Free-energy dissipative schemes for the Oldroyd-B model

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The Model: Oldroyd-B

A model for dilute polymer solutions: Oldroyd-B

Conservation of momentum:

$$\mathrm{Re}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}\right) = -\boldsymbol{\nabla} p + (1-\varepsilon)\Delta\mathbf{u} + \operatorname{div}\boldsymbol{\tau}$$

Conservation of mass:

$$\operatorname{div}\mathbf{u} = 0$$

Constitutive equation (Maxwell fluid):

$$\frac{\partial \boldsymbol{\tau}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\boldsymbol{\tau} - (\boldsymbol{\nabla}\mathbf{u})\boldsymbol{\tau} - \boldsymbol{\tau}(\boldsymbol{\nabla}\mathbf{u})^T = -\frac{1}{\mathrm{Wi}}\boldsymbol{\tau} + \frac{\varepsilon}{\mathrm{Wi}}\left[\boldsymbol{\nabla}\mathbf{u} + (\boldsymbol{\nabla}\mathbf{u})^T\right]$$

$\mathbf{u}$ - velocity field, p - hydrostatic pressure, $\boldsymbol{\tau}$ - extra-stress tensor
$\mathrm{Re} = \frac{\rho U L}{\eta}$, $\varepsilon = \frac{\eta_p}{\eta}$, $\mathrm{Wi} = \frac{\lambda U}{L} := \frac{t_{elastic}}{t_{flow}}$ (Weissenberg number)

Local existence

Assume $\mathbf{x} \in \Omega$ bounded in $\mathbb{R}^d$ ($d = 2$ or 3).
Provide initial data $\mathbf{u}(t = 0), \boldsymbol{\tau}(t = 0)$.
Assume Homogeneous Dirichlet boundary conditions on $\mathbf{u}$.
Local in time existence theorem for smooth enough initial data.
No global existence theorem.

Numerical difficulties

Problems as $\mathrm{Wi} \nearrow O(1)$:

- Numerical instability causing divergence
- Results are mesh-dependent

Possible reasons:

- Analytical difficulties: apart from some cases, there may be no steady-state solution, or an unstable solution.
- Bad numerical scheme

Goal

Goal: write a "good" numerical scheme which verifies a free energy inequality.
Consequence: no spurious free energy is created.
Analytical outputs: global existence of discrete solutions.

Conformation tensor

Definition: conformation tensor

$$\boldsymbol{\sigma} = \mathbf{I} + \frac{\mathrm{Wi}}{\varepsilon}\boldsymbol{\tau}$$

$\boldsymbol{\sigma}$ is symmetric positive definite (spd)

Rewrite system: Oldroyd-B-$\boldsymbol{\sigma}$

$$\begin{aligned}
\mathrm{Re}\left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right] &= -\nabla p + (1-\varepsilon)\Delta\mathbf{u} + \frac{\varepsilon}{\mathrm{Wi}}\,\mathrm{div}\,\boldsymbol{\sigma} \\
\mathrm{div}\,\mathbf{u} &= 0 \\
\frac{\partial \boldsymbol{\sigma}}{\partial t} + (\mathbf{u}\cdot\nabla)\boldsymbol{\sigma} - (\nabla\mathbf{u})^T\boldsymbol{\sigma} - \boldsymbol{\sigma}(\nabla\mathbf{u}) &= -\frac{\boldsymbol{\sigma} - \mathbf{I}}{\mathrm{Wi}}
\end{aligned}$$

For a smooth solution with $\boldsymbol{\sigma}(t=0)$ spd, $\boldsymbol{\sigma}(t)$ is spd at all time.

Improved stability of equations

Better numerical stability when using a logarithmic transformation of $\boldsymbol{\sigma}$ [Fattal, Kupferman]:

$$\psi = \ln \boldsymbol{\sigma}$$

(well-defined since $\boldsymbol{\sigma}$ is spd.)
Question: does this improved stability show up in our analysis?

