Small time existence of the flow of a viscoelastic fluid with a free boundary

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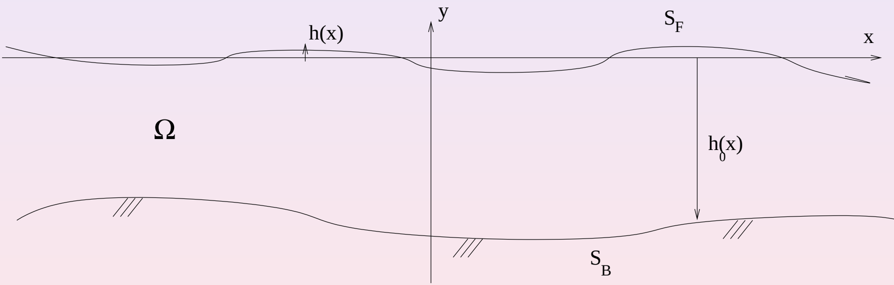
Geometry



Eulerian equations

$$
\begin{aligned}
&\mathrm{Re}\,(\underline{v}_t + \underline{v}.\underline{\nabla}\,\underline{v}) - (1-\varepsilon)\Delta\underline{v} + \underline{\nabla}p - \mathrm{div}\,\underline{\underline{\tau}} &&= 0 &&\text{in } \Omega(t)\\
&\mathrm{div}\;\underline{v} &&= 0 &&\text{in } \Omega(t)\\
&\underline{\underline{\tau}} + \mathrm{We}\frac{\mathcal{D}_a[\underline{v}]\underline{\underline{\tau}}}{\mathcal{D}t} - 2\varepsilon\underline{\underline{D}}[\underline{v}] &&= 0 &&\text{in } \Omega(t),\\
&-p\underline{n} + 2(1-\varepsilon)\underline{\underline{D}}[\underline{v}]\cdot\underline{n} + \underline{\underline{\tau}}\cdot\underline{n} - \alpha H\underline{n} + g_0 x_2\,\underline{n} &&= 0 &&\text{on } S_F(t),\\
&\underline{v} &&= 0 &&\text{on } S_B,\\
&\underline{v}(x,t=0) &&= \underline{u}_0(x) &&\text{in } \Omega,\\
&\underline{\underline{\tau}}(x,t=0) &&= \underline{\underline{\sigma}}_0(x) &&\text{in } \Omega,
\end{aligned}
$$

where

$$
\begin{aligned}
\frac{\mathcal{D}_a[\tilde{\underline{v}}]\tilde{\underline{\underline{\tau}}}}{\mathcal{D}\tilde{t}} &= \frac{\partial\tilde{\underline{\underline{\tau}}}}{\partial\tilde{t}} + \tilde{\underline{v}}.\tilde{\nabla}\tilde{\underline{\underline{\tau}}} - g_a(\underline{\underline{\tilde{\nabla} v}}, \tilde{\underline{\underline{\tau}}})\\
g_a(\underline{\underline{\nabla v}}, \underline{\underline{\tau}}) &= \tfrac{a-1}{2}\left(\underline{\underline{\nabla v}}^T\underline{\underline{\tau}} + \underline{\underline{\tau}}\,\underline{\underline{\nabla v}}\right) + \tfrac{a+1}{2}\left(\underline{\underline{\tau}}\,\underline{\underline{\nabla v}}^T + \underline{\underline{\nabla v}}\,\underline{\underline{\tau}}\right),
\end{aligned}
$$

Some notations

$$\begin{array}{rccc} \overline{\eta}(.,t): & \Omega & \rightarrow & \Omega(t) \\ & X & \mapsto & \overline{\eta}(X,t), \end{array}$$

and

$$\overline{\eta}(X,t) = X + \eta(X,t).$$

and

$$\begin{array}{rcl} \text{Lagrangian} & & \text{Eulerian} \\ \underline{u}(X,t) & = & \underline{v}(\overline{\eta}(X,t),t), \\ q(X,t) & = & p(\overline{\eta}(X,t),t), \\ \underline{\underline{\sigma}}(X,t) & = & \underline{\underline{\tau}}(\overline{\eta}(X,t),t), \end{array}$$

