Metastability in Stochastic Dynamics, Paris

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Lectures on Metastability

Based on collaborations with:

M. Eckhoff, Véronique Gayrard, M. Klein, A. Bianchi, D. Ioffe, and many others....;

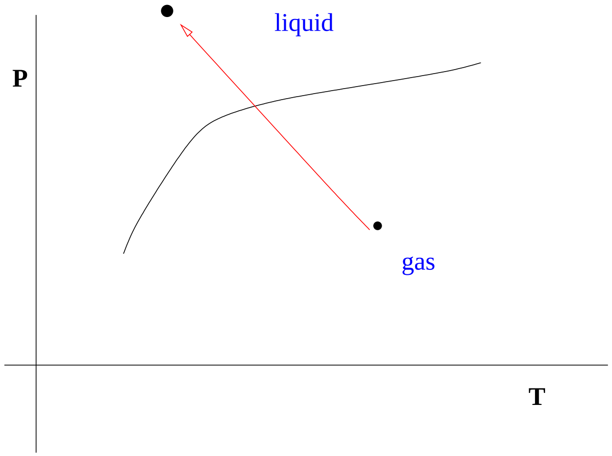
hausdorff center for mathematics

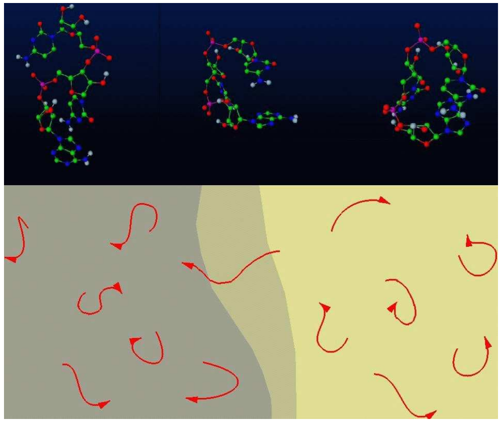
Plan

1. Metastability: basic ideas and approaches
2. Metastability: potential theoretic approach
3. The random field Curie Weiss model

Metastability: basic ideas

Metastability is a common phenomenon in non-linear dynamics. In particular, it is related to the dynamics of first order phase transitions:





Thermally activated transitions between comformations

If the parameters of a systems are changed rapidly across the line of a first order phase transition, the system will persist for a long time in a metastable state before transiting rapidly to the new equilibrium state under the influence of random fluctuations.

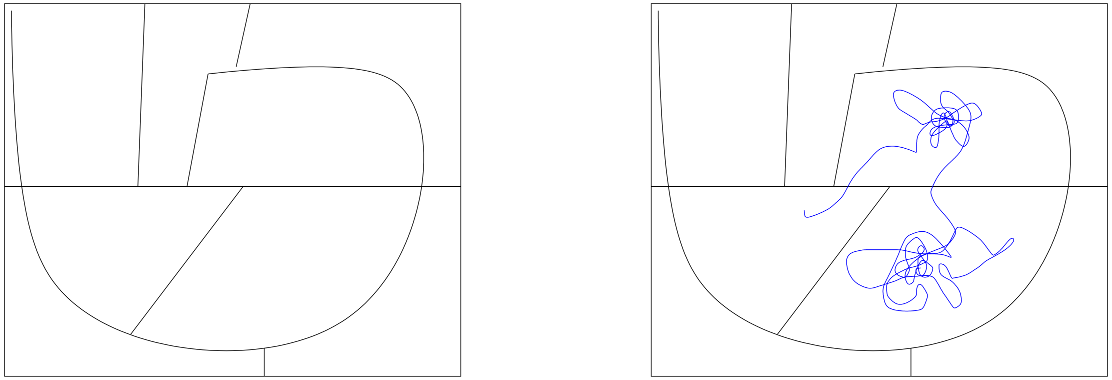
Real life examples are numerous:
Conformational transitions in molecules, catalytic chemical reactions, global climate changes, market crashes, etc.....

Metastability: phenomenological description

Two main characteristics of metastability are:

▷ Quasi-invariant subspaces S_i and

▷ multiple, separated time scales:



▷ Local equilibrium (='metastable state') S_i reached on a fast scale.

▷ Transitions between S_i and S_j occur on much longer metastable time scales.

▷ Due to speparation of time-scales, exponential law of metastable exit times.

Common context: Markov Processes

Main examples:

- ▷ Markov chains with finite (or countable) state space.
- ▷ The classical example of a diffusion process with gradient drift on $\Omega \subset \mathbb{R}^d$, e.g. the stochastic differential equation

