On the well-posedness of Stochastic Lagrangian Models

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The equation of 9.00 am

For any arbitrary finite $T > 0$, we are interested in $((X_t, U_t);\ 0 \leq t \leq T)$, solving

$$\begin{cases} X_t = X_0 + \displaystyle\int_0^t U_s\, ds, \\ U_t = U_0 + \displaystyle\int_0^t B[X_s;\rho_s]ds + \sigma W_t - \sum_{0<s\leq t} 2\left(U_{s^-} \cdot n_{\mathcal{D}}(X_s)\right) n_{\mathcal{D}}(X_s)\mathbf{1}_{\{X_s \in \partial\mathcal{D}\}}, \\ \rho(t) \text{ is the probability density of } (X_t, U_t) \text{ for all } t \in (0, T], \end{cases} \tag{1}$$

$(W_t, t \geq 0)$ is a standard $\mathbb{R}^d$-Brownian motion ;
The diffusion $\sigma > 0$ is a constant ;
$\mathcal{D}$ is an open bounded domain of $\mathbb{R}^d$.

This study is a joint work with Jean Francois Jabir (University of Valparaiso) ;
Preprint on arXiv.

The nonlinear coefficient

The drift coefficient $B : \mathcal{D} \times L^1(\mathcal{D} \times \mathbb{R}^d) \to \mathbb{R}^d$

$$B[x;\psi] = \begin{cases} \dfrac{\displaystyle\int_{\mathbb{R}^d} b(v)\psi(t,x,v)dv}{\displaystyle\int_{\mathbb{R}^d} \psi(t,x,v)dv}, \text{ whenever } \displaystyle\int_{\mathbb{R}^d} \psi(t,x,v)dv \neq 0, \\ 0 \text{ otherwise,} \end{cases} \tag{2}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is a given measurable function.

Formally the function $(t,x) \mapsto B[x;\rho(t)]$ in (1) corresponds to the conditional expectation

$$(t,x) \mapsto \mathbb{E}[b(U_t)/X_t = x]$$

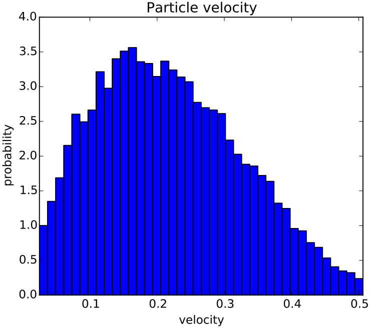
and the velocity equation in (1) rewrites

$$U_t = U_0 + \int_0^t \mathbb{E}[b(U_s)/X_s]ds + \sigma W_t + K_t.$$

Motivation

The law of $(X_t, U_t) = (x_t, y_t, z_t, u_t, v_t, w_t)$ is a local probability distribution of physical quantities (within the meaning of the statistical approach to turbulence)

$$\rho(t,x,y,z,v,u,w)dxdydzdudvdw$$


Particle velocity

Computation of local moment fields :

$$\langle u\rangle(t,x,y,z) = \frac{\int_{\mathcal{D}\times\mathbb{R}^3} u\ \rho(t,x,y,z,v,u,w)dudvdw}{\int_{\mathcal{D}\times\mathbb{R}^3} \rho(t,x,y,z,v,u,w)dudvdw} = \mathbb{E}\left[u_t/X_t = (x,y,z)\right]$$

$$\langle uv\rangle(t,x,y,z) = \frac{\int_{\mathcal{D}\times\mathbb{R}^3} uv\ \rho(t,x,y,z,v,u,w)dudvdw}{\int_{\mathcal{D}\times\mathbb{R}^3} \rho(t,x,y,z,v,u,w)dudvdw} = \mathbb{E}\left[u_t\,v_t/X_t = (x,y,z)\right]$$

Lagrangian modeling for turbulent flow

On a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider the fluid particle state vector $(\mathrm{X}_t, \mathrm{U}_t, \psi_t)$ satisfying

$$\begin{aligned}
d\mathrm{X}_t =& \mathrm{U}_t dt, \\
d\mathrm{U}_t =& \left[-\frac{1}{\varrho} \nabla_x \langle \mathscr{P} \rangle (t, \mathrm{X}_t) + \nu \triangle_x \langle \mathscr{U} \rangle (t, \mathrm{X}_t) \right] dt \\
& - G(t, X_t) \left(\mathrm{U}_t - \langle \mathscr{U} \rangle (t, \mathrm{X}_t) \right) dt + \sqrt{C(t, \mathrm{X}_t) \varepsilon(t, \mathrm{X}_t)} dW_t, \\
d\psi_t =& D_1(t, \mathrm{X}_t, \psi_t) dt + D_2(t, \mathrm{X}_t, \psi_t) d\widetilde{W}_t.
\end{aligned}$$

$\left(W, \widetilde{W}\right)$ is a $4D$-Brownian motion.

- $\langle \mathscr{U}^{(i)} \rangle (t, x)$, $\langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle (t, x)$ are computed as conditional expectations.
- ε, $C,\ G,\ D_1,\ D_2$ are determined by the RANS closure.

Statistical approach of turbulent flows

The Reynolds averages (or ensemble averages) are expectations :

$$\langle \mathscr{U} \rangle(t,x) := \int_\Omega \mathscr{U}(t,x,\omega) d\mathbb{P}(\omega).$$

Reynolds decomposition

$$\mathscr{U}(t,x,\omega) = \langle \mathscr{U} \rangle(t,x) + \mathbf{u}(t,x,\omega),$$
$$\mathscr{P}(t,x,\omega) = \langle \mathscr{P} \rangle(t,x) + \mathbf{p}(t,x,\omega)$$

The random field $\mathbf{u}(t,x,\omega)$ is the turbulent part of the velocity.