Some notations 2

$$
\begin{aligned}
(\mathrm{d}\overline{\eta})_{ij} &= \frac{\partial \overline{\eta}_i}{\partial X_j}(X,t) = \overline{\eta}_{i,j}(X,t),\\
(\underline{\underline{\xi}}) &= \underline{\underline{(\mathrm{d}\overline{\eta})}}^{-1}(X,t);\\
\underline{N}(X) &= (-h'(X_1),1)/\sqrt{1+h'^2},\\
\underline{\mathcal{N}}(X,t) &= (N_1 - \partial_{\underline{\tau}}\eta_2, N_2 + \partial_{\underline{\tau}}\eta_1),
\end{aligned}
$$

where $\partial_{\underline{\tau}} = (1+h'^2)^{-1/2}\partial_{X_1}$.

$$\begin{aligned}
&\mathrm{Re}\, u_{i,t} - (1-\varepsilon)\overline{\xi}_{kj}\partial_k(\overline{\xi}_{lj} u_{i,l}) + \overline{\xi}_{ki}\partial_k q - \sigma_{ij,k}\overline{\xi}_{kj} &= 0,\\
&\overline{\xi}_{kj} u_{j,k} &= 0,\\
&\sigma_{ij} + \mathrm{We}\left(\frac{\partial \sigma_{ij}}{\partial t} - \frac{a-1}{2}(\overline{\xi}_{li} u_{k,l}\sigma_{kj} + \sigma_{ik} u_{k,l}\overline{\xi}_{lj})\right.\\
&\qquad \left. - \frac{a+1}{2}(\sigma_{ik}\overline{\xi}_{lk} u_{j,l} + u_{i,l}\overline{\xi}_{lk}\sigma_{kj})\right) - \varepsilon(u_{i,k}\overline{\xi}_{kj} + u_{j,k}\overline{\xi}_{ki}) &= 0,\\
&-q\underline{\mathcal{N}}_i + (1-\varepsilon)(\overline{\xi}_{kj} u_{i,k} + \overline{\xi}_{ki} u_{j,k})\underline{\mathcal{N}}_j + \sigma_{ij}\underline{\mathcal{N}}_j +\\
&\qquad g_0(h(X_1) + \eta_2(X_1,t))\underline{\mathcal{N}}_i -\\
&\qquad \alpha\partial_{\underline{\tau}}\left((1+(\Phi+h')^2)^{\frac{-1}{2}}\begin{pmatrix}1\\ \Phi+h'\end{pmatrix}\right) &= 0,\\
&\Phi_t - \frac{(\partial_{\underline{\tau}}\underline{u})\cdot\underline{\mathcal{N}}}{\mathcal{N}_2^2} &= 0,
\end{aligned}$$

with the conditions

$$\Phi(t=0)=0;\ \underline{u}(X,t=0)=\underline{u}_0(X),\ \underline{\underline{\sigma}}(X,t=0)=\underline{\underline{\sigma}}_0(X),\ \underline{u}=0 \text{ on } S_B.$$

Operators

$\overline{\eta}(X,t) = X + \eta(X,t) \simeq X$ so

$$\left(\underline{\underline{(\mathrm{d}\overline{\eta})}}^{-1} =\right) \overline{\underline{\underline{\xi}}} = \underline{\underline{Id}} + \underline{\underline{\xi}} \simeq \underline{\underline{Id}},$$

From

$$P(\xi, \underline{u}, q, \phi, \underline{\underline{\sigma}}) = (0,0,0,0,0,\underline{u}_0, \underline{\underline{\sigma}}_0),$$

for $\underline{u}$ vanishing on S_B and $\Phi(t=0) = 0$.

