Existence and Approximation of Global Weak Solutions to some Regularized Dumbbell Models for Dilute Polymers

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The Standard Dumbbell Polymer Model

Polymer chains, which are suspended in a solvent, are assumed not to interact with each other; i.e. a dilute polymer.

The solvent is an incompressible, viscous, isothermal Newtonian fluid in a bounded $\Omega \subset \mathbb{R}^d$, $d = 2$ or 3, with Lipschitz boundary $\partial\Omega$.

Set $\Omega_T := \Omega \times (0, T]$, $\qquad \partial\Omega_T^* := \partial\Omega \times (0, T]$.

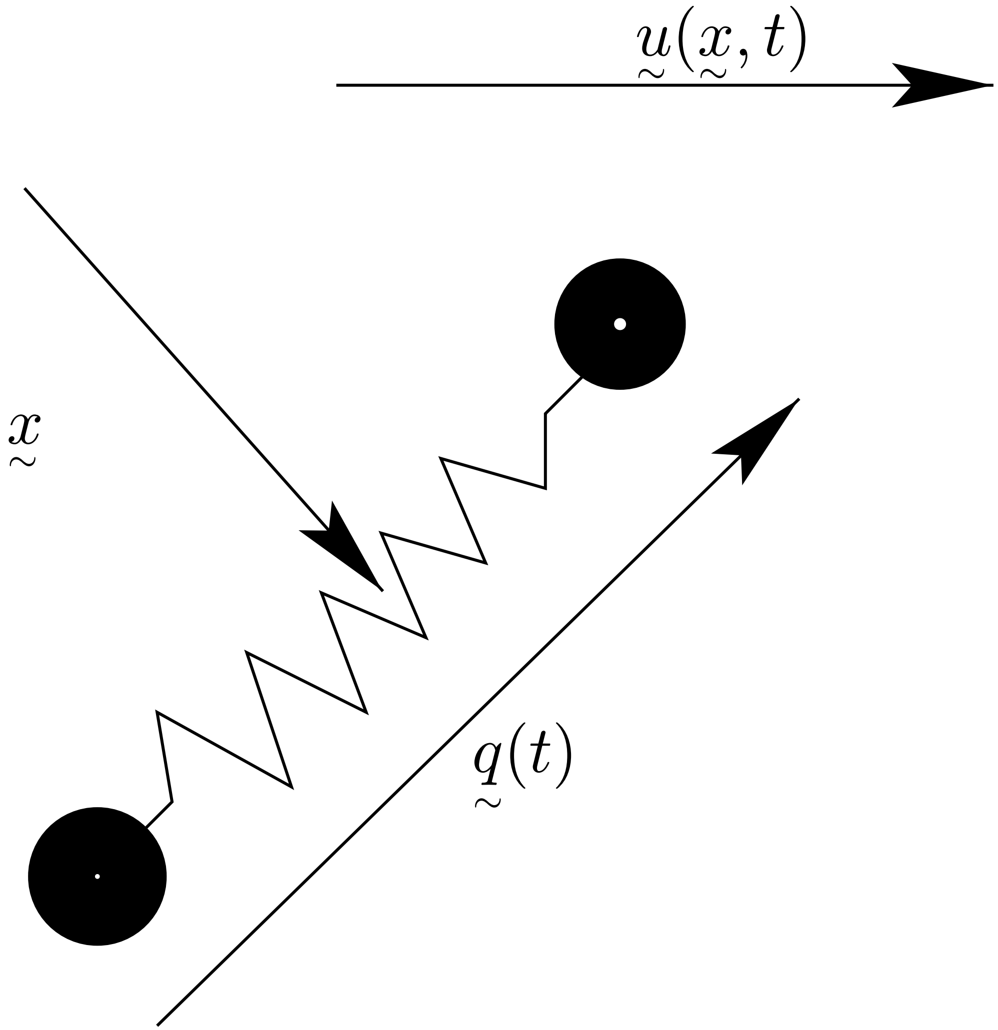
Hence the Navier-Stokes equations in which the symmetric *extra-stress* tensor $\underset{\approx}{\tau}$ (i.e. the polymeric part of the Cauchy stress tensor), appears as a source term:

Find the velocity field $\underset{\sim}{u}(\underset{\sim}{x},t) \in \mathbb{R}^d$ and the pressure $p(\underset{\sim}{x},t) \in \mathbb{R}$ of the solvent s.t.

$$\begin{aligned}
\frac{\partial \underset{\sim}{u}}{\partial t} + (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{u} - \nu\,\Delta_x \underset{\sim}{u} + \underset{\sim}{\nabla}_x\, p = \underset{\sim}{f} + \underset{\sim}{\nabla}_x \cdot \underset{\approx}{\tau} \quad & \text{in } \Omega_T, \\
\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u} = 0 \quad & \text{in } \Omega_T, \\
\underset{\sim}{u} = \underset{\sim}{0} \quad & \text{on } \partial\Omega_T^*, \\
\underset{\sim}{u}(\underset{\sim}{x},0) = \underset{\sim}{u}^0(\underset{\sim}{x}) \quad & \forall \underset{\sim}{x} \in \Omega;
\end{aligned}$$

where $\nu \in \mathbb{R}_{>0}$ is the given viscosity of the solvent, and $\underset{\sim}{f}$ is a given body force.

Here for simplicity, we assume a no slip boundary condition.



Noninteracting polymer chains modelled by using dumbbells.
A dumbbell is a pair of beads connected with an elastic spring, and is characterized by its centre of mass, $\underset{\sim}{x}$, and its elongation vector $\underset{\sim}{q}(t)$. A very simple model.

$\psi(\underset{\sim}{x}, \underset{\sim}{q}, t) \in \mathbb{R}$ is a probability density function

(the probability at time t of there being a dumbbell with centre of mass at $\underset{\sim}{x}$ and elongation $\underset{\sim}{q}$)

and satisfies the Fokker-Planck equation

$$\frac{\partial \psi}{\partial t} + (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x)\psi + \underset{\sim}{\nabla}_q \cdot ((\underset{\approx}{\nabla}_x \underset{\sim}{u})\, \underset{\sim}{q}\, \psi) = \frac{1}{2\lambda} \underset{\sim}{\nabla}_q \cdot (\underset{\sim}{\nabla}_q \psi + U' \underset{\sim}{q}\, \psi) \qquad \text{in } \Omega_T \times D,$$

$$\frac{1}{2\lambda}(\underset{\sim}{\nabla}_q \psi + U' \underset{\sim}{q}\, \psi) \cdot \underset{\sim}{n}_{\partial D} = (\underset{\approx}{\nabla}_x \underset{\sim}{u})\, \underset{\sim}{q}\, \psi \cdot \underset{\sim}{n}_{\partial D} \qquad \text{on } \Omega_T \times \partial D,$$

$$\psi(\underset{\sim}{x}, \underset{\sim}{q}, 0) = \psi^0(\underset{\sim}{x}, \underset{\sim}{q}) \geq 0 \qquad \forall (\underset{\sim}{x}, \underset{\sim}{q}) \in \Omega \times D;$$

where $\underset{\sim}{n}_{\partial D}$ is $\perp$ to ∂D, and $\int_D \psi^0(\underset{\sim}{x}, \underset{\sim}{q})\, \mathrm{d}\underset{\sim}{q} = 1$ for a.e. $\underset{\sim}{x} \in \Omega$.

b.c. $\Rightarrow$ $\int_D \psi(\underset{\sim}{x}, \underset{\sim}{q}, t)\, \mathrm{d}\underset{\sim}{q} = 1$ for a.e. $(\underset{\sim}{x}, t) \in \Omega_T$.

Here $\lambda > 0$ is the elastic relaxation constant of the fluid.

$D \subset \mathbb{R}^d$, $d = 2$ or 3: the set of admissible elongation vectors $\underset{\sim}{q}$.

U is the potential for the elastic force $\underset{\sim}{F} : D \mapsto \mathbb{R}^d$ of the dumbbell spring (U' strictly monotonic increasing):

$$\underset{\sim}{F}(\underset{\sim}{q}) := U'(\tfrac{1}{2}|\underset{\sim}{q}|^2)\,\underset{\sim}{q}.$$

On introducing the normalised Maxwellian:

$$M(\underset{\sim}{q}) := e^{-U(\frac{1}{2}\,|\underset{\sim}{q}|^2)} \Big/ \int_D e^{-U}\,\mathrm{d}\underset{\sim}{q}$$

$$\Rightarrow \quad \underset{\sim}{\nabla}_q \cdot (\underset{\sim}{\nabla}_q\,\psi + U'\,\underset{\sim}{q}\,\psi) \equiv \underset{\sim}{\nabla}_q \cdot \left(M\,\underset{\sim}{\nabla}_q \left(\frac{\psi}{M} \right) \right).$$

EXAMPLES:

Hookean case: $D = \mathbb{R}^d$,

$U(s) = s \quad \Rightarrow \quad U'(s) = 1$ and $e^{-U(\frac{1}{2}|\underset{\sim}{q}|^2)} = e^{-\frac{1}{2}|\underset{\sim}{q}|^2}$.

b.c. on ∂D replaced by decay conditions as $|\underset{\sim}{q}| \to \infty$.

Note that $M(\underset{\sim}{q}) \propto e^{-\frac{1}{2}|\underset{\sim}{q}|^2} \to 0$ as $|\underset{\sim}{q}| \to \infty$.

FENE (Finitely Extensible Nonlinear Elastic) case:

$D = B(\underset{\sim}{0}, b^{\frac{1}{2}})$,

$U(s) = -\frac{b}{2}\ln(1 - \frac{2\,s}{b}) \quad \Rightarrow \quad U'(s) = (1 - \frac{2\,s}{b})^{-1}$,

$$M(\underset{\sim}{q}) \propto e^{-U(\frac{1}{2}|\underset{\sim}{q}|^2)} = \left(1 - \frac{|\underset{\sim}{q}|^2}{b}\right)^{\frac{b}{2}} \quad \Rightarrow \quad M = 0 \text{ on } \partial D.$$

Note that $b \to \infty \quad \Rightarrow$ Hookean case.

Finally, the symmetric extra stress tensor, due to the dumbbells, on the RHS of the Navier-Stokes equations is

$$\underset{\approx}{\tau}(\psi) := \mu\,(\underset{\approx}{C}(\psi) - \rho(\psi)\,\underset{\approx}{I})\,,$$

Kramers expression.

Here $\mu \in \mathbb{R}_{>0}$ depends on the Boltzmann constant and temperature, $\underset{\approx}{I}$ is the unit $d \times d$ tensor, and

$$\underset{\approx}{C}(\psi)(\underset{\sim}{x},t) := \int_D \psi(\underset{\sim}{x},\underset{\sim}{q},t)\,U'(\tfrac{1}{2}|\underset{\sim}{q}|^2)\,\underset{\sim}{q}\,\underset{\sim}{q}^{\top}\,\mathrm{d}\underset{\sim}{q}$$

and

$$\rho(\psi)(\underset{\sim}{x},t) := \int_D \psi(\underset{\sim}{x},\underset{\sim}{q},t)\,\mathrm{d}\underset{\sim}{q}.$$

We denote the above coupled Navier-Stokes/Fokker-Planck system for $\underset{\sim}{u}(\underset{\sim}{x}, t)$ and $\psi(\underset{\sim}{x}, \underset{\sim}{q}, t)$ as (P).

(a Microscopic-Macroscopic Polymer Model)

The term that causes all the mathematical difficulties in establishing the existence of global-in-time weak solutions is the drag term

$$\underset{\sim}{\nabla}_q \cdot ((\underset{\approx}{\nabla}_x \underset{\sim}{u}) \underset{\sim}{q} \psi)$$

in the Fokker-Planck equation

$$\frac{\partial \psi}{\partial t} + (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x) \psi + \underset{\sim}{\nabla}_q \cdot ((\underset{\approx}{\nabla}_x \underset{\sim}{u}) \underset{\sim}{q} \psi)$$
$$= \frac{1}{2\lambda} \underset{\sim}{\nabla}_q \cdot \left(M \underset{\sim}{\nabla}_q \left(\frac{\psi}{M} \right) \right) \qquad \text{in } \Omega_T \times D\,.$$

A mathematically simpler model is the Corotational model.