$$dX_\epsilon(t) = -\nabla F(X_\epsilon(t))dt + \sqrt{2\epsilon}dW(t)$$

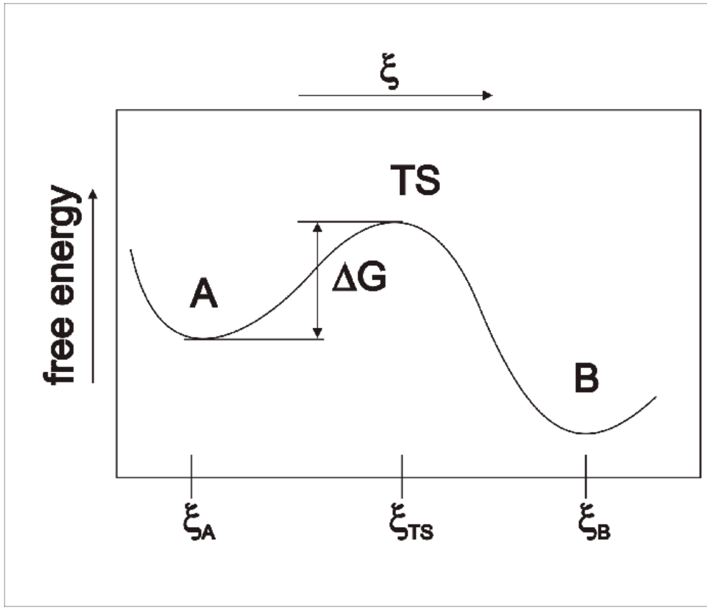
- ▷ Stochastic dynamics of lattice spin systems/lattice gases

The canonical model

1940's: Kramers and Eyring: reaction path theory:

A transition from a metastable is described by the motion of the system along a canonical reaction-path in a free-energy landscape under the driving force of noise:

$$dX_t = F'(X_t)dt + \sqrt{2\epsilon}dB_t$$



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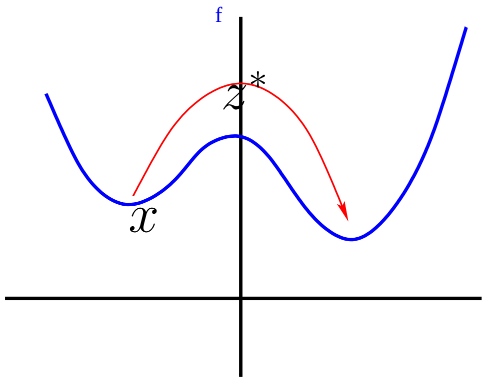
The double well

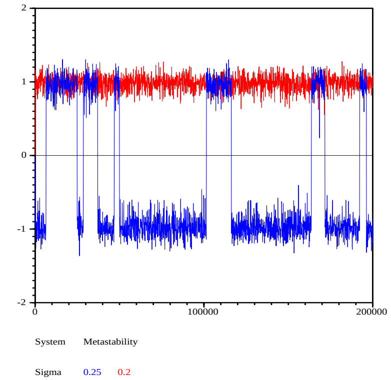
Kramers one-dimensional diffusion model allows quantitative computations: E.g., Kramers obtained the Eyring-Kramers formula for the mean transition time:

$$\mathbb{E}\tau = \frac{2\pi}{\sqrt{-F''(z^*)F''(x)}} e^{[F(z^*)-F(x)]/\epsilon}$$



Further features:

- ▷ Arrival in stable state after many unsuccessful trials;
- ▷ Time of last exit very short compared to $\mathbb{E}\tau$;
- ▷ $\tau/\mathbb{E}\tau$ exponentially distributed



Some big questions remain:

- ▷ What is F for a specific system?
- ▷ How can the simple $1d$ sde model be justified in examples?

Freidlin-Wentzell theory

The most influential work was the study by Mark I. Freidlin and Alexander D. Wentzell on the theory of Small Random Perturbations of Dynamical Systems. This is concerned with generalisations of Kramers equation to higher dimension, namely SDE's

$$dX_t = g(X_t)dt + \sqrt{2\epsilon}(h(X_t), dB_t)$$

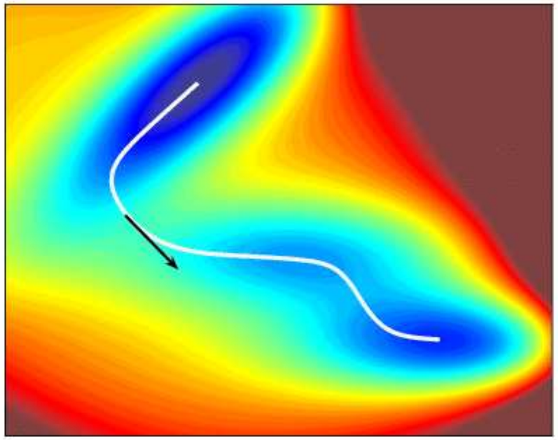
where the unperturbed system, $\frac{d}{dt}X_t = g(X_t)$ has multiple attractors.

Wentzell-Freidlin theory is based on the theory of large deviations on path space.

It expresses the probability for a solution, X^ϵ, to be in the vicinity of an specified path, $\gamma : a \to b$, in term of an action functional

$$\mathbb{P}\left[\|X^\epsilon - \gamma\|_\infty \leq \delta\right] \sim \exp\left(-\frac{1}{\epsilon}\int_0^T L(\gamma(t), \gamma'(t))dt\right)$$

This allows to estimate the success probability for an escape from one attractor to another on the leading exponential scale.

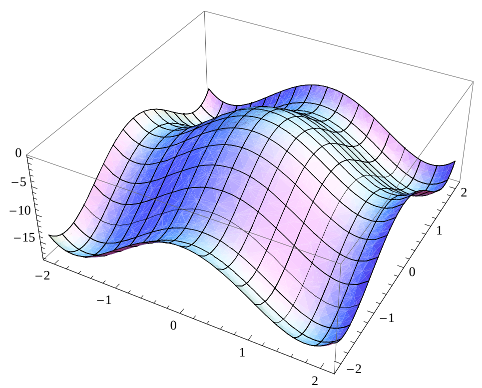


This allows the reduction of the analysis of the metastable behaviour of the system to that of a Markov chain with exponentially small transition probabilities.

