A combinatorial proof of a theorem of Freund

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Abstract

In 1989, Robert W. Freund published an article about generalizations of the Sperner lemma for triangulations of n-dimensional polytopes, when the vertices of the triangulations are labelled with points of $\mathbb{R}^n$. For $y \in \mathbb{R}^n$, the generalizations ensure, under various conditions, that there is at least one simplex containing y in the convex hull of its labels. Moreover, if y is generic, there is generally a parity assertion, which states that there is actually an odd number of such simplices.

For one of these generalizations, contrary to the others, neither a combinatorial proof, nor the parity assertion were established. Freund asked whether a corresponding parity assertion could be true and proved combinatorially.

The aim of this paper is to give a positive answer, using a technique which can be applied successfully to prove several results of this type in a very simple way. We prove actually a more general version of this theorem. This more general version was published by van der Laan, Talman and Yang in 2001, who proved it in a non-combinatorial way, without the parity assertion.

Key Words: chain map; combinatorial proof; Freund's theorem; labelling of a polytope; Sperner's lemma; triangulation.

1 Introduction

Let A be a set of m elements of $\mathbb{R}^n$ defining a polytope $P = \{x \in \mathbb{R}^n : a^T x \leq 1, a \in A\}$ having m facets. For $y \in \mathbb{R}^n$, we define

$$D_y := \big\{(S,T) : S,T \subseteq A \text{ and } y \in \mathrm{conv}((-S) \cup T)\big\},$$

where $\mathrm{conv}(X)$ denotes the convex hull of X, and where $(-X)$ denotes the set $\{-a : a \in X\}$.

Theorem 1 (Freund's Theorem) *Let T be a triangulation of the n-dimensional polytope P and let λ be a map $V(\mathsf{T}) \to A$. Then for each $y \in \mathrm{int}(\mathrm{conv}(A))$, there exist a simplex $\sigma \in \mathsf{T}$ such that $\big(\lambda(V(\sigma)), \mathrm{car}(\sigma)\big) \in D_y$.*

$V(\sigma)$ is the set of vertices of σ and $\mathrm{car}(\sigma)$, the *carrier* of σ, is the set of $a \in A$ such that $a^T x = 1$ for all $x \in \sigma$. Moreover, by triangulation T of P, we mean a geometric simplicial complex whose underlying space is P. Thus the vertex set of the triangulation contains the vertex set of P, but can be much larger.

This theorem is one of many generalizations of Sperner's lemma proposed in a paper of Robert M. Freund published in 1989 ([1]).

In 2001, Van der Laan, Talman and Yang ([2]) generalized this theorem. The generalization deals with any full dimensional polytope P in $\mathbb{R}^n$ with m facets. It should be noticed

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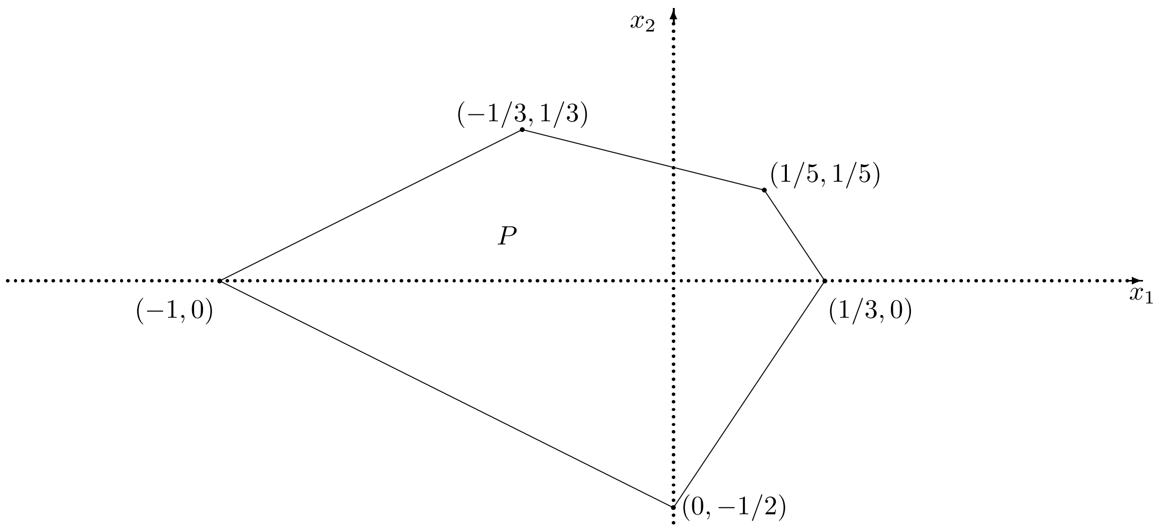