Incompressible Navier Stokes equation in $\mathbb{R}^3$, for the velocity field $(\mathscr{U}^{(1)}, \mathscr{U}^{(2)}, \mathscr{U}^{(3)})$ and the pressure $\mathscr{P}$, with constant mass density ϱ

$$\partial_t \mathscr{U} + (\mathscr{U} \cdot \nabla)\mathscr{U} = \nu \triangle \mathscr{U} - \frac{1}{\varrho} \nabla \mathscr{P}, \;\; t > 0, \; x \in \mathbb{R}^3,$$
$$\nabla \cdot \mathscr{U} = 0, \;\; t \geq 0, \; x \in \mathbb{R}^3,$$
$$\mathscr{U}(0,x) = \mathscr{U}_0(x), \; x \in \mathbb{R}^3.$$

The Reynolds averaged NS Equation for the mean velocity : RANS Equation

Assuming Reynolds decomposition, we obtain the unclosed equation with constant mass density ϱ

$$\partial_t \langle \mathscr{U}^{(i)} \rangle + \sum_{j=1}^{3} \langle \mathscr{U}^{(j)} \rangle \partial_{x_j} \langle \mathscr{U}^{(i)} \rangle + \sum_{j=1}^{3} \partial_{x_j} \langle \mathbf{u}^{(i)} \mathbf{u}^{(j)} \rangle = \nu \triangle \langle \mathscr{U}^{(i)} \rangle - \frac{1}{\varrho} \partial_{x_i} \langle \mathscr{P} \rangle,$$

$$\nabla \cdot \langle \mathscr{U} \rangle = 0,\ t \geq 0,\ x \in \mathbb{R}^3,$$

$$\langle \mathscr{U} \rangle (0,x) = \langle \mathscr{U}_0 \rangle (x),\ x \in \mathbb{R}^3,$$

where $(\langle \mathbf{u}^{(i)} \mathbf{u}^{(j)} \rangle = \langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle - \langle \mathscr{U}^{(i)} \rangle \langle \mathscr{U}^{(j)} \rangle, i, j)$ is Reynolds stress tensor.
Depending of the physics, a turbulent closure of the Reynolds stress is associated to the RANS Equation.
Direct modeling of the Reynolds stress, or turbulent viscosity model, ...

kinetic turbulent energy $k(t,x) := \sum_{i=1}^{3} \frac{1}{2} \langle \mathbf{u}^{(i)} \mathbf{u}^{(i)} \rangle (t,x)$

and

pseudo-dissipation $\varepsilon(t,x) := \nu \sum^{3} \sum^{3} \langle \partial_{x_k} \mathbf{u}^{(i)} \partial_{x_k} \mathbf{u}^{(i)} \rangle (t,x).$

An alternative viewpoint to compute the Reynolds stress [by Stephen B. Pope]

Let $\rho_E(t,x;V)$ be the probability density function (PDF) of the random field $\mathscr{U}(t,x)$, then

$$\langle \mathscr{U}^{(i)} \rangle (t,x) = \int_{\mathbb{R}^3} V^{(i)} \rho_E(t,x;V) dV,$$

$$\langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle (t,x) = \int_{\mathbb{R}^3} V^{(i)} V^{(j)} \rho_E(t,x;V) dV.$$

The closure problem is reported on the PDE satisfied by the probability density function ρ_E.

In a series of papers (see e.g. Pope 85), Stephen B. Pope propose to model the PDF ρ_E with a Lagrangian description of the flow,
or equivalently with the Lagrangian probability density function (the PDF method).

Fluid particle model family

On a given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, consider the fluid particle state vector $(\mathrm{X}_t, \mathrm{U}_t, \psi_t)$ satisfying

$$\begin{aligned} d\mathrm{X}_t =& \mathrm{U}_t dt, \\ d\mathrm{U}_t =& \left[-\frac{1}{\varrho} \nabla_x \langle \mathscr{P} \rangle (t, \mathrm{X}_t) + \nu \triangle_x \langle \mathscr{U} \rangle (t, \mathrm{X}_t) \right] dt \\ & - G(t, X_t) \left(\mathrm{U}_t - \langle \mathscr{U} \rangle (t, \mathrm{X}_t) \right) dt + \sqrt{C(t, \mathrm{X}_t) \varepsilon(t, \mathrm{X}_t)} dW_t, \\ d\psi_t =& D_1(t, \mathrm{X}_t, \psi_t) dt + D_2(t, \mathrm{X}_t, \psi_t) d\widetilde{W}_t. \end{aligned}$$

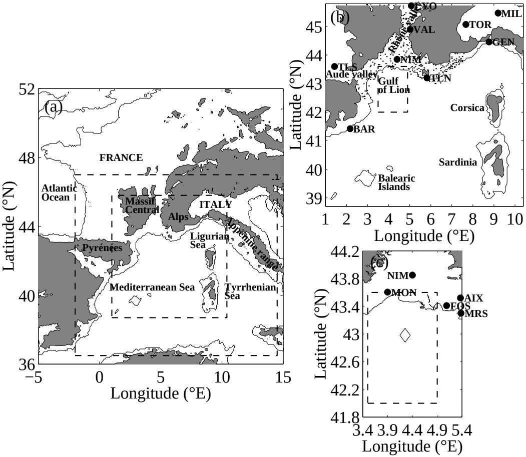
$\left(W, \widetilde{W}\right)$ is a $4D$-Brownian motion.

One needs to

- compute de Eulerian fields $\langle \mathscr{U}^{(i)} \rangle (t, x)$, $\langle \mathscr{U}^{(i)} \mathscr{U}^{(j)} \rangle (t, x)$.
- determine ε, C, G, D_1, D_2 by the RANS closure.

Numerical Experiments : comparison with wind measures

In [Bernardin Bossy Chauvin Drobinski Rousseau Salameh 2009]



The MM5 model is run for 3 days between March 23rd and 25th, 1998 over the 3 nested domains with 3 with respective horizontal resolutions of 27, 9 and $3\ km$.

The initial and boundary conditions are taken from the ECMWF reanalyses.

◇ represents the location of the buoy ASIS.

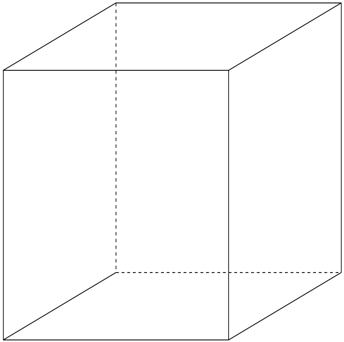
The numerical framework

Our computational domain $\mathcal{D}$.
For example, a given cell of the NWP solver.