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for $\underline{u}$ vanishing on S_B and $\Phi(t=0) = 0$.

$$\begin{aligned} P(\underline{\underline{\xi}}, \underline{u}, q, \phi, \underline{\underline{\sigma}}) &= P(0,0,0,0,0) + P_1(\underline{u}, q, \phi, \underline{\underline{\sigma}}) + E(\underline{\underline{\xi}}, \underline{u}, q, \phi, \underline{\underline{\sigma}}) \\ &= (0,0,0,0,0, \underline{u}_0, \underline{\underline{\sigma}}_0), \end{aligned}$$

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for $\underline{u}$ vanishing on S_B and $\Phi(t=0) = 0$.

$$\begin{aligned} P(\underline{\underline{\xi}}, \underline{u}, q, \phi, \underline{\underline{\sigma}}) &= P(0,0,0,0,0) + P_1(\underline{u}, q, \phi, \underline{\underline{\sigma}}) + E(\underline{\underline{\xi}}, \underline{u}, q, \phi, \underline{\underline{\sigma}}) \\ &= (0,0,0,0,0, \underline{u}_0, \underline{\underline{\sigma}}_0), \end{aligned}$$

We define

$$P_2[\underline{u}^0, \underline{\underline{\sigma}}^0](\underline{u}, q, \phi, \underline{\underline{\sigma}}) := P_1(\underline{u}^0 + \underline{u}, q^0 + q, \phi^0 + \phi, \underline{\underline{\sigma}}^0 + \underline{\underline{\sigma}}) - P_1(\underline{u}^0, q^0, \phi^0, \underline{\underline{\sigma}}^0$$

Spaces

$$K^r(\Omega \times (0,T)) = L^2(0,T;H^r(\Omega)) \bigcap H^{r/2}(0,T;L^2(\Omega)),$$
$$K^r(S_F \times (0,T)) = L^2(0,T;H^r(S_F)) \bigcap H^{r/2}(0,T;L^2(S_F)),$$

$$\begin{aligned} X^r_T(\Omega \times (0,T)) = \quad & \{(\underline{u}, q, \phi, \underline{\underline{\sigma}})/ \\ & \underline{u} \in K^{r+2} \text{ and } \underline{u} = 0 \text{ on } S_B \times (0,T); \\ & \nabla q \in K^r \text{ and } q|_{S_F} \in H^{\frac{r}{2}+\frac{1}{4}}(0,T;L^2(S_F)); \\ & \partial_{\underline{\tau}}\phi \in L^2(0,T;H^{r+\frac{1}{2}}(S_F)) \\ & \qquad \phi_t \in K^{r+\frac{1}{2}}(S_F \times (0,T)) \\ & \qquad \phi(0) = 0; \\ & \underline{\underline{\sigma}} \in K^{r+1}(\Omega \times (0,T)) \ \}. \end{aligned}$$

Spaces 2

The image space Y_T^r is

$$
\begin{aligned}
Y_T^r(\Omega \times (0,T)) = \; & \{(\underline{f}, a, \underline{\underline{m}}, g, k, \underline{u}_0, \underline{\underline{\sigma}}_0)/ \\
& \underline{f} \in K^r(\Omega \times (0,T)), \\
& a \in L^2(0,T; H^{r+1}(\Omega)) \bigcap H^{\frac{r}{2}+1}(0,T, {}_0H^{-1}(\Omega)), \\
& \underline{\underline{m}} \in K^{r+1}(\Omega \times (0,T)), \\
& g, k \in K^{r+\frac{1}{2}}(S_F \times (0,T)), \\
& \underline{u}_0, \underline{\underline{\sigma}}_0 \in H^{r+1}(\Omega)\},
\end{aligned}
$$

where ${}_0H^{-1}(\Omega)$ is the dual space of ${}^0H^1 = \{p \in H^1 / p \equiv 0 \text{ on } S_F\}$.