Figure 1: The polytope P used for the illustration of Theorem 2.

that for any point x_0 in the interior of P, there is a set A of m elements of $\mathbb{R}^n$ such that $P = \{x \in \mathbb{R}^n : a^T x \leq 1 + a^T x_0, a \in A\}$. A polytope P in this representation is said to be in the *standard form*. For $y \in \mathbb{R}^n$ and C a nonempty finite collection of elements of $\mathbb{R}^n$, we define:

$$E_y := \big\{(S, T) : S \subseteq C, T \subseteq A \text{ and } y \in \text{conv}(S \cup T)\big\}.$$

The theorem of van der Laan, Talman and Yang is then the following:

Theorem 2 *Let* T *be a triangulation of the* n*-dimensional polytope* P *in standard form and* $\lambda : V(\mathsf{T}) \to C$ *be a labelling of the vertices of* T*. Then for each* $y \in \text{int}(\text{conv}(A))$*, there exists a simplex* σ *of* T *such that* $\big(\lambda(V(\sigma)), \text{car}(\sigma)\big) \in E_y$*.*

Theorem 1 is then the special case $C = -A$.

Let's give an example. Let P be the polytope of Figure 1 and let T be a triangulation of P (see Figure 2). For $x_0 = 0$, we have $a_1 = (3, 2)$, $a_2 = (1, 4)$, $a_3 = (-1, 2)$, $a_4 = (-1, -2)$ and $a_5 = (3, -2)$.

We define

$$C := \{c_1, c_2, \ldots, c_5\},$$

where $c_1 = (7, 2)$, $c_2 = (7, 4)$, $c_3 = (6, 9)$, $c_4 = (5, 14)$ and $c_5 = (5, 2)$ (see Figure 3).

For $z = (1, 1)$, there are three simplices σ of T such that $\big(\lambda(V(\sigma)), \text{car}(\sigma)\big) \in E_z$: $\{a\}$, $\{f, k\}$ and $\{k, p\}$.

For $y = (1, -2)$, there is one simplex σ de T such that $\big(\lambda(V(\sigma)), \text{car}(\sigma)\big) \in E_y$: $\{u\}$. Note that in this case, y is not in the interior of conv(A), but as we will see, the theorem remains true (see Theorem 3 below).

Our aim is to give a combinatorial proof of a strengthening of Theorem 2 (Theorem 3), which asserts that there are an odd number of such simplices σ, if y is generic. According to Ziegler ([6]), a combinatorial proof in a topological context is a proof using no simplicial approximation, no homology, no continuous map.

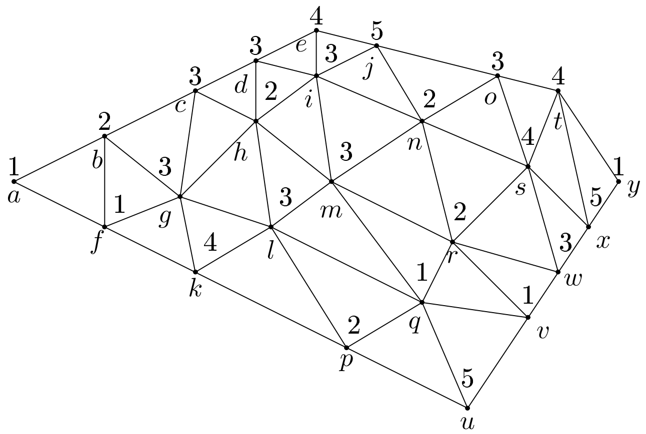

Figure 2: The triangulation T of P and the labelling used in the illustration of Theorem 2, the names of the vertices of T are indicated below the vertices, the index i of the labels c_i above.

