Dynamic Depletion of Vortex Stretching and Non-Blowup of the 3-D Incompressible Euler Equations

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The Clay Millennium Problem

For 3-D Navier-Stokes equations

$$u_t + u \cdot \nabla u = -\nabla p + \Delta u,$$

$$\nabla \cdot u = 0,$$

in $\mathbb{R}^3$ or with periodic boundary condition, prove the global well-posedness or provide a counter-example to show there is a finite time singularity, with initial value

$$u(x, 0) = u_0(x).$$

Without the diffusive term: Euler equations.

Difference between 2D and 3D

In vorticity formation, $\omega = \nabla \times u$, we have

- 2D:
$$\omega_t + u \cdot \nabla\omega = 0$$
- 3D:
$$\omega_t + u \cdot \nabla\omega = \omega \cdot \nabla u$$
- $\omega \cdot \nabla u$: Vorticity stretching!

Historical Review: Leray's result

J. Leray (1934) proved for domain as $\mathbb{R}^3$

1. *There exists a $T^* > 0$ such that the Cauchy problem has a unique smooth solution with "reasonal properties at ∞";*
2. *There exists at least one global weak solution satisfying a natural energy inequality. Moreover, the weak solutions coincide with the smooth solution in $\mathbb{R}^3 \times (0, T^*)$;*
3. *If $(0, T^*)$ is the maximal interval of the existence of the smooth solution, then for each $p > 3$, there exists $\epsilon_p > 0$ such that*
$$\left(\int_{\mathbb{R}^3} |u|^p dx \right)^{1/p} \geq \frac{\epsilon_p}{(T^* - t)^{\frac{p-3}{2p}}}$$
as $t \to T^*$.
4. *For a given weak solution, there exists a closed set $S \in (0, +\infty)$ of measure zero such that the solution is smooth in $\mathbb{R}^3 \times ((0, \infty) \backslash S)$.*

Historical Review: Leray-Hopf weak solution

In the domain $Q_T := \mathbb{R}^3 \times (0, T)$, the vector field u in $L_\infty(0, T; L_2) \bigcap L_2(0, T; W_2^1)$ satisfys:

1. the function $t \to \int u(x, t) \cdot w(x)dx$ can be continuously extended to $[0, T]$ for any $w \in L_2$;

2. $$\int_{Q_T} (-u\cdot\partial_t w - u\otimes u : \nabla w + \nabla u : \nabla w)dxdt = 0, \forall w \in W_0^\infty(Q_T);$$

3. $$\frac{1}{2}\int_{\mathbb{R}^3} |u(x,t)|^2 dx + \int_{Q_T} |\nabla u|^2 dxdt \leq \frac{1}{2}\int_{\mathbb{R}^3} |u_0|^2 dx, \forall t \in [0, T];$$

4. $$\|u(\cdot, t) - u_0(\cdot)\|_2 \to 0, t \to 0.$$

Historical Review: The $L_{p,q}$ theory

If the initial value satisfies certain condition, and for some $T > 0$ *the velocity field* u *satisfyes the Ladyzhenskaya-Prodi-Serrin condition*

$$u \in L_{p,q}(\mathbb{R}^3 \times (0, T))$$

with

$$\frac{3}{p} + \frac{2}{q} = 1, \quad p \in (3, +\infty).$$

Then there is a smooth function in $\mathbb{R}^3 \times (0, T]$, *where*

$$\|v\|_{L_{p,q}} := \begin{cases} \left(\int_0^T \|v(\cdot, t)\|_p^q dt \right)^{1/q}, & q \in [1, +\infty) \\ esssup_{t \in (0,T)} \|v(\cdot, t)\|_s, & q = +\infty. \end{cases}$$

The Substantial Gap and Current Best Results

- Standard imbeddings give that the functions of the Leray-Hopf class satisfy

$$\frac{3}{p} + \frac{2}{q} = \frac{3}{2}, \qquad p \in [2,6].$$

- localization in x: Scheffer, Di Perna, Majda, Lin etc. *(Caffarelli-Kohn-Nirenberg, 1983) Let E be the singular set of u, then $\mathcal{P}_{5/3}(E) = 0$, where*

$$\mathcal{P}_K(E) := lim_{\delta\to 0+}\mathcal{P}_{K,\delta}(E),$$

$$\mathcal{P}_{K,\delta}(E) = inf\left\{r_i^K; Q_{r_1}, Q_{r_2}, \cdots \text{ cover } E, r_i < \delta\right\},$$

and $Q_r = B_r \times I_r$ with B_r a ball of radius r and I_r an interval of length r^2.