Main result

Theorem

Let $0 < r < 1/2$, the height functions h and $(h_0 - \lim_{\pm\infty} h_0)$ be in $H^{r+5/2}(\mathbb{R})$, $(\underline{u}_0, \underline{\underline{\sigma}}_0)$ be in $H^{r+1} \times H^{r+1}_{sym}$ and the compatibility conditions div $\underline{u}_0 = 0$ in Ω, $\underline{u}_0 = 0$ on S_B be satisfied. Then there exists $T_0 > 0$ depending on the data ; $r, \Omega, \underline{u}_0, \underline{\underline{\sigma}}_0, h, h_0, We, \varepsilon, a$ and there exists $(\underline{u}, q, \phi, \underline{\underline{\sigma}}) \in X^r_{T_0}$ solution of the Lagrangian system. Under the same hypothesis, the Eulerian system admits a solution with $\overline{\eta} \in H^1(0, T; H^{2+r}(\Omega)) \bigcap H^{2+\frac{r}{2}}(0, T; L^2(\Omega))$.

First step

We define

$$P_2[\underline{u}^0, \underline{\underline{\sigma}}^0](\underline{u}, q, \phi, \underline{\underline{\sigma}}) := P_1(\underline{u}^0+\underline{u}, q^0+q, \phi^0+\phi, \underline{\underline{\sigma}}^0+\underline{\underline{\sigma}})-P_1(\underline{u}^0, q^0, \phi^0, \underline{\underline{\sigma}}^0$$

and we want to solve

$$P_2[\underline{u}^0, \underline{\underline{\sigma}}^0](\underline{u}, q, \phi, \underline{\underline{\sigma}}) = (\underline{f}, 0, \underline{\underline{m}}, 0, 0, 0, 0).$$

The system P_2

$$
\begin{aligned}
&\mathrm{Re}\frac{\partial \underline{u}}{\partial t} - (1-\varepsilon)\Delta \underline{u} + \underline{\nabla} q - \mathrm{div}\underline{\underline{\sigma}} && = \underline{f},\\
&\mathrm{div}\underline{u} && = 0,\\
&\underline{\underline{\sigma}} + \mathrm{We}\left(\frac{\partial \underline{\underline{\sigma}}}{\partial t} - \underline{\underline{g}}(\underline{\underline{\nabla u}}, \underline{\underline{\sigma}}) - \underline{\underline{g}}(\underline{\underline{\nabla u}}_1, \underline{\underline{\sigma}}) - \underline{\underline{g}}(\underline{\underline{\nabla u}}, \underline{\underline{\tau}}_1)\right) - 2\varepsilon \underline{\underline{D}}[\underline{u}] && = \underline{\underline{m}},\\
&-q\underline{N} + 2(1-\varepsilon)\underline{\underline{D}}[\underline{u}]\cdot \underline{N} - \alpha \partial_{\underline{\tau}}(\phi \underline{N}) + \underline{\underline{\sigma}}\cdot \underline{N} && = 0,\\
&\phi_t - \partial_{\underline{\tau}} \underline{u} \cdot \underline{N} && = 0,\\
&\underline{u}(t=0) && = 0,\\
&\underline{\underline{\sigma}}(t=0) && = 0,
\end{aligned}
$$

The sequence $(\underline{\underline{\sigma}})$

$$\begin{cases} \underline{\underline{\sigma}}^{n+1} + \mathrm{We}\left(\dfrac{\partial \underline{\underline{\sigma}}^{n+1}}{\partial t} - \underline{\underline{g}}_a(\underline{\underline{\nabla u}}^n, \underline{\underline{\sigma}}^{n+1}) - \underline{\underline{g}}_a(\underline{\underline{\nabla u}}_1, \underline{\underline{\sigma}}^{n+1}) - \right. \\ \qquad \left. \underline{\underline{g}}_a(\underline{\underline{\nabla u}}^n, \underline{\underline{\tau}}_1)\right) = 2\varepsilon \underline{\underline{D}}[\underline{u}^n] + \underline{\underline{m}} \\ \underline{\underline{\sigma}}^{n+1}(0) = 0. \end{cases}$$