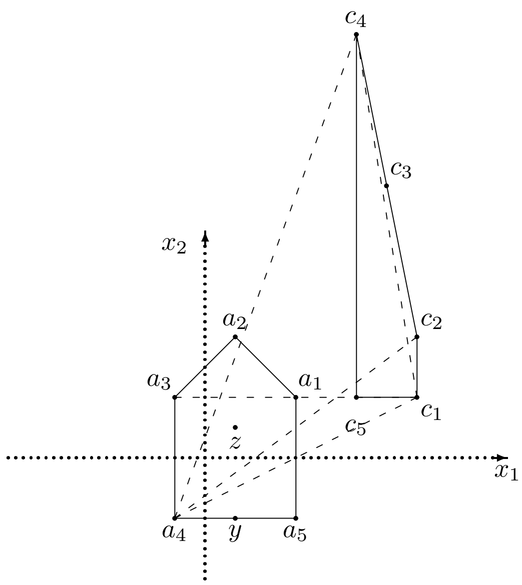

Figure 3: The sets A and C used in the illustration of Theorem 2, as well as y and z.

The existence of such a combinatorial proof and the oddness of the number of simplices σ for Theorem 1 were presented as an open question in the paper of Freund [1], who proved it using properties of continuous mappings, as the Brouwer fixed-point theorem. Van der Laan and al. proved their theorem with the use of the Kakutani theorem.

The result of the present paper is the answer of this open question. Moreover, it presents a simple method to prove results of this kind.

2 Definitions, notations and tools

$\binom{X}{\leq k}$ is the set of subsets of X having at most k elements. For a sequence $a_0, \ldots, a_i, \ldots, a_k$, the sequence $a_0, \ldots, \hat{a}_j, \ldots, a_k$ is the same sequence with the a_j missing. If X and Y are two finite sets, $X \uplus Y$ denotes the disjoint union of X and Y: $X \uplus Y := X \times \{1\} \cup Y \times \{2\}$. $\mathrm{int}(X)$ denotes the interior of X.

2.1 Simplices, complexes and chains

We give here a short introduction to the notions of simplices, complexes and chains. For a more complete study of this subject, see the book of Munkres [3]. We work with chains with coefficients in GF(2), thus we will not introduce the notion of an oriented simplex.

2.1.1 Simplices and simplicial complexes

An *abstract simplicial complex* is a collection K of subsets of a finite ground set X with the property that $\sigma' \subseteq \sigma \in \mathsf{K}$ implies $\sigma' \in \mathsf{K}$. We define the *dimension* of K: $\dim(\mathsf{K}) = \max\{|\sigma| - 1 : \sigma \in \mathsf{K}\}$. The sets in K are called *(abstract) simplices* and the dimension of a simplex σ is $\dim(\sigma) = |\sigma| - 1$. If $\dim(\sigma) = d$, we say that σ is a d-simplex. $\emptyset$ has dimension -1.

The elements of K (resp. the subsets of a simplex σ) are called *faces*. A p-face of K is a face of K of dimension p. 0-faces are called *vertices*, and 1-faces *edges*. The set of the formers is denoted by $V(\mathsf{K})$, and the set of the latters by $E(\mathsf{K})$. $\emptyset$ is the only -1-face of K. For a p-simplex σ, the *facets* are the simplices $\sigma' \subseteq \sigma$ of dimension $p-1$. The set of p-faces is denoted by K_p.

A *pseudomanifold* is an abstract n-dimensional simplicial complex whose any $(n-1)$-simplex is contained in at most two n-simplices. If each $(n-1)$-simplex is contained in exactly two n-simplices, we say that the pseudomanifold is *without boundary*.

A *geometric simplex* is the convex hull σ of affinely independent points, the *vertices* $V(\sigma)$ of σ. The dimension $\dim(\sigma)$ of σ is then the number of vertices minus 1.

A *geometric simplicial complex* is a collection C of geometric simplices such that (i) the faces of the simplices are in C, (ii) the intersection of two simplices is a face of both.

The *underlying space* of C is denoted by $||\mathsf{C}||$ and defined by $||\mathsf{C}|| := \cup_{\sigma \in \mathsf{C}} \sigma$.

It is easy to see that the vertex sets of the simplices of a geometric simplicial complex C is an abstract simplicial complex: $\{V(\sigma) : \sigma \in \mathsf{C}\}$ is an abstract simplicial complex.

A *triangulation* of a polytope P is a geometric simplicial complex T such that $||\mathsf{T}|| = P$.

Let K and L be two abstract simplicial complexes. A *simplicial map* of K into L is a mapping $f : V(\mathsf{K}) \to V(\mathsf{L})$ that maps simplices to simplices, i.e., such that $f(\sigma) \in \mathsf{L}$ whenever $\sigma \in \mathsf{K}$.