Boundary condition :

$$\forall x \in \partial\mathcal{D},\ \langle \mathscr{U} \rangle(t,x) = V_{\text{ext}}(t,x)$$

(V_{ext} is the $MM5$ forcing.)



The guidance with an external velocity field

The Downscaling method

Let $\mathcal{D}$ be an open set of $\mathbb{R}^3$, and a velocity V_{ext} given at $\partial\mathcal{D}$:

$$\begin{cases} d\mathrm{X}_t = & \mathrm{U}_t dt, \\ d\mathrm{U}_t = & \Big[-\frac{1}{\varrho} \nabla \langle \mathscr{P} \rangle (t, \mathrm{X}_t) \\ & \quad - \left(\frac{1}{2} + \frac{3}{4} C_0\right) \frac{\varepsilon(t, \mathrm{X}_t)}{k(t, \mathrm{X}_t)} \left(\mathrm{U}_t - \langle \mathscr{U} \rangle (t, \mathrm{X}_t)\right) \Big] dt \\ & + \sqrt{C_0 \varepsilon(t, \mathrm{X}_t)} dW_t \\ & + \sum_{0 < s \leq t} 2 \left(V_{\text{ext}}(s, \mathrm{X}_s) - \mathrm{U}_{s^-}\right) \mathbf{1}_{\{\mathrm{X}_s \in \partial\mathcal{D}\}}. \end{cases}$$

The jump term should ensure that

$$\langle \mathscr{U} \rangle (t, x) = V_{\text{ext}}(t, x), \forall x \in \partial\mathcal{D}.$$

The guidance with an external velocity field

Boundary condition

$\forall x \in \partial\mathcal{D}$,

$$\langle \mathcal{U} \rangle(t,x) = V_{\text{ext}}(t,x).$$

$$\frac{\int_{\mathbb{R}^d} v\rho_\ell(t,x,v)dv}{\int_{\mathbb{R}^d} \rho_\ell(t,x,v)dv} = V_{\text{ext}}(t,x).$$

$$\begin{aligned} & \int_{\mathbb{R}^d} v\rho_\ell(t,x,v,)dv = \int_{\mathbb{R}^d} V_{\text{ext}}(t,x)\rho_\ell(t,x,v)dv \\ \Leftrightarrow \quad & \int_{\mathbb{R}^d} v\rho_\ell(t,x,v,)dv = \int_{\mathbb{R}^d} v\rho_\ell(t,x,v+2(V_{\text{ext}}(t,x)-v))dv \\ \Uparrow \quad & \end{aligned}$$

$$\rho_\ell(t,x,v,) = \rho_\ell(t,x,v+2(V_{\text{ext}}(t,x)-v)), \ \forall v \in \mathbb{R}^d$$

leads to specular boundary condition with jump on $\partial\mathcal{D}$ for ρ_ℓ ...

The equation of 9.15 am

$$\begin{cases} X_t = X_0 + \int_0^t U_s\, ds, \\ U_t = U_0 + \int_0^t B[X_s;\rho_s] ds + \sigma W_t - \sum_{0<s\leq t} 2\left(U_{s^-}\cdot n_{\mathcal{D}}(X_s)\right) n_{\mathcal{D}}(X_s) \mathbf{1}_{\{X_s\in\partial\mathcal{D}\}}, \\ \rho(t) \text{ is the probability density of } (X_t, U_t) \text{ for all } t\in(0,T], \end{cases}$$

$Q_T = (0,T)\times\mathcal{D}\times\mathbb{R}$ and $\Sigma_T = (0,T)\times\partial\mathcal{D}\times\mathbb{R}$

Definition 1. Trace of the density along Σ_T

$\gamma(\rho) : \Sigma_T \longrightarrow \mathbb{R}$ is the trace of $(\rho(t);\ t\in[0,T])$ along Σ_T if it is nonnegative and satisfies, for all t in $(0,T]$, f in $C_c^\infty\left(\overline{Q_t}\right)$:

$$\begin{aligned} &\int_{\Sigma_t} (u\cdot n_{\mathcal{D}}(x))\,\gamma(\rho)(s,x,u) f(s,x,u)\, ds\, d\sigma_{\partial\mathcal{D}}(x)\, du \\ &= -\int_{\mathcal{D}\times\mathbb{R}^d} f(t,x,u)\rho_t(x,u)\, dx\, du + \int_{\mathcal{D}\times\mathbb{R}^d} f(0,x,u)\rho_0(x,u)\, dx\, du \\ &\quad + \int_{Q_t} \left(\partial_s f + u\cdot\nabla_x f + B[\cdot;\rho_\cdot]\cdot\nabla_u f + \frac{\sigma^2}{2}\triangle_u f\right)(s,x,u)\rho_s(x,u)\, ds\, dx\, du \end{aligned} \tag{3}$$

and, for $dt\otimes d\sigma_{\partial\mathcal{D}}$ a.e. (t,x) in $(0,T)\times\partial\mathcal{D}$,

$$\int_{\mathbb{R}^d} |(u\cdot n_{\mathcal{D}}(x))|\gamma(\rho)(t,x,u)\, du + \infty, \qquad \int_{\mathbb{R}^d} \gamma(\rho)(t,x,u)\, du > 0. \tag{4}$$

Main Theorem

Under (*H*), there exists a unique solution in law to (1) in Π_ω.

$$\Pi_\omega := \big\{ \mathcal{Q}, \text{ probability measure on } \mathcal{C}([0,T];\overline{\mathcal{D}}) \times \mathbb{D}([0,T];\mathbb{R}^d), \text{ s.t.,} \\ \text{for all } t \in [0,T],\ \rho_t = \mathcal{Q} \circ (x(t), u(t))^{-1} \in L^2(\omega; \mathcal{D} \times \mathbb{R}^d) \big\}.$$

