Viscoplastic flows and free surface.

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Outline of this talk

- I) Shallow-water type equations from Navier-Stokes equations.
- I.a) Formal derivation: Effect of boundary conditions and Reynolds number order.
- I.b) Mathematical justifications and open questions.
- I.c) Global weak solutions for viscous shallow-water system.
- II) Shallow-water type equations with thresholds terms.
- II.a) Effect of contact angle: some physical works.
- II.b) Compressible viscoplastic constitutive law :
  Numerical difficulties, mathematical difficulties.
- III) Other models used in avalanches simulation.

Shallow water equations from Navier-Stokes equations

SHALLOW–WATER TYPE SYSTEMS:

Used in various applications (river flows, ocean, MHD.....)

Obtained in a simple version from elementary principles in 1871, De St Venant.

Described flow: vertical mean value of the horizontal flow components.

Various models obtained from $\neq$ systems : Euler-Irrotationnel, Navier-Stokes, ...

From 'Euler-Irrotationnel': *see.* D. Lannes *et. al*.

Here we focus on viscous effects and/or threshold effects through recent works.

Shallow water equations from Navier-Stokes equations

Formal derivation from Navier-Stokes :

Strongly depends on boundary conditions choices.

- a) No slip conditions.
- b) Friction conditions (which BC ? Wall laws!).
- c) Mixing of the both?

Strongly depends on the order of the Reynolds number.

- i) Reynolds order 1.
- ii) Reynolds linked to the aspect ratio (which link?).

Examples.

- cas a)-i) See for instance J.–P. Vila (2007),
- cas b)-ii) See for Instance J.–F. Gerbeau, B. Perthame (2001) ($\mathrm{Re} = O(1/\varepsilon)$).

Shallow water equations from Navier-Stokes equations

Denoting $U = (u, w)$ the velocity of the flow governed by Navier-Stokes and $v = \left(\int_0^{h(t,x)} u(t,\xi)\, d\xi \right) / h(t,x)$ the vertical mean value of the horizontal component.

Asymptotic at order 2 !!

Slip boundary condition.

See: J.–F. Gerbeau, B. Perthame (*DCDS*, 2001) with a flat bottom.

$$\begin{cases} h_t + (hv)_x = 0, \\ (hv)_t + (h\, v^2 + \dfrac{h^2}{2})_x - \overline{\kappa} h\, h_{xxx} - 4\varepsilon \partial_x (h \partial_x v) + \dfrac{r_0 v}{1 + \varepsilon h} = 0. \end{cases}$$

No slip boundary condition with slop.

see: J.–P. Vila (2007) with $c = \cos\theta$ and $s = \sin\theta$, θ slop angle.

$$\begin{cases} h_t + (hv)_x = 0, \\ (hv)_t + (\dfrac{6}{5} h\, v^2 + c\, \dfrac{h^2}{2} - \dfrac{(2s)^2}{75} h^5)_x - \overline{\kappa} h\, h_{xxx} = \dfrac{1}{\varepsilon} \big(2s\, h - \dfrac{3\, v}{h}\big). \end{cases}$$

If we want to see viscous effect, go next order!

Shallow water equations from Navier-Stokes equations

Slip boundary condition (J.-F. Gerbeau, B. Perthame)

$$\begin{cases} \partial_z^2 u^0 = 0, \\ \partial_z u^0|_{z=0} = 0, \partial_z u^0|_{z=h} = 0. \end{cases}$$

$\Longrightarrow$ $u^0(t,x)$ only

$\Longrightarrow$ need dynamics for $u^0(t,x)$ (Hyperbolic Shallow-Water equations)

$\Longrightarrow$ Next order (viscous term). $\Longrightarrow$ Need order 2 to close the system since $u^1|_{z=0}$ is *a priori* unknown.

