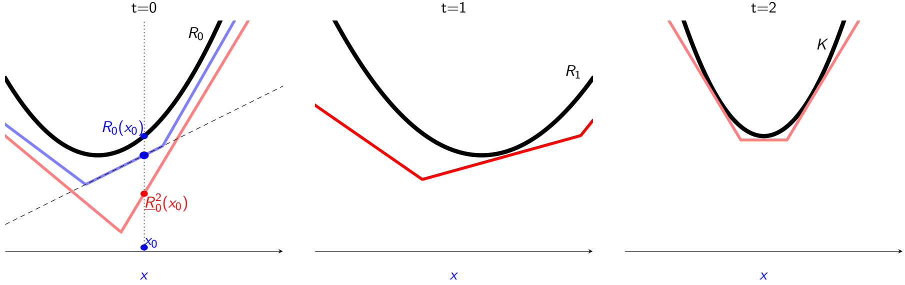

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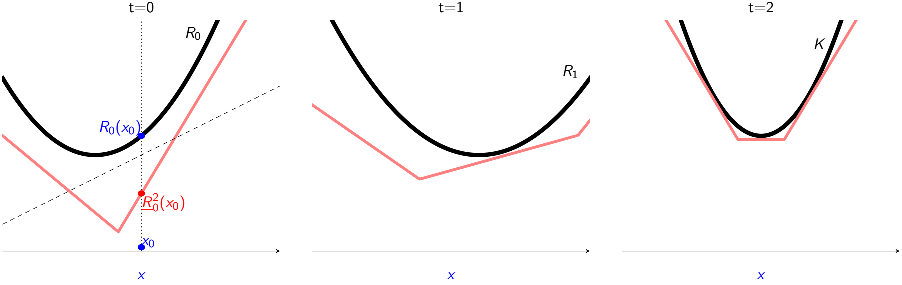

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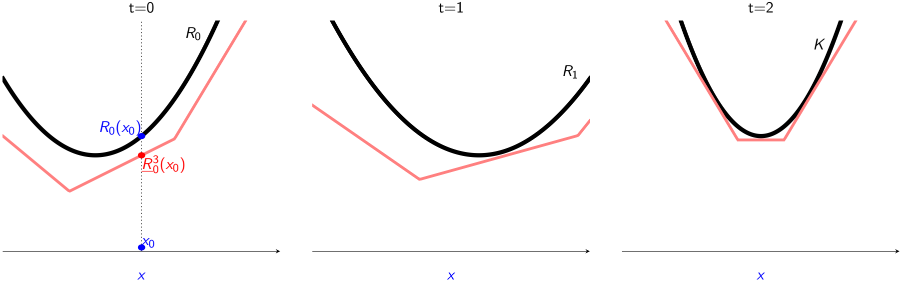

Data: Initial point x_0

Set $\underline{R}_t^{(0)} \equiv -\infty$

for $k \in \mathbb{N}$ **do**

```
// Forward Pass : compute a set of trial points
```
$\{x_t^k\}_{t \in [\![0,T]\!]}$

Draw a noise scenario $\omega^k \in \Omega$;

Set $x_0^k = x_0$;

for $t : 0 \to T$ **do**

select $\boldsymbol{x}_{t+1}^k \in S_t(\underline{R}_{t+1}^k)(x_t^k)$;

set $x_{t+1}^k = \boldsymbol{x}_{t+1}^k(\omega^k)$;

end

```
// Backard Pass : refine the lower-approx at trial points
```

Set $\underline{R}_T^{k+1} = K$;

for $t : T-1 \to 0$ **do**

$\beta_t^{k+1} = \mathcal{B}_t(\underline{R}_{t+1}^{k+1})(x_t^k)$; `// computing cut coefficients`

$\lambda_t^{k+1} \in \partial \mathcal{B}_t(\underline{R}_{t+1}^{k+1})(x_t^k)$;

$\beta_t^{k+1} := \theta_t^{k+1} - \langle \lambda_t^{k+1}, \overline{x}_t^k \rangle$;

set $\mathcal{C}_t^{k+1} : x \mapsto \langle \lambda_t^{k+1}, x \rangle + \beta_t^{k+1}$; `// new cut`

$\underline{R}_t^{k+1} := \max\{\underline{R}_t^k, \mathcal{C}_t^{k+1}\}$; `// update lower approximation`

end

end

Absract SDDP convergence

Theorem

Assume that Ω is finite, $R(x_0)$ is finite, and $\{\mathcal{B}_t\}_t$ is compatible. Further assume that, for all $t \in [\![0, T]\!]$ there exists compact sets X_t such that, for all k, $x_t^k \in X_t$ (e.g. $\mathcal{B}_t$ have compact domain).