2.1.2 Chains

Let K be an abstract simplicial complex. The *chain complex* $\mathcal{C}(\mathsf{K})$ is :

$$\ldots \to C_3(\mathsf{K}) \xrightarrow{\partial} C_2(\mathsf{K}) \xrightarrow{\partial} C_1(\mathsf{K}) \xrightarrow{\partial} C_0(\mathsf{K}) \to \ldots,$$

where $C_p(\mathsf{K})$ is the free abelian group of all formal linear combinations of p-faces of K with coefficients in GF(2). Any element c of $C_k(\mathsf{K})$ is called a *k-chain*.

We define the *boundary operator* ∂ for a simplicial complex K as follows: ∂ is a homomorphism of free groups: $C_p(\mathsf{K}) \to C_{p-1}(\mathsf{K})$ and if σ is a p-simplex $[v_0, \ldots, v_p]$, $p \geq 1$, $\partial\sigma := \sum_{i=0}^{p}[v_0, \ldots, \hat{v}_i, \ldots, v_p]$.

The boundary operator satisfies:

$$\partial\partial = \partial^2 = 0 \tag{1}$$

because it is obviously true for simplices ($[\ldots, \hat{v}_i, \ldots, \hat{v}_j, \ldots]$ arises twice).

A *chain map* ν is a collection of homomorphisms $\nu_p : C_p(\mathsf{K}) \to C_p(\mathsf{L})$ such that

$$\partial\nu_p = \nu_{p-1}\partial \tag{2}$$

for all p. If f is a simplicial map of K to L, we define a collection $f_\#$ of homomorphisms $f_{\#p} : C_p(\mathsf{K}) \to C_p(\mathsf{L})$ by defining it on simplices as follows: for σ a p-simplex, we have

$$f_{\#p}(\sigma) = \begin{cases} f(\sigma) \text{ if } \dim f(\sigma) = p \\ 0 \text{ otherwise.} \end{cases}$$

$f_\#$ is then a chain map (it can be easily checked, again, first for simplices).

2.1.3 The main tool: is the point in the simplex?

Let X be a set of points in $\mathbb{R}^n$. Let Δ be an abstract simplicial complex whose ground set is X. We define then the following function ψ_g, which checks if the point g is in the convex hull of a given n-simplex:

$$\psi_g : \sigma \in \Delta_n \mapsto \begin{cases} 1 \text{ if } g \in \mathrm{conv}(\sigma) \\ 0 \text{ if not,} \end{cases}$$

and the function $\phi_{(g,\vec{b})}$, which checks if the half-line $(g, \vec{b})$ (emanating from g and whose direction is given by $\vec{b}$) intersects the convex hull of a given $(n-1)$-simplex:

$$\phi_{(g,\vec{b})} : \tau \in \Delta_{n-1} \mapsto \begin{cases} 1 \text{ if } (g, \vec{b}) \text{ intersects } \mathrm{conv}(\tau) \\ 0 \text{ if not.} \end{cases}$$

Let us emphasize that σ and τ are here abstract simplices, whose vertices are points in $\mathbb{R}^n$. Extending those two maps by linearity, we get two linear forms: ψ_g on $C_n(\Delta)$ and $\phi_{(g,\vec{b})}$ on $C_{n-1}(\Delta)$. We have then the following property:

Claim 1 *Let X be a set of points in $\mathbb{R}^n$. Let Δ an (abstract) simplicial complex whose ground set is X, and let $g \in \mathbb{R}^n$ and $\vec{b} \in \mathbb{R}^n$ (both generic). The following equality holds: $\psi_g = \phi_{(g,\vec{b})}\partial$.*

Proof: The equality is true for a unique n-simplex, thus it is true for any n-chain, by linearity. ∎

2.2 Polarity for polytopes

We do not reprove the results of this section: they are common knowledge (see the book of Ziegler [5]).

Let P be a polytope in $\mathbb{R}^n$ containing the point 0 in its interior. If m is the number of facets, we can represent P in its standard form $P = \{x \in \mathbb{R}^n : a_i^T x \leq 1, i = 1, \ldots, m\}$ where the a_i live in $\mathbb{R}^n$.

The polytope $P^\triangle := \mathrm{conv}\{a_1, \ldots, a_m\}$ is called the polar of P.