No slip condition (J.-P. Vila)

$$\begin{cases} -\partial_z^2 u^0 = 2s, \\ u^0|_{z=0} = 0, \partial_z u^0|_{z=h} = 0. \end{cases}$$

$\Longrightarrow$ $u^0(t,x)$ depends explicitely on h (Nusselt profile).

$\Longrightarrow$ Thin film equation (see BERTOZZI, PUGH *et al.*).

$\Longrightarrow$ Need next order to get the shallow water dynamic.

$\Longrightarrow$ Need order 2 to close the system since $\partial_z u^1|_{z=0}$ is *a priori* unknown.

Shallow water equations from Navier-Stokes equations

Mathematical justification of the formal derivations.

1st mathematical justification and derivation of the J.–P. Vila model:
Non zero capillarity and lateral periodic condition !! : D.B., P. Noble (*Methods of Anal. and Appl.*, 2007), (strong solution, 2D->1D).

Global existence for free surface on slope: T. Nishida, Y. Teramoto, H.A. Win. J. Math. Kyoto Univ. (1993).

Mathematical justification and derivation of the J.–F. Gerbeau, B. Perthame model:
D.B., P. Noble: large almost 2D initial data, from primitive equations, still in progress.

In the spirit of thin-domain Navier-Stokes equations: D. Iftimie, G. Raugel, G. Sell, R. Temam, M. Ziane etc...

Open problems: Mathematical justification without surface tension, dynamical shore boundary conditions.

Shallow water equations from Navier-Stokes equations

Global existence of weak solutions for viscous degenerate shallow-water system.
An interesting mathematical structure for a model as Gerbeau-Perthame model (D.B., B. Desjardins, CMP2003)!!

Simplification $\Longrightarrow$ Expressions for $r_0 = 0$ and $\overline{\kappa} = 0$:
1) Energy equality :

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}h|v|^2 + |h|^2\right) + \int_\Omega 4\varepsilon|\partial_x v|^2 = 0.$$

2) A mathematical entropy (*cf.* works D.B., B. Desjardins on St. Venant.)

$$\frac{d}{dt}\int_\Omega \left(\frac{1}{2}h|v - 2\varepsilon\partial_x \log h|^2 + |h|^2\right) + \int_\Omega |\partial_x h|^2 = 0.$$

If we control $\partial_x\sqrt{h}$ initially, we control it all the time..... and prove global existence of weak solutions!!

Remark: We find a specific velocity $v - 2\varepsilon\partial_x \log h$! What it means physically?? It comes from degenerate viscosity!! How to conserve it numerically??

Some examples where such quantities appear: Fick law, Low Mach number..... see recent works by Brenner (MIT): bi-velocity model!!!

Shallow water equations from Navier-Stokes equations

Shallow-water with thresholds terms

Thresholds phenomena?

Used in various applications: dense flows (lava, mud, snow), Landslide......

Different sources (earthquakes, precipitation ...), various scales, comparaison.

Not too much data : essentially on deposites.

Threshold = (chgt of state) : Below nothing, above yes.

Threshold = rest state friction angle δ given or plasticity stress tensor.

rest state friction angle = typical angle for granular state
$\Longrightarrow$ reproduces flow behavior, deposite shape, morphological structure.

Shallow-water with thresholds terms

Physical works.

Some simulations by A. Mangeney *et al.* calibrating through friction angle δ, $\mu = \tan\delta$:

- Fei Tsui, Shum Wan (Hong-Kong) : $\delta = 26^{\text{o}}, 18^{\text{o}}$
- Six des eux froides (Suisse) : $\delta = 17^{\text{o}}$
- Frank (Canada) : $\delta = 14^{\text{o}}$
- Boxing day (Motserrat) : $\delta = 15^{\text{o}}$
- Ophir Chasma, Candor Chasma, Ganges Chasma (Mars) : $\delta = 9.8^{\text{o}}, 9.9^{\text{o}}, 9.4\text{o}$.

$\Longrightarrow$ large variability of friction angles..............

Shallow-water with thresholds terms

Threshold terms under which form?

Examples.