- Escauriaza, Seregin, Sverak (2004)

  $L_{3,\infty}$ *case.*

It is Really Hard!

Charles Feffeman: *Let me end with a few words about the significance of the problems posed here. Fluids are important and hard to understand. There are many fascinating problems and conjectures about the behavior of solutions of the Euler and Navier-Stokes equations. Since we don't even know whether these solutions exist, our understanding is at a very primitive level. Standard methods from PDE appear inadequate to settle the problem. Instead, we probably need some deep, new ideas.*

Some Well-known Criteria

- Beale-Kato-Majda (pure algebraic, Euler equations):
  If
  $$\int_0^T \|\omega(\cdot,t)\|_\infty dt < \infty,$$
  the solution is classical.
- Constantin-Fefferman (pure geometric, NS):
  The regularity of the direction of vorticity can introduce depletion in the vorticity growth, thus the regularity of the solution.
- Interpolation of these two criteria $\cdots$

Search for Potential Singularities though DNS

DNS: Direct Numerical Simulation
The first candidate: Taylor-Green (1937) vortex:

$$\begin{aligned} u_1 &= \sin x \cos y \cos z, \\ u_2 &= -\cos x \sin y \cos z, \\ u_3 &= 0. \end{aligned}$$

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Brachet et al. (1991): only mildly increasement in the maximum vorticity until the previously conjectured singularity time!

R. Kerr (1993): x-y plane: dividing plane; x-z plane: symmetry plane;
anti-parallel: $\vec{\omega}(x, y, z) = -\vec{\omega}(x, y, -z)$;

$$\begin{aligned} \omega_x(x, y, z) &= -\omega_x(x, -y, z), \quad (1) \\ \omega_y(x, y, z) &= \omega_y(x, -y, z), \quad (2) \\ \omega_z(x, y, z) &= -\omega_z(x, -y, z). \quad (3) \end{aligned}$$

Anti-Parallel Vortex Tube

Three steps in preparing the initial value:

1. define the vortex core;
2. define the vortex vector;
3. rescale the initial profile and filter;

Profile of initial value

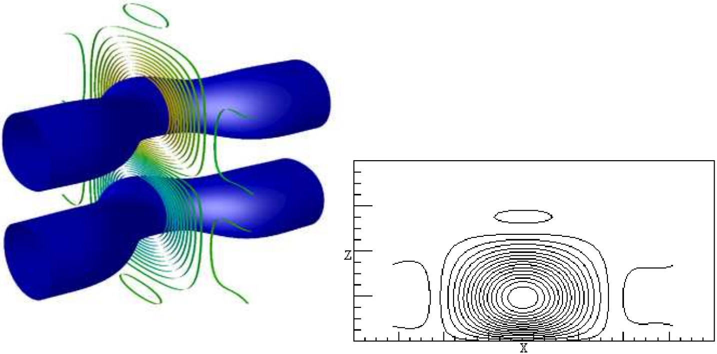


Figure: The 3D vortex tube and axial vorticity on the symmetry plane for initial value.

Why this configuration?

- A numerical study was presented in 1993 (R. Kerr, Phys. Fluids), and concluded that this configuration will develop a finite blowup according to his analysis.
- This initial value was titled as "*the most attractive candidates for potential singular behavior*" of the 3D Euler equations. (Majda and Bertozzi, Vorticity and Incompressible Flow, Cambridge University Press, pp187, 2002);

The Phys. Fluids computation and result

- maximum vorticity blowup as $(T-t)^{-1}$;
- maximum velocity blowup as $(T-t)^{-1/2}$;
- the blowup structure sized as $(T-t)\times\sqrt{T-t}\times\sqrt{T-t}$;
- relative straight vortex lines;
- Chebyshev in z direction;

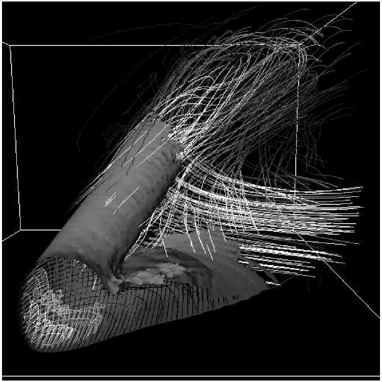


Figure: From: R.Kerr, Euler singularities and turbulence, 19th ICTAM Kyoto '96, 1997, pp57-70.