Proposition 1 *The following properties hold:*

1. *0 is in the interior of* $P^{\triangle}$.
2. $P^{\triangle\triangle} = P$.
3. *for* F *a face of* P,

$$F \mapsto F^{\diamond} := \text{conv}\{a_i : i \in \{1, \dots, m\}, a_i^T x = 1 \textit{ for all } x \in F\}$$

   (a) *is one-to-one.*
   (b) *is such that* $\dim(F) + \dim(F^{\diamond}) = n - 1$.
   (c) *is such that* $F^{\diamond\diamond} = F$.

In particular, the vertices of $P^{\triangle}$ are the a_i, $\emptyset^{\diamond} = P^{\triangle}$ and $P^{\diamond} = \emptyset$.

3 The main result

The proof of our generalization of Theorem 2 consists principally in applying the commutation result of Claim 1 on the chain which is the formal sum of all full-dimensional simplices of a certain pseudomanifold without boundary. This pseudomanifold, denoted by L, is defined by the following lemma.

Lemma 1 *Let* P *be a* n*-dimensional polytope. Suppose that* P *and* $P^{\triangle}$ *are triangulated independently by* T *and* T' *respectively. The abstract simplicial complex*

$$\mathsf{L} := \{V(\sigma) \uplus V(\tau) : \sigma \in \mathsf{T}, \tau \in \mathsf{T}', \sigma \subseteq F, \tau \subseteq F^{\diamond}, F \textit{ is a face of } P\}$$

is then a pseudomanifold without boundary.

We emphasize that P is a face of itself. Figure 4 is a simple illustration of this lemma. P and $P^{\triangle}$ are there triangles with vertex set respectively $V(P) = \{v_1, v_2, v_3\}$ and $V(P^{\triangle}) = \{e_1^{\diamond}, e_2^{\diamond}, e_3^{\diamond}\}$ (the edges of P being $\{e_1, e_2, e_3\}$); T (resp. T') is the collection of subsets of $V(P)$ (resp. $V(P^{\triangle})$); L is a 2-dimensional simplicial complex, homeomorphic to a 2-dimensional sphere, and whose maximal simplices are $\{v_1, v_2, v_3\}$, $\{e_1^{\diamond}, e_2^{\diamond}, e_3^{\diamond}\}$, and simplices of the form $\{v_i, v_j, e_k^{\diamond}\}$ and $\{v_i, e_j^{\diamond}, e_k^{\diamond}\}$ with $\{i, j, k\} = \{1, 2, 3\}$.

Proof of Lemma 1: Let σ and τ be two simplices such that $V(\sigma) \uplus V(\tau)$ is a $(n-1)$-simplex of L, and let F be the face carrying σ. Note that the roles played by σ and F on one hand are symmetric to the roles played by τ and $F^{\diamond}$ on the other hand.

Let $k := \dim(\sigma)$. We have then the following relations (still valid if one of the simplices is empty): $\dim(\tau) = n - k - 2$, $\dim(F) \geq k$, because F carries σ, and $\dim(F^{\diamond}) \geq n - k - 2$. Furthermore, by polarity, $\dim(F) + \dim(F^{\diamond}) = n - 1$. Substituting this equality in the last inequality, we get: $\dim(F) \leq k + 1$. Thus either $\dim(F) = \dim(\sigma)$, and then $\dim(F^{\diamond}) = \dim(\tau) + 1$, or $\dim(F) = \dim(\sigma) + 1$, and then $\dim(F^{\diamond}) = \dim(\tau)$.

Since these two possibilities are symmetric, we only have to check one of them. So, let us suppose that $\dim(F) = k$. A n-simplex of L containing $V(\sigma) \uplus V(\tau)$ has either the form $V(\sigma') \uplus V(\tau)$, with $\dim(\sigma') = k + 1$, or the form $V(\sigma) \uplus V(\tau')$, with $\dim(\tau') = n - k - 1$.

1. if τ is carried by a $(n - k - 2)$-face $G^{\diamond}$ of $F^{\diamond}$:

   (a) there is exactly one n-simplex in L of the form $V(\sigma') \uplus V(\tau)$, $\sigma \subseteq \sigma'$: indeed, if such a σ' exists, it has to be contained in a face F' of dimension $\geq k + 1$, and $F'^{\diamond}$ has to have dimension $\leq n - k - 2$, and has to contain τ. Hence, $F' = G$ and only one such a σ' exists: it is the $(k + 1)$-simplex of T included in G and containing σ.