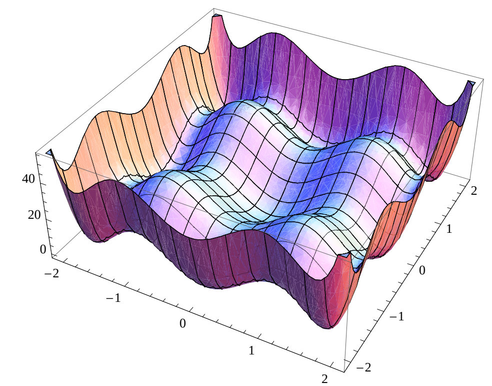
The drawback of this approach are the rather imprecise exponential estimates, that miss, e.g., the prefactor in the Eyring-Kramers formula.

Spectral approach

In the context of diffusion models (multidimensional Kramers equation), analytic methods from quantum mechanics have been used to compute asymptotic expansions in the diffusivity (ϵ) (Matkovski-Schuss, Buslov-Makarov, Maier-Stein).



$\Leftrightarrow$



$$-L = -\epsilon\Delta + \nabla F \cdot \nabla \qquad H = -\epsilon^2\Delta + \frac{1}{4}\|\nabla F\|^2 - \frac{\epsilon}{2}\Delta F$$

Rigorous analysis of eigenvalues and eigenfunctions via Witten complex by Klein, Helffer, Nier [2005] (aee also Matthieu (95), Miclo (95)).

- **Strong point:** in principle gives full asymptotic expansions in ϵ.
- **Drawback:** so far only applicable for reversible diffusion processes.

Our approach

Main objectives:

- ▷ Model independent structural results
- ▷ Link properties of the invariant measure to those of dynamics
- ▷ Good (precise!) calculability of model dependent quantities

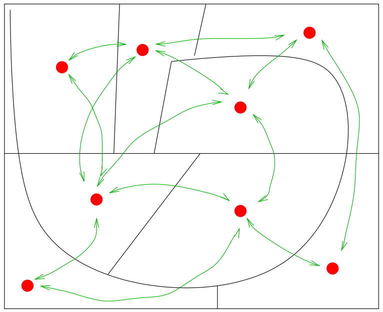
This has led us to

- ▷ Systematic use of potential theoretic ideas
- ▷ Focus on capacities
- ▷ Systematic use of variational principles

Limitations:

- ▷ So far useful only in reversible Markov processes

Definition of metastability



A key feature in our approach is to represent quasi-invariant subsets by a single point (or small ball, B_i), and to consider the transitions between them.

A good definition should characterize the travel times,
$\tau_B \equiv \inf\{t > 0 : X(t) \in B\}$

A family of Markov processes is metastable, if there exists a collection of disjoint sets B_i, such that

$$\frac{\sup_{x \notin \cup_i B_i} \mathbb{E}_x \tau_{\cup_i B_i}}{\inf_i \inf_{x \in B_i} \mathbb{E}_x \tau_{\cup_{k \neq i} B_k}} = o(1)$$

but involve only well-computable quantities.

Martingale and Dirichlet problems

Markov processes are conveniently characterised as solutions to a martingale problem associated to a closed, linear dissipative operator, Λ, called generator.

Martingale problem: *A stochastic process, X_t, is a Markov process, if for a (sufficiently large) class of functions, f,*

$$f(X_t) - f(X_0) - \int_0^t Lf(X_s)ds$$

is a martingale. [In discrete time, replace the sum by an integral].

Dirichlet problem: *Let L be a dissipative linear operator on S, and $D \subset S$. Let $g, k, u : S \to \mathbb{R}$ be continuous functions. Can we find a continuous function $f : S \to \mathbb{R}$, such that*

$$\begin{aligned} -(Lf)(x) + k(x)f(x) &= g(x), \forall x \in D \\ f(x) &= u(x), \forall x \in D^c. \end{aligned}$$

Under suitable conditions, the martingale problem provides a stochastic solution of the Dirichlet problem:

Stochastic solution: *Let* $\tau_{D^c} \equiv \inf\{t > 0 : X_t \notin D\}$*. If*

$$\mathbb{E}_x \tau_{D^c} e^{-\inf_{x\in D}(k(x))\tau_{D^c}} < \infty, \quad \forall x \in D,$$

then

$$\begin{aligned} f(x) &= \mathbb{E}_x \Bigg[u(X_{\tau_{D^c}}) \exp\left(-\int_0^{\tau_{D^c}} k(X_s)ds\right) \\ &+ \int_0^{\tau_{D^c}} g(X_s) \exp\left(-\int_0^t k(X_s)ds\right) dt \Bigg] \end{aligned}$$

Representation for probabilitic quantities

Equilibrium potential: Let $A, B \subset S$ disjoint then

$$h_{A,B}(x) \equiv \mathbb{P}_x\left(\tau_A < \tau_B\right)$$

for $x \notin A \cup B$, solves

$$\begin{aligned} Lh_{A,B}(x) &= 0, \quad x \in S \setminus (A \cup B), \\ h_{A,B}(x) &= 1, \quad x \in A, \\ h_{A,B}(x) &= 0, \quad x \in B \end{aligned}$$

Mean entrance time: For $B \subset S$,

$$w(x) \equiv \mathbb{E}_x \tau_B$$

solves

$$\begin{aligned} -(Lw_B)(x) &= 1, \quad x \notin B, \\ w_B(x) &= 0, \quad x \in B. \end{aligned}$$

Equilibrium measure:

$$e_{A,B}(dx) \equiv -Lh_{A,B}(x), x \in A$$

or

$$h_{A,B}(x) = \int_A G_B(x,y) e_{A,B}(dy)$$

with G_B Green function.