Moreover the time-marginal densities $(\rho_t, t \in [0,T])$ is in $V^1(\omega, Q_T)$ and admits a trace $\gamma(\rho)$ in the sense of Definition 1 which satisfies the no-permeability boundary condition

$$\mathbb{E}\{(U_t \cdot n_{\mathcal{D}}(X_t))/X_t = x\} = \frac{\int_{\mathbb{R}^d} (u \cdot n_{\mathcal{D}}(x)) \gamma(\rho)(t,x,u)\, du}{\int_{\mathbb{R}^d} \gamma(\rho)(t,x,u)\, du} = 0, \ \ dt \otimes d\sigma_{\partial\mathcal{D}}\text{-a.e. on } (0,T)$$

or equivalently the specular boundary condition :

$$\gamma(\rho)(t,x,u) = \gamma(\rho)(t,x,u - 2(u \cdot n_{\mathcal{D}}(x)) n_{\mathcal{D}}(x)), \ dt \otimes d\sigma_{\partial\mathcal{D}} \otimes du\text{-a.e. on } (0,T) \times \partial\mathcal{D} \times \mathbb{R}^d.$$

The hypotheses (*H*)

(H_{Langevin}) for the construction of the linear Langevin process
(H_{MVFP}) for the well-posedness of the Vlasov-Fokker-Planck equation

- (H_{Langevin})-(i) (X_0, U_0) is distributed according to the initial law μ_0 having its support in $\mathcal{D} \times \mathbb{R}^d$ and such that $\int_{\mathcal{D}\times\mathbb{R}^d} \left(|x|^2 + |u|^2\right) \mu_0(dx, du) < +\infty$.
- (H_{Langevin})-(ii) The boundary $\partial\mathcal{D}$ is a compact $\mathcal{C}^3$ submanifold of $\mathbb{R}^d$.

- (H_{MVFP})-(i) $b : \mathbb{R}^d \to \mathbb{R}^d$ is a bounded measurable function.
- (H_{MVFP})-(ii) The initial law μ_0 has a density ρ_0 in the weighted space $L^2(\omega, \mathcal{D} \times \mathbb{R}^d)$ with $\omega(u) := (1 + |u|^2)^{\frac{\alpha}{2}}$ for some $\alpha > d \vee 2$.
- (H_{MVFP})-(iii) There exist two measurable functions $\underline{P}_0, \overline{P}_0 : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ such that

$$0 < \underline{P}_0(|u|) \le \rho_0(x,u) \le \overline{P}_0(|u|), \text{ a.e. on } \mathcal{D} \times \mathbb{R}^d;$$

$$\text{and} \quad \int_{\mathbb{R}^d} (1 + |u|)\omega(u)\overline{P}_0^2(|u|)du < +\infty.$$

Notation

$$\begin{aligned}
&Q_t := (0,t)\times\mathcal{D}\times\mathbb{R}^d, \\
&\Sigma^+ := \left\{(x,u)\in\times\partial\mathcal{D}\times\mathbb{R}^d \text{ s.t. } (u\cdot n_{\mathcal{D}}(x))>0\right\}, \qquad \Sigma_t^+ := (0,t)\times\Sigma^+, \\
&\Sigma^- := \left\{(x,u)\in\times\partial\mathcal{D}\times\mathbb{R}^d \text{ s.t. } (u\cdot n_{\mathcal{D}}(x))<0\right\}, \qquad \Sigma_t^- := (0,t)\times\Sigma^-, \\
&\Sigma^0 := \left\{(x,u)\in\partial\mathcal{D}\times\mathbb{R}^d \text{ s.t. } (u\cdot n_{\mathcal{D}}(x))=0\right\}, \qquad \Sigma_t^0 := (0,t)\times\Sigma^0,
\end{aligned}$$

$\Sigma_T := (0,T)\times\partial\mathcal{D}\times\mathbb{R}^d$ endowing with $d\lambda_{\Sigma_T} := dt\otimes d\sigma_{\partial\mathcal{D}}(x)\otimes du$.
$d\sigma_{\partial\mathcal{D}}$ is the surface measure on $\partial\mathcal{D}$.
The weighted Lebesgue space (with $\omega(u) := (1+|u|^2)^{\frac{\alpha}{2}}$)

$$L^2(\omega,Q_t) := \left\{\psi : Q_t\to\mathbb{R}\ ;\ \sqrt{\omega}\psi\in L^2(Q_t)\right\}, \text{ with } \|\psi\|^2_{L^2(\omega,Q_t)} = \|\sqrt{\omega}\psi\|_{L^2(Q_t)}$$

The weighted Sobolev space

$$V_1(\omega,Q_T) = \mathcal{C}\left([0,T];L^2(\omega,\mathcal{D}\times\mathbb{R}^d)\right)\cap L^2((0,T)\times\mathcal{D};H^1(\mathbb{R}^d)),$$

equipped with the norm

$$\|\phi\|^2_{V_1(\omega,Q_T)} = \max_{t\in[0,T]}\left\{\int_{\mathcal{D}\times\mathbb{R}^d}\omega(u)\,|\phi(t,x,u)|^2\,dx\,du\right\}+\int_{Q_T}\omega(u)|\nabla_u\phi(t,x,u)|^2\,dt\,dx\,du.$$

Plan of the proof

- Phase 1. Construct of a solution to the linear Confined Langevin equation.
- Phase 2. Construct a density ρ solution of the conditional McKean Vlasov Fokker Planck equation, with specular condition.
- Phase 3. Add the drift $B[x;\rho]$ to the Langevin process constructed in Phase 1 et show that the result is the unique solution of our confined stochastic Lagrangian model.

Sketch of the Phase 1 -a)

Prove the well-posedness of the confined linear Langevin equation : there exists a unique solution, defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t;\, t \in [0,T]), \mathcal{P})$ endowed with a Brownian motion W, to

$$\begin{cases} X_t = x_0 + \displaystyle\int_0^t U_s\, ds, \\ U_t = u_0 + \sigma W_t - 2 \displaystyle\sum_{0 \le s \le t} (U_{s^-} \cdot n_{\mathcal{D}}(X_s))\, n_{\mathcal{D}}(X_s) \mathbf{1}_{\{X_s \in \partial\mathcal{D}\}}, \ \forall t \in [0,T], \end{cases} \tag{5}$$

for any $(x_0, u_0) \in (\mathcal{D} \times \mathbb{R}^d) \cup (\Sigma \setminus \Sigma^0)$.

