Combinatorial approach to Kneser graphs

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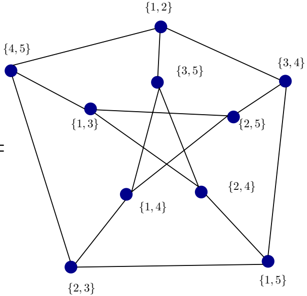
CERMICS, Optimisation et Systèmes

Combinatorial proof of the Lovász-Kneser theorem

Kneser graphs

$\mathrm{KG}(5,2) =$



m, ℓ two integers s.t. $m \geq 2\ell$.

Kneser graph $\mathrm{KG}(m,\ell)$:

$$V(\mathrm{KG}(m,\ell)) = \binom{[m]}{\ell}$$

$$E(\mathrm{KG}(m,\ell))) = \left\{ AB : A, B \in \binom{[m]}{\ell},\ A \cap B = \emptyset \right\}$$

Lovász-Kneser theorem

Theorem
$\chi(\mathrm{KG}(m,\ell)) = m - 2\ell + 2.$

$\chi(\mathrm{KG}(m,\ell)) \leq m - 2\ell + 2$ (easy: explicit coloring).

Matoušek proposed in 2003 a combinatorial (yet still topological) proof.

Octahedral Tucker lemma

Lemma

Let $\lambda : \{+, -, 0\}^m \setminus \{\mathbf{0}\} \to \{\pm 1, \ldots, \pm k\}$ *s.t.*

- $\lambda(-\boldsymbol{x}) = -\lambda(\boldsymbol{x})$ *for every* $\boldsymbol{x}$
- $\lambda(\boldsymbol{x}) + \lambda(\boldsymbol{y}) \neq 0$ *for every* $\boldsymbol{x} \preceq \boldsymbol{y}$

Then $k \geq m$.

$$\boldsymbol{x} = (x_1, \ldots, x_m) \preceq \boldsymbol{y} = (y_1, \ldots, y_m) \quad \text{if} \quad x_i \neq 0 \Rightarrow y_i = x_i$$

Matoušek's proof

- ⋆ $c : \binom{[m]}{\ell} \to [t]$ proper coloring of $\mathrm{KG}(m, \ell)$ with t colors.

- ⋆ Extension for any $U \subseteq [m]$: $c(U) = \max\{c(A) : \ A \subseteq U, \ |A| = \ell\}$.

- ⋆ $\boldsymbol{x}^+ = \{i : \ x_i = +\}$ and $\boldsymbol{x}^- = \{i : \ x_i = -\}$

- ⋆ $$\lambda(\boldsymbol{x}) = \begin{cases} |\boldsymbol{x}| & \text{if } |\boldsymbol{x}| \leq 2\ell - 2, \min(\boldsymbol{x}^+) < \min(\boldsymbol{x}^-) \\ -|\boldsymbol{x}| & \text{if } |\boldsymbol{x}| \leq 2\ell - 2, \min(\boldsymbol{x}^-) < \min(\boldsymbol{x}^+) \\ & \\ c(\boldsymbol{x}^+) + 2\ell - 2 & \text{if } |\boldsymbol{x}| \geq 2\ell - 1, c(\boldsymbol{x}^+) > c(\boldsymbol{x}^-) \\ -c(\boldsymbol{x}^-) - 2\ell + 2 & \text{if } |\boldsymbol{x}| \geq 2\ell - 1, c(\boldsymbol{x}^-) > c(\boldsymbol{x}^+) \end{cases}$$

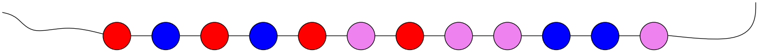
An interlude: the splitting necklace theorem

Two thieves and a necklace

n beads, t types of beads, a_i (even) beads of each type.

Two thieves: Alice and Bob.

Beads fixed on the string.