The sequence $(\underline{u}, q, \phi)$

and then

$$
\begin{aligned}
\mathrm{Re}\frac{\partial \underline{u}^{n+1}}{\partial t} - (1-\varepsilon)\Delta \underline{u}^{n+1} + \underline{\nabla} q^{n+1} &= \underline{f} + \mathrm{div}\underline{\underline{\sigma}}^{n+1}, \\
\mathrm{div}\, \underline{u}^{n+1} &= 0, \\
-q^{n+1}\underline{N} + 2(1-\varepsilon)\underline{\underline{D}}[\underline{u}^{n+1}] \cdot \underline{N} - \alpha \partial_{\underline{\tau}}(\phi^{n+1}\underline{N}) &= -\underline{\underline{\sigma}}^{n+1} \cdot \underline{N}, \\
\phi_t^{n+1} - (\partial_{\underline{\tau}} \underline{u}^{n+1}) \cdot \underline{N} &= 0, \\
\phi^{n+1}(t=0) &= 0, \\
\underline{u}^{n+1}(t=0) &= 0, \\
\underline{u}^{n+1} &= 0 \text{ on } S_B.
\end{aligned}
$$

Uniform (in n) estimates

Proof by induction of the existence of $T_0 > 0, 0 < V < 1, S > 0$ such that $\forall n$

$$
\begin{cases}
\mid \underline{\sigma}^n \mid_{H^{1+r}} (t) \leq S, \\
\mid \underline{u}^n, q^n, \phi^n, 0 \mid_{X_T^r} \leq V, \\
\mid \underline{\underline{\nabla u}}^n \mid_{L^2(0,T_0;H^{1+r})} \leq V, \\
\mid \underline{\underline{\nabla u}}_1 \mid_{L^2(0,T_0;H^{1+r})} \leq 1, \\
C\sqrt{T_0}(V + \mid \underline{\underline{\nabla u}}_1 \mid_{L^2(0,T_0;H^{1+r})}) < 1, \\
C(V + \mid \underline{\underline{m}} \mid_{L^2(0,T_0;H^{1+r})}) < S, \\
C \mid \underline{f} \mid_{H^{r,\frac{r}{2}}(0,T_0)} < V/5 \\
CT^{\epsilon'}(V + \mid \underline{\underline{m}} \mid_{L^2(0,T_0;H^{1+r})}) < V/5, \\
CT^{\epsilon'}(V + \mid \underline{\underline{m}} \mid_{L^2(0,T_0;H^{1})}) < V/5, \\
CT^{\epsilon'}(V + \mid \underline{\underline{m}} \mid_{L^2(0,T_0;H^{\frac{1}{2}})}) < V/5.
\end{cases}
$$

One uniform estimate

Scalar product of the definition of $\underline{\underline{\sigma}}^{n+1}$ with $\underline{\underline{\sigma}}^{n+1}$ in H^{1+r} :

$$
\begin{aligned}
|\, \underline{\underline{\sigma}}^{n+1} \,|_{1+r}^2 + \frac{\mathrm{We}}{2} \frac{\mathrm{d} \,|\, \underline{\underline{\sigma}}^{n+1} \,|_{1+r}^2}{\mathrm{d}\, t} \leq C \Big[2\varepsilon \,|\, \underline{\underline{\nabla u}}^n \,|_{1+r} + |\, \underline{\underline{m}} \,|_{1+r} + \\
C\mathrm{We}(|\, \underline{\underline{\nabla u}}^n \,|_{1+r} |\, \underline{\underline{\sigma}}^{n+1} \,|_{1+r} + |\, \underline{\underline{\nabla u}}_1 \,|_{1+r} |\, \underline{\underline{\sigma}}^{n+1} \,|_{1+r} + \\
+ |\, \underline{\underline{\nabla u}}^n \,|_{1+r} |\, \underline{\underline{\tau}}_1 \,|_{1+r}) \Big] \,|\, \underline{\underline{\sigma}}^{n+1} \,|_{1+r} \,.
\end{aligned}
$$

$$(1 - C\mathrm{We}(|\ \underline{\underline{\nabla u}}^n\ |_{1+r} + |\ \underline{\underline{\nabla u}}_1\ |_{1+r}))\ |\ \underline{\underline{\sigma}}^{n+1}\ |_{1+r}^2 + \frac{\mathrm{We}}{2}\frac{\mathrm{d}\ |\ \underline{\underline{\sigma}}^{n+1}\ |_{1+r}^2}{\mathrm{d}\, t} \le C\left[(2\varepsilon + C\mathrm{We}\ |\ \underline{\underline{\tau}}_1\ |_{1+r})\ |\ \underline{\underline{\nabla u}}^n\ |_{1+r} + |\ \underline{\underline{m}}\ |_{1+r}\right]\ |\ \underline{\underline{\sigma}}^{n+1}\ |_{1+r}\ .$$