Except in very simple cases there are no easy solutions of the Dirichlet problem. So why is this useful?

A Markov process is called reversible with respect to a measure μ, if its generator, L, is self-adjoint on the space $L^2(S,\mu)$.
This entails a number of remarkably powerful facts.

Dirichlet form:

$$\mathcal{E}(f,g) \equiv -\int_S \mu(dx) f(x)(Lg)(x)$$

positive quadratic form.
Capacity:

$$\text{cap}(A,B) \equiv \int_A \frac{\mu(dx)}{dx} e_{A,B}(dx) = \mathcal{E}\left(h_{A,B}, h_{A,B}\right)$$

For A, B disjoint,

$$\int_A \nu_{A,B}(dz)\mathbb{E}_z\tau_B = \frac{1}{\text{cap}(A,B)} \int \mu(dy) h_{A,B}(y)$$

Dirichlet priciple:

$$\text{cap}(A,B) = \inf_{h \in \mathcal{H}_{A,B}} \mathcal{E}(h,h)$$

$$\mathcal{H}_{A,B} = \{h : h(x) = 1, x \in A, h(x) = 0, x \in B\}$$

$\Rightarrow$ upper bounds!

Monotonicity:

Since the Dirichlet form can be written as an integral (sum) over positive terms, lower bounds can be obtained by removing terms

Diffusions: Here $L = -\epsilon\Delta + \nabla F(x)\nabla$,

$$\mathcal{E}(h,h) = \frac{\epsilon}{2}\int e^{-F(x)/\epsilon}\left(\nabla h(x), \nabla h(x)\right) dx$$

Lower bound by reducing integration domain to tube and gradient by directional derivative

$$\mathcal{E}(h,h) \geq \frac{\epsilon}{2}\int_{\circ_{\sqrt{\epsilon}}} dx_{|} \int_{-1}^{1} e^{-F(\gamma(t)+x_{|})/\epsilon}\left(\frac{d}{dt}h(\gamma(t)+x_{|})\right)^2$$

Example: discrete chains

In discrete state space, the Dirichlet form can be written as

$$\mathcal{E}(f,g) = \frac{1}{2}\sum_{x,y}\mu(x)p(x,y)\left(f(x)-f(y)\right)\left(g(x)-g(y)\right)$$

Best lower bounds use a variational principle from Berman and Konsowa [1990]:

Let $f : \mathcal{E} \to \mathbb{R}_+$ be a non-negative unit flow from $A \to B$, i.e. a function on edges such that

- ▷ $\sum_{a\in A}\sum_b f(a,b) = 1$
- ▷ for any a, $\sum_b f(b,a) = \sum_b f(a,b)$ (Kirchhoff's law).

Set $q^f(a,b) \equiv \frac{f(a,b)}{\sum_b f(a,b)}$, and let the initial distribution for $a \in A$ be $F(a) \equiv \sum_b f(a,b)$.

This defines a Markov chain on paths $\mathcal{X} : A \to B$, with law $\mathbb{P}^f$.

***Theorem* 1.** *For any non-negative unit flow, f, one has that, for $\mathcal{X} = (a_0, a_1, \ldots, a_{|\mathcal{X}|})$,*

$$\textit{cap}(A,B) = \sup_{f:A\to B} = \mathbb{E}^f_{\mathcal{X}} \left[\sum_{\ell=0}^{|\mathcal{X}|-1} \frac{f(a_\ell, a_{\ell+1})}{\mu(a_\ell) p(a_\ell, a_{\ell+1})} \right]^{-1}$$

Equality is reached for the harmonic flow

$$f(a,b) = \frac{1}{\text{cap}(A,B)} \mu(a) p(a,b) \left[h_{H,B}(b) - h_{A,B}(a) \right]_+$$

In the simplest (but already rich) setting of finite state Markov chain, application of potential theory yields a very clear and simple picture.

Definition

A Markov processes X_t is ρ-metastable with respect to the set of metastable points $\mathcal{M} \subset S$, if

$$\frac{\sup_{x\in\mathcal{M}} \mathbb{P}_x[\tau_{\mathcal{M}\setminus x} < \tau_x]}{\inf_{z\notin\mathcal{M}} \mathbb{P}_z[\tau_{\mathcal{M}} < \tau_z]} \leq \rho \ll |S|^{-1}$$

$$\Leftrightarrow$$

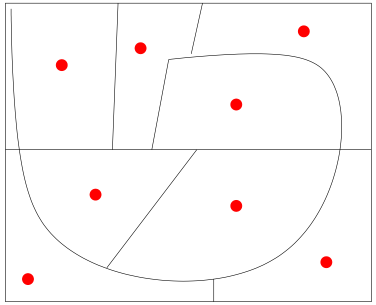
$$\frac{\inf_{x\in\mathcal{M}} \text{cap}(x, \mathcal{M} \setminus x)/\mu(x)}{\sup_{z\notin\mathcal{M}} \text{cap}(z, \mathcal{M})/\mu(z)} \leq \rho \ll |S|^{-1}$$

Note: we will always think of a sequence of processes X_n, where $\rho = \rho_n$, but also possibly $S = S_n$; our definition says that $\rho_n|S_n| \to 0$.