- A) Coulomb type friction term,
- B) Viscoplastic constitutive law for example.

Results :

- case B)-a)-i) / E. Fernandez-Nieto, P. Noble, J.–P. Vila. In progress (2008),
- case B)-b)-i) / D.B., E. Fernandez-Nieto, I. Ionescu, P. Vigneaux, *Adv. Math. Fluid Mech.* (2009).
  To appear *Adv. Math Fluid Mech.* (2008).
- cas A) bilayers / E. Fernandez-Nieto, F. Bouchut, D.B., M. Castro, A. Mangeney, *J. Comput Physics* (2008).

Shallow-water with thresholds terms

Which modeling and numerical problems?

Viscoplastic COMPRESSIBLE system: Mixing finite-volume/Augmented Lagrangian.
$\Longrightarrow$ Generalisation of the Fortin-Glowinski method to the compressible case!!
Which viscoplastic model ? Take into account other phenomena: Elasticity, fluidity ?

Other applications : Perforation......DGA....

COMPRESSIBLE system coupled with coulomb term. Models with threshold term in only one momentum equation (term linked to the granulat friction angle) $\Longrightarrow$ Pb since interaction solid/fluid and fluid/solid $\Longrightarrow$ How to discretize such term in an iterative scheme? Granular medium in water? what time and space scales ?

Some references regarding finite volume/ Shallow-Water : F. Bouchut, M. Castro-Diaz *et al.*, Th. Gallouët, R.J. Leveque, F. Marche.

Shallow-water with thresholds terms

What kind of mathematical problems?

- 1) Shallow-water type models derivation from models with threshold and free surface.
- 2) Well posedness of derived systems.
- 3) Stability, long-time behavior.
- 4) Security criterium, time stopping estimate.

1) Really open since Newtonian case justification recent.

2)-3) The procedure to prove global existence of weak solutions à la Leray due P.–L. Lions does not work. Use of BD type entropy? Strong solutions à la Hoff ?
Some Russian interesting refs: Mamontov (weak solution with linear pressure, multi-D model), V. Shelukhin et I. Basov (strong solution on 1D model, asymptotic limit from non-newtonian flow).

4) Some works have been done related to vicoplastic solids in some admissible motions: T. Lachand-Robert, I. Ionescu *et al.*, G. Carlier (Cheeger sets).

Shallow-water with thresholds terms

Shallow water and Bingham

$$
\left\{
\begin{aligned}
&\partial_t h + (hv)_x = 0,\\
&\int_0^L h(\partial_t v + v\partial_x v)(\psi - v)\,dx + \int_0^L \beta v(\psi - v)\,dx + \int_0^L 4\eta h \partial_x v \partial_x(\psi - v)\,dx\\
&+ \int_0^L Bh\sqrt{2}(|\partial_x \psi| - |\partial_x v|)\,dx \geqslant \frac{-1}{\mathrm{Fr}^2}\int_0^L h \sin\theta(\psi - v)\,dx\\
&\qquad\qquad + \frac{\varepsilon}{\mathrm{Fr}^2}\int_0^L \frac{\cos\theta}{2} h^2(\partial_x\psi - \partial_x v)\,dx.
\end{aligned}
\right.
$$

If $B = 0$ then viscous-shallow water similar to Gerbeau-Perthame.

Case B)-b)-i) : We get such model from variational formulation of incompressible Bingham flows with free surface and adequate test functions choices.

Shallow-water with thresholds terms

Incompressible visco-plastic Bingham equations

Constitutive law

Cauchy tensor: $\sigma = -p\mathrm{Id} + \tau.$

Bingham model (1992):

$$\begin{cases} |\tau| \leqslant B \text{ when } |D(u)| = 0, \\ \tau = 2\eta D(u) + B\dfrac{D(u)}{|D(u)|} \text{ when } |D(u)| \neq 0. \end{cases}$$

The case $B = 0$ gives Newtonian fluid.