Why working on this problem?

- A local vortex line geometric criteria by Den-Hou-Yu(2005): $L(t)$: the arclength of a vortex line segment L_t around the maximum vorticity, if
  1. the velocity field along L_t is bounded by $C_U(T-t)^{-\alpha}$ for some $\alpha < 1$;
  2. $C_L(T-t)^{\beta} \leq L(t) \leq C_0/\max_{L_t}(|\kappa|,\ |\nabla \cdot \vec{\xi}|)$,

  then for some $\beta < 1-\alpha$, then the solution of the 3D Euler equations remains regular up to T. When $\beta = 1-\alpha$, if the solution will be regular depends on an algebraic inequality of C_U, C_0 and C_L.
- Thus the blowup scenario described by Kerr falls into the critical case.
- Those constants in the criteria became important to judge if the blowup is theoretically possible, while such information is not available from Kerr's numerical result;

Numerical Method, Software and Hardware

- Pseudo-spectral method and Runge-Kutta scheme;
- High order Fourier smoothing method for dealiasing;
- *LSSC-II* in the Institute of Computational Mathematics and Scientific/Engineering Computing of Chinese Academy of Sciences;
  Shenteng 6800 in the Super Computing Center of Chinese Academy of Sciences (special thanks to Prof Linbo Zhang of CAS);
- Maximal memory consumption: about 120 Gb;
  Time consumption for one computation: over 300 hours;
  Mean data transfer speed on the network: over 2Gb/s;
- FFTW 3.1 for DFT, DST and DCT;
  MPI as the parallel interface;

High Order Fourier Smoothing

- For 3D computation, the effective modes increased from 29% to about 50% (2/3 dealiasing (Orszag, 1977) contributes an increasement from 12.5% to 29%);
- The exponential decay of the numerical error in the distance to the under-resolved point observed (Hou and Li, 2006);
- Solution more closed to the FDM or FVM solutions when under-resolved than other filtering method (Grauer et al, 2007);

Numerical Resolutions

- We adopted a sequences of different resolutions for resolution study, including $768 \times 512 \times 1536$, $1024 \times 768 \times 2048$ and $1536 \times 1024 \times 3072$.
- Since the solution appears to be most singular in the z direction, we allocate twice as many grid points along the z direction than along the x direction. The solution is least singular in the y direction.
- In our computations, two typical ratios in the resolution along the x, y and z directions are $3:2:6$ and $4:3:8$.

Three stages of the solution behavier

1. $t \in [0, 12)$, the maximum vorticity grows only exponentially in time;
2. $t \in [12, 17)$, maximum vorticity is slightly slower than double exponential in time;
3. $t \in [17, 19)$, the growth of the maximum vorticity may well slow down and deviate from double exponential growth;

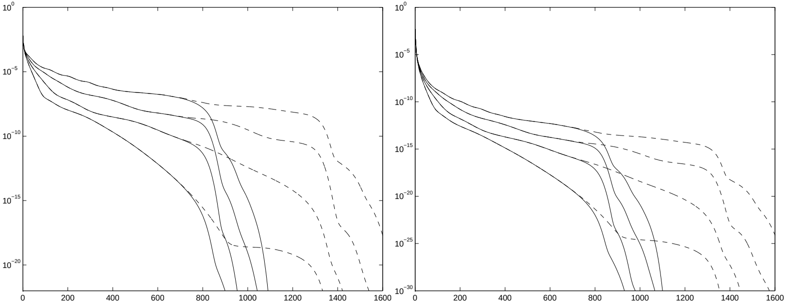
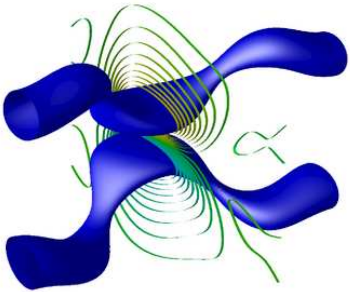


Figure: Convergence study for enstrophy(left) and energy(right) spectra using different resolutions. The dashed lines and the solid lines are the spectra on resolution $1536 \times 1024 \times 3072$ and $1024 \times 768 \times 2048$, respectively. The times for the lines from bottom to top are $t = 16, 17, 18, 19$.