Splitting the tensor $\quad \underset{\approx}{\nabla}_x \underset{\sim}{v} = \underset{\approx}{D}(\underset{\sim}{v}) + \underset{\approx}{\omega}(\underset{\sim}{v})$

into its symmetric and skew-symmetric parts

$$\underset{\approx}{D}(\underset{\sim}{v}) = \tfrac{1}{2}\,[\underset{\approx}{\nabla}_x \underset{\sim}{v} + (\underset{\approx}{\nabla}_x \underset{\sim}{v})^\top\,], \quad \underset{\approx}{\omega}(\underset{\sim}{v}) = \tfrac{1}{2}\,[\underset{\approx}{\nabla}_x \underset{\sim}{v} - (\underset{\approx}{\nabla}_x \underset{\sim}{v})^\top\,]\,,$$

the difficult drag term is written as

$$\underset{\sim}{\nabla}_q \cdot (\underset{\approx}{\zeta}(\underset{\sim}{u})\,\underset{\sim}{q}\,\psi)\,.$$

The two cases are then

(i) the noncorotational case $\quad \underset{\approx}{\zeta}(\underset{\sim}{v}) = \underset{\approx}{\nabla}_x \underset{\sim}{v},$

or (ii) the corotational case $\quad \underset{\approx}{\zeta}(\underset{\sim}{v}) = \underset{\approx}{\omega}(\underset{\sim}{v})\,.$

(i) is the original, difficult, case.

(ii) is mathematically easier, (physical justification ?).

In the Hookean case, as $U' = 1$, one can eliminate $\psi(\underset{\sim}{x}, \underset{\sim}{q}, t)$ leading to a closed macroscopic model (Oldroyd-B model) for $\underset{\sim}{u}(\underset{\sim}{x}, t), \rho(\underset{\sim}{x}, t)$ and $\underset{\approx}{\tau}(\underset{\sim}{x}, t)$:

Navier-Stokes for $\underset{\sim}{u}$ with extra stress tensor $\underset{\approx}{\tau}$ plus

$$\frac{\partial \rho}{\partial t} + (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x)\, \rho = 0 \qquad \text{in } \Omega_T\,,$$

$$\lambda \frac{\delta \underset{\approx}{\tau}}{\delta t} + \underset{\approx}{\tau} = \mu\, \lambda\, \rho \left[\underset{\approx}{\zeta}(\underset{\sim}{u}) + [\underset{\approx}{\zeta}(\underset{\sim}{u})]^\top \right] \qquad \text{in } \Omega_T\,;$$

where

$$\frac{\delta \underset{\approx}{\tau}}{\delta t} := \frac{\partial \underset{\approx}{\tau}}{\partial t} + (\underset{\sim}{u} \cdot \underset{\sim}{\nabla}_x)\, \underset{\approx}{\tau} - \left[\underset{\approx}{\zeta}(\underset{\sim}{u})\, \underset{\approx}{\tau} + \underset{\approx}{\tau}\, [\underset{\approx}{\zeta}(\underset{\sim}{u})]^\top \right]$$

is the upper-convected time derivative.

$\int_D \psi^0(\underset{\sim}{x}, \underset{\sim}{q})\, \mathrm{d}\underset{\sim}{q} = 1$ for a.e. $\underset{\sim}{x} \in \Omega \quad \Rightarrow \quad \rho(\underset{\sim}{x}, t) \equiv 1.$

Lions & Masmoudi (2001) have shown the existence of global-in-time weak solutions to the COROTATIONAL Oldroyd-B model.

Lions & Masmoudi (2007) have shown the existence of global-in-time weak solutions to the COROTATIONAL FENE model.

I.e. in both cases $\underset{\approx}{\zeta}(\underset{\sim}{v}) = \underset{\approx}{\omega}(\underset{\sim}{v}) = \frac{1}{2}\,[\underset{\approx}{\nabla}_x\,\underset{\sim}{v} - (\underset{\approx}{\nabla}_x\,\underset{\sim}{v})^{\top}\,]\,.$

To the best of our knowledge,

there are NO proofs of existence of global-in-time weak solutions to

(i) the original Oldroyd-B model,

(ii) the original FENE model,

i.e. $\underset{\approx}{\zeta}(\underset{\sim}{v}) = \underset{\approx}{\nabla}_x \underset{\sim}{v}$, in the literature.

There do exist various local-in-time results.

Throughout we will consider, for mathematical simplicity, a slightly different FENE model with $\underset{\sim}{x}$-diffusion in the Fokker-Planck equation, with a corresponding no flux boundary condition.

In addition, we will work with $\widehat{\psi} := \frac{\psi}{M}$, as opposed to ψ.

For a given $\varepsilon > 0$.

$(\mathbf{P}_\varepsilon)$ Find $\underset{\sim}{u}_\varepsilon(\underset{\sim}{x}, t) \in \mathbb{R}^d$ and $p_\varepsilon(\underset{\sim}{x}, t) \in \mathbb{R}$ s.t.

$$\frac{\partial \underset{\sim}{u}_\varepsilon}{\partial t} + (\underset{\sim}{u}_\varepsilon \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{u}_\varepsilon - \nu\,\Delta_x\,\underset{\sim}{u}_\varepsilon + \underset{\sim}{\nabla}_x\,p_\varepsilon$$

$$= \underset{\sim}{f} + \underset{\sim}{\nabla}_x \cdot \underset{\approx}{\tau}(M\,\widehat{\psi}_\varepsilon) \qquad \text{in } \Omega_T,$$

$$\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}_\varepsilon = 0 \qquad \text{in } \Omega_T,$$

$$\underset{\sim}{u}_\varepsilon = \underset{\sim}{0} \qquad \text{on } \partial\Omega_T^*,$$

$$\underset{\sim}{u}_\varepsilon(\underset{\sim}{x}, 0) = \underset{\sim}{u}^0(\underset{\sim}{x}) \qquad \forall \underset{\sim}{x} \in \Omega;$$

where

$$\underset{\approx}{\tau}(M\,\widehat{\psi}_\varepsilon) = \mu\,(\underset{\approx}{C}(M\,\widehat{\psi}_\varepsilon) - \rho(M\,\widehat{\psi}_\varepsilon)\,\underset{\approx}{I})\,;$$

and $\widehat{\psi}_\varepsilon(\underset{\sim}{x}, \underset{\sim}{q}, t) \in \mathbb{R}$ is s.t.

$$M\,\frac{\partial \widehat{\psi}_\varepsilon}{\partial t} + (\underset{\sim}{u}_\varepsilon \cdot \underset{\sim}{\nabla}_x)(M\,\widehat{\psi}_\varepsilon) + \underset{\sim}{\nabla}_q \cdot (\underset{\approx}{\zeta}(\underset{\sim}{u}_\varepsilon)\,\underset{\sim}{q}\,M\,\widehat{\psi}_\varepsilon)$$

$$= \frac{1}{2\,\lambda}\,\underset{\sim}{\nabla}_q \cdot (M\,\underset{\sim}{\nabla}_q\,\widehat{\psi}_\varepsilon) + \varepsilon\,M\,\Delta_x\,\widehat{\psi}_\varepsilon \qquad \text{in } \Omega_T \times D,$$

$$M\left[\frac{1}{2\,\lambda}\,\underset{\sim}{\nabla}_q\,\widehat{\psi}_\varepsilon - [\underset{\approx}{\zeta}(\underset{\sim}{u}_\varepsilon)\,\underset{\sim}{q}]\,\widehat{\psi}_\varepsilon\right] \cdot \underset{\sim}{n}_{\partial D} = 0 \qquad \text{on } \Omega_T \times \partial D,$$

$$\varepsilon\,M\,\underset{\sim}{\nabla}_x\,\widehat{\psi}_\varepsilon \cdot \underset{\sim}{n}_{\partial\Omega} = 0 \qquad \text{on } \partial\Omega_T^* \times D,$$

$$M\,\widehat{\psi}_\varepsilon(\underset{\sim}{x}, \underset{\sim}{q}, 0) = \psi^0(\underset{\sim}{x}, \underset{\sim}{q}) \geq 0 \qquad \forall(\underset{\sim}{x}, \underset{\sim}{q}) \in \Omega \times D\,;$$

where $\underset{\sim}{n}_{\partial D}$ is $\perp$ to ∂D, and $\underset{\sim}{n}_{\partial\Omega}$ is $\perp$ to $\partial\Omega$.

The inclusion of $\varepsilon\, M\, \Delta_x\, \widehat{\psi}_\varepsilon$ can be justified.

It does appear in the derivation of the model, but is usually dropped because ε is very small.

Corresponding Oldroyd-B model in the Hookean case for $\underset{\sim}{u}_\varepsilon(\underset{\sim}{x},t), \rho_\varepsilon(\underset{\sim}{x},t)$ and $\underset{\approx}{\tau}_\varepsilon(\underset{\sim}{x},t)$:

Navier-Stokes for $\underset{\sim}{u}_\varepsilon$ with extra stress tensor $\underset{\approx}{\tau}_\varepsilon$ plus

$$\frac{\partial \rho_\varepsilon}{\partial t} + (\underset{\sim}{u}_\varepsilon \cdot \underset{\sim}{\nabla}_x)\, \rho_\varepsilon - \varepsilon\, \Delta_x\, \rho_\varepsilon = 0 \qquad \text{in } \Omega_T\,,$$

$$\lambda\left(\frac{\delta \underset{\approx}{\tau}_\varepsilon}{\delta t} - \varepsilon\, \Delta_x\, \underset{\approx}{\tau}_\varepsilon\right) + \underset{\approx}{\tau}_\varepsilon = \mu\, \lambda\, \rho_\varepsilon \left[\, \underset{\approx}{\zeta}(\underset{\sim}{u}_\varepsilon) + [\underset{\approx}{\zeta}(\underset{\sim}{u}_\varepsilon)]^\top \right] \qquad \text{in } \Omega_T\,.$$

$\int_D \psi^0(\underset{\sim}{x},\underset{\sim}{q})\, \mathrm{d}\underset{\sim}{q} = 1$ for a.e. $\underset{\sim}{x} \in \Omega \quad \Rightarrow \quad \rho_\varepsilon(\underset{\sim}{x},t) \equiv 1.$

Formal Energy Bounds for (P_ε):

Testing the Navier-Stokes equation with $\underset{\sim}{u}_\varepsilon$, integrating over $\Omega \Rightarrow$

$$
\frac{1}{2}\,\frac{d}{dt}\left[\int_\Omega |\underset{\sim}{u}_\varepsilon|^2 \,\mathrm{d}\underset{\sim}{x}\right] + \nu \int_\Omega |\underset{\approx}{\nabla}_x\, \underset{\sim}{u}_\varepsilon|^2 \,\mathrm{d}\underset{\sim}{x} - \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{u}_\varepsilon \,\mathrm{d}\underset{\sim}{x}
$$

$$
= - \int_\Omega \underset{\approx}{\tau}(M\,\widehat{\psi}_\varepsilon) : \underset{\approx}{\nabla}_x\, \underset{\sim}{u}_\varepsilon \,\mathrm{d}\underset{\sim}{x}
$$

$$
= -\mu \int_\Omega \underset{\approx}{C}(M\,\widehat{\psi}_\varepsilon) : \underset{\approx}{\nabla}_x\, \underset{\sim}{u}_\varepsilon \,\mathrm{d}\underset{\sim}{x}
$$

$$
\leq \frac{\nu}{2} \int_\Omega |\underset{\approx}{\nabla}_x\, \underset{\sim}{u}_\varepsilon|^2 \,\mathrm{d}\underset{\sim}{x} + \frac{\mu^2}{2\,\nu} \int_\Omega |\underset{\approx}{C}(M\,\widehat{\psi}_\varepsilon)|^2 \,\mathrm{d}\underset{\sim}{x}\,.
$$

We will consider the Oldroyd-B model separately

(i.e. the Hookean case, $D = \mathbb{R}^d$).

Here we consider only the FENE macroscopic/microscopic model:

$D = B(\underset{\sim}{0}, b^{\frac{1}{2}})$, $\qquad U(s) = -\frac{b}{2}\,\ln(1 - \frac{2\,s}{b}) \qquad \Rightarrow$

$$M(\underset{\sim}{q}) \propto \left(1 - \frac{|\underset{\sim}{q}|^2}{b}\right)^{\frac{b}{2}} \quad \text{and} \quad M = 0 \text{ on } \partial D.$$

We will assume throughout that $b > 2$, which implies that

$$\int_D M\ \left[1 + U^2 + |U'|^2\right]\ \mathrm{d}\underset{\sim}{q} < \infty\,.$$

Introducing the weighted Sobolev norm (degenerate weight M)

$$\|\widehat{\varphi}\|_{H^1(\Omega\times D;M)} :=$$
$$\left\{\int_{\Omega\times D} M \left[|\widehat{\varphi}|^2 + \left|\underset{\sim}{\nabla}_q \widehat{\varphi}\right|^2 + \left|\underset{\sim}{\nabla}_x \widehat{\varphi}\right|^2\right] \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right\}^{\frac{1}{2}},$$

we set

$$\widehat{X} \equiv H^1(\Omega \times D; M)$$
$$:= \left\{\widehat{\varphi} \in L^1_{\mathrm{loc}}(\Omega \times D) : \|\widehat{\varphi}\|_{H^1(\Omega\times D;M)} < \infty\right\}.$$

One can show, for example, that

$C^\infty(\overline{\Omega \times D})$ is dense in $\widehat{X}$,

the embedding $L^2(\Omega \times D; M) \hookrightarrow H^1(\Omega \times D; M)$ is compact.