Shallow-water with thresholds terms

Incompressible Bingham type visco-plastic equations

$$\partial_t u - \mathrm{div}\sigma = f, \qquad \mathrm{divu} = 0.$$

If fixed domain Ω: minimization problem with non differentiable convex energy.

$$J(u) = \int_\Omega 2\eta |D(u)|^2 + \int_\Omega B|D(u)| - \int_\Omega f \cdot u$$

Problem:

$$Min_{v \in H_0^1} J(v).$$

Difficulties: Non-differentiable energy $\Longrightarrow$ Variational inequality.

Numerical strategy :
Augmented Lagrangian Algorithm + mesh adaptation (*cf.* P. Saramito (Grenoble)).

Shallow-water with thresholds terms

Standart Bingham

$$\begin{cases} |\tau| \leqslant B \text{ when } |D(u)| = 0, \\ \tau = \eta D(u) + B\dfrac{D(u)}{|D(u)|} \text{ when } |D(u)| \neq 0 \end{cases}$$

equivalent to

$$\max\big(0, \frac{|\tau| - B}{|\tau|}\big)\tau = \eta_m D(u).$$

Remark: We have written a shallow-water/Bingham type model for order 1 Reynolds number and slip boundary condition at order *i.e.* B)-b)-i). The asymptotic is realized on the variational inequality.

Remark: For model assuming order 1 Reynolds number and no slip boundary condition (*i.e.* B)-a)-i)). See E. Fernandez-Nieto, P. Noble, J.P. Vila (2009).

Shallow-water with thresholds terms

Difficulty coming from viscoplastic compressible equations : Finite volume and Augmented Lagrangian method.

Simple idea : The numerical flux has to take into account the several steps of the Augmented Lagrangian method (Saddle point calculus in (V, q, μ) then iterative method).

- Linear system associated to the velocity problem with right-hand side term.
- Minimization problem associated to Lagrangian multiplier (explicit calculus).

Stationary states have to be preserved on the linear system associated to the velocity problem, we get a necesary and sufficient condition on the iterates which push to choose adequate numerical fluxes.

Shallow-water with thresholds terms

An idea of how it works !!

In the iterative scheme: Let V_0^n, H^n, μ^n and q^n be given for $k = 0$.

1) Calculus of q^{k+1}

2) Calculus of V^{k+1} as solution of linear ODE with right hand side

3) Update of μ^{k+1}.

4) Loop on μ^k.

At convergence, we get the velocity at time t^{n+1}. We let $V_0^{n+1} = V^{k+1}$, $\mu^{n+1} = \mu^k$, $q^{n+1} = q^{k+1}$.

Shallow-water with thresholds terms

In the iterative calculus we get the following expression at the right hand-ide of the velocity pb that we denote b at time t^n :

$$-H^n\left[\partial_x\left(\frac{\overline{\rho_0}(V_0^n)^2}{2}+\frac{\varepsilon}{\mathrm{Fr}^2}H^n\overline{\rho_0}\cos\theta\right)-\overline{\rho_0}\mathrm{St}\frac{V_0^n}{\Delta t}-\frac{1}{\mathrm{Fr}^2}\overline{\rho_0}\sin\theta\right].$$

$$+\partial_x\big(H^n(\mu^k-rq^{k+1})\big).$$

From the height equation, we also have the term evaluated at time $t=t^n$:

$$-St\frac{H^n}{\Delta t}+\frac{\partial(H^nV_0^n)}{\partial x}.$$

We define the flux

$$F(W)=\begin{pmatrix}\overline{\rho}V^2/2+\varepsilon H\overline{\rho_0}\cos\theta/\mathrm{Fr}^2\\ HV\end{pmatrix},\quad \text{with}\quad W=\begin{pmatrix}H\\ V\end{pmatrix},$$

and by ϕ a numerical flux approaching F.