The first stage



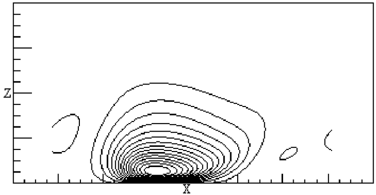


Figure: The 3D vortex tube and axial vorticity on the symmetry plane when $t = 6$.

Maximum vorticity growth

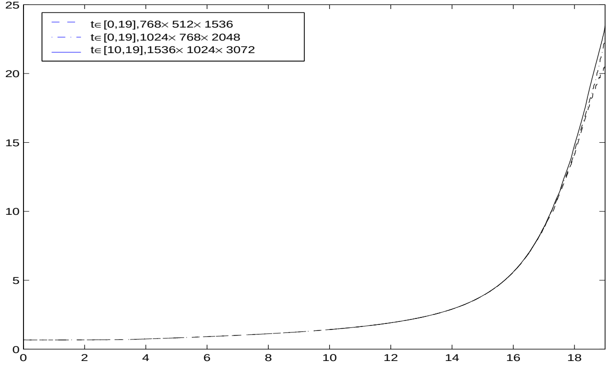


Figure: The maximum vorticity $\|\omega\|_\infty$ in time using different resolutions.

Maximum vorticity growth

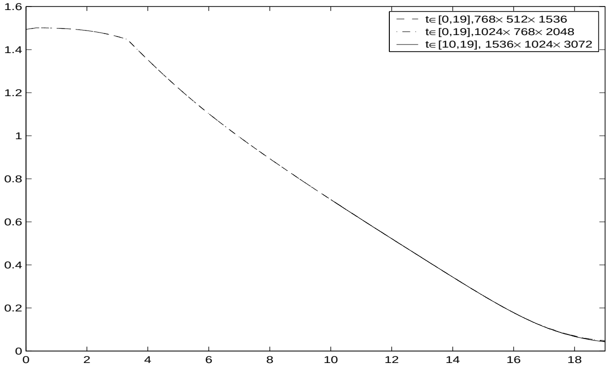


Figure: The inverse of maximum vorticity $\|\omega\|_\infty$ in time using different resolutions.

Maximum vorticity growth

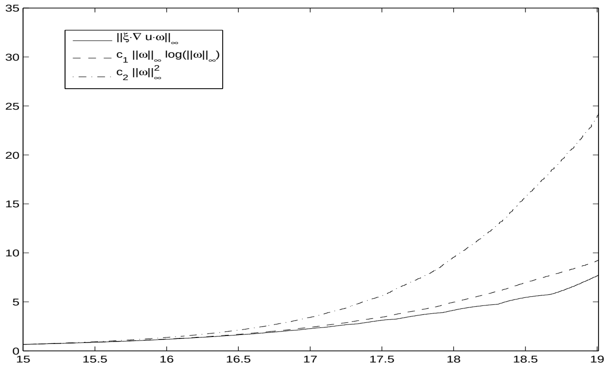


Figure: Study of the vortex stretching term in time, resolution $1536 \times 1024 \times 3072$. We take $c_1 = 1/8.128$, $c_2 = 1/23.24$ to match the same starting value for all three plots.

Maximum vorticity growth

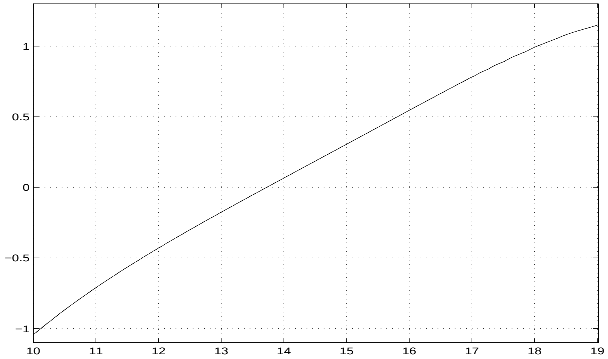


Figure: The plot of $\log \log \|\omega\|_\infty$ vs time, resolution $1536 \times 1024 \times 3072$.