For all $\widehat{\varphi} \in \widehat{X}$, we have that

$$\int_\Omega |\underset{\approx}{C}(M\,\widehat{\varphi})|^2 \,\mathrm{d}\underset{\sim}{x}$$

$$= \int_\Omega \sum_{i=1}^{d} \sum_{j=1}^{d} \left(\int_D M\,\widehat{\varphi}\, U'\, q_i\, q_j \,\mathrm{d}\underset{\sim}{q} \right)^2 \mathrm{d}\underset{\sim}{x}$$

$$\leq d \left(\int_D M\, |U'|^2\, |\underset{\sim}{q}|^4 \,\mathrm{d}\underset{\sim}{q} \right) \left(\int_{\Omega\times D} M\, |\widehat{\varphi}|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \right)$$

$$\leq C \left(\int_{\Omega\times D} M\, |\widehat{\varphi}|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \right) < \infty\,.$$

Multiplying the Fokker-Planck equation with $\widehat{\psi}_\varepsilon$,

integrating over $\Omega \times D \Rightarrow$

$$\frac{1}{2}\,\frac{d}{dt}\left[\int_{\Omega\times D} M\,|\widehat{\psi}_\varepsilon|^2\;\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right]$$

$$+\frac{1}{2\,\lambda}\int_{\Omega\times D} M\,|\underset{\sim}{\nabla}_q\,\widehat{\psi}_\varepsilon|^2\;\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$

$$+\,\varepsilon\int_{\Omega\times D} M\,|\underset{\sim}{\nabla}_x\,\widehat{\psi}_\varepsilon|^2\;\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$

$$=\int_{\Omega\times D} M\,(\underset{\approx}{\zeta}(\underset{\sim}{u}_\varepsilon)\,\underset{\sim}{q}\,\widehat{\psi}_\varepsilon)\;\cdot\;\underset{\sim}{\nabla}_q\,\widehat{\psi}_\varepsilon\;\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,.$$

Corotational case (skew-symmetric $\underset{\approx}{\zeta}$)

$$\underset{\approx}{\zeta}(\underset{\sim}{v}) = \underset{\approx}{\omega}(\underset{\sim}{v}) \qquad \Rightarrow \qquad \underset{\sim}{q}^{\top}\underset{\approx}{\omega}(\underset{\sim}{v})\,\underset{\sim}{q} = 0 \qquad \forall \underset{\sim}{q} \in \mathbb{R}^d .$$

Hence we have for all $\widehat{\varphi} \in \widehat{X}$ and $\underset{\sim}{v} \in [W^{1,\infty}(\Omega)]^d$ that

$$\begin{aligned}
\int_{\Omega\times D} M\,(\underset{\approx}{\omega}(\underset{\sim}{v})\,\underset{\sim}{q}\,\widehat{\varphi}) \cdot \underset{\sim}{\nabla}_q\,\widehat{\varphi}\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}
&= \tfrac{1}{2}\int_{\Omega\times D} M\,(\underset{\approx}{\omega}(\underset{\sim}{v})\,\underset{\sim}{q}) \cdot \underset{\sim}{\nabla}_q\,(\widehat{\varphi}^2)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} \\
&= \tfrac{1}{2}\int_{\Omega\times\partial D} M\,(\underset{\approx}{\omega}(\underset{\sim}{v})\,\underset{\sim}{q}) \cdot \underset{\sim}{n}_{\partial D}\,\widehat{\varphi}^2\,\mathrm{d}s\,\mathrm{d}\underset{\sim}{x} \\
&\quad + \tfrac{1}{2}\int_{\Omega\times D} M\,(\underset{\sim}{q}^{\top}\underset{\approx}{\omega}(\underset{\sim}{v})\,\underset{\sim}{q})\,U'\,\widehat{\varphi}^2\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = 0\,,
\end{aligned}$$

since $\underset{\sim}{n}_{\partial D} = \dfrac{\underset{\sim}{q}}{|\underset{\sim}{q}|}$, $\underset{\sim}{\nabla}_q M = -M\,U'\,\underset{\sim}{q}$ and $\mathrm{trace}(\underset{\approx}{\omega}(\underset{\sim}{v})) = 0$.

Hence in the Corotational case, we have the formal estimates:

$$\frac{d}{dt}\left[\int_\Omega |\underset{\sim}{u}_\varepsilon|^2 \,\mathrm{d}\underset{\sim}{x}\right] + \nu \int_\Omega |\underset{\approx}{\nabla}_x \underset{\sim}{u}_\varepsilon|^2 \,\mathrm{d}\underset{\sim}{x} - \tfrac{1}{2}\int_\Omega \underset{\sim}{f}\cdot \underset{\sim}{u}_\varepsilon \,\mathrm{d}\underset{\sim}{x}$$
$$\leq \frac{\mu^2}{\nu}\int_\Omega |\underset{\approx}{C}(M\,\widehat{\psi}_\varepsilon)|^2 \,\mathrm{d}\underset{\sim}{x} \leq C \int_{\Omega\times D} M\,|\widehat{\psi}_\varepsilon|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,;$$

$$\frac{d}{dt}\left[\int_{\Omega\times D} M\,|\widehat{\psi}_\varepsilon|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right] + \frac{1}{\lambda}\int_{\Omega\times D} M\,|\underset{\sim}{\nabla}_q \widehat{\psi}_\varepsilon|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$
$$+\, 2\,\varepsilon \int_{\Omega\times D} M\,|\underset{\sim}{\nabla}_x \widehat{\psi}_\varepsilon|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} = 0.$$

The above can be made rigorous, and one can easily establish the existence of global-in-time weak solutions for (P_ε) in the Corotational case.

One can also easily construct Finite Element approximations, and prove convergence to (P_ε) in the Corotational case; see B. & Süli (2009).

The Noncorotational case.

The trick is to choose the testing procedure so as to cancel the extra stress term in the Navier-Stokes equation with the drag term in the Fokker-Planck equation;

see e.g. B., Schwab & Süli (2005);
Jourdain, Lelièvre, Le Bris & Otto (2006); Lin, Liu & Zhang (2007).

As before for the Navier-Stokes equations tested with $\underset{\sim}{u}_{\varepsilon}$, we have that

$$\frac{1}{2}\,\frac{d}{dt}\left[\int_{\Omega}|\underset{\sim}{u}_{\varepsilon}|^2\,\mathrm{d}\underset{\sim}{x}\right]+\nu\int_{\Omega}|\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_{\varepsilon}|^2\,\mathrm{d}\underset{\sim}{x}$$
$$=\int_{\Omega}\underset{\sim}{f}\cdot\underset{\sim}{u}_{\varepsilon}\,\mathrm{d}\underset{\sim}{x}-\mu\int_{\Omega}\underset{\approx}{C}(M\,\widehat{\psi}_{\varepsilon}):\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_{\varepsilon}\,\mathrm{d}\underset{\sim}{x}\,.$$

Let $\mathcal{F}(s) := s\,(\ln s - 1) + 1 \in \mathbb{R}_{\geq 0}$ for $s \geq 0$.

Multiplying the Fokker-Planck equation with $\mathcal{F}'(\widehat{\psi}_\varepsilon) \equiv \ln \widehat{\psi}_\varepsilon$, assumes that $\widehat{\psi}_\varepsilon > 0$, integrating over $\Omega \times D$ $\Rightarrow$

$$\frac{d}{dt}\left[\int_{\Omega\times D} M\,\mathcal{F}(\widehat{\psi}_\varepsilon)\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right] + \frac{1}{2\,\lambda}\int_{\Omega\times D} M\,\underset{\sim}{\nabla}_q\,\widehat{\psi}_\varepsilon \cdot \underset{\sim}{\nabla}_q\,[\mathcal{F}'(\widehat{\psi}_\varepsilon)]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x} + \varepsilon\int_{\Omega\times D} M\,\underset{\sim}{\nabla}_x\,\widehat{\psi}_\varepsilon \cdot \underset{\sim}{\nabla}_x\,[\mathcal{F}'(\widehat{\psi}_\varepsilon)]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$

$$= \int_{\Omega\times D} M\,\widehat{\psi}_\varepsilon\,[(\underset{\approx}{\nabla}_x\,\underset{\sim}{u}_\varepsilon)\,\underset{\sim}{q}] \cdot \underset{\sim}{\nabla}_q\,[\mathcal{F}'(\widehat{\psi}_\varepsilon)]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\,.$$

Note that $\mathcal{F}''(s) = s^{-1} > 0$ for $s > 0$.

Noting that $\widehat{\psi}_\varepsilon \, \underset{\sim}{\nabla}_q \, [\mathcal{F}'(\widehat{\psi}_\varepsilon)] = \underset{\sim}{\nabla}_q \, \widehat{\psi}_\varepsilon$, $\underset{\sim}{\nabla}_q \, M = -M \, U' \, \underset{\sim}{q}$, $M = 0$ on ∂D and $\underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}_\varepsilon = 0 \quad \Rightarrow$

$$\begin{aligned}
&\int_{\Omega \times D} M \, \widehat{\psi}_\varepsilon \, [(\underset{\approx}{\nabla}_x \, \underset{\sim}{u}_\varepsilon) \, \underset{\sim}{q}] \cdot \underset{\sim}{\nabla}_q \, [\mathcal{F}'(\widehat{\psi}_\varepsilon)] \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \\
&\quad = \int_{\Omega \times D} M \, [(\underset{\approx}{\nabla}_x \, \underset{\sim}{u}_\varepsilon) \, \underset{\sim}{q}] \cdot \underset{\sim}{\nabla}_q \, \widehat{\psi}_\varepsilon \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \\
&\quad = \int_{\Omega \times D} M \, U' \, \underset{\sim}{q} \cdot [(\underset{\approx}{\nabla}_x \, \underset{\sim}{u}_\varepsilon) \, \underset{\sim}{q}] \, \widehat{\psi}_\varepsilon \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \\
&\quad = \int_{\Omega} \underset{\approx}{C}(M \, \widehat{\psi}_\varepsilon) : \underset{\approx}{\nabla}_x \, \underset{\sim}{u}_\varepsilon \, \mathrm{d}\underset{\sim}{x} \, ,
\end{aligned}$$

on recalling that

$$\underset{\approx}{C}(M \, \widehat{\psi}_\varepsilon)(\underset{\sim}{x}, t) = \int_D M \, \widehat{\psi}_\varepsilon(\underset{\sim}{x}, \underset{\sim}{q}, t) \, U'(\tfrac{1}{2}|\underset{\sim}{q}|^2) \, \underset{\sim}{q} \, \underset{\sim}{q}^{\top} \, \mathrm{d}\underset{\sim}{q} \, .$$

To make the above rigorous, and for computational purposes, we replace the convex $\mathcal{F} \in C^\infty(\mathbb{R}_{>0})$ for any $\delta \in (0,\, 1)$ and $L > 1$ by the convex $\mathcal{F}_\delta^L \in C^{2,1}(\mathbb{R})$:

$$\mathcal{F}_\delta^L(s) \quad := \begin{cases} \frac{s^2-\delta^2}{2\,\delta} + (\ln\delta - 1)\, s + 1 & s \leq \delta \\ \mathcal{F}(s) \equiv s\,(\ln s - 1) + 1 & \delta \leq s \leq L \\ \frac{s^2-L^2}{2\,L} + (\ln L - 1)\, s + 1 & L \leq s \end{cases},$$

$$\Rightarrow \quad [\mathcal{F}_\delta^L]'(s) = \begin{cases} \frac{s}{\delta} + \ln\delta - 1 & s \leq \delta \\ \ln s & \delta \leq s \leq L \\ \frac{s}{L} + \ln L - 1 & L \leq s \end{cases},$$

$$\Rightarrow \quad [\mathcal{F}_\delta^L]''(s) = \begin{cases} \delta^{-1} & s \leq \delta \\ s^{-1} & \delta \leq s \leq L \\ L^{-1} & L \leq s \end{cases}.$$

Let

$$\beta_\delta^L(s) := [[\mathcal{F}_\delta^L]''(s)]^{-1} = \begin{cases} \delta & s \leq \delta \\ s & \delta \leq s \leq L \\ L & L \leq s \end{cases} .$$

Let $\{\underset{\sim}{u}_{\varepsilon,\delta}^L, \widehat{\psi}_{\varepsilon,\delta}^L\}$ solve $(\mathrm{P}_{\varepsilon,\delta}^L)$, which is (P_ε) with the drag term

$$\underset{\sim}{\nabla}_q \cdot ((\underset{\approx}{\nabla}_x \underset{\sim}{u}_\varepsilon) \underset{\sim}{q} M \widehat{\psi}_\varepsilon)$$

replaced by

$$\underset{\sim}{\nabla}_q \cdot ((\underset{\approx}{\nabla}_x \underset{\sim}{u}_{\varepsilon,\delta}^L) \underset{\sim}{q} M \beta_\delta^L(\widehat{\psi}_{\varepsilon,\delta}^L)) .$$