Shallow-water with thresholds terms

Approximation of b^i :

$$b_i = -H_i^n \left[\frac{\phi^{V^n}_{i+1/2} - \phi^{V^n}_{i-1/2}}{\Delta x} - \frac{1}{\mathrm{Fr}^2} \overline{\rho_0} \sin\theta - St \frac{\overline{\rho_0} V_0^n}{\Delta t} \right] + \frac{G^V_{i-1/2} + G^V_{i+1/2}}{2}$$

with

$$G^V_{i+1/2} = H^n_{i+1/2} \frac{\mu^k_{i+1} - r q^{k+1}_{i+1} - (\mu^k_i - r q^{k+1}_i)}{\Delta x}.$$

Approximation of H^{n+1} :

$$St H_i^{n+1} = St H_i^n + \frac{\Delta t}{\Delta x} (\phi^H_{i+1/2} - \phi^H_{i-1/2}).$$

Shallow-water with thresholds terms

The source term due to the topography has to be taken into account in ϕ^H . If we note

$$G_{topo} = \begin{pmatrix} 0 \\ -\frac{1}{\mathrm{Fr}^2}\overline{\rho_0}\sin\theta \end{pmatrix}$$

then ϕ^H is defined as the first component of

$$\phi_{topo,i+1/2} = \frac{F(W_i) + F(W_{i+1})}{2} - \frac{1}{2}D_{i+1/2}(W_{i+1} - W_i - A^{-1}_{i+1/2}G_{topo}).$$

Shallow-water with thresholds terms

Due to source terms associated to the Augmented-Lagrangian, $\mu + r\,q$, ϕ_H have to take into account them. If the algorithm tops at index k_e, we approach the terms by $\mu^{ke+1} + r\,q^{ke+1}$. And we define ϕ^H as first component of

$$\phi_{\mu,q,i+1/2} = \frac{F(W_i) + F(W_{i+1})}{2} -$$

$$\frac{1}{2} D_{i+1/2}(W_{i+1} - W_i - A^{-1}_{i+1/2}(G_{topo,i+1/2} + G_{\mu,q,i+1/2})).$$

where

$$G_{\mu,q,i+1/2} = \begin{pmatrix} 0 \\ H^n_{i+1/2}(\mu^{ke+1}_{i+1} - rq^{ke+1}_{i+1}) - H^n_{i-1/2}(\mu^{ke+1}_i - rq^{ke+1}_i). \end{pmatrix}.$$

Shallow-water with thresholds terms

Properties: We show that if initially, for all $x \in [0, L]$

$$\mu(x) = \frac{1}{\mathrm{Fr}^2}\overline{\rho_0}\sin\theta(x - L/2) - \frac{\varepsilon}{\mathrm{Fr}^2}\overline{\rho_0}\cos\theta(H(x) - H(L/2)), \qquad q(x) = 0$$

then the scheme preserves exactly the stationary solutions.

Shallow-water with thresholds terms

Various numerical schemes:

- 1) What happens with bad flux discretization?
- 2) What is the influence of the Bingham number?
- 3) What happens on a stationary solution if we change Bingham number?
- 4) What happens for a big bump?

Detail : See D. Bresch, E. Fernandez-Nieto, I. Ionescu, P. Vigneaux. Augmented lagrangian method and compressible viscoplastic flows: application to shallow dense avalanches. To appear *Adv. Math. Fluid Mech.* (2008).

Shallow-water with thresholds terms

Other elasto-viscoplastic models

A mixed Oldroyd-Bingham type model:

see P. Saramito, *J. Non Newtonian Fluid Mech* (2007).

$\Longrightarrow$ Generalisation of Schwedoff model.

$$\begin{cases} \sigma = -p\mathrm{Id} - 2\eta D(u) + \tau, \\ \lambda D_t \tau + \max\big(0, \dfrac{|\tau_d| - B}{|\tau_d|}\big)\tau = 2\eta_m D(u) \\ D_t\tau = \partial_t \tau + u \cdot \nabla \tau + W(u)\tau - \tau W(u) - a[D(u)\tau + \tau D(u)] \end{cases}$$

with $a \in [-1,1]$, τ_d the deviatoric part of τ.