A micro-macro model: Hookean dumbbells

The Oldroyd-B model is formally equivalent to the following micro-macro model:

Hookean dumbbells

Navier-Stokes equations +

$$
\begin{aligned}
\boldsymbol{\tau} &= \frac{\varepsilon}{\mathrm{Wi}}\left(\int_{\mathbb{R}^d} \mathbf{q}\otimes\mathbf{q}\,\psi(t,\mathbf{x},\mathbf{q})d\mathbf{q} - \mathbf{I}\right) \\
\frac{\partial\psi}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\psi &= \mathrm{div}_{\mathbf{q}}\left(\left((\boldsymbol{\nabla}\mathbf{u})\,\mathbf{q} - \frac{1}{2\mathrm{Wi}}\mathbf{q}\right)\psi\right) + \frac{1}{2\mathrm{Wi}}\Delta_{\mathbf{q}}\psi
\end{aligned}
$$

q - extension vector of dumbbells, ψ - distribution of the dumbbells.

Entropy and kinetic energy

In the micro-macro context, it is natural to work with the entropy:

$$H(t) = \int_\Omega \int_{\mathbb{R}^d} \psi(t, \mathbf{x}, \mathbf{q}) \ln \frac{\psi(t, \mathbf{x}, \mathbf{q})}{\psi_\infty(\mathbf{q})}$$

where $\psi_\infty(\mathbf{q}) = \frac{exp(-|\mathbf{q}|^2)}{\int_{\mathbb{R}^d} exp(-|\mathbf{q}|^2)}$ is the equilibrium distribution.
Compute and obtain a macroscopic quantity:

$$H(t) = \frac{1}{2} \int_\Omega \mathrm{tr}(\boldsymbol{\sigma} - \ln \boldsymbol{\sigma} - \mathbf{I})$$

Define the kinetic energy:

$$E_{\mathrm{kinetic}}(t) = \frac{\mathrm{Re}}{2} \int_\Omega |\mathbf{u}|^2$$

Free energy

Definition: free energy

Let $(\mathbf{u}, p, \boldsymbol{\sigma})$ be a smooth solution to Oldroyd-B-$\boldsymbol{\sigma}$. Define

$$F(\mathbf{u}, \boldsymbol{\sigma}) = \underbrace{\frac{\mathrm{Re}}{2}\int |\mathbf{u}|^2}_{\text{kinetic energy}} + \underbrace{\frac{\varepsilon}{2\mathrm{Wi}}\int_\Omega \mathrm{tr}(\boldsymbol{\sigma} - \ln \boldsymbol{\sigma} - \mathbf{I})}_{\text{entropy}}$$

$\boldsymbol{\sigma}$ spd $\Rightarrow F \geq 0$ always.

Free energy dissipation

A free energy equality

Let $(\mathbf{u}, p, \boldsymbol{\sigma})$ be a smooth solution to Oldroyd-B-$\boldsymbol{\sigma}$, then:

$$\frac{d}{dt}F(\mathbf{u},\boldsymbol{\sigma}) + \underbrace{(1-\varepsilon)\int_\Omega |\nabla \mathbf{u}|^2 + \frac{\varepsilon}{2\mathrm{Wi}^2}\int_\Omega \mathrm{tr}(\boldsymbol{\sigma} + \boldsymbol{\sigma}^{-1} - 2\mathbf{I})}_{\text{dissipative terms}\geq 0} = 0$$

$\Rightarrow$ F decreases in time.

Consequence : there exists $C > 0$ s.t.

$$F(\mathbf{u},\boldsymbol{\sigma}) \leq F(\mathbf{u}(t=0),\boldsymbol{\sigma}(t=0))\exp(-Ct)$$

Useful to characterize long-time asymptotics of solutions. [Jourdain, Le-Bris, Lelièvre, Otto 06].

A remark: classical energy estimate

The classical energy estimate can be used, but to obtain exponential decay, the assumption $\det \boldsymbol{\sigma}(t = 0) > 1$ is needed.

Numerical framework: finite elements

In order to work in the F.E. framework, we need a variational formulation.
Then we need to be able to recover the free energy equality.