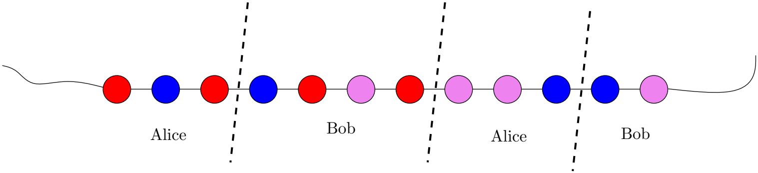


Fair splitting = each thief gets $a_i/2$ beads of type i

The splitting necklace theorem

Theorem (Alon, Goldberg, West, 1985-1986)

There is a fair splitting of the necklace with at most t cuts.



t is tight

t cuts are sometimes necessary:

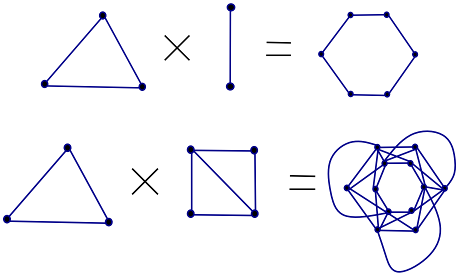
An interlude: Hedetniemi's conjecture for Kneser graphs

Product of graphs and Hedetniemi's conjecture

Let G, H be two graphs.

Categorical product $G \times H$:

$$V(G \times H) = V(G) \times V(H)$$
$$E(G \times H) = \{(u,v)(u',v') : \ uu' \in E(G), \ vv' \in E(H)\}$$



Hedetniemi's conjecture (1966)

$$\chi(G \times H) = \min(\chi(G), \chi(H))$$

Hedetniemi and Kneser

Theorem (Hell, 1980)

Hedetniemi's conjecture is true for pairs of Kneser graphs.

"Direct proof" by Valencia-Pabon and Vrecia.

Circular chromatic number of Kneser graphs

Circular chromatic number

Let $G = (V, E)$ be a graph.

(p, q)-coloring. Mapping $c : V \to [p]$ s.t.

$$q \leq |c(u) - c(v)| \leq p - q \quad \text{for every } uv \in E,$$

($p \geq q \geq 1$ are two integers.)

Circular chromatic number.

$$\chi_c(G) = \inf\{p/q : \ G \text{ admits a } (p, q)\text{-coloring}\}.$$

Known facts.

- $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$
- the infimum is a minimum ($\chi_c(G)$ is rational)

Question. For which G do we have $\chi_c(G) = \chi(G)$?

Circular chromatic number of Kneser graphs

The following has been conjectured by Johnson, Holroyd, and Stahl in 1997.

Theorem (Chen 2011)

$$\chi_c(\mathrm{KG}(m,\ell)) = \chi(\mathrm{KG}(m,\ell))$$

For even m, already proved by Simonyi, Tardos, and M..

Chen's proof is combinatorial: extension of Matoušek's proof.

Chen's lemma

$K^*_{t,t} = K_{t,t} -$ a perfect matching.

Lemma

Any proper coloring of $\mathrm{KG}(m,\ell)$ *with* $m-2\ell+2$ *colors contains a colorful copy of* $K^*_{m-2\ell+2,m-2\ell+2}$.

$\chi_c(\mathrm{KG}(m,\ell)) = \chi(\mathrm{KG}(m,\ell))$ is then a direct consequence.

Octahedral Ky Fan lemma

Lemma

Let $\lambda : \{+,-,0\}^m \setminus \{\mathbf{0}\} \to \{\pm 1, \dots, \pm k\}$ *s.t.*

- $\lambda(-\boldsymbol{x}) = -\lambda(\boldsymbol{x})$ *for every* $\boldsymbol{x}$
- $\lambda(\boldsymbol{x}) + \lambda(\boldsymbol{y}) \neq 0$ *for every* $\boldsymbol{x} \preceq \boldsymbol{y}$

Then there is at least one negatively alternating m-chain.

Negatively alternating m-chain: $\boldsymbol{x}^1 \preceq \cdots \preceq \boldsymbol{x}^m$ with

$$\lambda(\{\boldsymbol{x}^1, \dots, \boldsymbol{x}^m\}) = \{-j_1, +j_2, \dots, (-1)^m j_m\} \quad \text{and} \quad 1 \leq j_1 < j_2 < \cdots < j_m.$$

$$\boldsymbol{x} = (x_1, \dots, x_m) \preceq \boldsymbol{y} = (y_1, \dots, y_m) \quad \text{if} \quad x_i \neq 0 \Rightarrow y_i = x_i$$

Proof of Chen's lemma

⋆ $c : \binom{[m]}{\ell} \to [t]$ proper coloring of $\mathrm{KG}(m,\ell)$ with $t = m - 2\ell + 2$ colors.