See also paper of S. Benito, C.–H. Bruneau, T. Colin, C. Gay, F. Molino. *Eur. Phys. J.E.* (2008).

Question : Quid St-Venant type model derivation from such kind of model?

Interest: Take into account the elastic character for instance for lava. Important also in cosmetic.....

Shallow-water with thresholds terms

Stratified model with a fluid ρ_1 and a granular medium ρ_s

H1) Fluid and granular material immiscible and "Euler" for both.

$$\rho_2 = (1 - \psi_0)\rho_s + \psi_0 \rho_1 .$$

H2) Anisotropy in the pressure tensors which has a part due to fluid and one due to material: Iverson-Delinger law (2001) for granular material.

H3) Continuity normal part and friction interface tangential part.

H4) At bottom, tangential part on P^2 with Coulomb law taking into account P^1 (Archimede.)

Shallow-water with thresholds terms

$$
\begin{cases}
\partial_t h_1 + \partial_x (h_1 v_1) = 0, \\
\partial_t (h_1 v_1) + \partial_x \left(h_1 v_1^2 + g h_1^2 (\cos\theta)/2 \right) \\
\qquad = -g h_1 b_x + g(\sin\theta)\theta_x h_1^2/2 - g h_1 \partial_x ((\cos\theta) h_2) + fric(v_1, v_2)/\rho_1, \\
\partial_t h_2 + \partial_x (h_2 v_2) = 0, \\
\partial_t (h_2 v_2) + \partial_x \left(h^2 v_2^2 + \Lambda_2 g h_2^2 (\cos\theta)/2 \right) \\
\qquad = -g h_2 \partial_x b - r g h_2 \Lambda_1 \partial_x (h_1 \cos\theta) - fric(v_1, v_2)/\rho_2 + g h_2^2 (\sin\theta) \partial_x \theta/2 + \tau
\end{cases}
$$

where g gravité, $\rho_2 = (1-\psi_0)\rho_s + \psi_0 \rho_1$, $r = \rho_1/\rho_2$, $r_s = \rho_s/\rho_2$, $\Lambda_1 = \lambda_1 + K(1-\lambda_1)$, $\Lambda_2 = r\lambda_2 + K(1 - r\Lambda_2)$ where K mesures anisotropy.

The friction term τ of Coulomb type is defined by :

If $|\tau| \geqslant \sigma_c$ then $\tau = -g h_2 \left((\cos\theta)(1-r) + v_2^2 \theta_x \right) (\tan\delta_0) v_2/|v_2|$ elsewhere $v_2 = 0$.

Shallow-water with thresholds terms

I) This model possesses a dissipative entropy inequality:

II) It preserves the stationary-state

$$v_1 = v_2 = 0, \qquad b + (h_1 + h2)\cos\theta = \text{cst}$$

$$|(\Lambda_2 - r\Lambda_1)\partial_x(b + h_2(\cos\theta)) + (1-\Lambda_2)(\partial_x b - h_2(\sin\theta)\partial_x\theta/2)| \leqslant (1-r)\tan\delta_0.$$

Remark : For $K = 1$, we get

$$|\partial_x(b + h_2\cos\theta)| \leqslant \tan\delta_0.$$

Shallow-water with thresholds terms

Finite volume scheme preserving stationary states!!
Discretized system following Roe Scheme.
Scheme in two steps in order to the friction Coulomb term:
1) Calculus of $W_i^* = (h_{1,i}^*, Q_{1,i}^*, h_{2,i}^*, Q_{2,i}^*)$ by flux of generalized Roe type.

$$W_i^* = W_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{DF}_{i-1/2}^{n,+} + \mathcal{DF}_{i+1/2}^{n,-}\right).$$

2) Then we define $W_i^{n_1} = [h_{1,i}, q_{1,i}, h_{2,i}, q_{2,i}^{n+1}$ with $q_{2,i}^{n+1}]$ defined following threshold compared to $q_{2,i}^*$.