The confined Brownian motion primitive in the half line

Starting from $(\mathrm{X}_0, \mathrm{U}_0)$ with $\mathrm{X}_0 > 0$, and (B_t) Brownian motion in $\mathbb{R}$,

$$\mathscr{Y}_t = \mathrm{X}_0 + \int_0^t \mathscr{V}_s ds, \qquad \mathscr{V}_t = \mathrm{U}_0 + B_t.$$

Set $\mathrm{X}_t = |\mathscr{Y}_t|$,

$\mathrm{U}_t = \mathscr{V}_t \mathscr{S}_{t^+}$, with $\mathscr{S}_t := sign(\mathscr{Y}_t)$.

Lemma B. & Jabir 2011

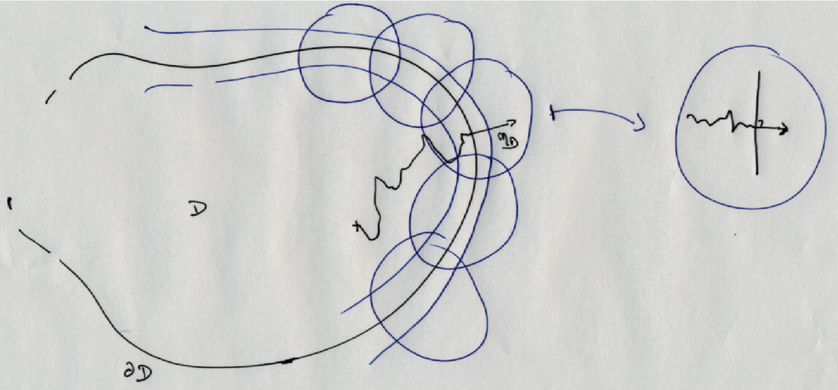
If ρ_0 has its support in $(0, +\infty) \times \mathbb{R}$, then $\mathscr{S}_t$ jumps a countable number of times, and U_t solves

$$\mathrm{U}_t = \mathrm{U}_0 + W_t - 2 \sum_{0<s\leq t} \mathrm{U}_{s-} \mathbf{1}_{\{\mathrm{X}_s = 0\}} \text{ a.s.}$$

where W_t is a Brownian motion.

(Lachal 97 : Passage time of the Brownian motion primitive at 0)

Construction of the confined Langevin process, by local straightening of the boundary



This method impose that ∂D is a compact submanifold of class $\mathcal{C}^3$. The local straightening is then $\mathcal{C}^2$, enough to apply the Ito formula to the process in the new system of coordinates.

Phase 1-b) On the semigroup of the confined Langevin process

For some test function $\psi : \mathcal{D} \times \mathbb{R}^d \to \mathbb{R}^+$, for all $(x,u) \in (\mathcal{D} \times \mathbb{R}^d) \cup (\Sigma \setminus \Sigma^0)$, we define

$$\Gamma^{\psi}(t,x,u) := \mathbb{E}_{\mathbb{P}}\left[\psi(X_t^{x,u}, U_t^{x,u})\right], \qquad (6)$$

where $((X_t^{x,u}, U_t^{x,u});\ t \in [0,T])$ is the solution of (5) starting from $(0,x,u)$

Proposition

Assume (H_{Langevin}). For all nonnegative $\psi \in \mathcal{C}_c^{\infty}(\mathcal{D} \times \mathbb{R}^d)$, Γ^{ψ} is a nonnegative function that belongs to $L^2((0,T) \times \mathcal{D}; H^1(\mathbb{R}^d))$ and satisfies the energy equality :

$$\|\Gamma^{\psi}(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \sigma^2 \|\nabla_u \Gamma^{\psi}\|^2_{L^2(Q_t)} = \|\psi\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)},\ \forall\, t \in (0,T)$$

Furthermore, $\Gamma^{\psi}(t)$ is solution in the sense of distributions of

$$\begin{cases} \partial_t \Gamma^{\psi} - (u \cdot \nabla_x \Gamma^{\psi}) - \frac{\sigma^2}{2} \triangle_u \Gamma^{\psi} = 0, \text{ on } Q_T, \\ \Gamma^{\psi}(0,x,u) = \psi(x,u), \text{ on } \mathcal{D} \times \mathbb{R}^d, \\ \Gamma^{\psi}(t,x,u) = \Gamma^{\psi}(t,x,u - 2(u \cdot n_{\mathcal{D}}(x)) n_{\mathcal{D}}(x)), \text{ on } \Sigma_T^+. \end{cases}$$

and $\forall p \in (1, +\infty)$, $\quad \|\Gamma^{\psi}(t)\|^p_{L^p(\mathcal{D}\times\mathbb{R}^d)} \leq \|\psi\|^p_{L^p(\mathcal{D}\times\mathbb{R}^d)}, \quad \forall\, t \in (0,T)$

Main ingredient

Theorem

Assume (H_{Langevin}). Given two nonnegative functions $f_0 \in L^2(\mathcal{D}\times\mathbb{R}^d)\cap\mathcal{C}_b(\mathcal{D}\times\mathbb{R}^d)$ and $q \in L^2(\Sigma^+_T)\cap\mathcal{C}_b(\Sigma^+_T)$, there exists a unique nonnegative function $f \in \mathcal{C}_b^{1,1,2}(Q_T)\cap\mathcal{C}(\overline{Q_T}\setminus\Sigma^0_T)\cap L^2((0,T)\times\mathcal{D};H^1(\mathbb{R}^d))$ solution to

$$\begin{cases} \partial_t f(t,x,u) - (u\cdot\nabla_x f(t,x,u)) - \frac{\sigma^2}{2}\triangle_u f(t,x,u) = 0, & \text{for all } (t,x,u)\in Q_T, \\ f(0,x,u) = f_0(x,u), \ \ \text{for all } (x,u)\in\mathcal{D}\times\mathbb{R}^d, \\ f(t,x,u) = q(t,x,u), \ \ \text{for all } (t,x,u)\in\Sigma^+_T. \end{cases}$$