Velocity profile

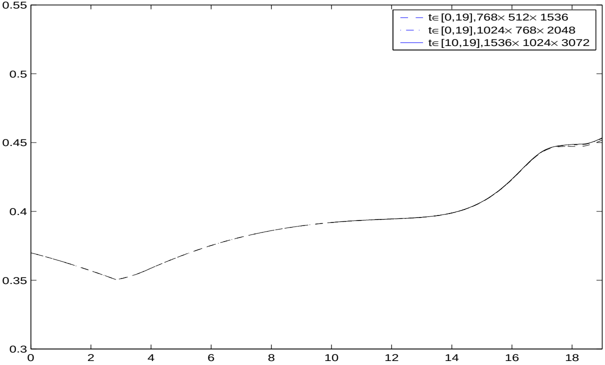


Figure: Maximum velocity $\|\vec{u}\|_\infty$ in time using different resolutions.

The local geometric criteria applies

Recall the local geometric criteria:

1. the velocity field along L_t is bounded by $C_U(T-t)^{-\alpha}$ for some $\alpha < 1$;
2. $C_L(T-t)^{\beta} \le L(t) \le C_0/\max_{L_t}(|\kappa|,\ |\nabla \cdot \vec{\xi}|)$,

then for some $\beta < 1-\alpha$, then the solution of the 3D Euler equations remains regular up to T. When $\beta = 1-\alpha$, if the solution will be regular depends on an algebraic inequality of C_U, C_0 and C_L.

- For Kerr's data, $\alpha = 1/2$, $L(t) = \sqrt{T-t}$;
- For our computation, α can be 0 since the velocity field is bounded.

Let $\vec{\xi} \triangleq \vec{\omega}/|\vec{\omega}|$, we have

$$\frac{\partial}{\partial t}|\vec{\omega}| + (\vec{u} \cdot \nabla)|\vec{\omega}| = \vec{\xi} \cdot M \cdot \vec{\omega}, \quad (4)$$

where $M \triangleq \frac{1}{2}(\nabla u + \nabla^T u)$ is the deformation tensor and λ_i $(i = 1, 2, 3)$ is the i-th eigenvalue of M, then

- $\vec{\xi}$ align to the 3rd eigenvector: Dangerous!
- $\vec{\xi}$ align to the 2nd eigenvector: Unknown and depended on the 2nd eigenvalue;
- $\vec{\xi}$ align to the 1st eigenvector: Safe.

time	$\|\omega\|$	λ_1	θ_1	λ_2	θ_2	λ_3	θ_3
16.012	5.628	-1.508	89.992	0.206	0.007	1.302	89.998
16.515	7.016	-1.864	89.995	0.232	0.010	1.631	89.990
17.013	8.910	-2.322	89.998	0.254	0.006	2.066	89.993
17.515	11.430	-2.630	89.969	0.224	0.085	2.415	89.920
18.011	14.890	-3.625	89.969	0.257	0.036	3.378	89.979
18.516	19.130	-4.501	89.966	0.246	0.036	4.274	89.984
19.014	23.590	-5.477	89.966	0.247	0.034	5.258	89.994

Table: The alignment of the vorticity vector and the eigenvectors of M around the point of maximum vorticity with resolution $1536 \times 1024 \times 3072$. Here, θ_i is the angle between the i-th eigenvector of M and the vorticity vector. One can see that the vorticity vector is aligned very well with the second eigenvector of M.

Conclusion Remarks on the Anti-Parallel Vortex Tube

- There are no finite time blow-up at the alleged time by the former computation;
- The numerical computations demonstrate a very subtle dynamic depletion of vortex stretching.
- The maximum vorticity is shown to grow no faster than double exponential in time;
- The velocity field and the enstrophy are shown to be bounded throughout the computations.
- The local geometric regularity of vortex lines seems to be responsible for this dynamic depletion of vortex stretching.

Thank you!

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