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A smooth solution to Oldroyd-B-$\boldsymbol{\sigma}$ $(\mathbf{u}, p, \boldsymbol{\sigma})$ satisfies

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$$\underbrace{-p\,\mathrm{div}\,\mathbf{v} + q\,\mathrm{div}\,\mathbf{u}}_{\text{NS terms}}$$

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$$\underbrace{+ \left(\frac{\partial\boldsymbol{\sigma}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\sigma}\right) : \phi - ((\nabla\mathbf{u})\boldsymbol{\sigma} + \boldsymbol{\sigma}(\nabla\mathbf{u})^T) : \phi + \frac{1}{\mathrm{Wi}}(\boldsymbol{\sigma} - \mathbf{I}) : \phi}_{\text{constitutive eq. terms}} \Bigg),$$

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for all sufficiently smooth test functions $(\mathbf{v}, q, \boldsymbol{\phi})$.

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for all sufficiently smooth test functions $(\mathbf{v}, q, \phi)$.
The free energy equality is recovered with
$(\mathbf{v}, q, \phi) = (\mathbf{u}, p, \frac{\varepsilon}{2\mathrm{Wi}}(\mathbf{I} - \boldsymbol{\sigma}^{-1})) \Rightarrow \boldsymbol{\sigma}^{-1}$ is a test function.

A "simple" choice of F.E.

Scott-Vogelius mixed finite element space for $(\mathbf{u}_h, p_h)$:

- $\mathbf{u}_h \in (\mathbb{P}_2)^2$
- $p_h \in \mathbb{P}_{1,disc}$

Good because $\operatorname{div} \mathbf{u}_h(\mathbf{x}) = 0, \forall \mathbf{x} \in \Omega$.
For meshes built in a certain way, this F.E. satisfies the Babuška-Brezzi inf-sup condition.
For $\boldsymbol{\sigma}_h$, assume simply

- $\boldsymbol{\sigma}_h \in (\mathbb{P}_0)^3 \Longrightarrow \boldsymbol{\sigma}_h^{-1}$ can be used as test function.

Main challenges

- Discretize the advection term $(\mathbf{u} \cdot \boldsymbol{\nabla})\boldsymbol{\sigma}$:
  1. Method of characteristics
  2. Discontinuous Galerkin method
- Recover a free energy dissipation.

Discrete problem

$$0 = \int_\Omega (\mathrm{stuff}_h^n)$$

Local existence

Show local existence in time, then define

Definition: discrete free energy

The free energy for the solution $(\mathbf{u}_h^n, p_h^n, \boldsymbol{\sigma}_h^n)$ is:

$$F_h^n = F(\mathbf{u}_h^n, \boldsymbol{\sigma}_h^n) = \frac{\mathrm{Re}}{2} \int_\Omega |\mathbf{u}_h^n|^2 + \frac{\varepsilon}{2\mathrm{Wi}} \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}_h^n - \ln \boldsymbol{\sigma}_h^n - \mathbf{I})$$

Recovering a free energy dissipation (method of characteristics)

The problematic terms are the time derivatives.

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1. Treatment of $\mathbf{u}$ derivative:

$\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u}\right) \cdot \mathbf{v}$ is discretized as $\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t} + \mathbf{u}_h^n \cdot \nabla \mathbf{u}_h^{n+1}\right) \cdot \mathbf{v}$.

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$$= \int_\Omega \left(\frac{|\mathbf{u}_h^{n+1}|^2 - |\mathbf{u}_h^n|^2}{2\Delta t} + \frac{|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|^2}{2\Delta t} + \mathbf{u}_h^n \cdot \nabla \frac{|\mathbf{u}_h^{n+1}|^2}{2}\right)$$

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(since $\operatorname{div}\mathbf{u}_h^n(\mathbf{x}) = 0, \forall \mathbf{x}\in\Omega$).

Recovering a free energy dissipation (method of characteristics)

2. Treatment of σ derivative:
$\left(\frac{\partial \boldsymbol{\sigma}}{\partial t} + \mathbf{u} \cdot \nabla \boldsymbol{\sigma}\right) : \phi$ is discretized as $\left(\frac{\boldsymbol{\sigma}_h^{n+1} - \boldsymbol{\sigma}_h^n \circ X^n(t^n)}{\Delta t}\right) : \phi$ where

$$\begin{cases} \frac{d}{dt} X^n(t,x) = \mathbf{u}_h^n(X^n(t,x)), \quad \forall t \in [t^n, t^{n+1}], \\ X^n(t^{n+1}, x) = x. \end{cases}$$

$\operatorname{div} \mathbf{u}_h^n(\mathbf{x}) = 0, \forall \mathbf{x} \in \Omega \Rightarrow X^n(t)$ is a mapping with constant Jacobian $(= 1)$, $\forall t \in [t^n, t^{n+1}]$.