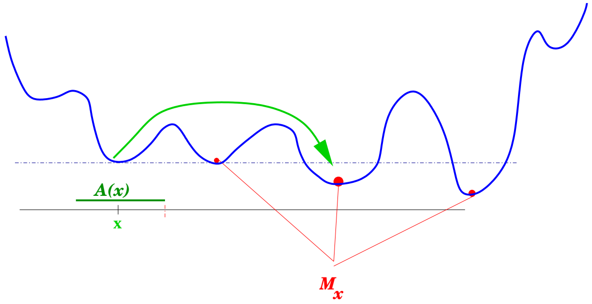
Consequences of the definition

Basins of Attraction: $x \in \mathcal{M}$,

$$A(x) = \{y \in \Gamma : \mathbb{P}_y(\tau_x = \tau_\mathcal{M}) = \max_{z \in \mathcal{M}} \mathbb{P}_y(\tau_z = \tau_\mathcal{M})\}$$



Metastable exits: $\mathbb{E}_x \tau_{\mathcal{M}_x}$ where $x \in \mathcal{M}$ and $\mathcal{M}_x$ are all $y \in \mathcal{M}$ s.t. $\mu(y) \geq \mu(x)$.



Theorem 1:[BEGK '02] If a reversible Markov process with finite state space is metastable with metastable set $\mathcal{M}$, then:

$$\mathbb{E}_x \tau_{\mathcal{M}_x} = \frac{\mu(A(x))}{\text{cap}(x, \mathcal{M}_x)} (1 + o(1))$$

and

$$\mathbb{P}_x[\tau_{\mathcal{M}_x} / \mathbb{E}_x \tau_{\mathcal{M}_x} > t] = (1 + o(1)) e^{-t(1+o(1))}$$

For each $x \in \mathcal{M}$ there exists one eigenvalue, λ_x, of L, s.t.

$$\lambda_x = \frac{1}{\mathbb{E}_x \tau_{\mathcal{M}_x}} (1 + o(1))$$

The corresponding right-eigenfunctions, ϕ_i, satisfy s.t.

$$\phi_x(y) \sim \mathbb{P}_y(\tau_x < \tau_{\mathcal{M}_x}) + o(1)$$

[all under some non-degeneracy hypothesis]

The magic formula:

$$\mathbb{E}_x \tau_B = \frac{1}{\text{cap}(x,B)} \sum_{y \in S} \mu(y) h_{x,A}(y)$$

Renewal inequality:

$$h_{A,B}(x) \leq \frac{\text{cap}(x,B)}{\text{cap}(x,A)}$$

Ultrametric triangle inequality: For any x, y, z,

$$\text{cap}(x,y) \geq \frac{1}{3} \min\left(\text{cap}(x,z), \text{cap}(y,z)\right)$$

Variational representations to estimate capacities.

Eigenvalues.

The connection between metastable behaviour and the existence of small eigenvalues of the generator of the Markov process has been realised for a very long time. Some key references are Davies, Wentzell, Freidlin-Wentzell, Mathieu, Miclo, Gaveau-Schulman, etc.

The key idea of our approach is the observation that if f_λ is an eigenfunction of L, then it must be representable as the solution of a boundary value problem with suitable boundary conditions on $\mathcal{M}$:

$$(L-\lambda)f(y) = 0, \quad y \in \Omega \setminus \mathcal{M}; \qquad f(x) = f_\lambda(x), \quad x \in \mathcal{M} \qquad (*)$$

This allows to characterize all eigenvalues that are smaller than the smallest eigenvalue, λ^0, of the Dirichlet operator in $\Omega \setminus \mathcal{M}$.
A priori estimates. The first step consists in showing that the matrix $L^{\mathcal{M}}$ with Dirichlet conditions in all the points in $\mathcal{M}$ is not very small:

Lemma: (Donsker-Varadhan) Let λ^0 denote the infimum of the spectrum of $-L^{\mathcal{M}}$. Then

$$\lambda^0 \geq \frac{1}{\sup_{x\in\Omega} \mathbb{E}_z \tau_{\mathcal{M}}}$$

Let us denote by $\mathcal{E}_{\mathcal{M}}(\lambda)$ the $|\mathcal{M}| \times |\mathcal{M}|$- matrix with elements

$$(\mathcal{E}_{\mathcal{M}}(\lambda))_{xy} \equiv e^{\lambda}_{z,\mathcal{M}\setminus z}(x)$$

Lemma A number $\lambda < \lambda^0$ is an eigenvalue of the matrix L if and only if

$$\det \mathcal{E}_{\mathcal{M}}(\lambda) = 0$$

Anticipating that we are interested in small λ, we want can use perturbation theory to find the roots of this non-linear characteristic equations.