Multiplying Fokker-Planck in $(\mathrm{P}^L_{\varepsilon,\delta})$ with $[\mathcal{F}^L_\delta]'(\widehat{\psi}^L_{\varepsilon,\delta})$,
integrating over $\Omega \times D$,
noting that $\quad \beta^L_\delta(\widehat{\psi}^L_{\varepsilon,\delta})\, \underset{\sim}{\nabla}_q \left[[\mathcal{F}^L_\delta]'(\widehat{\psi}^L_{\varepsilon,\delta}) \right] = \underset{\sim}{\nabla}_q\, \widehat{\psi}^L_{\varepsilon,\delta}\,,$
$\underset{\sim}{\nabla}_q\, M = -M\, U'\, \underset{\sim}{q} \quad$ and $\quad \underset{\sim}{\nabla}_x \cdot \underset{\sim}{u}^L_{\varepsilon,\delta} = 0 \quad \Rightarrow$

$$\frac{d}{dt}\left[\int_{\Omega\times D} M\, \mathcal{F}^L_\delta(\widehat{\psi}^L_{\varepsilon,\delta})\, \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}\right]$$
$$+ \frac{1}{2\,\lambda} \int_{\Omega\times D} M\, \underset{\sim}{\nabla}_q\, \widehat{\psi}^L_{\varepsilon,\delta} \cdot \underset{\sim}{\nabla}_q \left[[\mathcal{F}^L_\delta]'(\widehat{\psi}^L_{\varepsilon,\delta}) \right] \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}$$
$$+ \varepsilon \int_{\Omega\times D} M\, \underset{\sim}{\nabla}_x\, \widehat{\psi}^L_{\varepsilon,\delta} \cdot \underset{\sim}{\nabla}_x \left[[\mathcal{F}^L_\delta]'(\widehat{\psi}^L_{\varepsilon,\delta}) \right] \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}$$
$$= \int_{\Omega\times D} M\, \beta^L_\delta(\widehat{\psi}^L_{\varepsilon,\delta})\, [(\underset{\approx}{\nabla}_x\, \underset{\sim}{u}^L_{\varepsilon,\delta})\, \underset{\sim}{q}] \cdot \underset{\sim}{\nabla}_q \left[[\mathcal{F}^L_\delta]'(\widehat{\psi}^L_{\varepsilon,\delta}) \right] \mathrm{d}\underset{\sim}{q}\, \mathrm{d}\underset{\sim}{x}$$
$$= \int_\Omega \underset{\approx}{C}(M\, \widehat{\psi}^L_{\varepsilon,\delta}) : \underset{\approx}{\nabla}_x\, \underset{\sim}{u}^L_{\varepsilon,\delta}\, \mathrm{d}\underset{\sim}{x}\,.$$

Note that $[\mathcal{F}_\delta^L]'' \geq L^{-1}$, and

$$\mathcal{F}_\delta^L(s) \geq \begin{cases} \frac{s^2}{2\,\delta} & \text{if } s \leq 0\,, \\ \frac{s^2}{4\,L} - C(L) & \text{if } s \geq 0\,. \end{cases}$$

Let $\mathcal{G} : \widehat{X}' \mapsto \widehat{X}$, (duality with respect to the M weight)

be s.t. $\mathcal{G}\,\widehat{\eta}$ is the unique solution of

$$\int_{\Omega \times D} M \left[(\mathcal{G}\,\widehat{\eta})\,\widehat{\varphi} + \underset{\sim}{\nabla}_q\,(\mathcal{G}\,\widehat{\eta}) \cdot \underset{\sim}{\nabla}_q\,\widehat{\varphi} + \underset{\sim}{\nabla}_x\,(\mathcal{G}\,\widehat{\eta}) \cdot \underset{\sim}{\nabla}_x\,\widehat{\varphi} \right] \mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$
$$= \langle M\,\widehat{\eta}, \widehat{\varphi} \rangle_{\widehat{X}} \qquad \forall \widehat{\varphi} \in \widehat{X}\,,$$

where $\langle M\,\cdot, \cdot \rangle_{\widehat{X}}$ is the duality pairing between $\widehat{X}$ and $\widehat{X}'$.

Let

$$\underset{\sim}{H} := \{\underset{\sim}{w} \in [L^2(\Omega)]^d : \underset{\sim}{\nabla}_x \cdot \underset{\sim}{w} = 0\}\,,$$
$$\underset{\sim}{V} := \{\underset{\sim}{w} \in [H^1_0(\Omega)]^d : \underset{\sim}{\nabla}_x \cdot \underset{\sim}{w} = 0\}\,,$$

$\underset{\sim}{V}'$ the dual of $\underset{\sim}{V}$ and $\langle \cdot, \cdot \rangle_V$ the duality pairing between $\underset{\sim}{V}'$ and $\underset{\sim}{V}$.

Let $\underset{\sim}{S} : \underset{\sim}{V}' \mapsto \underset{\sim}{V}$ be s.t. $\underset{\sim}{S}\,\underset{\sim}{v}$ is the unique solution of the Helmholtz-Stokes problem

$$\int_\Omega \left[\underset{\sim}{S}\,\underset{\sim}{v} \cdot \underset{\sim}{w} + \underset{\approx}{\nabla}_x (\underset{\sim}{S}\,\underset{\sim}{v}) : \underset{\approx}{\nabla}_x \underset{\sim}{w}\right] \mathrm{d}\underset{\sim}{x} = \langle \underset{\sim}{v}, \underset{\sim}{w} \rangle_V \quad \forall \underset{\sim}{w} \in \underset{\sim}{V}\,.$$

Hence $\|\underset{\sim}{S} \cdot \|_{H^1(\Omega)}$ is a norm on $\underset{\sim}{V}'$.

Assumptions:

$\partial\Omega \in C^{0,1}$, $\quad \underset{\sim}{u}^0 \in \underset{\sim}{H}$, $\quad M^{\frac{1}{2}}\,\widehat{\psi}^0 \equiv M^{-\frac{1}{2}}\,\psi^0 \in L^2(\Omega \times D)$ with $\widehat{\psi}^0 \geq 0$ and $\quad \underset{\sim}{f} \in L^2(0,T;\underset{\sim}{V}')$.

Noncorotational case, assuming that $\widehat{\psi}^0 \leq L$, we obtain

$$\sup_{t\in(0,T)} \left[\int_\Omega |\underset{\sim}{u}^L_{\varepsilon,\delta}|^2 \,\mathrm{d}\underset{\sim}{x}\right] + \nu \int_{\Omega_T} |\underset{\approx}{\nabla}_x\, \underset{\sim}{u}^L_{\varepsilon,\delta}|^2 \,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \leq C,$$

$$\sup_{t\in(0,T)} \left[\int_{\Omega\times D} M\, |[\widehat{\psi}^L_{\varepsilon,\delta}]_-|^2 \,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\right] \leq C\,\delta,$$

where C is a constant depending on the data $\underset{\sim}{u}^0$, $\widehat{\psi}^0$ and $\underset{\sim}{f}$ (dependence suppressed from now on);

In addition, testing

the Fokker-Planck equation with (a) $\widehat{\psi}^L_{\varepsilon,\delta}$ and (b) $\mathcal{G}\, \dfrac{\partial \widehat{\psi}^L_{\varepsilon,\delta}}{\partial t}$,

and the Navier-Stokes equation with $\underset{\sim}{S}\, \dfrac{\partial \underset{\sim}{u}^L_{\varepsilon,\delta}}{\partial t}$;

we obtain that

$$\int_0^T \left\| \frac{\partial \underset{\sim}{u}^L_{\varepsilon,\delta}}{\partial t} \right\|^{\frac{4}{d}}_{\underset{\sim}{V}'} \mathrm{d}t + \sup_{t\in(0,T)} \left[\int_{\Omega\times D} M \, |\widehat{\psi}^L_{\varepsilon,\delta}|^2 \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \right]$$

$$+ \frac{1}{\lambda} \int_0^T \int_{\Omega\times D} M \left| \underset{\sim}{\nabla}_q \, \widehat{\psi}^L_{\varepsilon,\delta} \right|^2 \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t$$

$$+ \varepsilon \int_0^T \int_{\Omega\times D} M \left| \underset{\sim}{\nabla}_x \, \widehat{\psi}^L_{\varepsilon,\delta} \right|^2 \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t$$

$$+ \sup_{t\in(0,T)} \left[\int_\Omega |\underset{\approx}{C}(M \, \widehat{\psi}^L_{\varepsilon,\delta})|^2 \, \mathrm{d}\underset{\sim}{x} \right]$$

$$+ \int_0^T \| \frac{\partial \widehat{\psi}^L_{\varepsilon,\delta}}{\partial t} \|^{\frac{4}{d}}_{\widehat{X}'} \, \mathrm{d}t \leq C(L,T) \, .$$

The testing (a) and (b) require the cut-off $\beta^L_\delta(\cdot)$, as opposed to $\beta_\delta(\cdot)$, in the drag term of the Fokker-Planck equation.

One can pass to the limit $\delta \to 0$, to obtain e.g. that

$$M^{\frac{1}{2}}\, \widehat{\psi}^L_{\varepsilon,\delta} \to M^{\frac{1}{2}}\, \widehat{\psi}^L_{\varepsilon} \geq 0 \quad \text{strongly in } L^2(0,T;L^2(\Omega \times D)),$$

$$M^{\frac{1}{2}}\, \beta^L_\delta(\widehat{\psi}^L_{\varepsilon,\delta}) \to M^{\frac{1}{2}}\, \beta^L(\widehat{\psi}^L_{\varepsilon}) \quad \text{strongly in } L^2(0,T;L^2(\Omega \times D));$$

where

$$\beta^L(s) := \begin{cases} s & s \leq L \\ L & L \leq s \end{cases} .$$

The above can be made rigorous, and one can establish the existence of global-in-time weak solutions for (P^L_ε) in the noncorotational case.

Noncorotational case for given $\varepsilon \in (0,1]$ and $L > 1$:

$(\mathrm{P}_{\varepsilon}^{L})$ Find $\underset{\sim}{u}_{\varepsilon}^{L} \in L^{\infty}(0,T;[L^2(\Omega)]^d) \cap L^2(0,T;\underset{\sim}{V}) \cap W^{1,\frac{4}{d}}(0,T;\underset{\sim}{V}')$ and $\widehat{\psi}_{\varepsilon}^{L} \in L^2(0,T;\widehat{X}) \cap W^{1,\frac{4}{d}}(0,T;\widehat{X}')$, with $\widehat{\psi}_{\varepsilon}^{L} \geq 0$, $M^{\frac{1}{2}}\,\widehat{\psi}_{\varepsilon}^{L} \in L^{\infty}(0,T;L^2(\Omega \times D))$ and $\underset{\approx}{C}(M\,\widehat{\psi}_{\varepsilon}^{L}) \in L^{\infty}(0,T;[L^2(\Omega)]^{d\times d})$, such that $\underset{\sim}{u}_{\varepsilon}^{L}(\cdot,0) = \underset{\sim}{u}^{0}(\cdot)$, $\widehat{\psi}_{\varepsilon}^{L}(\cdot,\cdot,0) = \widehat{\psi}^{0}(\cdot,\cdot)$ and

$$\int_0^T \left\langle \frac{\partial \underset{\sim}{u}_{\varepsilon}^{L}}{\partial t}, \underset{\sim}{w} \right\rangle_V \mathrm{d}t + \int_{\Omega_T} \left[\left[(\underset{\sim}{u}_{\varepsilon}^{L} \cdot \underset{\sim}{\nabla}_x) \underset{\sim}{u}_{\varepsilon}^{L} \right] \cdot \underset{\sim}{w} + \nu \underset{\approx}{\nabla}_x \underset{\sim}{u}_{\varepsilon}^{L} : \underset{\approx}{\nabla}_x \underset{\sim}{w} \right] \mathrm{d}\underset{\sim}{x}\,\mathrm{d}t$$

$$= \int_0^T \langle \underset{\sim}{f}, \underset{\sim}{w} \rangle_V \,\mathrm{d}t - \mu \int_{\Omega_T} \underset{\approx}{C}(M\,\widehat{\psi}_{\varepsilon}^{L}) : \underset{\approx}{\nabla}_x \underset{\sim}{w}\,\mathrm{d}\underset{\sim}{x}\,\mathrm{d}t \qquad \forall \underset{\sim}{w} \in L^{\frac{4}{4-d}}(0,T;\underset{\sim}{V});$$

$$\int_0^T \left\langle M \frac{\partial \widehat{\psi}_\varepsilon^L}{\partial t}, \widehat{\varphi} \right\rangle_{\widehat{X}} \mathrm{d}t$$

$$+ \int_0^T \int_{\Omega \times D} M \left[\varepsilon \underset{\sim}{\nabla}_x \widehat{\psi}_\varepsilon^L - \underset{\sim}{u}_\varepsilon^L \widehat{\psi}_\varepsilon^L\right] \cdot \underset{\sim}{\nabla}_x \widehat{\varphi} \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t$$

$$+ \int_0^T \int_{\Omega \times D} M \left[\frac{1}{2\lambda} \underset{\sim}{\nabla}_q \widehat{\psi}_\varepsilon^L - (\underset{\approx}{\nabla}_x \underset{\sim}{u}_\varepsilon^L) \underset{\sim}{q} \, \beta^L(\widehat{\psi}_\varepsilon^L)\right] \cdot \underset{\sim}{\nabla}_q \widehat{\varphi} \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t$$

$$= 0 \quad \forall \widehat{\varphi} \in L^{\frac{4}{4-d}}(0, T; \widehat{X}).$$

In addition, we have that

$$\sup_{t \in (0,T)} \left[\int_\Omega |\underset{\sim}{u}_\varepsilon^L|^2 \, \mathrm{d}\underset{\sim}{x}\right] + \nu \int_{\Omega_T} |\underset{\approx}{\nabla}_x \underset{\sim}{u}_\varepsilon^L|^2 \, \mathrm{d}\underset{\sim}{x} \, \mathrm{d}t \le C,$$

i.e. independent of ε and L; see B. & Süli (2008).