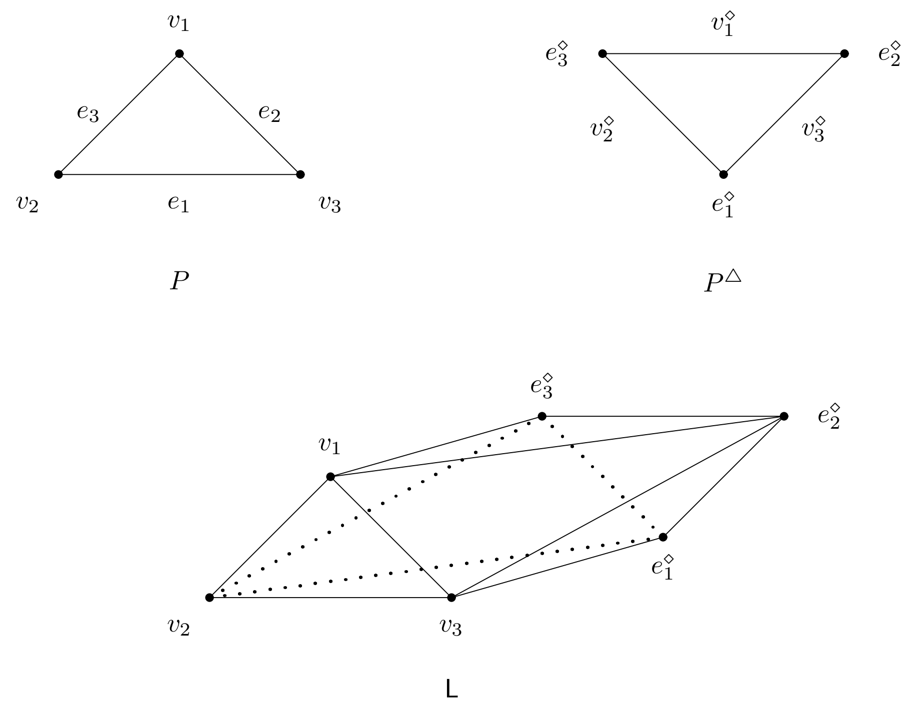

Figure 4: An illustration of Lemma 1 with P a triangle, T the set of faces of P and T' the set of faces of $P^\triangle$.

   (b) there is exactly one n-simplex in L of the form $V(\sigma) \uplus V(\tau')$, $\tau \subseteq \tau'$: indeed, if such a τ' exists, it has to be contained in a face $G'^\Diamond$ of dimension $\geq n-k-1$, and G' has to have dimension $\leq k$, and has to contain σ. Hence, $G' = F$ and only one such a τ' exists: it is the $(n-k-1)$-simplex of T' included in $F^\Diamond$ and containing τ.

2. if τ is not carried by a $(n-k-2)$-face:

   (a) there is no n-simplex in L of the form $V(\sigma') \uplus V(\tau)$, $\sigma \subseteq \sigma'$: indeed, if such a σ' exists, it has to be contained in a face F' of dimension $\geq k+1$, and the polar of F' has to have dimension $\leq n-k-2$, and has to contain τ. Such a face does not exist by assumption.

   (b) there are exactly two n-simplices in L of the form $V(\sigma) \uplus V(\tau')$, $\tau \subseteq \tau'$: indeed, if such a τ' exists, it has to be contained in a face $G'^\Diamond$ of dimension $\geq n-k-1$, and G' has to have dimension $\leq k$, and has to contain σ, which implies $G' = F$. Exactly two such τ' exist: it is the $(n-k-1)$-simplices of T' included in $F^\Diamond$ and containing τ.

In both cases 1. and 2., the $(n-1)$-simplex $V(\sigma) \uplus V(\tau)$ is contained in exactly two n-simplices of L. Thus L is a pseudomanifold without boundary. ∎

The same kind of pseudomanifold appears in the paper of Freund (Lemma 3 of [1]: the pseudomanifold introduced by Freund is defined similarly, but requires moreover that τ is in the boundary of $P^\triangle$; it implies that the boundary of the pseudomanifold is a triangulation of $\partial P^\triangle$).

With this lemma and the Claim 1, we are now able to prove the following theorem, which is a strengthening of Theorem 2 when y is generic. Note that we have deleted the assumption that y is in the interior of the convex hull of the points of A; it is not necessary.