Then, $(\underline{R}_t^k)_{k\in\mathbb{N}}$ is a non-decreasing sequence of lower approximations of R_t, and $\lim_k \underline{R}_0^k(x_0) = R_0(x_0)$, for $t \in [\![0, T-1]\!]$.

Further, the cuts coefficients generated remain in a compact set.

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Fenchel transform of LBO

Theorem

Assume that the pair $(\mathcal{B}, R)$ satisfy the RCR assumption, R being proper polyhedral, and $\mathcal{B}$ compact (i.e. $\mathcal{G}$ is compact valued with compact domain).

Then $\mathcal{B}(R)$ is a proper function and we have that

$$[\mathcal{B}(R)]^\star = \mathcal{B}^\ddagger(R^\star)$$

where $\mathcal{B}^\ddagger$ is an explicitely given LBO.

Dual LBO

More precisely we have

$$\mathcal{B}^{\ddagger}(Q) : \lambda \mapsto \inf_{\mu \in \mathcal{L}^0(\mathbb{R}^{n_x}), \nu \in \mathcal{L}^0(\mathbb{R}^{n_c})} \quad \mathbb{E}\Big[-\mu^\top h + Q(\nu)\Big]$$

$$\begin{aligned} s.t. \quad & T^\top \mathbb{E}\big[\mu\big] + \lambda = 0 \\ & \mathcal{W}_u^\dagger(\mu) = C \\ & \mathcal{W}_y^\dagger(\mu) = \nu \\ & \mu \leq 0 \,, \end{aligned}$$

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Recursion over dual value function

Denote $\mathcal{D}_t := V_t^\star$.

Theorem

Then

$$\begin{cases} \mathcal{D}_T &= K^\star \,, \\ \mathcal{D}_t &= \mathcal{B}^\ddagger_{t,L_{t+1}}(\mathcal{D}_{t+1}) \qquad \forall t \in [\![0, T-1]\!] \end{cases}$$

where $\mathcal{B}^\ddagger_{t,L_{t+1}} := \mathcal{B}^\ddagger_t + \mathbb{I}_{\|\lambda_{t+1}\|_\infty \leq L_{t+1}}$.

This is a Bellman recursion on $\mathcal{D}_t$ instead of V_t.
Further, under easy technical assumptions, $\left\{\mathcal{B}^\ddagger_{t,L_{t+1}}\right\}_{t\in[\![0,T]\!]}$ is a compatible sequence of LBOs, where V_t is L_t-Lipschitz.

Data: Initial primal point x_0, Lipschitz bounds $\{L_t\}_{t\in[\![0,T]\!]}$

for $k \in \mathbb{N}$ **do**

`// Forward Pass : compute a set of trial points` $\{\lambda_t^{(k)}\}_{t\in[\![0,T]\!]}$

Compute $\lambda_0^k \in \arg\max_{\|\lambda_0\|_\infty \le L_0} \left\{ x_0^\top \lambda_0 - \underline{\mathcal{D}}_0^k(\lambda_0) \right\}$;

for $t : 0 \to T$ **do**

select $\boldsymbol{\lambda}_{t+1}^k \in \arg\min \mathcal{B}_t^\ddagger(\underline{\mathcal{D}}_{t+1}^k)(\lambda_t^k)$;

and draw a realization λ_{t+1}^k of $\boldsymbol{\lambda}_{t+1}^k$;

end

`// Backard Pass : refine the lower-approx at trial points`

Set $\underline{\mathcal{D}}_T^k = K^\star$. ;

for $t : T-1 \to 0$ **do**

$\overline{\theta}_t^{k+1} := \mathcal{B}_{t,L_{t+1}}^\ddagger(\underline{\mathcal{D}}_{t+1}^{k+1})(\lambda_t^k)$; `// computing cut coefficients`

$\overline{x}_t^{k+1} \in \partial\mathcal{B}_{t,L_{t+1}}^\ddagger(\underline{\mathcal{D}}_{t+1}^{k+1})(\lambda_t^k)$;

$\overline{\beta}_t^{k+1} := \overline{\theta}_t^{k+1} - \langle \lambda_t^k, \overline{x}_t^{k+1} \rangle$;

$\mathcal{C}_t^{k+1} : \lambda \mapsto \langle \overline{x}_t^{k+1}, \lambda \rangle + \overline{\beta}_t^{k+1}$;

$\underline{\mathcal{D}}_t^{k+1} = \max\left(\underline{\mathcal{D}}_t^k, \mathcal{C}_t^{k+1}\right)$; `// update lower approximation`

end

If some stopping test is satisfied STOP ;

end

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Converging upper bound and stopping test