For the Corotational case one can consider a very general numerical approximation, as it is easy to mimic the testing procedure for (P_ε). Not so easy for the Noncorotational case, which we state specifically here.

Finite Element Approximation:

Let Ω be a convex polytope (for ease of exposition).

Let $\mathcal{T}_x^h$ be a partitioning of Ω into open ACUTE simplices κ_x.

$\overline{\Omega} \equiv \bigcup_{\kappa_x \in \mathcal{T}_x^h} \overline{\kappa_x}\,, \quad h_{\kappa_x} := \mathrm{diam}(\kappa_x)\,, \quad h_x := \max_{\kappa_x \in \mathcal{T}_x^h} h_{\kappa_x}\,.$

Let $\mathcal{T}_q^h$ be a partitioning of $D \equiv B(\underset{\sim}{0}, b^{\frac{1}{2}})$ into open ACUTE simplices κ_q, with possibly one curved edge/face ($d = 2/3$).

$\overline{D} \equiv \bigcup_{\kappa_q \in \mathcal{T}_q^h} \overline{\kappa_q}\,, \quad h_{\kappa_q} := \mathrm{diam}(\kappa_q)\,, \quad h_q := \max_{\kappa_q \in \mathcal{T}_q^h} h_{\kappa_q}\,.$

Assume both partitionings, $\mathcal{T}_x^h$ and $\mathcal{T}_q^h$, are quasi-uniform.

(Acute $\equiv$ Non-obtuse, i.e. right angles allowed.)

$\mathbb{P}_k^x$ and $\mathbb{P}_k^q$ polynomials of degree k or less in $\underset{\sim}{x}$ and $\underset{\sim}{q}$, respectively.

The lowest order Taylor-Hood element for the pressure/velocity:

$$R_h := \{\eta_h \in C(\overline{\Omega}) : \eta_h \,|_{\kappa_x} \in \mathbb{P}_1^x \quad \forall \kappa_x \in \mathcal{T}_x^h\}\,,$$

$$\underset{\sim}{W}_h := \{\underset{\sim}{w}_h \in [C(\overline{\Omega})]^d : \underset{\sim}{w}_h \,|_{\kappa_x} \in [\mathbb{P}_2^x]^d \quad \forall \kappa_x \in \mathcal{T}_x^h$$

$$\text{and} \quad \underset{\sim}{w}_h = 0 \text{ on } \partial\Omega\} \subset [H_0^1(\Omega)]^d\,,$$

$$\underset{\sim}{V}_h := \{\underset{\sim}{v}_h \in \underset{\sim}{W}_h : \int_\Omega (\underset{\sim}{\nabla}_x \cdot \underset{\sim}{v}_h)\, \eta_h \,\mathrm{d}\underset{\sim}{x} = 0 \quad \forall \eta_h \in R_h\}\,.$$

R_h and $\underset{\sim}{W}_h$ satisfy the LBB inf-sup condition

$$\sup_{\underset{\sim}{w}_h \in \underset{\sim}{W}_h} \frac{\int_\Omega (\underset{\sim}{\nabla}_x \cdot \underset{\sim}{w}_h)\, r_h \,\mathrm{d}\underset{\sim}{x}}{\|\underset{\sim}{w}_h\|_{H^1(\Omega)}} \geq C_0 \,\|r_h\|_{L^2(\Omega)} \qquad \forall r_h \in R_h\,;$$

Hence for all $\underset{\sim}{v} \in \underset{\sim}{V}$, $\exists \{\underset{\sim}{v}_h\}_{h>0}$, $\underset{\sim}{v}_h \in \underset{\sim}{V}_h$, such that

$$\lim_{h\to 0} \|\underset{\sim}{v} - \underset{\sim}{v}_h\|_{H^1(\Omega)} = 0.$$

Set

$$\widehat{X}_h^x := \{\widehat{\varphi}_h^x \in C(\overline{\Omega}) : \widehat{\varphi}_h^x \,|_{\kappa_x} \in \mathbb{P}_1^x \quad \forall \kappa_x \in \mathcal{T}_x^h\} \subset R_h\,,$$
$$\widehat{X}_h^q := \{\widehat{\varphi}_h^q \in C(\overline{D}) : \widehat{\varphi}_h^q \,|_{\kappa_q} \in \mathbb{P}_1^q \quad \forall \kappa_q \in \mathcal{T}_q^h\},$$
$$\widehat{X}_h := \widehat{X}_h^x \otimes \widehat{X}_h^q \subset H^1(\Omega \times D) \subset \widehat{X}\,.$$

To mimic the energy bound, we require $\forall \underset{\sim}{v}_h \in \underset{\sim}{V}_h$, $\widehat{\varphi}_h \in \widehat{X}_h$ that

$$\int_\Omega (\underset{\sim}{\nabla}_x \cdot \underset{\sim}{v}_h)(\underset{\sim}{x})\; \widehat{\varphi}_h(\underset{\sim}{x},\underset{\sim}{q})\,\mathrm{d}\underset{\sim}{x} = 0 \qquad \text{for any } \underset{\sim}{q} \in \overline{D}\,.$$

Mimic the testing procedure for $(\mathsf{P}^L_{\varepsilon,\delta})$ in the Noncorotational case: $\underset{\sim}{u}^L_{\varepsilon,\delta}$ for Navier-Stokes, $[\mathcal{F}^L_\delta]'(\widehat{\psi}^L_{\varepsilon,\delta})$ for Fokker-Planck.

Finite element discretization of the Noncorotational case is tricky as

$$[\mathcal{F}^L_\delta]'(\widehat{\varphi}_h) \notin \widehat{X}_h \qquad \text{for } \widehat{\varphi}_h \in \widehat{X}_h.$$

Let $\pi_h : C(\overline{\Omega \times D}) \mapsto \widehat{X}_h$ be the interpolation operator s.t.

$$(\pi_h\, \widehat{\varphi})(\underset{\sim}{P}^{(x)}_i, \underset{\sim}{P}^{(q)}_j) = \widehat{\varphi}(\underset{\sim}{P}^{(x)}_i, \underset{\sim}{P}^{(q)}_j)$$

for all vertices $\{\underset{\sim}{P}^{(x)}_i\}_{i=1}^{I_x}$ of $\mathcal{T}^h_x$ and $\{\underset{\sim}{P}^{(q)}_j\}_{j=1}^{I_q}$ of $\mathcal{T}^h_q$.

We require also the local interpolation operators

$$\pi_{h,\kappa_x\times\kappa_q} \equiv \pi_h \,\big|_{\kappa_x\times\kappa_q} \qquad \forall \kappa_x \in \mathcal{T}_x \quad \forall \kappa_q \in \mathcal{T}_q\,.$$

We extend these to vector functions, denoted by $\underset{\sim}{\pi}_h$ and $\underset{\sim}{\pi}_{h,\kappa_x\times\kappa_q}$.

For any $\widehat{\varphi}_h \in \widehat{X}_h$, and for all $\kappa_x \in \mathcal{T}_x \quad \kappa_q \in \mathcal{T}_q$

$$\underset{\approx}{\Xi}_{\delta}^{L,(x)}(\widehat{\varphi}_h) \mid_{\kappa_x \times \kappa_q} \in [\mathbb{P}_1^q]^{d \times d}, \qquad \underset{\approx}{\Xi}_{\delta}^{L,(q)}(\widehat{\varphi}_h) \mid_{\kappa_x \times \kappa_q} \in [\mathbb{P}_1^x]^{d \times d}$$

are s.t.

$$\pi_{h,\kappa_x \times \kappa_q} \left[\underset{\approx}{\Xi}_{\delta}^{L,(x)}(\widehat{\varphi}_h) \underset{\sim}{\nabla}_x \left[\pi_h [[\mathcal{F}_\delta^L]'(\widehat{\varphi}_h)] \right] \right] = \underset{\sim}{\nabla}_x \widehat{\varphi}_h,$$

$$\pi_{h,\kappa_x \times \kappa_q} \left[\underset{\approx}{\Xi}_{\delta}^{L,(q)}(\widehat{\varphi}_h) \underset{\sim}{\nabla}_q \left[\pi_h [[\mathcal{F}_\delta^L]'(\widehat{\varphi}_h)] \right] \right] = \underset{\sim}{\nabla}_q \widehat{\varphi}_h .$$

$\underset{\approx}{\Xi}_{\delta}^{L,(x)}(\widehat{\varphi}_h)$ and $\underset{\approx}{\Xi}_{\delta}^{L,(q)}(\widehat{\varphi}_h)$ are approximations of

$$\beta_\delta^L(\widehat{\varphi}_h) \underset{\approx}{I} \equiv [[\mathcal{F}_\delta^L]''(\widehat{\varphi}_h)]^{-1} \underset{\approx}{I} = \begin{cases} \delta \underset{\approx}{I} & \text{if} \quad \widehat{\varphi}_h \leq \delta \\ \widehat{\varphi}_h \underset{\approx}{I} & \text{if} \quad \widehat{\varphi}_h \in [\delta, L] \\ L \underset{\approx}{I} & \text{if} \quad \widehat{\varphi}_h \geq L \end{cases} .$$

So the above are discrete analogues of $\beta_\delta^L(\widehat{\varphi}) \underset{\sim}{\nabla}_x [[\mathcal{F}_\delta^L]'(\widehat{\varphi})] = \underset{\sim}{\nabla}_x \widehat{\varphi}$ and $\beta_\delta^L(\widehat{\varphi}) \underset{\sim}{\nabla}_q [[\mathcal{F}_\delta^L]'(\widehat{\varphi})] = \underset{\sim}{\nabla}_q \widehat{\varphi}$.

Note that for all $\underset{\sim}{v} \in \underset{\sim}{V}$ and $\underset{\sim}{w},\ \underset{\sim}{z} \in [H^1(\Omega)]^d$

$$\int_\Omega \Big((\underset{\sim}{v} \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{w} \Big) \cdot \underset{\sim}{z}\, \mathrm{d}\underset{\sim}{x}$$

$$\equiv \tfrac{1}{2} \int_\Omega \Big[\Big((\underset{\sim}{v} \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{w} \Big) \cdot \underset{\sim}{z} - \Big((\underset{\sim}{v} \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{z} \Big) \cdot \underset{\sim}{w} \Big]\, \mathrm{d}\underset{\sim}{x}$$

$$\approx \tfrac{1}{2} \int_\Omega \Big[\Big((\underset{\sim}{v}_h \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{w}_h \Big) \cdot \underset{\sim}{z}_h - \Big((\underset{\sim}{v}_h \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{z}_h \Big) \cdot \underset{\sim}{w}_h \Big]\, \mathrm{d}\underset{\sim}{x}$$

for $\underset{\sim}{v}_h \in \underset{\sim}{V}_h,\ \ \underset{\sim}{w}_h,\ \underset{\sim}{z}_h \in \underset{\sim}{W}_h$.

Note that the above vanishes if $\underset{\sim}{w}_h = \underset{\sim}{z}_h$, which is not necessarily true for the direct approximation

$$\int_\Omega \Big((\underset{\sim}{v}_h \cdot \underset{\sim}{\nabla}_x)\underset{\sim}{w}_h \Big) \cdot \underset{\sim}{z}_h\, \mathrm{d}\underset{\sim}{x}\,, \qquad \text{as } \underset{\sim}{V}_h \not\subset \underset{\sim}{V}\,.$$

Let $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ be a partitioning of $[0, T]$ into time steps $\Delta t_n = t_n - t_{n-1}$, $n = 1 \to N$.