Details : See E.D. Fernandez-Nieto, F. Bouchut, D. Bresch, M.J. Castro-Diaz, A. Mangeney. A new Savage–Hutter type model for submarine avalanches and generated tsunami. *J. Comput. Physics* 227, (2008), 7720-7754.

Shallow-water with thresholds terms

Various numerical tests:

- 1) Sub-aerial landslide.
- 2) Generation of "Tsunamis” and propagation : Inspired from paper Heinrich-Piatanesi-Hèbert on 1998 Papaa New Guinea event.

Other models

Kazhikov-smagulov type model:

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0, \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u) + \mathcal{D} + \nabla P = 0, \\ u = v + \alpha \nabla \log \rho, \qquad \mathrm{div} v = 0, \end{cases}$$

where $\mathcal{D}$ is the viscosity term.

See J. Etienne, E. Hopfinger, P. Saramito for simulation: Annals of Glaciology, (2004).

Global existence of weak solutions for adequate viscosity : $-\alpha \mathrm{div}(\rho D(u))$.
D.B., E. Essoufi, M. Sy (2003).

Other models

Biphasic compressible system.

$$\begin{cases} \partial_t(\alpha^\pm\rho^\pm) + \operatorname{div}(\alpha^\pm\rho^\pm u^\pm) = 0, \\ \partial_t(\alpha^\pm\rho^\pm u^\pm) + \operatorname{div}(\alpha^\pm\rho^\pm u^\pm \otimes u^\pm) + \alpha^\pm\nabla p, \\ \qquad\qquad = \operatorname{div}(\alpha^\pm\tau^\pm) + \sigma^\pm\alpha^\pm\rho^\pm\nabla\Delta(\alpha^\pm\rho^\pm), \\ \alpha^+ + \alpha^- = 1, \qquad p = p^\pm(\rho^\pm), \qquad \tau^\pm = 2\mu^\pm D(u^\pm) + \lambda^\pm \operatorname{div} u^\pm \mathrm{Id}, \\ (\alpha^\pm\rho^\pm)|_{t=0} = R_0^\pm, \quad (\alpha^\pm\rho^\pm u^\pm)|_{t=0} = m_0^\pm \\ R_0^\pm \geqslant 0, \qquad \alpha_0^\pm \in [0,1] \text{ tel que } \alpha_0^- + \alpha_0^- = 1, \\ |m_0^\pm|^2/R_0^\pm = 0 \text{ sur } \{x \in \Omega : R_0^\pm(x) = 0\}. \end{cases}$$

See D. Dutykh for numerical simulation.

Other models

Non conservative model and non-hyperbolic model without viscosity.

D.B., B. Desjardins, J.M. Ghidaglia, E. Grenier. Submitted (2008).

Result. Let $\mu^{\pm}(\rho^{\pm}) = \mu^{\pm}\rho^{\pm}$, $\lambda^{\pm} = 0$ and $p(\rho^{\pm}) = a^{\pm}(\rho^{\pm})^{\gamma^{\pm}}$ with $1 < \gamma^{\pm} < 6$. Standart hypothesis on initial data + $\nabla\sqrt{R_0^{\pm}}$ in $(L^2(R^3))^3$. There exists a global weak solution of the biphasic system.

Compactness (integrability) in pressure term $\Longrightarrow$ constraints on $\gamma^{\pm}$.

Weak solutions formulation similar to D. Bresch, B. Desjardins, C.K. Lin. CPDE (2001).

Other results in this paper: local existence of strong sol, invariant sets, spectral analysis.....

Similar to bi-layers shallow-water system.

Other models

For more details on recent results related to viscoplastic fluids:
see J. Non-Newtonian Fluid Mech, volume 142 (2007), Guest editors: N.J. Balmforth, I. Friguard.

Herschel-Bulkley, power laws fluids, suspension models......