and so

$$\frac{\mathrm{d}\left(e^{2\int_0^t \frac{(1/2 - C\mathrm{We}(|\ \underline{\underline{\nabla u}}^n\ |_{1+r} + |\ \underline{\underline{\nabla u}}_1\ |_{1+r}))}{\mathrm{We}} ds}\ |\ \underline{\underline{\sigma}}^{n+1}\ |_{1+r}^2 (t)\right)}{\mathrm{d}\, t} \le C(\varepsilon, \underline{\underline{\tau}}_1, \mathrm{We})\left(|\ \underline{\underline{\nabla u}}^n\ |_{1+r}^2 + |\ \underline{\underline{m}}\ |_{1+r}^2\right) \times e^{2\int_0^t \frac{(1/2 - C\mathrm{We}(|\ \underline{\underline{\nabla u}}^n\ |_{1+r} + |\ \underline{\underline{\nabla u}}_1\ |_{1+r}))}{\mathrm{We}} ds},$$

$$
\begin{aligned}
&\mid \underline{\underline{\sigma}}^{n+1} \mid_{1+r}^2 (t) \leq \\
&\quad \int_0^t e^{-2\int_s^t \frac{(1/2 - C\mathrm{We}(\mid \underline{\underline{\nabla u}}^n \mid_{1+r} + \mid \underline{\underline{\nabla u}}_1 \mid_{1+r}))}{\mathrm{We}} dt'} \times \\
&\quad \times (C \mid \underline{\underline{\nabla u}}^n \mid_{1+r}^2 + \mid \underline{\underline{m}} \mid_{1+r}^2) ds.
\end{aligned}
$$

Thanks to the induction property ;

$$
\begin{aligned}
\mid \underline{\underline{\sigma}}^{n+1} \mid_{1+r} (t) &\leq C \left(\mid \nabla \underline{u}^n \mid_{L^2(0,T;H^{1+r})} + \mid \underline{\underline{m}} \mid_{L^2(0,T;H^{1+r})} \right) \\
&\leq C \left(V + \mid \underline{\underline{m}} \mid_{L^2(0,T;H^{1+r})} \right),
\end{aligned}
$$

Same computations for the contractance.
$(\underline{u}^n, q^n, \phi^n, \underline{\underline{\sigma}}^n)$ is of Cauchy type in X_T^r and P_2 is solved with simple rhs.

Let $\underline{\underline{\tau}}_1$ such that

$$\begin{cases} \underline{\underline{\tau}}_1 + \mathrm{We}\dfrac{\partial \underline{\underline{\tau}}_1}{\partial t} = 0 \\ \underline{\underline{\tau}}_1(0, X) = \underline{\underline{\sigma}}_0(X) \ \ \forall X. \end{cases}$$

and $\underline{u}_1, p, \Psi$ such that

$$\begin{cases} -p\underline{N} + 2(1-\varepsilon)\underline{\underline{D}}[\underline{u}_1]\cdot\underline{N} - \alpha\partial(\Psi\underline{N}) & = -\underline{\underline{\tau}}_1 \cdot \underline{N} + g \text{ on } S_F \times (0, T) \\ \Psi_t - \partial_{\underline{\tau}}\underline{u}_1 \cdot \underline{N} & = k \ \text{ on } S_F \times (0, T) \\ \underline{u}_1 & = 0 \ \text{ on } S_B \times (0, T) \\ \underline{u}_1(t=0) & = \underline{u}_0(X) \ \text{ in } \Omega \\ \mathrm{div}\underline{u}_1 & = a \ \text{ in } \Omega. \end{cases}$$

Then $(\underline{u}_1 + \underline{u}, p + q, \Psi + \phi, \underline{\underline{\tau}}_1 + \underline{\underline{\sigma}}) \in X_T^r$ and solves P_1 (full).