Recovering a free energy dissipation (method of characteristics)

Lemma 1:

Let $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ be two symmetric positive definite matrices. Then

$$\mathrm{tr}\left((\boldsymbol{\sigma}-\boldsymbol{\tau})\boldsymbol{\tau}^{-1}\right) = \mathrm{tr}(\boldsymbol{\sigma}\boldsymbol{\tau}^{-1} - \mathbf{I}) \geq \mathrm{tr}\left(\ln \boldsymbol{\sigma} - \ln \boldsymbol{\tau}\right),$$

Recovering a free energy dissipation with the method of characteristics

Use as test function $\phi = \mathbf{I} - (\boldsymbol{\sigma}_h^{n+1})^{-1}$, and use Lemma 1:

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$$\int_\Omega \left(\boldsymbol{\sigma}_h^{n+1} - \boldsymbol{\sigma}_h^n \circ X^n(t^n)\right) : (\mathbf{I} - (\boldsymbol{\sigma}_h^{n+1})^{-1})$$

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$$= \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}_h^{n+1}) - \mathrm{tr}(\boldsymbol{\sigma}_h^n \circ X^n(t^n)) + \mathrm{tr}\left([\boldsymbol{\sigma}_h^n \circ X^n(t^n)][\boldsymbol{\sigma}_h^{n+1}]^{-1} - \mathbf{I}\right)$$

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$$\geq \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}_h^{n+1}) - \mathrm{tr}(\boldsymbol{\sigma}_h^n \circ X^n(t^n)) + \mathrm{tr}\ln(\boldsymbol{\sigma}_h^n \circ X^n(t^n)) - \mathrm{tr}\ln(\boldsymbol{\sigma}_h^{n+1})$$

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&\int_\Omega \left(\boldsymbol{\sigma}_h^{n+1} - \boldsymbol{\sigma}_h^n \circ X^n(t^n)\right) : (\mathbf{I} - (\boldsymbol{\sigma}_h^{n+1})^{-1}) \\
&= \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}_h^{n+1}) - \mathrm{tr}(\boldsymbol{\sigma}_h^n \circ X^n(t^n)) + \mathrm{tr}\left([\boldsymbol{\sigma}_h^n \circ X^n(t^n)][\boldsymbol{\sigma}_h^{n+1}]^{-1} - \mathbf{I}\right) \\
&\geq \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}_h^{n+1}) - \mathrm{tr}(\boldsymbol{\sigma}_h^n \circ X^n(t^n)) + \mathrm{tr}\ln(\boldsymbol{\sigma}_h^n \circ X^n(t^n)) - \mathrm{tr}\ln(\boldsymbol{\sigma}_h^{n+1}) \\
&\qquad = \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}_h^{n+1} - \ln \boldsymbol{\sigma}_h^{n+1}) - \mathrm{tr}(\boldsymbol{\sigma}_h^n - \ln \boldsymbol{\sigma}_h^n)
\end{aligned}$$

Recovering a free energy dissipation with the method of characteristics

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$$\int_\Omega \left(\boldsymbol{\sigma}_h^{n+1} - \boldsymbol{\sigma}_h^n \circ X^n(t^n)\right) : (\mathbf{I} - (\boldsymbol{\sigma}_h^{n+1})^{-1})$$

$$= \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}_h^{n+1}) - \mathrm{tr}(\boldsymbol{\sigma}_h^n \circ X^n(t^n)) + \mathrm{tr}\left([\boldsymbol{\sigma}_h^n \circ X^n(t^n)][\boldsymbol{\sigma}_h^{n+1}]^{-1} - \mathbf{I}\right)$$

$$\geq \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}_h^{n+1}) - \mathrm{tr}(\boldsymbol{\sigma}_h^n \circ X^n(t^n)) + \mathrm{tr}\ln(\boldsymbol{\sigma}_h^n \circ X^n(t^n)) - \mathrm{tr}\ln(\boldsymbol{\sigma}_h^{n+1})$$

$$= \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}_h^{n+1} - \ln \boldsymbol{\sigma}_h^{n+1}) - \mathrm{tr}(\boldsymbol{\sigma}_h^n - \ln \boldsymbol{\sigma}_h^n)$$

(since X^n has Jacobian equal to 1).