⋆ Extension for any $U \subseteq [m]$: $c(U) = \max\{c(A) : \ A \subseteq U, \ |A| = \ell\}$.

⋆ $\boldsymbol{x}^+ = \{i : \ x_i = +\}$ and $\boldsymbol{x}^- = \{i : \ x_i = -\}$

⋆ $$\lambda(\boldsymbol{x}) = \begin{cases} |\boldsymbol{x}| & \text{if } |\boldsymbol{x}| \leq 2\ell - 2, \ \min(\boldsymbol{x}^+) < \min(\boldsymbol{x}^-) \\ -|\boldsymbol{x}| & \text{if } |\boldsymbol{x}| \leq 2\ell - 2, \ \min(\boldsymbol{x}^-) < \min(\boldsymbol{x}^+) \\ & \\ c(\boldsymbol{x}^+) + 2\ell - 2 & \text{if } |\boldsymbol{x}| \geq 2\ell - 1, \ c(\boldsymbol{x}^+) > c(\boldsymbol{x}^-) \\ -c(\boldsymbol{x}^-) - 2\ell + 2 & \text{if } |\boldsymbol{x}| \geq 2\ell - 1, \ c(\boldsymbol{x}^-) > c(\boldsymbol{x}^+) \end{cases}$$

Two m-chains

The negatively alternating m-chain

$$\boldsymbol{x}^1 \prec \cdots \prec \boldsymbol{x}^m \qquad \lambda(\boldsymbol{x}^i) = (-1)^i i$$

Another m-chain

$$\boldsymbol{y}^1 \prec \cdots \prec \boldsymbol{y}^m \qquad \lambda(\boldsymbol{y}^i) = (-1)^i i \quad \text{if } i \neq 2\ell - 2$$

$$\boldsymbol{y}^{2\ell-2} = -\boldsymbol{x}^{2\ell-2}$$

$$\boldsymbol{x}^{(2\ell-2+i)-} = S \cup \{a_1, a_3, a_5, \ldots, a_i\} \qquad i \text{ is odd}$$

$$\boldsymbol{x}^{(2\ell-2+i)+} = T \cup \{a_2, a_4, a_6, \ldots, a_i\} \qquad i \text{ is even}$$

$$\boldsymbol{y}^{(2\ell-2+i)-} = T \cup \{b_1, b_3, b_5, \ldots, b_i\} \qquad i \text{ is odd}$$

$$\boldsymbol{y}^{(2\ell-2+i)+} = S \cup \{b_2, b_4, b_6, \ldots, b_i\} \qquad i \text{ is even}$$

The lemma follows.

Octahedral Ky Fan lemma

Existence of the two chains $\boldsymbol{x}^1 \prec \cdots \prec \boldsymbol{x}^m$ and $\boldsymbol{y}^1 \prec \cdots \prec \boldsymbol{y}^m$ proved via

Lemma

Let $\lambda : \{+,-,0\}^m \setminus \{\mathbf{0}\} \to \{\pm 1, \ldots, \pm k\}$ *s.t.*

- $\lambda(-\boldsymbol{x}) = -\lambda(\boldsymbol{x})$ *for every* $\boldsymbol{x}$
- $\lambda(\boldsymbol{x}) + \lambda(\boldsymbol{y}) \neq 0$ *for every* $\boldsymbol{x} \preceq \boldsymbol{y}$

Then there is an odd number of negatively alternating m-chains.

Defining

$$\mu(\boldsymbol{y}) = \begin{cases} \lambda(\boldsymbol{y}) & \text{if } \boldsymbol{y} \notin \{-\boldsymbol{x}^{2\ell-2}, \boldsymbol{x}^{2\ell-2}\} \\ -\lambda(\boldsymbol{y}) & \text{otherwise,} \end{cases}$$

makes the job.

An interlude: circular chromatic Hedetniemi's conjecture for Kneser graphs

Hedetniemi, Kneser, and circular chromatic number

Theorem

$$\chi_c(\mathrm{KG}(m,\ell) \times \mathrm{KG}(m',\ell')) = \min(\chi_c(\mathrm{KG}(m,\ell), \chi_c(\mathrm{KG}(m',\ell')).$$

Conjecture

$$\chi_c(G \times H) = \min(\chi_c(G), \chi_c(H)).$$