Theorem 3 *Let* T *be a triangulation of the* n*-dimensional polytope* P *in standard form and let* C *be a nonempty finite collection of elements of* $\mathbb{R}^n$*. Let* $\lambda : V(\mathsf{T}) \to C$ *be a labelling of the vertices of* T*. Then for each* $y \in \mathrm{conv}(A)$*, there exists a simplex* σ *of* T *such that* $\big(\lambda(V(\sigma)), \mathrm{car}(\sigma)\big) \in E_y$*. Moreover, if* y *is generic, the number of such simplices is odd.*

Proof of Theorem 3: By translation, it is sufficient to prove the theorem for the case $x_0 = 0$. We can assume that y is generic: indeed, if not, take a generic y' close to y in the interior of $\mathrm{conv}(A)$; such an y' exists for obvious measure reasons and if σ is such that $\big(\lambda(V(\sigma)), \mathrm{car}(\sigma)\big) \in E_{y'}$, then we have also $\big(\lambda(V(\sigma)), \mathrm{car}(\sigma)\big) \in E_y$.

By definition, $P^\triangle = \mathrm{conv}(A)$. Let T' be a triangulation of $P^\triangle$ using only vertices of $P^\triangle$: thus $V(\mathsf{T}') = A$. We consider the simplicial complex L defined as in Lemma 1. This lemma ensures that L is a n-dimensional pseudomanifold without boundary.

Let c be the formal sum of all n-simplices of L. Because of Lemma 1, any $(n-1)$-simplex of L is in an even number of n-simplices of L. This means $\partial c = 0$. We extend λ on the vertices of L: if $v \in V(\mathsf{T})$, v keeps its label $\lambda(v)$; if $v \in V(\mathsf{T}')$, we put $\lambda(v) := v$.

Because of Claim 1, with $X := A \cup C$, $\Delta := \binom{X}{\leq n+1}$, $g := y$ and $\vec{b}$ generic, we have $\psi_y(\lambda_\# c) = \phi_{(y,\vec{b})}(\partial \lambda_\# c) = \phi_{(y,\vec{b})}(\lambda_\# \partial c) = 0$, which means that there is an even number of n-simplices of L containing y in the convex hull of their images. But there is exactly one n-simplex $\emptyset \uplus V(\tau)$ of L containing y in the convex hull of its image because T' is a triangulation of $P^\triangle$. Thus, there is an odd number of n-simplices $V(\sigma) \uplus V(\tau)$ in $\mathsf{L} - (\emptyset \uplus \mathsf{T}')$ that have y in the convex hull of their images. Since $\lambda(V(\tau)) = V(\tau) \subseteq \mathrm{car}(\sigma)$, this means that $\big(\lambda(V(\sigma)), \mathrm{car}(\sigma)\big) \in E_y$.

It remains to show that there is an odd number of such simplices σ. We know that there is an odd number of pairs $(\sigma, \tau) \in \mathsf{T} \times \mathsf{T}'$ such that $\sigma \neq \emptyset$ and such that $y \in \mathrm{int}\big(\mathrm{conv}\big(\lambda(V(\sigma)) \cup V(\tau)\big)\big)$. Let (σ, τ) be such a pair, and let F and $F^\diamond$ be the supporting faces respectively of σ and τ (we have $\dim(\sigma) = \dim(F)$ and $\dim(\tau) = \dim(F^\diamond)$). If we show that there is no simplex τ' such that (σ, τ') satisfies the same conditions as (σ, τ), the conclusion will follows. But this is direct: the $n+1$ points of $\lambda(V(\sigma)) \cup V(\tau)$ are affinely independent; the interiors of τ and τ', if τ' exists, are disjoint, and τ and τ' generate the same affine subspace; thus the interiors of $\mathrm{conv}\big(\lambda(V(\sigma)) \cup V(\tau)\big)$ and $\mathrm{conv}\big(\lambda(V(\sigma)) \cup V(\tau')\big)$ are disjoint as well. ∎

4 Conclusion

The open question of R. Freund is now solved, but the method presented here is interesting for itself, since it can be used to prove other results of this kind, in particular the other generalizations of Freund presented in [1]. We do not give the details here because combinatorial proofs of those theorems were known, and because the techniques to use are the same as those presented here: in particular, the use of Claim 1 to obtain the parity of the number of n-simplices of a pseudomanifold containing a given vector in its convex hull from considerations on the $(n-1)$-simplices of the boundary.

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