In addition, for the Langevin process $(x^{x,u}_t, u^{x,u}_t;\, t\in[0,T])$ starting from $(x,u)\in\mathcal{D}\times\mathbb{R}^d$ at $t=0$ and $\beta^{x,u} := \inf\{t>0\,;\, x^{x,u}_t\in\partial\mathcal{D}\}$, we have

$$f(t,x,u) = \mathbb{E}_{\mathbb{P}}\left[f_0(x^{x,u}_t, u^{x,u}_t)\mathbf{1}_{\{t\,\leq\,\beta^{x,u}\}}\right] + \mathbb{E}_{\mathbb{P}}\left[q(t-\beta^{x,u}, x^{x,u}_{\beta^{x,u}}, u^{x,u}_{\beta^{x,u}})\mathbf{1}_{\{t\,>\,\beta^{x,u}\}}\right]$$

Furthermore, for all $t\in(0,T)$, f satisfies

$$\|f(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|f\|^2_{L^2(\Sigma^-_t)} + \sigma^2\|\nabla_u f\|^2_{L^2(Q_t)} = \|f_0\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|q\|^2_{L^2(\Sigma^+_t)},$$

$$\|f(t)\|^p_{L^p(\mathcal{D}\times\mathbb{R}^d)} + \|f\|^p_{L^p(\Sigma^-_t)} + \sigma^2 p(p-1)\|\nabla_u f\|^p_{L^p(Q_t)} \leq \|f_0\|^p_{L^p(\mathcal{D}\times\mathbb{R}^d)} + \|q\|^p_{L^p(\Sigma^+_t)}.$$

Interior regularity

or $\alpha \in (0,1)$, we further denote by D_x^α the fractional derivative w.r.t. x-variables, defined as the fractional Laplace operator of order α

$$D_x^\alpha = (-\triangle_x)^{\alpha/2}.$$

Theorem Bouchut 2002

Let $h \in L^2(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)$. Assume that $\phi \in L^2(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d)$, such that $\nabla_u \phi \in (L^2(\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d))^d$, satisfies (in the sense of distributions)

$$\partial_t \phi + (u \cdot \nabla_x \phi) - \frac{\sigma^2}{2} \triangle_u \phi = h, \text{ in } \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^d. \tag{7}$$

Then there exists a positive constant C

$$\|\partial_t \phi + (u \cdot \nabla_x \phi)\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} + \frac{\sigma^2}{2} \|\triangle_u \phi\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} \leq C(d) \|h\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)},$$
$$\|\nabla_u D_x^{1/3} \phi\|^2_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} + \|D_x^{2/3} \phi\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)} \leq C(d) \|h\|_{L^2(\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d)}.$$

Continuity up and along Σ^-

Feynman-Kac formula (by the interior regularity) :

$$f(t,x,u) = \mathbb{E}_{\mathbb{P}}\left[f_0(x_t^{x,u}, u_t^{x,u})\mathbf{1}_{\{t \leq \beta^{x,u}\}}\right] + \mathbb{E}_{\mathbb{P}}\left[q(t-\beta^{x,u}, x_{\beta^{x,u}}^{x,u}, u_{\beta^{x,u}}^{x,u})\mathbf{1}_{\{t > \beta^{x,u}\}}\right].$$

The continuity of f up to Σ_T^- will follows from the continuity of $(y,v) \mapsto (\beta^{y,v}, x_t^{y,v}, u_t^{y,v})$.

$\mathbb{P}$-almost surely, for all $t \geq 0$, the flow $(y,v) \mapsto (x_t^{y,v}, u_t^{y,v})$ is continuous on $\mathbb{R}^d \times \mathbb{R}^d$.

As $(y,v) \notin \Sigma_T^0 \cup \Sigma_T^+$, we have $\beta^{y,v} = \tau^{y,v} := \inf\{t > 0;\, x^{y,v} \notin \overline{\mathcal{D}}\}$.

To prove that $(y,v) \mapsto \tau^{y,v}$ is continuous up to Σ_T^-, follow the general argument of the continuity of exit time related to a flow of continuous processes given in Darling & Pardoux 1997.

Chained PDEs

We consider also the semigroup related to the stopped process :

$$\Gamma_n^{\psi}(t,x,u) = \mathbb{E}_{\mathbb{P}}\left[\psi(X^{x,u}_{t\wedge\tau_n^{x,u}}, U^{x,u}_{t\wedge\tau_n^{x,u}})\right],$$

where $\{\tau_n^{x,u}, n \in \mathbb{N})\}$ are the hitting times defined in the Main Theorem.

Corollary

Assume (H_{Langevin}). Then, for all nonnegative $\psi \in \mathcal{C}_c^{\infty}(\mathcal{D}\times\mathbb{R}^d)$ and all $n \in \mathbb{N}^*$, Γ_n^{ψ} is a nonnegative function in $\mathcal{C}_b^{1,1,2}(Q_T) \cap \mathcal{C}(\overline{Q_T} \setminus \Sigma^0)$ and satisfies

$$\begin{cases} \partial_t \Gamma_n^{\psi}(t,x,u) - (u\cdot\nabla_x\Gamma_n^{\psi}(t,x,u)) - \dfrac{\sigma^2}{2}\triangle_u\Gamma_n^{\psi}(t,x,u) = 0, \ \text{ on } Q_T, \\ \Gamma_n^{\psi}(0,x,u) = \psi(x,u), \ \text{ in } \mathcal{D}\times\mathbb{R}^d, \\ \Gamma_n^{\psi}(t,x,u) = \Gamma_{n-1}^{\psi}(t,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)), \ \text{ in } \Sigma_T^+. \end{cases}$$

In addition, the set $\{\Gamma_n^{\psi}, n \geq 1, \Gamma_0^{\psi} = \psi\}$ belongs to $L^2((0,T)\times\mathcal{D}; H^1(\mathbb{R}^d))$ and admits traces that satisfy the energy equality

$$\|\Gamma_n^{\psi}(t)\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \sigma^2\|\nabla_u\Gamma_n^{\psi}\|^2_{L^2(Q_t)} + \|\Gamma_n^{\psi}\|^2_{L^2(\Sigma^-)} = \|\psi\|^2_{L^2(\mathcal{D}\times\mathbb{R}^d)} + \|\Gamma_{n-1}^{\psi}\|^2_{L^2(\Sigma^-)}.$$