Beyond finite state space

The conditon $\rho_n|S_n|| \downarrow 0$ fails when:

- ▷ The state space is uncountable
  [Diffusions]
- ▷ Cardinality of state space grows too fast
  [lattice spin systems at finite temperature]

Reason in all cases is that processes will not hit single points fast enough.

Remedy: Replace points in definition of metastability be small balls.

Difficulty: Complicates all renewal arguments!

Needed: A priori control of regularity of harmonic functions and other solutions of Dirichlet problems.

Trivial examples: Mean field models

State space: $S_n = \{-1,1\}^n$ **Hamiltonian:**

$$H_n(\sigma) = -\frac{1}{2n}\sum_{i,j=1}^{n} \sigma_i\sigma_j = \frac{n}{2}\left(m_n(\sigma)\right)^2$$

where $m_n(\sigma) = \frac{1}{n}\sum_{i=1}^{n} \sigma_i$.

Rates:

$$p_n(\sigma,\sigma') = \exp\left(-\left[H_n(\sigma') - H_n(\sigma)\right]_+\right)$$

$m_\lambda(\sigma(t))$ again Markov chain on $\{-1,-1-2/N,\dots,1\}$;

- ▷ nearest neigbor random walk reversible with respect to measure $\exp(-\beta N F(x))$ with F free energy;
- ▷ explicitely solvable;

Thus here we essentially have exactly the situation imagined by Kramers and Eyring.

The real thing:

Whenever we are not in one of the two situations above, we have problems:

▷ There is no exact reduction to a finite dimensional system!

Still, we expect an effective description of the dynamics in terms of some meso-scopic coarse grained dynamics!

In the remainder of this talk I will explain how this idea can be implemented in a simple example.

Random Hamiltonian:

$$H_N(\sigma) \equiv -\frac{N}{2}\left(\frac{1}{N}\sum_{i=1}^{N}\sigma_i\right)^2 - \sum_{i=1}^{N} h_i\sigma_i.$$

h_i, $i \in \mathbb{N}$ are (bounded) i.i.d. random variables, $\sigma \in \{-1,1\}^N$.

Equilibrium properties: [see Amaro de Matos, Patrick, Zagrebnov (92), Külske (97)]

Gibbs measure: $\mu_{\beta,N}(\sigma) = \frac{2^{-N}e^{-\beta H_N(\sigma)}}{Z_{\beta,N}}$

Magnetization: $m_N(\sigma) \equiv \frac{1}{N}\sum_{i=1}^{N}\sigma_i$.

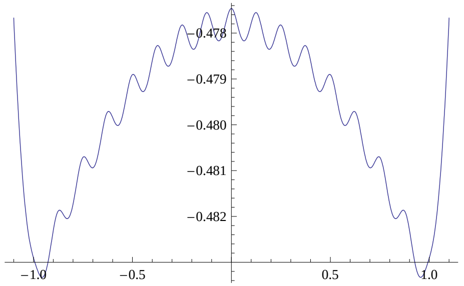
Induced measure: $\mathcal{Q}_{\beta,N} \equiv \mu_{\beta,N} \circ m_N^{-1}$. on the set $\Gamma_N \equiv \{-1, -1+2/N, \dots, +1\}$.

Using sharp large deviation estimates, one gets

$$Z_{\beta,N}\mathcal{Q}_{\beta,N}(m) = \sqrt{\frac{2I''_N(m)}{N\pi}} \exp\left\{-N\beta F_N(x)\right\}(1+o(1)),$$

where $F_N(x) \equiv \frac{1}{2}m^2 - \frac{1}{\beta}I_N(m)$ and $I_N(y)$ is the Legendre-Fenchel transform of

$$U_N(t) \equiv \frac{1}{N}\sum_{i\in\Lambda} \ln\cosh\left(t+\beta h_i\right)$$



Critical points: Solutions of $m^* = \frac{1}{N}\sum_{i\in\Lambda}\tanh(\beta(m^*+h_i))$.
Maxima if $\beta\mathbb{E}_h\left(1-\tanh^2(\beta(z^*+h))\right) > 1$.
Moreover, at critical points,

$$Z_{\beta,N}\mathcal{Q}_{\beta,N}(z^*) = \frac{\exp\left\{\beta N\left(-\frac{1}{2}(z^*)^2 + \frac{1}{\beta N}\sum_{i\in\Lambda}\ln\cosh\left(\beta(z^*+h_i)\right)\right)\right\}}{\sqrt{\frac{N\pi}{2}\left(\mathbb{E}_h\left(1-\tanh^2(\beta(z^*+h))\right)\right)}}(1+o(1))$$

We consider for definiteness discrete time Glauber dynamics with Metropolis transition probabilities

$$p_N(\sigma,\sigma') \equiv \frac{1}{N}\exp\left\{-\beta[H_N(\sigma')-H_N(\sigma)]_+\right\}$$

if σ and σ' differ on a single coordinate, and zero else.

We will be interested in transition times from a local minimum, m^*, to the set of "deeper" local minima,

$$M \equiv \{m : F_{\beta,N}(m) \leq F_{\beta,N}(m^*)\}.$$

Set $S[M] = \{\sigma \in S_N : m_N(\sigma) \in M\}$.