An other lifting

Lift the initial conditions and change of fields enables to solve :

$$P_2[\underline{u}_1, \underline{\underline{\sigma}}_1](\underline{u}, q, \phi, \underline{\underline{\sigma}}) = (\underline{f}, a, \underline{\underline{m}}, g, k, 0, 0)$$

Notice : initial vanishing conditions.

The error is small and contracting

Theorem

Let $0 < r < 1/2$, $(\underline{u}^0, q^0, \phi^0, \underline{\underline{\sigma}}^0) \in X^r_{T_0}$ *and* $(\underline{u}, q, \phi, \underline{\underline{\sigma}}) \in B_{X^{r*}_T}(0, R)$. *There exists* $\epsilon' > 0$ *and* $0 < T_0' \leq T_0$ *depending on* $(\underline{u}^0, q^0, \phi^0, \underline{\underline{\sigma}}^0)$ *and* R*, such that if* $0 < T < T_0'$*, then* $E(\underline{u}^0 + \underline{u}, q^0 + q, \phi^0 + \phi, \underline{\underline{\sigma}}^0 + \underline{\underline{\sigma}})$ *is in the space* $Y^r_T(\Omega)$ *and the following estimates hold :*

$$
\begin{aligned}
&| E^i(\underline{u}^0 + \underline{u}, q^0 + q, \phi^0 + \phi, \underline{\underline{\sigma}}^0 + \underline{\underline{\sigma}}) |_{(Y^r_T)_i} \leq CT^{\epsilon'} \quad i \neq 2\\
&| E^2(\underline{u}^0 + \underline{u}, q^0 + q, \phi^0 + \phi, \underline{\underline{\sigma}}^0 + \underline{\underline{\sigma}}) - \\
&\qquad\qquad E^2(\underline{u}^0, q^0, \phi^0, \underline{\underline{\sigma}}^0) |_{(Y^r_T)_2} \leq CT^{\epsilon'}.
\end{aligned}
$$

See next slide.

The error is small and contracting

Theorem

Same assumptions as before. In addition, let $(\underline{u}', q', \phi', \underline{\underline{\sigma}}') \in X_T^*$ *also. The operator* E *is contracting :*

$$
\begin{aligned}
&| \, E(\underline{u}^0 + \underline{u}, q^0 + q, \phi^0 + \phi, \underline{\underline{\sigma}}^0 + \underline{\underline{\sigma}}) - \\
&E(\underline{u}^0 + \underline{u}', q^0 + q', \phi^0 + \phi', \underline{\underline{\sigma}}^0 + \underline{\underline{\sigma}}') \, |_{Y_T^r} \leq \\
&\qquad\qquad CT^{\epsilon'} \, | \, \underline{u} - \underline{u}', q - q', \phi - \phi', \underline{\underline{\sigma}} - \underline{\underline{\sigma}}' \, |_{X_T^r}
\end{aligned}
$$

with constants C *that depend on* $\varepsilon, a, We, r, R, (\underline{u}^0, q^0, \phi^0, \underline{\underline{\sigma}}^0)$*, but not on* T *provided* $T \leq T_0'$*.*

Let us remind :

$$\begin{aligned} P(\xi, \underline{u}, q, \phi, \underline{\underline{\sigma}}) &= P(0,0,0,0,0) + P_1(\underline{u}, q, \phi, \underline{\underline{\sigma}}) + E(\xi, \underline{u}, q, \phi, \underline{\underline{\sigma}}) \\ &= (0,0,0,0,0,\underline{u}_0, \underline{\underline{\sigma}}_0) \end{aligned}$$

Let $(\underline{u}^0, q^0, \phi^0, \underline{\underline{\sigma}}^0)$ be such that :

$$P_1(\underline{u}^0, q^0, \phi^0, \underline{\underline{\sigma}}^0) = (0,0,0,0,0,\underline{u}_0, \underline{\underline{\sigma}}_0) - P(0,0,0,0,0),$$

Let $(\underline{u}, q, \phi, \underline{\underline{\sigma}}) := (\underline{u}^0 + \underline{u}, q^0 + q, \phi^0 + \phi, \underline{\underline{\sigma}}^0 + \underline{\underline{\sigma}})$.