Sketch of the Phase 2 : On the conditional McKean-Vlasov-Fokker-Planck equation

We construct a probability density function satisfying (in the sense of distribution) :

$$\partial_t\rho + (u\cdot\nabla_x\rho) + (B[\cdot\,;\rho]\cdot\nabla_u\rho) - \frac{\sigma^2}{2}\triangle_u\rho = 0 \text{ in } (0,T)\times\mathcal{D}\times\mathbb{R}^d,$$

$$\rho(0,x,u) = \rho_0(x,u) \text{ on } \mathcal{D}\times\mathbb{R}^d,$$

$$\gamma(\rho)(t,x,u) = \gamma(\rho)(t,x,u-2(u\cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x)) \text{ on } (0,T)\times\partial\mathcal{D}\times\mathbb{R}^d,$$

Definition : Maxwellian distribution

For given $a\in\mathbb{R},\ \mu>0,\ P_0\in L^1(\mathbb{R}^d)$, such that $P_0\geq 0$ on $\mathbb{R}^d$, a Maxwellian distribution with parameters (a,μ,P_0) is a function $P:\mathbb{R}^+\times\mathbb{R}^d\to\mathbb{R}^+$ such that

$$P(t,u) = \exp\{at\}\,[m(t,u)]^{\mu}\,, \tag{8}$$

where $m:\mathbb{R}^+\times\mathbb{R}^d\to\mathbb{R}^+$ is defined by $m(t,u) = (G(\sigma^2 t) * P_0^{\frac{1}{\mu}})(u)$, with $G(t,u) = \left(\frac{1}{2\pi t}\right)^{\frac{d}{2}}\exp\{\frac{-|u|^2}{2t}\}$.

Maxwelian bounds for the linear Vlasov-FP equation

Consider the unique solution in $V_1(\omega, Q_T)$ to the linear problem

$$\begin{cases} \mathcal{T}(S) + (B \cdot \nabla_u S) - \dfrac{\sigma^2}{2} \triangle_u S = 0 \text{ in } Q_T, \\ S(0,x,u) = \rho_0(x,u) \text{ on } \mathcal{D} \times \mathbb{R}^d, \\ \gamma^-(S)(t,x,u) = q(t,x,u) \text{ on } \Sigma_T^- . \end{cases} \tag{9}$$

Proposition

Assume (H_{MVFP}). For $B \in L^\infty((0,T) \times \mathcal{D}; \mathbb{R}^d)$, for $(\underline{P}_0, \overline{P}_0)$ as in (H_{MVFP})-(iii), let $\left(\underline{p}, \overline{p}\right)$ be a couple Maxwellian distributions with parameters $\left(\underline{a}, \underline{\mu}, \underline{P}_0\right)$ and $\left(\overline{a}, \overline{\mu}, \overline{P}_0\right)$ satisfying

$(a1)$ $\underline{\mu} > 1$, and $\overline{\mu} \in (\frac{1}{2}, 1)$.

$(a2)$ $\underline{a} \leq \dfrac{-\underline{\mu}}{2\sigma^2(\underline{\mu} - 1)} \|B\|^2_{L^\infty((0,T)\times\mathcal{D};\mathbb{R}^d)}$, and $\overline{a} \geq \dfrac{\overline{\mu}}{2\sigma^2(1 - \overline{\mu})} \|B\|^2_{L^\infty((0,T)\times\mathcal{D};\mathbb{R}^d)}$.

Then

$(d1)$ $\sup_{t\in[0,T]} \int_{\mathbb{R}^d} (1 + |u|)\omega(u) \left|\overline{p}(t,u)\right|^2 du < +\infty,$ and $\inf_{t\in[0,T]} \int_{\mathbb{R}^d} \underline{p}(t,u) du > 0.$

$(d2)$ If $\underline{p} \leq q \leq \overline{p}$, on Σ_T^-, then $\underline{p} \leq S \leq \overline{p}$, a.e. on Q_T, and $\underline{p} \leq \gamma^+(S) \leq \overline{p}$, on Σ_T^+.

Add the specular condition + the nonlinear term $B[x;\rho]$

Theorem

Under (H_{MVFP}), there exists a function $\rho \in V_1(\omega, Q_T)$, and there exist $\gamma^+(\rho)$, $\gamma^-(\rho)$ defined on Σ_T^+ and Σ_T^- respectively, with $\gamma^\pm(\rho) \in L^2(\omega, \Sigma_T^\pm)$, s.t. $\forall t \in (0,T]$, $\forall \psi \in \mathcal{C}_c^\infty(\overline{Q_t})$,

$$\begin{aligned}
&\int_{Q_t} \left(\rho \mathcal{T}(\psi) + \psi \left(B[\cdot\,;\rho] \cdot \nabla_u \rho \right) + \frac{\sigma^2}{2} \left(\nabla_u \psi \cdot \nabla_u \rho \right) \right) (s,x,u)\, ds\, dx\, du \\
&= \int_{\mathcal{D}\times\mathbb{R}^d} \rho(t,x,u)\psi(t,x,u)\, dx\, du - \int_{\mathcal{D}\times\mathbb{R}^d} \rho_0(x,u)\psi(0,x,u)\, dx\, du \\
&\quad + \int_{\Sigma_t^+} (u \cdot n_{\mathcal{D}}(x))\gamma^+(\rho)(s,x,u)\psi(s,x,u) d\lambda_{\Sigma_T}(s,x,u) \\
&\quad + \int_{\Sigma_t^-} (u \cdot n_{\mathcal{D}}(x))\gamma^+(\rho)(s,x,u - 2(u \cdot n_{\mathcal{D}}(x))n_{\mathcal{D}}(x))\psi(s,x,u) d\lambda_{\Sigma_T}(s,x,u).
\end{aligned}$$

In addition, there exist a couple of Maxwellian distributions $\left(\overline{P},\ \underline{P}\right)$ such that

$$\underline{P} \le \rho \le \overline{P},\ \text{a.e. on } Q_T,$$

$$\underline{P} \le \gamma^\pm(\rho) \le \overline{P},\ \lambda_{\Sigma_T}\text{-a.e. on } \Sigma_T^\pm,$$