$\Delta t := \max_{n=1\to N} \Delta t_n$.

We assume that

$$\Delta t_n \leq C\, \Delta t_{n-1}, \qquad n = 2 \to N, \qquad \text{as } \Delta t \to 0_+ \,.$$

Let

$$\underset{\sim}{f}^n(\cdot) := \frac{1}{\Delta t_n} \int_{t_{n-1}}^{t_n} \underset{\sim}{f}(\cdot, t)\, \mathrm{d}t, \qquad n = 1 \to N;$$

where we now assume that $\underset{\sim}{f} \in L^2(0, T; ([H_0^1(\Omega)]^d)')$ as opposed to $\underset{\sim}{f} \in L^2(0, T; \underset{\sim}{V}')$ as $\underset{\sim}{V}_h \not\subset \underset{\sim}{V}$, but $\underset{\sim}{V}_h \subset [H_0^1(\Omega)]^d$.

Approximation of the Initial Data:

Let $\underset{\sim}{u}^{L,0}_{\varepsilon,\delta,h} \in \underset{\sim}{V}_h$ and $\widehat{\psi}^{L,0}_{\varepsilon,\delta,h} \in \widehat{X}_h$ be such that

$$\int_\Omega \left[\underset{\sim}{u}^{L,0}_{\varepsilon,\delta,h} \cdot \underset{\sim}{w}_h + \Delta t_0 \, \underset{\approx}{\nabla}_x \, \underset{\sim}{u}^{L,0}_{\varepsilon,\delta,h} : \underset{\approx}{\nabla}_x \, \underset{\sim}{w}_h\right] \mathrm{d}\underset{\sim}{x} = \int_\Omega \underset{\sim}{u}^0 \cdot \underset{\sim}{w}_h \, \mathrm{d}\underset{\sim}{x} \qquad \forall \underset{\sim}{w}_h \in \underset{\sim}{V}_h \, ,$$

$$\int_{\Omega \times D} M \, \pi_h \left[\widehat{\psi}^{L,0}_{\varepsilon,\delta,h} \, \widehat{\varphi}_h\right] \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} = \int_{\Omega \times D} M \, \widehat{\psi}^0 \, \widehat{\varphi}_h \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \qquad \forall \widehat{\varphi}_h \in \widehat{X}_h \, ;$$

where Δt_0 is such that $\Delta t_1 \leq C \, \Delta t_0$ as $\Delta t \to 0$.

It follows from our assumptions on $\underset{\sim}{u}^0$ and ψ^0 that

$$\int_\Omega \left[|\underset{\sim}{u}^{L,0}_{\varepsilon,\delta,h}|^2 + \Delta t_0 \, |\underset{\approx}{\nabla}_x \, \underset{\sim}{u}^{L,0}_{\varepsilon,\delta,h}|^2\right] \mathrm{d}\underset{\sim}{x} \leq C$$

and $\qquad 0 \leq \widehat{\psi}^{L,0}_{\varepsilon,\delta,h} \leq L \, .$

Our numerical approximation of $(\mathsf{P}^{L}_{\varepsilon,\delta})$ is then:

$(\mathbf{P}^{L,\Delta t}_{\varepsilon,\delta,h})$ For $n = 1 \to N$, given $\{\underset{\sim}{u}^{L,n-1}_{\varepsilon,\delta,h}, \widehat{\psi}^{L,n-1}_{\varepsilon,\delta,h}\} \in \underset{\sim}{V}_h \times \widehat{X}_h$,

find $\{\underset{\sim}{u}^{L,n}_{\varepsilon,\delta,h}, \widehat{\psi}^{L,n}_{\varepsilon,\delta,h}\} \in \underset{\sim}{V}_h \times \widehat{X}_h$ s.t.

$$\int_\Omega \left[\frac{\underset{\sim}{u}^{L,n}_{\varepsilon,\delta,h} - \underset{\sim}{u}^{L,n-1}_{\varepsilon,\delta,h}}{\Delta t_n} \cdot \underset{\sim}{w}_h + \nu \underset{\approx}{\nabla}_x \underset{\sim}{u}^{L,n}_{\varepsilon,\delta,h} : \underset{\approx}{\nabla}_x \underset{\sim}{w}_h \right] \mathrm{d}\underset{\sim}{x}$$

$$+ \frac{1}{2} \int_\Omega \left[\left((\underset{\sim}{u}^{L,n-1}_{\varepsilon,\delta,h} \cdot \underset{\sim}{\nabla}_x) \underset{\sim}{u}^{L,n}_{\varepsilon,\delta,h} \right) \cdot \underset{\sim}{w}_h - \left((\underset{\sim}{u}^{L,n-1}_{\varepsilon,\delta,h} \cdot \underset{\sim}{\nabla}_x) \underset{\sim}{w}_h \right) \cdot \underset{\sim}{u}^{L,n}_{\varepsilon,\delta,h} \right] \mathrm{d}\underset{\sim}{x}$$

$$= \langle \underset{\sim}{f}^n, \underset{\sim}{w}_h \rangle_{H^1_0(\Omega)} - \mu \int_\Omega \underset{\approx}{C}(M\, \widehat{\psi}^{L,n}_{\varepsilon,\delta,h}) : \underset{\approx}{\nabla}_x \underset{\sim}{w}_h \, \mathrm{d}\underset{\sim}{x} \qquad \forall \underset{\sim}{w}_h \in \underset{\sim}{V}_h,$$

$$\int_{\Omega\times D} M\,\pi_h \left[\frac{\widehat{\psi}^{L,n}_{\varepsilon,\delta,h} - \widehat{\psi}^{L,n-1}_{\varepsilon,\delta,h}}{\Delta t_n}\,\widehat{\varphi}_h + \varepsilon\,\underset{\sim}{\nabla}_x\,\widehat{\psi}^{L,n}_{\varepsilon,\delta,h}\cdot\underset{\sim}{\nabla}_x\,\widehat{\varphi}_h\right]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$

$$+\frac{1}{2\,\lambda}\int_{\Omega\times D} M\,\pi_h\left[\underset{\sim}{\nabla}_q\,\widehat{\psi}^{L,n}_{\varepsilon,\delta,h}\cdot\underset{\sim}{\nabla}_q\,\widehat{\varphi}_h\right]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$

$$=\int_{\Omega\times D} M\,\left((\underset{\approx}{\nabla}_x\,\underset{\sim}{u}^{L,n}_{\varepsilon,\delta,h})\,\underset{\sim}{q}\right)\cdot\underset{\sim}{\pi}_h\left[\underset{\approx}{\Xi}^{L,(q)}_{\delta}(\widehat{\psi}^{L,n}_{\varepsilon,\delta,h})\,\underset{\sim}{\nabla}_q\,\widehat{\varphi}_h\right]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}$$

$$+\int_{\Omega\times D} M\,\underset{\sim}{u}^{L,n}_{\varepsilon,\delta,h}\cdot\underset{\sim}{\pi}_h\left[\underset{\approx}{\Xi}^{L,(x)}_{\delta}(\widehat{\psi}^{L,n}_{\varepsilon,\delta,h})\,\underset{\sim}{\nabla}_x\,\widehat{\varphi}_h\right]\,\mathrm{d}\underset{\sim}{q}\,\mathrm{d}\underset{\sim}{x}\qquad\forall\widehat{\varphi}_h\in\widehat{X}_h.$$

Here π_h and $\underset{\sim}{\pi}_h$ are really $\pi_{h,\kappa_x\times\kappa_q}$ and $\underset{\sim}{\pi}_{h,\kappa_x\times\kappa_q}$ on each $\kappa_x\times\kappa_q$ of $\Omega\times D$.

Hence the approximations $\underset{\sim}{u}^{L,n}_{\varepsilon,\delta,h}$ and $\widehat{\psi}^{L,n}_{\varepsilon,\delta,h}$ at time level t_n to the velocity field and the probability distribution satisfy a coupled nonlinear system.

Scheme satisfies a discrete analogue of the above energy bound,

choose $\underset{\sim}{w}_h \equiv \underset{\sim}{u}^{L,n}_{\varepsilon,\delta,h}$ and $\widehat{\varphi}_h \equiv \pi_h[\,[\mathcal{F}^L_\delta]'(\widehat{\psi}^{L,n}_{\varepsilon,\delta,h})]$.

Exploiting this, existence of $\underset{\sim}{u}^{L,n}_{\varepsilon,\delta,h}$ and $\widehat{\psi}^{L,n}_{\varepsilon,\delta,h}$ at time level t_n follows for any $\Delta t_n > 0$ from a Brouwer fixed point theorem.

To prove convergence, we need more stability bounds.

We require the L^2 projector $\underset{\sim}{Q}_h : \underset{\sim}{V} \mapsto \underset{\sim}{V}_h$ defined by

$$\int_\Omega (\underset{\sim}{v} - \underset{\sim}{Q}_h \underset{\sim}{v}) \cdot \underset{\sim}{w}_h \, \mathrm{d}\underset{\sim}{x} = 0 \qquad \forall \underset{\sim}{w}_h \in \underset{\sim}{V}_h.$$

Ω convex and $\mathcal{T}_x^h$ quasi-uniform $\Rightarrow$ $\underset{\sim}{Q}_h$ is uniformly H^1 stable; that is,

$$\|\underset{\sim}{Q}_h \underset{\sim}{v}\|_{H^1(\Omega)} \leq C \, \|\underset{\sim}{v}\|_{H^1(\Omega)} \qquad \forall \underset{\sim}{v} \in \underset{\sim}{V}.$$

In addition, we require $\widetilde{Q}_h^M : \widehat{X} \mapsto \widehat{X}_h$ such that

$$\int_{\Omega \times D} M \, \pi_h[(\widetilde{Q}_h^M \widehat{\psi}) \, \widehat{\varphi}_h] \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} = \int_{\Omega \times D} M \, \widehat{\psi} \, \widehat{\varphi}_h \, \mathrm{d}\underset{\sim}{q} \, \mathrm{d}\underset{\sim}{x} \qquad \forall \widehat{\varphi}_h \in \widehat{X}_h.$$

One can show that

$$\|\widetilde{Q}_h^M \widehat{\psi}\|_{\widehat{X}}^2 \leq C \, \|\widehat{\psi}\|_{\widehat{X}}^2 \qquad \forall \widehat{\psi} \in \widehat{X}.$$

(Obviously, the degeneracy of M makes this very delicate.)

For these stability results, choose

$$\widehat{\varphi}_h \equiv \widehat{\psi}^{L,n}_{\varepsilon,\delta,h}\,, \qquad \widehat{\varphi}_h \equiv \widetilde{Q}^M_h \left[\mathcal{G}\left(\frac{\widehat{\psi}^{L,n}_{\varepsilon,\delta,h} - \widehat{\psi}^{L,n-1}_{\varepsilon,\delta,h}}{\Delta t_n}\right)\right]$$

$$\underset{\sim}{w}_h \equiv \underset{\sim}{Q}_h \left[\underset{\sim}{S}\left(\frac{\underset{\sim}{u}^{L,n}_{\varepsilon,\delta,h} - \underset{\sim}{u}^{L,n-1}_{\varepsilon,\delta,h}}{\Delta t_n}\right)\right].$$

Finally, one can prove that a subsequence of

$\{\{\underset{\sim}{u}^L_{\varepsilon,\delta,h}, \widehat{\psi}^L_{\varepsilon,\delta,h}\}\}_{\delta>0,h>0,\Delta t>0}$ converges to $\{\underset{\sim}{u}^L_\varepsilon, \widehat{\psi}^L_\varepsilon\}$ as $\delta,\ h,\ \Delta t \to 0_+$,

where $\{\underset{\sim}{u}^L_\varepsilon, \widehat{\psi}^L_\varepsilon\}$ solves (P^L_ε),

but with the convective term $\underset{\sim}{u}^L_\varepsilon \cdot \underset{\sim}{\nabla}_x\, \widehat{\psi}^L_\varepsilon$ replaced by $\underset{\sim}{u}^L_\varepsilon \cdot \underset{\sim}{\nabla}_x \left[\beta^L(\widehat{\psi}^L_\varepsilon)\right]$.