A discrete free energy inequality

Discrete free energy inequality

Let $(\mathbf{u}_h^n, p_h^n, \boldsymbol{\sigma}_h^n)_{0\le n\le N_T}$ be a solution to the discrete problem, such that $\boldsymbol{\sigma}_h^n$ is spd. Then, the free energy of the solution $(\mathbf{u}_h^n, p_h^n, \boldsymbol{\sigma}_h^n)$: satisfies:

$$F_h^{n+1} - F_h^n + \int_\Omega \frac{\mathrm{Re}}{2}|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|^2 + \Delta t \int_\Omega (1-\varepsilon)|\nabla \mathbf{u}_h^{n+1}|^2 \\ + \Delta t \frac{\varepsilon}{2\mathrm{Wi}^2}\int_\Omega \mathrm{tr}\left(\boldsymbol{\sigma}_h^{n+1} + (\boldsymbol{\sigma}_h^{n+1})^{-1} - 2I\right) \le 0.$$

In particular, the sequence $(F_h^n)_{0\le n\le N_T}$ is non-increasing.

Same free energy inequality for the DG method

The log transformation

The same treatment can be applied to the log-transformed system. We obtain an equivalent free energy inequality:

log free energy inequality

For a solution $(\mathbf{u}_h^n, p_h^n, \psi_h^n)$, the free energy is

$$F_h^n = F(\mathbf{u}_h^n, e^{\psi_h^n}) = \frac{\mathrm{Re}}{2}\int_\Omega |\mathbf{u}_h^n|^2 + \frac{\varepsilon}{2\mathrm{Wi}}\int_\Omega \mathrm{tr}(e^{\psi_h^n} - \psi_h^n - \mathbf{I}),$$

and it satisfies

$$F_h^{n+1} - F_h^n + \int_\Omega \frac{\mathrm{Re}}{2}|\mathbf{u}_h^{n+1} - \mathbf{u}_h^n|^2 + \Delta t\int_\Omega (1-\varepsilon)|\nabla \mathbf{u}_h^{n+1}|^2 + \Delta t\frac{\varepsilon}{2\mathrm{Wi}^2}\int_\Omega \mathrm{tr}\left(e^{\psi_h^{n+1}} + e^{-\psi_h^{n+1}} - 2I\right) \leq 0.$$

We obtained a "good" scheme

As a consequence, we obtain some numerical stability:

1. Global existence in time and uniqueness of discrete solutions for Δt small enough in the σ formulation.
2. In the case of the log formulation, global existence for any Δt.

Comments and further work

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MERCI!

Properties needed to obtain free energy estimate

Advection discretized by:	Characteristics	DG
Requirements for $\mathbf{u}_h$:	$\operatorname{div} \mathbf{u}_h = 0$ ($\Rightarrow \det(\nabla_{\mathbf{x}} X^n) \equiv 1$) ($\Rightarrow (\mathbf{u}_h \cdot \mathbf{n})\|_{E_j}$ well defined)	$\int_\Omega q \operatorname{div} \mathbf{u}_h = 0$, $\forall q \in \mathbb{P}_0$ and $(\mathbf{u}_h \cdot \mathbf{n})\|_{E_j}$ well defined

Table: Summary of the arguments with $(\mathbf{u}_h, p_h, \boldsymbol{\sigma}_h)$ or $(\mathbf{u}_h, p_h, \psi_h)$ in $(\mathbb{P}_2)^d \times \mathbb{P}_{1,disc} \times (\mathbb{P}_0)^{\frac{d(d+1)}{2}}$

Discretized problem (characteristics method)