$\overline{P}$ and $\underline{P}$ satisfy the specular boundary condition , and for all $t \in (0,T]$,

$$\sup \int (1+|u|)\omega(u) \left(\overline{P}(t,u)\right)^2 du < +\infty, \qquad \inf \int P(t,u)\, du > 0.$$

Sketch of the Phase 3

under $(\mathbb{P}, (\widetilde{w}(t);\ t \in [0,T])$, the canonical process $((x(t), u(t));\ t \in [0,T])$ satisfies

$$\begin{cases} x(t) = x(0) + \displaystyle\int_0^t u(s)\, ds, \\ u(t) = u(0) + \sigma \widetilde{w}(t) - \displaystyle\sum_{0<s\leq t} 2 \left(u(s^-) \cdot n_{\mathcal{D}}(x(s)) \right) n_{\mathcal{D}}(x(s)) \mathbf{1}_{\{x(s) \,\in\, \partial\mathcal{D}\}}, \end{cases}$$

Consider $\rho^{\mathsf{FP}} \in V_1(\omega, Q_T)$ solution to the MVFP eq.

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left\{ \frac{1}{\sigma} \int_0^T B[x(t); \rho^{\mathsf{FP}}(t)]\, d\widetilde{w}(t) - \frac{1}{2\sigma^2} \int_0^T \left| B[x(t); \rho^{\mathsf{FP}}(t)] \right|^2 dt \right\}.$$

Then, according to Girsanov Theorem, $((x(t), u(t));\ t \in [0,T])$ satisfies $\mathbb{Q}$-a.s.,

$$\begin{cases} x(t) = x(0) + \int_0^t u(s)\, ds, \\ u(t) = u(0) + \int_0^t B[x(s); \rho^{\mathsf{FP}}(s)]\, ds + \sigma w(t) - \displaystyle\sum_{0<s\leq t} 2 \left(u(s^-) \cdot n_{\mathcal{D}}(x(s)) \right) n_{\mathcal{D}}(x(s)) \mathbf{1}_{\{x} \end{cases}$$

where $(w(t) := \widetilde{w}(t) - \int_0^t B[x(s); \rho^{\mathsf{FP}}(s)]\, ds;\ t \in [0,T])$ is a $\mathbb{R}^d$-valued $\mathbb{Q}$-Brownian motion, and $\mathbb{Q}(x(0) \in dx, u(0) \in du) = \rho_0(x,u) dx du$.

The mild equation

$\rho \in \mathcal{C}([0,T]; L^2(\mathcal{D} \times \mathbb{R}^d))$ is a solution to the following linear mild equation if, for all $t \in (0,T]$, for all $\psi \in \mathcal{C}_c^\infty(\mathcal{D} \times \mathbb{R}^d)$,

$$\langle \psi, \rho(t) \rangle = \left\langle \Gamma^\psi(t), \rho_0 \right\rangle + \int_0^t \left\langle \nabla_u \Gamma^\psi(t-s), B[\cdot\,; \rho^{\mathsf{FP}}(s)]\rho(s) \right\rangle ds, \qquad (10)$$

Proposition

(i) There exists at most one solution in $\mathcal{C}([0,T]; L^2(\mathcal{D} \times \mathbb{R}^d))$ to the linear mild equation (10).

(ii) The weak solution $(\rho^{\mathsf{FP}}(t); t \in [0,T])$ to MVFP equation is solution to the mild equation (10).

(iii) The time marginal $\mathbb{Q} \circ (x(t), u(t))^{-1}$ admits a density $\rho(t) \in L^2(\omega, \mathcal{D} \times \mathbb{R}^d)$ which is solution to the mild equation (10).

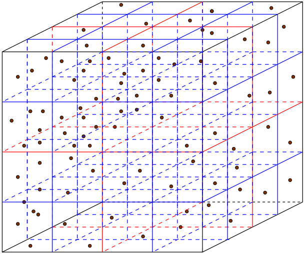
Numerical method : stochastic particle algorithm

The PIC method

The computational space is divided in cells

We use a *Particle in cell* (PIC) technique to compute the Eulerian fields like $\langle \mathscr{U}^{(i)} \rangle (t,x)$.

We compute the Eulerian fields (mean fields) at the center of each sub-cell only.



We introduce N_p particles $(\mathrm{X}_t^{k,N_p}, \mathrm{U}_t^{k,N_p})$ in $\mathcal{D}$.

Each cell $\mathcal{C}$ contains N_{pc} particles by constant mass density constraint.

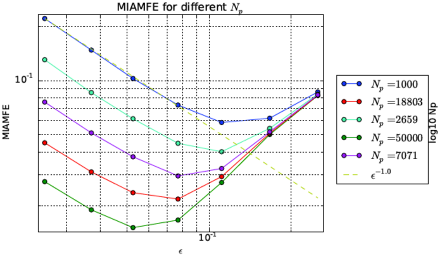
$K(.,x) = \mathbf{1}(.,\mathcal{C}(x))$.

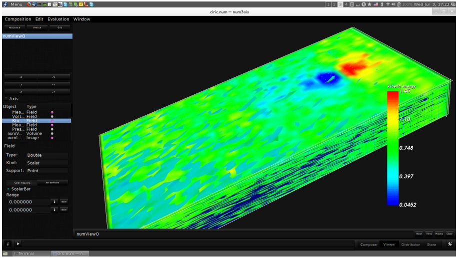
$$\langle F(\mathscr{U}) \rangle (t,x) \quad \simeq \quad \frac{1}{N_p} \sum_{k=1}^{N_p} F\left(\mathrm{U}_t^{k,N_p}\right) K(\mathrm{X}_t^{k,N_p}, x) \Big/ \frac{1}{N_p} \sum_{k=1}^{N_p} K(\mathrm{X}_t^{k,N_p}, x)$$

$$\sum^{N_p} K(\mathrm{X}_t^{k,N_p}, x) = N_{pc}$$

Connected projects

- PhD Thesis of Laurent Violeau on the numerical analysis of SLM. First rate of convergence result on the fluid particle algorithm for various cases of conditional expectation estimators.
- winpos : development of wind farm simulator (with Inria.Chile)





Field

Type: Double

Kind: Scalar

Support: Point

Range

0.000000

0.000000