The final proof

We look for $(\underline{u}, q, \phi, \underline{\underline{\sigma}}) \in X_T^r$ with $\underline{u}(t=0) = 0, \underline{\underline{\sigma}}(t=0) = 0$ such that :

$$
\begin{aligned}
&P_1(\underline{u}^0 + \underline{u}, q^0 + q, \phi^0 + \phi, \underline{\underline{\sigma}}^0 + \underline{\underline{\sigma}}) + \\
&+ E(\xi(\underline{u}^0 + \underline{u}), \underline{u}^0 + \underline{u}, q^0 + q, \phi^0 + \phi, \underline{\underline{\sigma}}^0 + \underline{\underline{\sigma}}) = \\
&= P_1(\underline{u}^0, q^0, \phi^0, \underline{\underline{\sigma}}^0)
\end{aligned}
$$

The final proof

We look for $(\underline{u}, q, \phi, \underline{\underline{\sigma}}) \in X_T^r$ with $\underline{u}(t=0)=0, \underline{\underline{\sigma}}(t=0)=0$ such that :

$$P_1(\underline{u}^0+\underline{u}, q^0+q, \phi^0+\phi, \underline{\underline{\sigma}}^0+\underline{\underline{\sigma}})+$$
$$+E(\xi(\underline{u}^0+\underline{u}), \underline{u}^0+\underline{u}, q^0+q, \phi^0+\phi, \underline{\underline{\sigma}}^0+\underline{\underline{\sigma}}) =$$
$$= P_1(\underline{u}^0, q^0, \phi^0, \underline{\underline{\sigma}}^0)$$

which is equivalent to

$$P_2[\underline{u}^0, \underline{\underline{\sigma}}^0](\underline{u}, q, \phi, \underline{\underline{\sigma}}) = -E(\xi(\underline{u}^0+\underline{u}), \underline{u}^0+\underline{u}, q^0+q, \phi^0+\phi, \underline{\underline{\sigma}}^0+\underline{\underline{\sigma}}).$$

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or

$$\begin{aligned}(\underline{u}, q, \phi, \underline{\underline{\sigma}}) &= P_2^{-1}[\underline{u}^0, \underline{\underline{\sigma}}^0](-E(\xi(\underline{u}^0+\underline{u}), \underline{u}^0+\underline{u}, q^0+q, \phi^0+\phi, \underline{\underline{\sigma}}^0+\\&=F(\underline{u}, q, \phi, \underline{\underline{\sigma}}).\end{aligned}$$

Thank you for your attention

The equations

The constitutive equation :

$$\underline{\underline{\sigma}} + \mathrm{We}\left(\frac{\partial \underline{\underline{\sigma}}}{\partial t} - \underline{\underline{g}}_a(\underline{\underline{\nabla u}}, \underline{\underline{\sigma}})\right) - 2\varepsilon \underline{\underline{D}}[\underline{u}] = \underline{\underline{m}}$$

No loss of regularity.

A crucial lemma

In an algebra, product is continuous.

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$$| \, 1 \, |_{H^{\frac{1+r}{2}}(0,T)} \leq C(\mathcal{A}) \, | \, 1 \, |_{H^{\frac{1+r}{2}}(0,T)} | \, 1 \, |_{H^{\frac{1+r}{2}}(0,T)} \, .$$

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Lemma

Let X a Hilbert space, $0 \leq s \leq 2$, such that $s - \frac{1}{2}$ is not integer. There exists a bounded extension operator from $\left\{ \underline{u} \in H^s(0, T; X), \partial_t^k \underline{u}(0) = 0,\ 0 \leq k < s - \frac{1}{2} \right\}$ in $H^s(\mathbb{R}^+; X)$. The boundedness constant C does not depend on $T \leq T_0$.

Thank you for your attention.