We have

$$\underline{V}_t^k \leq V_t$$

and

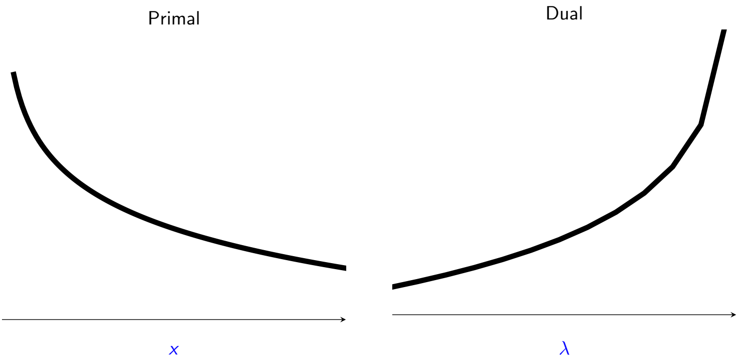
$$\underline{\mathcal{D}}_t^k \leq \mathcal{D}_t \quad \Longrightarrow \quad \underbrace{\left(\underline{\mathcal{D}}_t^k\right)^\star}_{:\approx \overline{V}_t^k} \geq \left(\mathcal{D}_t^\star\right) = V_t^{\star\star} = V_t$$

Finally, we obtain

$$\underline{V}_0(x_0) \leq V_0(x_0) \leq \overline{V}_0(x_0).$$

Using the convergence of the abstract SDDP algorithm we show that this bounds are converging, yielding converging deterministic stopping tests.

Link between primal and dual approximations



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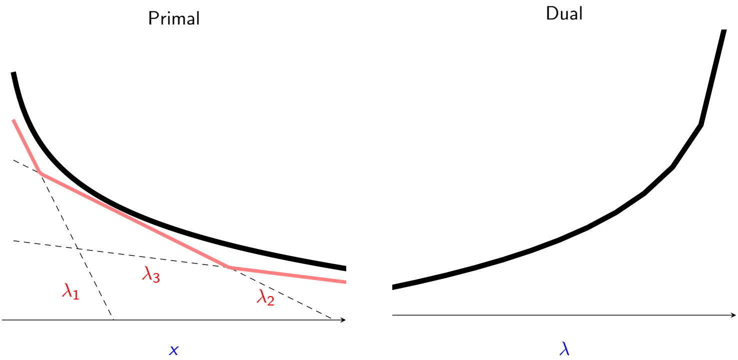

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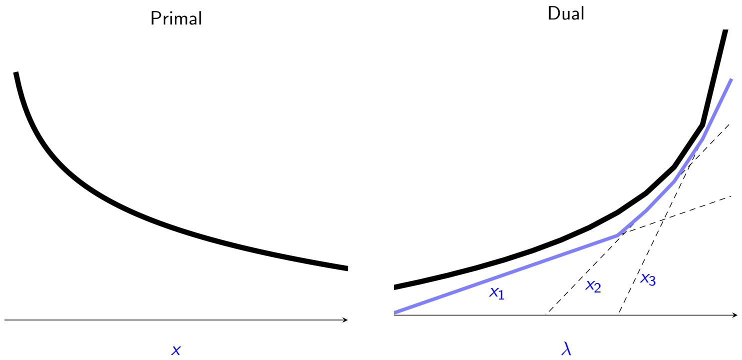

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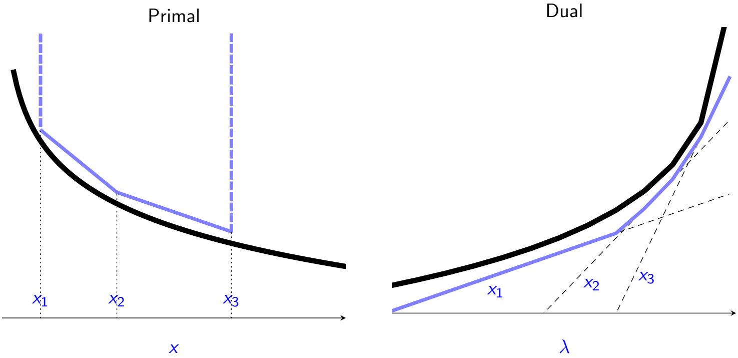

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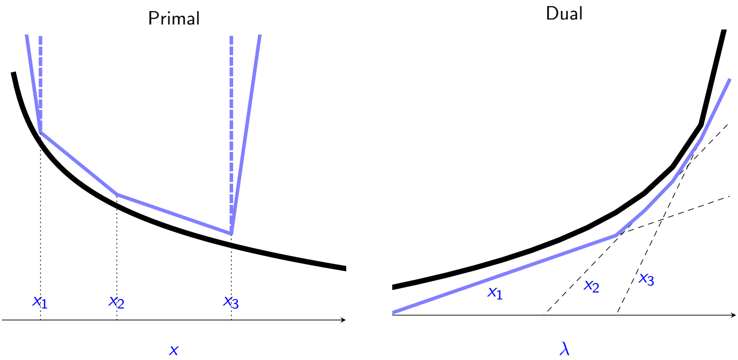