We need to define probability measures on $S[m^*]$ by

$$\nu_{m^*,M}(\sigma) = \frac{\mu_{\beta,N}(\sigma)\mathbb{P}_\sigma\left[\tau_{S[M]} < \tau_{S[m^*]}\right]}{\sum_{\sigma\in S[m^*]}\mu_{\beta,N}(\sigma)\mathbb{P}_\sigma\left[\tau_{S[M]} < \tau_{S[m^*]}\right]}.$$

Theorem 2. *Let m^* be a local minimum of $F_{\beta,N}$; let z^* be the critical point separating m^* from M.*

$$\mathbb{E}_{\nu_{m^*,M}}\tau_{S[M]} = \exp\left\{\beta N\left(F_N(m^*) - F_n(z^*)\right)\right\}$$
$$\times\frac{2\pi N}{\beta|\hat{\gamma}_1|}\sqrt{\frac{\beta\mathbb{E}_h\left(1-\tanh^2\left(\beta(z^*+h)\right)\right)-1}{1-\beta\mathbb{E}_h\left(1-\tanh^2\left(\beta(m^*+h)\right)\right)}}\left(1+o(1)\right),$$

where $\hat{\gamma}_1$ is the unique negative solution of the equation

$$\mathbb{E}_h\left[\frac{1-\tanh(\beta(z^*+h))}{\left[\beta\left(1+\tanh(\beta(z^*+h))\right)\right]^{-1}-\gamma}\right]=1.$$

Note that a naive approximation by a one-dimensional chain would give the same result except the wrong constant

$$\gamma = \frac{1}{\beta\mathbb{E}_h\left(1-\tanh^2\left(\beta(z^*+h)\right)\right)} - 1$$

The model was studied in

- ▷ F. den Hollander and P. dai Pra (JSP 1996) [large deviations, logarithmic asymptotics]
- ▷ P. Mathieu and P. Picco (JSP, 1998) [binary distribution; up to polynomial errors in N]
- ▷ A.B, M. Eckhoff, V. Gayrard, M. Klein (PTRF, 2001) [discrete distribution, up to multiplicative constants]

Both MP and BEGK made heavy use of exact mapping to finite-dimensional Markov chain!

The main goal of the present work was to show that potential theoretic methods allow to get sharp estimates (i.e. precise pre-factors of exponential rates) in spin systems at finite temperature when no symmetries are present. The RFCW model is the simplest model of this kind.

Remark on starting measure

The discusion above explains why it is natural in our formalism to get results for hitting times of the process started in the special measure ν_{m^*}.

Of course one would expect that in most cases, the same results hold unifomrly pointwise within a suitable set of in itial configurations, i.e. $\mathbb{E}_{\nu_{m^*}} \tau_{S[M]} \sim \mathbb{E}_\sigma \tau_{S[M]}$, for all $\sigma \in \mathcal{S}[m^*]$.

In our case, we can show this to be true using a rather elaborate coupling argument (BiBoIo, 2009). This allows also to prove that $\tau_{S[M]}/\mathbb{E}_{\nu_{m^*}} \tau_{S[M]}$ is asymptotically exponentially distributed.

Coarse graining

$I_\ell, \ell \in \{1, \ldots, n\}$: partition of the support of the distribution of the random field.

Random partition of the set $\Lambda \equiv \{1, \ldots, N\}$

$$\Lambda_k \equiv \{i \in \Lambda : h_i \in I_k\}$$

Order parameters

$$\boldsymbol{m}_k(\sigma) \equiv \frac{1}{N} \sum_{i \in \Lambda_k} \sigma_i$$

$$H_N(\sigma) = -NE(\boldsymbol{m}(\sigma)) + \sum_{\ell=1}^{n} \sum_{i \in I_\ell} \sigma_i \tilde{h}_i$$

where $\tilde{h}_i = h_i - \bar{h}_\ell$, $i \in \Lambda_\ell$. Note $|\tilde{h}_i| \leq c/n$;

$$E(\boldsymbol{x}) \equiv \frac{1}{2} \left(\sum_{\ell=1}^{n} \boldsymbol{x}_\ell \right)^2 + \sum_{\ell=1}^{n} \bar{h}_\ell \boldsymbol{x}_\ell$$

Equilibrium distribution of the variables $\boldsymbol{m}[\sigma]$

$$\mu_{\beta,N}(\boldsymbol{m}(\sigma) = \boldsymbol{x}) \equiv \mathcal{Q}_{\beta,N}(\boldsymbol{x})$$

Coarse grained Dirichlet form:

$$\widehat{\Phi}(g) \equiv \sum_{\boldsymbol{x},\boldsymbol{x}' \in \Gamma_N} \mathcal{Q}_{\beta,N}[\omega](\boldsymbol{x}) r_N(\boldsymbol{x},\boldsymbol{x}') \left[g(\boldsymbol{x}) - g(\boldsymbol{x}')\right]^2$$

with

$$r_N(\boldsymbol{x},\boldsymbol{x}') \equiv \frac{1}{\mathcal{Q}_{\beta,N}[\omega](\boldsymbol{x})} \sum_{\sigma:\boldsymbol{m}(\sigma)=\boldsymbol{x}} \mu_{\beta,N}[\omega](\sigma) \sum_{\sigma':\boldsymbol{m}(\sigma)=\boldsymbol{x}'} p(\sigma,\sigma').$$

Approximate harmonic functions

The key step in the proof of both upper and lower bounds is to find a function that is almost harmonic in a small neighborhood of the relevant saddle point. This will be given by

$$h(\sigma) = g(\boldsymbol{m}(\sigma)) = f((\boldsymbol{v}, (\boldsymbol{z}^* - \boldsymbol{m}(\sigma))))$$

for suitable vector $\boldsymbol{v} \in \mathbb{R}^n$ and $f : \mathbb{R} \to \mathbb{R}_+$

$$f(a) = \sqrt{\frac{\beta N \hat{\gamma}_1^{(n)}}{2\pi}} \int_{-\infty}^{a} e^{-\beta N |\hat{\gamma}_1| u^2/2} du.$$

This yields a straightforward upper bound for capacities which will turn out to be the correct answer, as $n \uparrow \infty$!