For a given $(\mathbf{u}_h^n, p_h^n, \boldsymbol{\sigma}_h^n)$, find $(\mathbf{u}_h^{n+1}, p_h^{n+1}, \boldsymbol{\sigma}_h^{n+1}) \in (\mathbb{P}_2)^2 \times \mathbb{P}_{1,disc} \times (\mathbb{P}_0)^3$ such that, for any test function $(\mathbf{v}, q, \phi) \in (\mathbb{P}_2)^2 \times \mathbb{P}_{1,disc} \times (\mathbb{P}_0)^3$,

$$
\begin{aligned}
0 = \int_\Omega \mathrm{Re}\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t} + \mathbf{u}_h^n \cdot \boldsymbol{\nabla}\mathbf{u}_h^{n+1}\right) \cdot \mathbf{v} - p_h^{n+1} \operatorname{div} \mathbf{v} + q \operatorname{div} \mathbf{u}_h^{n+1} \\
+ (1-\varepsilon)\boldsymbol{\nabla}\mathbf{u}_h^{n+1} : \boldsymbol{\nabla}\mathbf{v} + \frac{\varepsilon}{\mathrm{Wi}}\boldsymbol{\sigma}_h^{n+1} : \boldsymbol{\nabla}\mathbf{v} \\
+ \left(\frac{\boldsymbol{\sigma}_h^{n+1} - \boldsymbol{\sigma}_h^n \circ X^n(t^n)}{\Delta t}\right) : \phi - \left((\boldsymbol{\nabla}\mathbf{u}_h^{n+1})\boldsymbol{\sigma}_h^{n+1} + \boldsymbol{\sigma}_h^{n+1}(\boldsymbol{\nabla}\mathbf{u}_h^{n+1})^T\right) : \phi \\
+ \frac{1}{\mathrm{Wi}}(\boldsymbol{\sigma}_h^{n+1} - \mathbf{I}) : \phi.
\end{aligned}
$$

Discretized problem (DG)

For a given $(\mathbf{u}_h^n, p_h^n, \boldsymbol{\sigma}_h^n)$, find $(\mathbf{u}_h^{n+1}, p_h^{n+1}, \boldsymbol{\sigma}_h^{n+1}) \in (\mathbb{P}_2)^2 \times \mathbb{P}_{1,disc} \times (\mathbb{P}_0)^3$ such that, for any test function $(\mathbf{v}, q, \phi) \in (\mathbb{P}_2)^2 \times \mathbb{P}_{1,disc} \times (\mathbb{P}_0)^3$,

$$
\begin{aligned}
0 = \sum_{k=1}^{N_K} \int_{K_k} \mathrm{Re}\left(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t} + \mathbf{u}_h^n \cdot \nabla \mathbf{u}_h^{n+1}\right) \cdot \mathbf{v} - p_h^{n+1} \,\mathrm{div}\, \mathbf{v} + q \,\mathrm{div}\, \mathbf{u}_h^{n+1} \\
+ (1-\varepsilon) \nabla \mathbf{u}_h^{n+1} : \nabla \mathbf{v} + \frac{\varepsilon}{\mathrm{Wi}} \boldsymbol{\sigma}_h^{n+1} : \nabla \mathbf{v} \\
+ \left(\frac{\boldsymbol{\sigma}_h^{n+1} - \boldsymbol{\sigma}_h^n}{\Delta t}\right) : \phi - \left((\nabla \mathbf{u}_h^{n+1}) \boldsymbol{\sigma}_h^{n+1} + \boldsymbol{\sigma}_h^{n+1} (\nabla \mathbf{u}_h^{n+1})^T\right) : \phi \\
+ \frac{1}{\mathrm{Wi}} (\boldsymbol{\sigma}_h^{n+1} - \mathbf{I}) : \phi + \sum_{j=1}^{N_E} \int_{E_j} |\mathbf{u}_h^n \cdot \mathbf{n}| \, [\![\boldsymbol{\sigma}_h^{n+1}]\!] : \phi^+
\end{aligned}
$$

Discretization of the advection term in the DG method

$\left(\frac{\partial \boldsymbol{\sigma}}{\partial t} + \mathbf{u} \cdot \boldsymbol{\nabla} \boldsymbol{\sigma}\right) : \phi$ is discretized as

$$\left(\frac{\boldsymbol{\sigma}_h^{n+1} - \boldsymbol{\sigma}_h^n \circ X^n(t^n)}{\Delta t}\right) : \phi + \sum_{j=1}^{N_E} \int_{E_j} |\mathbf{u}_h^n \cdot \mathbf{n}| \, [\![\boldsymbol{\sigma}_h^{n+1}]\!] : \phi^+$$