Recall Hookean $\Rightarrow$ macroscopic Oldroyd-B model:

$(\mathbf{P}_\varepsilon)$ Find $\underset{\sim}{u}_\varepsilon(\underset{\sim}{x},t) \in \mathbb{R}^d$, $p_\varepsilon(\underset{\sim}{x},t) \in \mathbb{R}$ and $\underset{\approx}{\tau}_\varepsilon(\underset{\sim}{x},t) \in [\mathbb{R}]^{d\times d}_S$ s.t.

$$
\begin{aligned}
\frac{\partial \underset{\sim}{u}_\varepsilon}{\partial t} + (\underset{\sim}{u}_\varepsilon \cdot \underset{\sim}{\nabla})\underset{\sim}{u}_\varepsilon - \nu\,\Delta \underset{\sim}{u}_\varepsilon + \underset{\sim}{\nabla}\, p_\varepsilon &= \underset{\sim}{f} + \underset{\sim}{\nabla} \cdot \underset{\approx}{\tau}_\varepsilon && \text{in } \Omega_T,\\
\underset{\sim}{\nabla} \cdot \underset{\sim}{u}_\varepsilon &= 0 && \text{in } \Omega_T,\\
\underset{\sim}{u}_\varepsilon &= \underset{\sim}{0} && \text{on } \partial\Omega_T^*,\\
\underset{\sim}{u}_\varepsilon(\underset{\sim}{x},0) &= \underset{\sim}{u}^0(\underset{\sim}{x}) && \forall \underset{\sim}{x} \in \Omega,\\
\frac{\partial \underset{\approx}{\tau}_\varepsilon}{\partial t} + (\underset{\sim}{u}_\varepsilon \cdot \underset{\sim}{\nabla})\,\underset{\approx}{\tau}_\varepsilon + \frac{1}{\lambda}\underset{\approx}{\tau}_\varepsilon - \varepsilon\,\Delta \underset{\approx}{\tau}_\varepsilon &= \mu\,\left[(\underset{\approx}{\nabla}\,\underset{\sim}{u}_\varepsilon) + (\underset{\approx}{\nabla}\,\underset{\sim}{u}_\varepsilon)^\top\right] \\
&\quad + \left[(\underset{\approx}{\nabla}\,\underset{\sim}{u}_\varepsilon)\,\underset{\approx}{\tau}_\varepsilon + \underset{\approx}{\tau}_\varepsilon\,(\underset{\approx}{\nabla}\,\underset{\sim}{u}_\varepsilon)^\top\right] && \text{in } \Omega_T,\\
\underset{\approx}{\tau}_\varepsilon(\underset{\sim}{x},0) &= \underset{\approx}{\tau}^0(\underset{\sim}{x}) && \forall \underset{\sim}{x} \in \Omega.
\end{aligned}
$$

Setting $\underset{\approx}{\sigma}_{\varepsilon} := (\underset{\approx}{\tau}_{\varepsilon} + \mu\, \underset{\approx}{I}) \Rightarrow$

$(\mathbf{P}_{\varepsilon})$ Find $\underset{\sim}{u}_{\varepsilon}(\underset{\sim}{x}, t) \in \mathbb{R}^d$, $p_{\varepsilon}(\underset{\sim}{x}, t) \in \mathbb{R}$ and $\underset{\approx}{\sigma}_{\varepsilon}(\underset{\sim}{x}, t) \in [\mathbb{R}]_S^{d\times d}$ s.t.

$$\frac{\partial \underset{\sim}{u}_{\varepsilon}}{\partial t} + (\underset{\sim}{u}_{\varepsilon} \cdot \underset{\sim}{\nabla})\underset{\sim}{u}_{\varepsilon} - \nu\, \Delta \underset{\sim}{u}_{\varepsilon} + \underset{\sim}{\nabla}\, p_{\varepsilon} = \underset{\sim}{f} + \underset{\sim}{\nabla} \cdot \underset{\approx}{\sigma}_{\varepsilon} \qquad \text{in } \Omega_T,$$

$$\underset{\sim}{\nabla} \cdot \underset{\sim}{u}_{\varepsilon} = 0 \qquad \text{in } \Omega_T,$$

$$\underset{\sim}{u}_{\varepsilon} = \underset{\sim}{0} \qquad \text{on } \partial\Omega_T^*,$$

$$\underset{\sim}{u}_{\varepsilon}(\underset{\sim}{x}, 0) = \underset{\sim}{u}^0(\underset{\sim}{x}) \qquad \forall \underset{\sim}{x} \in \Omega,$$

$$\frac{\partial \underset{\approx}{\sigma}_{\varepsilon}}{\partial t} + (\underset{\sim}{u}_{\varepsilon} \cdot \underset{\sim}{\nabla})\, \underset{\approx}{\sigma}_{\varepsilon} + \frac{1}{\lambda}\, (\underset{\approx}{\sigma}_{\varepsilon} - \mu\, \underset{\approx}{I}) - \varepsilon\, \Delta \underset{\approx}{\sigma}_{\varepsilon} = (\underset{\approx}{\nabla}\, \underset{\sim}{u}_{\varepsilon})\, \underset{\approx}{\sigma}_{\varepsilon} + \underset{\approx}{\sigma}_{\varepsilon}\, (\underset{\approx}{\nabla}\, \underset{\sim}{u}_{\varepsilon})^{\top} \qquad \text{in } \Omega_T,$$

$$\underset{\approx}{\sigma}_{\varepsilon}(\underset{\sim}{x}, 0) = \underset{\approx}{\sigma}^0(\underset{\sim}{x}) \qquad \forall \underset{\sim}{x} \in \Omega.$$

Formal Energy Bounds for (P$_\varepsilon$): Hu & Lelièvre (2007)

Testing the Navier-Stokes equation with $\underset{\sim}{u}_\varepsilon$, integrating over $\Omega \Rightarrow$

$$\frac{1}{2}\frac{d}{dt}\left[\int_\Omega |\underset{\sim}{u}_\varepsilon|^2 \,\mathrm{d}\underset{\sim}{x}\right] + \nu \int_\Omega |\underset{\approx}{\nabla}\, \underset{\sim}{u}_\varepsilon|^2 \,\mathrm{d}\underset{\sim}{x} - \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{u}_\varepsilon \,\mathrm{d}\underset{\sim}{x} = -\int_\Omega \underset{\approx}{\sigma}_\varepsilon : \underset{\approx}{\nabla}\, \underset{\sim}{u}_\varepsilon \,\mathrm{d}\underset{\sim}{x}$$

Testing the stress equation with $\frac{1}{2}\,(\underset{\approx}{I} - \mu\,\mathcal{F}''(\underset{\approx}{\sigma}_\varepsilon))$, integrating over $\Omega \Rightarrow$
(assumes $\underset{\approx}{\sigma}_\varepsilon$ is positive definite, as $\mathcal{F}(s) := s(\ln s - 1) + 1 \Rightarrow \mathcal{F}''(s) = s^{-1}$)

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \mathrm{tr}(\underset{\approx}{\sigma}_\varepsilon - \mu\,\mathcal{F}'(\underset{\approx}{\sigma}_\varepsilon))\,\mathrm{d}\underset{\sim}{x} + \frac{1}{2}\int_\Omega \mathrm{tr}(\underset{\approx}{\sigma}_\varepsilon + \mu^2\,[\underset{\approx}{\sigma}_\varepsilon]^{-1} - 2\,\mu\,\underset{\approx}{I})\,\mathrm{d}\underset{\sim}{x}$$

$$- \frac{\mu\,\varepsilon}{2}\int_\Omega \underset{\approx}{\nabla}\,\underset{\approx}{\sigma}_\varepsilon :: \underset{\approx}{\nabla}\,[\mathcal{F}''(\underset{\approx}{\sigma}_\varepsilon)]\,\mathrm{d}\underset{\sim}{x} = \int_\Omega \underset{\approx}{\sigma}_\varepsilon : \underset{\approx}{\nabla}\,\underset{\sim}{u}_\varepsilon \,\mathrm{d}\underset{\sim}{x}.$$

For Finite Element Approximations of (P) mimicing the above formal energy estimate - see Boyaval, Lelièvre & Mangoubi (2009).

Based on piecewise constant approximation for $\underset{\approx}{\sigma}$, i.e.

$$\underset{\approx}{S}_h^0 := \{\underset{\approx}{\phi}_h \in [L^\infty(\Omega)]_S^{d\times d} : \underset{\approx}{\phi}_h \mid_\kappa \in [\mathbb{P}_0]_S^{d\times d} \quad \forall \kappa \in \mathcal{T}^h\}\,.$$

Hence $\qquad \underset{\approx}{\sigma}_h^n \in \underset{\approx}{S}_h^0 \Rightarrow \frac{1}{2}\,(\underset{\approx}{I} - \mu\,\mathcal{F}''(\underset{\approx}{\sigma}_h^n)) \in \underset{\approx}{S}_h^0.$

However, one has to ensure that $\underset{\approx}{\sigma}_h^n$ is positive definite.

For $\underset{\sim}{f} \equiv \underset{\sim}{0}$ and uniform time steps Δt, BLM show that for initial data $\{\underset{\sim}{u}_h^0, \underset{\approx}{\sigma}_h^0\}$ with $\underset{\approx}{\sigma}_h^0$ symmetric positive definite, then $\exists\ C_1(\underset{\sim}{u}_h^0, \underset{\approx}{\sigma}_h^0)$ such that for any $\Delta t < C_1$ $\{\underset{\sim}{u}_h^n, \underset{\approx}{\sigma}_h^n\}$ exists, is unique and $\underset{\approx}{\sigma}_h^n$ is positive definite.

B. & Boyaval (2009) show that $\underset{\approx}{\sigma}_h^n$ is positive definite, via regularization, for general $\underset{\sim}{f}$ and any time steps, Δt_n, $n = 1 \to N$.

A regular partitioning: $\overline{\Omega} := \bigcup_{\kappa \in \mathcal{T}_h} \overline{\kappa}$.

$$\underset{\sim}{W}_h := \{\underset{\sim}{w}_h \in [C(\overline{\Omega})]^d : \underset{\sim}{w}_h \,|_\kappa \in [\mathbb{P}_2]^d \quad \forall \kappa \in \mathcal{T}^h$$
$$\text{and} \quad \underset{\sim}{w}_h = 0 \text{ on } \partial\Omega\} \subset [H_0^1(\Omega)]^d ,$$
$$R_h^0 := \{\eta_h \in L^\infty(\Omega) : \eta_h \,|_\kappa \in \mathbb{P}_0 \quad \forall \kappa \in \mathcal{T}^h\} ,$$
$$\underset{\sim}{V}_h^0 := \{\underset{\sim}{v}_h \in \underset{\sim}{W}_h : \int_\Omega (\underset{\sim}{\nabla} \cdot \underset{\sim}{v}_h)\, \eta_h \, \mathrm{d}\underset{\sim}{x} = 0 \quad \forall \eta_h \in R_h^0\} ,$$
$$\underset{\approx}{S}_h^0 := \{\underset{\approx}{\phi}_h \in [L^\infty(\Omega)]_S^{d\times d} : \underset{\approx}{\phi}_h \,|_\kappa \in [\mathbb{P}_0]_S^{d\times d} \quad \forall \kappa \in \mathcal{T}^h\} .$$

$\underset{\sim}{W}_h \times R_h^0$ satisfy the LBB inf-sup condition. $\mathrm{tr}(\underset{\approx}{S}_h^0) \subset R_h^0$.

Recall $$\beta_\delta(s) \equiv [\mathcal{F}_\delta''(s)]^{-1} := \begin{cases} s & \delta \le s \\ \delta & s \le \delta \end{cases} .$$

$(\mathbf{P}^{\Delta t}_{\delta,h})$ For $n = 1 \to N$, given $\{\underset{\sim}{u}^{n-1}_{\delta,h}, \underset{\approx}{\sigma}^{n-1}_{\delta,h}\} \in \underset{\sim}{V}^0_h \times \underset{\approx}{S}^0_h$,

find $\{\underset{\sim}{u}^{n}_{\delta,h}, \underset{\approx}{\sigma}^{n}_{\delta,h}\} \in \underset{\sim}{V}^0_h \times \underset{\approx}{S}^0_h$ s.t.

$$\int_\Omega \Bigg[\left(\frac{\underset{\sim}{u}^n_{\delta,h} - \underset{\sim}{u}^{n-1}_{\delta,h}}{\Delta t_n} \right) \cdot \underset{\sim}{v}_h + \frac{1}{2} \left[\left((\underset{\sim}{u}^{n-1}_{\delta,h} \cdot \underset{\sim}{\nabla})\underset{\sim}{u}^n_{\delta,h} \right) \cdot \underset{\sim}{v}_h - \underset{\sim}{u}^n_{\delta,h} \cdot \left((\underset{\sim}{u}^{n-1}_{\delta,h} \cdot \underset{\sim}{\nabla})\underset{\sim}{v}_h \right) \right]$$

$$+ \nu \underset{\approx}{\nabla}\, \underset{\sim}{u}^n_{\delta,h} : \underset{\approx}{\nabla}\, \underset{\sim}{v}_h + \beta_\delta(\underset{\approx}{\sigma}^n_{\delta,h}) : \underset{\approx}{\nabla}\, \underset{\sim}{v}_h \Bigg] \, \mathrm{d}\underset{\sim}{x} = \langle \underset{\sim}{f}^n, \underset{\sim}{v}_h \rangle_{H^1_0(\Omega)} \qquad \forall \underset{\sim}{v}_h \in \underset{\sim}{V}^0_h,$$

$$\int_\Omega \Bigg[\left(\frac{\underset{\approx}{\sigma}^n_{\delta,h} - \underset{\approx}{\sigma}^{n-1}_{\delta,h}}{\Delta t_n} \right) : \underset{\approx}{\phi}_h - 2 \left((\underset{\approx}{\nabla}\, \underset{\sim}{u}^n_{\delta,h})\, \beta_\delta(\underset{\approx}{\sigma}^n_{\delta,h}) \right) : \underset{\approx}{\phi}_h + \frac{1}{\lambda} \left(\underset{\approx}{\sigma}^n_{\delta,h} - \mu\, \underset{\approx}{I} \right) : \underset{\approx}{\phi}_h \Bigg] \mathrm{d}\underset{\sim}{x}$$

$$+ \sum_{j=1}^{N_E} \int_{E_j} \left| \underset{\sim}{u}^{n-1}_{\delta,h} \cdot \underset{\sim}{n} \right| [\![\underset{\approx}{\sigma}^n_{\delta,h}]\!]_{\to \underset{\sim}{u}^{n-1}_{\delta,h}} : \underset{\approx}{\phi}_h^{+\underset{\sim}{u}^{n-1}_{\delta,h}} \, \mathrm{d}s = 0 \qquad \forall \underset{\approx}{\phi}_h \in \underset{\approx}{S}^0_h .$$

Discontinuous Galerkin approximation of the stress convection term.