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Inner Approximation

- $\overline{V}_t^k := \left[\underline{\mathcal{D}}_t^k\right]^\star$ which is lower than V_t on X_t
- Or

$$\overline{V}_t^k(x) = \min_{\sigma\in\Delta}\Big\{ -\sum_{\kappa=1}^{k}\sigma_\kappa\overline{\beta}_t^\kappa \quad\Big|\quad \sum_{\kappa=1}^{k}\sigma_\kappa\overline{x}_t^\kappa = x\Big\}$$

- The inner approximation can be computed by solving

$$\begin{aligned}\overline{V}_t^{k+1}(x) = \sup_{\lambda,\theta} \quad & x^\top\lambda - \theta \\ s.t. \quad & \theta \geq \left\langle \underline{x}_t^i \,, \lambda\right\rangle + \overline{\beta}_t^\kappa \qquad \forall\kappa\in[\![1,k]\!] \,.\end{aligned}$$

Inner Approximation - regularized

- $\overline{V}_t^k := \left[\underline{\mathcal{D}}_t^k\right]^\star \square (L_t \| \cdot \|_1)$ which is lower than V_t on X_t
- Or

$$\overline{V}_t^k(x) = \min_{y \in \mathbb{R}^{n_x}, \sigma \in \Delta} \left\{ L_t \|x - y\|_1 - \sum_{\kappa=1}^{k} \sigma_\kappa \overline{\beta}_t^\kappa \quad \middle| \quad \sum_{\kappa=1}^{k} \sigma_\kappa \overline{x}_t^\kappa = y \right\}$$

- The inner approximation can be computed by solving

$$\begin{aligned} \overline{V}_t^{k+1}(x) = \sup_{\lambda, \theta} \quad & x^\top \lambda - \theta \\ s.t. \quad & \theta \geq \langle \underline{x}_t^i , \lambda \rangle + \overline{\beta}_t^\kappa \qquad \forall \kappa \in [\![1, k]\!] \, . \\ & \|\lambda\|_\infty \leq L_t \end{aligned}$$

A converging strategy - with guaranteed payoff

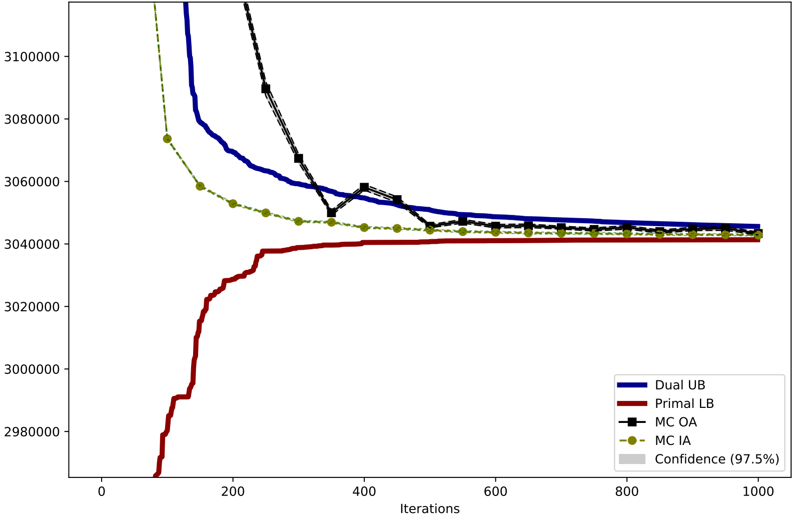
Theorem

Let $C_t^{IA,k}(x)$ be the expected cost of the strategy $\pi^{\overline{V}_t^k}$ when starting from state x at time t.
We have,

$$C_t^{IA,k}(x) \leq \overline{V}_t^k(x) , \quad \lim_k C_t^{IA,k}(x) = V_t(x)$$

Thus, the inner-approximation yields a new converging strategy, and we have an upper-bound on the (expected) value of this strategy.

Numerical results



Stopping test

	Dual stopping test		Statistical stopping test	
ε (%)	n it.	CPU time	n it.	CPU time
2.0	156	183s	250	618s
1.0	236	400s	300	787s
0.5	388	1116s	450	1429s
0.1	$>$ 1000	.	1000	5519s

Table: Comparing dual and statistical stopping criteria for different accuracy levels ε.

Conclusion

- We extend the SDDP algorithm to an abstract framework.
- Leveraging Fenchel conjugate we are able to show a dynamic recursion between dual Bellman value functions.
- We can apply SDDP to this dual recursion.
- This yields a converging exact upper bound on the value of the original problem, hence giving exact and converging stopping tests.
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More information :
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