Construction of flow

Lower bound on capacity uses the Berman-Konsowa variational principle. Again, care has to be taken in the construction of the flow only near the saddle point.

Two scale construction:

- ▷ Construct mesoscopic flow on variables m from approximate harmonic function used in upper bound. This gives good lower bound in the mesoscopic Dirichlet form.
- ▷ Construct microscopic flow for each mesoscopic path.
- ▷ Use the magnetic field is almost constant and averaging that conductance of most mesoscopic paths give the same values as in mesoscopic Dirichlet function.

This yields upper lower bound that differs from upper bound only by factor $1 + O(1/n))$.

Result for capacity

If $A = \{\sigma : \boldsymbol{m}_N(\sigma) = \boldsymbol{m}_1\}$, $B = \{\sigma : \boldsymbol{m}_N(\sigma) = \boldsymbol{m}_2\}$, and $\boldsymbol{z}^*$ is the essential saddle point connecting them, then

$$\text{cap}(A,B) = \mathcal{Q}_{\beta,N}(\boldsymbol{z}^*)\frac{\beta|\hat{\boldsymbol{\gamma}}_1^{(n)}|}{2\pi N}\left(\prod_{\ell=1}^{n}\sqrt{r_\ell}\right)\left(\frac{\pi N}{2\beta}\right)^{n/2}\frac{1}{\sqrt{\prod_{j=1}^{n}|\hat{\boldsymbol{\gamma}}_j|}}(1+O(\epsilon))$$

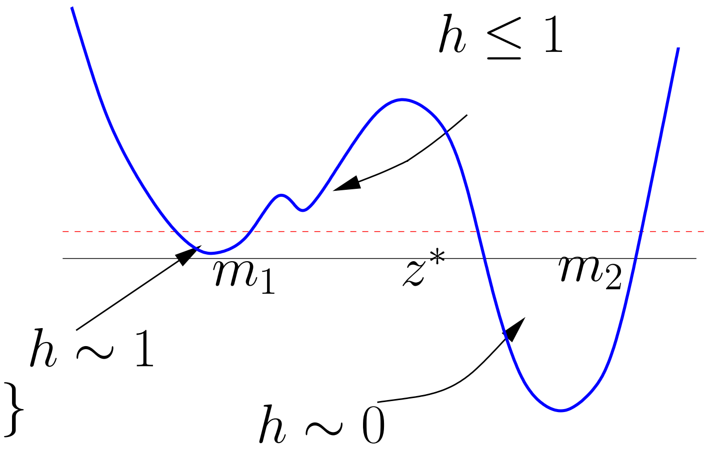
This can be re-written as:

***Theorem* 3.**

$$Z_{\beta,N}\textit{cap}(A,B) = \frac{\beta|\hat{\boldsymbol{\gamma}}_1^{(n)}|}{2\pi N}\frac{\exp\left\{\beta N\left(-\frac{1}{2}F_N(z^*)\right)\right\}(1+o(\epsilon))}{\sqrt{\beta\mathbb{E}_h\left(1-\tanh^2(\beta(z^*+h))\right)-1}}.$$

Final step in control of mean hitting times: Compute

$$\sum_{\sigma} \mu_{\beta,N}(\sigma) h_{A,B}(\sigma) \sim \mathcal{Q}_{\beta,N}([\rho + m_1, m_1 - \rho])$$



This requires to show that: $h_{A,B}(\sigma) \sim 1$, if σ near A, and

$h_{A,B}(\sigma) \leq \exp\left\{-\mathbb{N}(F_{\beta,N}(z^*) - F_{\beta,N}(m_N(\sigma)) - \delta)\right\}$

if $F_{\beta,N}(m_N(\sigma) \leq F_{b,N}(m_1)$.

Can be done using super-harmonic barrier function.

Future Challenges

Some further problems that have been or are being treated:

▷ Discrete diffusions (BEGK ´00)

▷ Glauber dynamics for Ising models, $T \downarrow 0$ (B, F. Manzo '01)

▷ Kawasaki dynamics, $T \downarrow 0$, (B,den Hollander, Nardi 05, B, dH, Spitoni 10)

▷ SPDE in $d = 1$ (Barret, B, Méléard 10, Barret 11)

▷ Ageing in disordered systems :

- Random Energy model (G. Ben Arous, B, V. Gayrard '03)
- REM - like trap model (B., A. Faggionato '05)
- Sinai's random walk (B, A. Faggionato, '08)

Major challenges:

▷ Extension of the general theory to infinite dimensional cases:

▷ Lattice models (Glauber, Kawasaki)

- at finite temperatures
- in infinite volume

▷ Infinite dimensional diffusion processes in $d > 1$?

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