For any $\delta \in (0, \frac{1}{2}]$ and

$\{\underset{\sim}{u}^0_{\delta,h}, \underset{\approx}{\sigma}^0_{\delta,h}\} = \{\underset{\sim}{u}^0_h, \underset{\approx}{\sigma}^0_h\} \in \underset{\sim}{V}^0_h \times \underset{\approx}{S}^0_h$ with $\underset{\approx}{\sigma}^0_h$ positive definite,

we prove existence of $\{\underset{\sim}{u}^n_{\delta,h}, \underset{\approx}{\sigma}^n_{\delta,h}\} \in \underset{\sim}{V}^0_h \times \underset{\approx}{S}^0_h$, $n = 1 \to N$.
Moreover $\{\underset{\sim}{u}^n_{\delta,h}, \underset{\approx}{\sigma}^n_{\delta,h}\}^N_{n=0}$ satisfy a discrete analogue of the δ regularized energy inequality, this yields that

$$\max_{n=0\to N} \int_\Omega \left[|\underset{\sim}{u}^n_{\delta,h}|^2 + \mathrm{tr}(|\underset{\approx}{\sigma}^n_{\delta,h}|) + \delta^{-1}\,\mathrm{tr}(|[\underset{\approx}{\sigma}^n_{\delta,h}]_-|) \right] + \sum_{n=1}^{N_T} \Delta t_n \int_\Omega \mathrm{tr}([\beta_\delta(\underset{\approx}{\sigma}^n_{\delta,h})]^{-1}) \leq C\,.$$

Hence the following subsequence results:

$$\underset{\sim}{u}^n_{\delta,h} \to \underset{\sim}{u}^n_h, \qquad \underset{\approx}{\sigma}^n_{\delta,h},\ \beta_\delta(\underset{\approx}{\sigma}^n_{\delta,h}) \to \underset{\approx}{\sigma}^n_h \qquad \text{as} \quad \delta \to 0_+$$

As $[\beta_\delta(\underset{\approx}{\sigma}^n_{\delta,h})]^{-1}\beta_\delta(\underset{\approx}{\sigma}^n_{\delta,h}) = \underset{\approx}{I}$, we have also that $\underset{\approx}{\sigma}^n_h$ is positive definite.

Ω a convex polytope, an Acute Quasi-Uniform partitioning:

$$\underset{\sim}{W}_h := \{\underset{\sim}{w}_h \in [C(\overline{\Omega})]^d : \underset{\sim}{w}_h \,|_\kappa \in [\mathbb{P}_2]^d \quad \forall \kappa \in \mathcal{T}^h$$

$$\text{and} \quad \underset{\sim}{w}_h = 0 \text{ on } \partial\Omega\} \subset [H^1_0(\Omega)]^d\,,$$

$$R^1_h := \{\eta_h \in C(\overline{\Omega}) : \eta_h \,|_\kappa \in \mathbb{P}_1 \quad \forall \kappa \in \mathcal{T}^h\}\,,$$

$$\underset{\sim}{V}^1_h := \{\underset{\sim}{v}_h \in \underset{\sim}{W}_h : \int_\Omega (\underset{\sim}{\nabla} \cdot \underset{\sim}{v}_h)\, \eta_h \,\mathrm{d}\underset{\sim}{x} = 0 \quad \forall \eta_h \in R^1_h\}\,,$$

$$\underset{\approx}{S}^1_h := \{\underset{\approx}{\phi}_h \in [C(\overline{\Omega})]^{d\times d}_S : \underset{\approx}{\phi}_h \,|_\kappa \in [\mathbb{P}_1]^{d\times d}_S \quad \forall \kappa \in \mathcal{T}^h\}\,.$$

Lowest order Taylor-Hood element $\underset{\sim}{W}_h \times R^1_h$ satisfies the LBB inf-sup condition. Also $\mathrm{tr}(\underset{\approx}{S}^1_h) \subset R^1_h$.

Let $\pi_h : C(\overline{\Omega}) \mapsto R^1_h$ be the interpolation operator,

extended to $\pi_h : [C(\overline{\Omega})]^{d\times d}_S \mapsto \underset{\approx}{S}^1_h$

$(\mathbf{P}^{L,\Delta t}_{\varepsilon,\delta,h})$ For $n = 1 \to N$, given $(\underset{\sim}{u}^{L,n-1}_{\varepsilon,\delta,h}, \underset{\approx}{\sigma}^{L,n-1}_{\delta,h}) \in \underset{\sim}{V}^1_h \times \underset{\approx}{S}^1_h$,

find $(\underset{\sim}{u}^{L,n}_{\varepsilon,\delta,h}, \underset{\approx}{\sigma}^{L,n}_{\varepsilon,\delta,h}) \in \underset{\sim}{V}^1_h \times \underset{\approx}{S}^1_h$ s.t.

$$\int_\Omega \left(\frac{\underset{\sim}{u}^{L,n}_{\varepsilon,\delta,h} - \underset{\sim}{u}^{L,n-1}_{\varepsilon,\delta,h}}{\Delta t_n} \right) \cdot \underset{\sim}{v}_h \, \mathrm{d}\underset{\sim}{x}$$

$$+ \frac{1}{2} \int_\Omega \left[\left((\underset{\sim}{u}^{L,n-1}_{\varepsilon,\delta,h} \cdot \underset{\sim}{\nabla}) \underset{\sim}{u}^{L,n}_{\varepsilon,\delta,h} \right) \cdot \underset{\sim}{v}_h - \underset{\sim}{u}^{L,n}_{\varepsilon,\delta,h} \cdot \left((\underset{\sim}{u}^{L,n-1}_{\varepsilon,\delta,h} \cdot \underset{\sim}{\nabla}) \underset{\sim}{v}_h \right) \right]$$

$$+ \int_\Omega \left[\nu \underset{\approx}{\nabla} \underset{\sim}{u}^{L,n}_{\varepsilon,\delta,h} : \underset{\approx}{\nabla} \underset{\sim}{v}_h + \pi_h[\beta^L_\delta(\underset{\approx}{\sigma}^{L,n}_{\varepsilon,\delta,h})] : \underset{\approx}{\nabla} \underset{\sim}{v}_h \right] \mathrm{d}\underset{\sim}{x} = \langle \underset{\sim}{f}^n, \underset{\sim}{v}_h \rangle_{H^1_0(\Omega)}$$

$$\forall \underset{\sim}{v}_h \in \underset{\sim}{V}^1_h,$$

$$\int_\Omega \pi_h \left[\left(\frac{\underset{\approx}{\sigma}^{L,n}_{\varepsilon,\delta,h} - \underset{\approx}{\sigma}^{L,n-1}_{\varepsilon,\delta,h}}{\Delta t_n} \right) : \underset{\approx}{\phi}_h + \frac{1}{\lambda} \left(\underset{\approx}{\sigma}^{L,n}_{\varepsilon,\delta,h} - \mu \underset{\approx}{I} \right) : \underset{\approx}{\phi}_h \right] \mathrm{d}\underset{\sim}{x}$$

$$+ \int_\Omega \left[\varepsilon \underset{\approx}{\nabla} \underset{\approx}{\sigma}^{L,n}_{\varepsilon,\delta,h} : \underset{\approx}{\nabla} \underset{\approx}{\phi}_h - 2 \underset{\approx}{\nabla} \underset{\sim}{u}^{L,n}_{\varepsilon,\delta,h} : \pi_h[\beta^L_\delta(\underset{\approx}{\sigma}^{L,n}_{\varepsilon,\delta,h}) \underset{\approx}{\phi}_h] \right] \mathrm{d}\underset{\sim}{x}$$

$$+ \int_\Omega \sum_{m=1}^{d} \sum_{p=1}^{d} [\underset{\sim}{u}^{L,n-1}_{\varepsilon,\delta,h}]_m \, \Lambda^L_{\delta,m,p}(\underset{\approx}{\sigma}^{L,n}_{\varepsilon,\delta,h}) : \frac{\partial \underset{\approx}{\phi}_h}{\partial \underset{\sim}{x}_p} \mathrm{d}\underset{\sim}{x} = 0$$

$$\forall \underset{\approx}{\phi}_h \in \underset{\approx}{S}^1_h \, .$$

$(\mathbf{P}_\varepsilon^L)$ Find $\underset{\sim}{u}_\varepsilon^L \in L^\infty(0,T;[L^2(\Omega)]^d) \cap L^2(0,T;\underset{\sim}{V}) \cap W^{1,\frac{4}{\vartheta}}(0,T;\underset{\sim}{V}')$ and $\underset{\approx}{\sigma}_\varepsilon^L \in L^\infty(0,T;[L^2(\Omega)]_S^{d\times d}) \cap L^2(0,T;[H^1(\Omega)]_S^{d\times d}) \cap H^1(0,T;([H^1(\Omega)]_S^{d\times d})')$ such that $\underset{\sim}{u}_\varepsilon^L(\cdot,0) = \underset{\sim}{u}^0(\cdot)$, $\underset{\approx}{\sigma}_\varepsilon^L(\cdot,0) = \underset{\approx}{\sigma}^0(\cdot)$ and

$$\int_0^T \left\langle \frac{\partial \underset{\sim}{u}_\varepsilon^L}{\partial t}, \underset{\sim}{v} \right\rangle_V \mathrm{d}t + \int_{\Omega_T} \left[\nu \underset{\approx}{\nabla}\, \underset{\sim}{u}_\varepsilon^L : \underset{\approx}{\nabla}\, \underset{\sim}{v} + \left[(\underset{\sim}{u}_\varepsilon^L \cdot \underset{\sim}{\nabla}) \underset{\sim}{u}_\varepsilon^L \right] \cdot \underset{\sim}{v} \right] \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t$$

$$= \int_0^T \langle \underset{\sim}{f}, \underset{\sim}{v} \rangle_{H_0^1(\Omega)} \mathrm{d}t - \int_{\Omega_T} \beta^L(\underset{\approx}{\sigma}_\varepsilon^L) : \underset{\approx}{\nabla}\, \underset{\sim}{v}\, dt \qquad \forall \underset{\sim}{v} \in L^{\frac{4}{4-\vartheta}}(0,T;\underset{\sim}{V});$$

$$\int_0^T \left\langle \frac{\partial \underset{\approx}{\sigma}_\varepsilon^L}{\partial t}, \underset{\approx}{\phi} \right\rangle_{H^1(\Omega)} \mathrm{d}t + \int_{\Omega_T} \left[(\underset{\sim}{u}_\varepsilon^L \cdot \underset{\sim}{\nabla}) [\beta^L(\underset{\approx}{\sigma}_\varepsilon^L)] : \underset{\approx}{\phi} + \varepsilon \underset{\approx}{\nabla}\, \underset{\approx}{\sigma}_\varepsilon^L :: \underset{\approx}{\nabla}\, \underset{\approx}{\phi} \right] \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t$$

$$= \int_{\Omega_T} \left[2\,(\underset{\sim}{\nabla}\, \underset{\sim}{u}_\varepsilon^L)\, \beta^L(\underset{\approx}{\sigma}_\varepsilon^L) - \frac{1}{\lambda}(\underset{\approx}{\sigma}_\varepsilon^L - \mu\, \underset{\approx}{I}) \right] : \underset{\approx}{\phi}\, \mathrm{d}\underset{\sim}{x}\, \mathrm{d}t$$

$$\forall \underset{\approx}{\phi} \in L^2(0,T;[H^1(\Omega)]_S^{d\times d}).$$