Discrete Splitting of the Necklace

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December 1, 2007

Abstract

This paper deals with direct proofs and combinatorial proofs of the famous Necklace theorem of Alon, Goldberg and West. The new results are a direct proof for the case of two thieves and three types of beads, and an efficient constructive proof for the general case with two thieves. This last proof uses a theorem of Ky Fan which is a version of Tucker's lemma concerning cubical complexes instead of simplicial complexes.

Key Words: constructive proof; cubical complex; splitting necklaces.

1 Introduction

A classical application of the Borsuk-Ulam theorem is the Splitting Necklace theorem (see the book of Matoušek [10]). Proved first by Goldberg and West in 1985 [9], this theorem got a simpler proof in 1986, found by Alon and West and using the Borsuk-Ulam theorem [4].

Two thieves have stolen a precious necklace, which has n beads. These beads belong to t different types. Assume that there is an even number of beads of each type, say $2a_i$ beads of type i, for each $i \in \{1, 2, \ldots, t\}$, where a_i is a nonzero integer. Remark that we have $2\sum_{i=1}^{t} a_i = n$.

The beads are fixed on an open chain made of gold. An example is given in Figure 1. The different types are encoded by the numbers 1,2,3.

As we do not know the exact value of each type of beads, a fair division of the necklace consists of giving the same number of beads of each type to each thief. The number of beads of each type is even, hence such a division is always possible: cut the chain at the $n-1$ possible positions. But the chain is made of gold! It is a shame that we damage it. Isn't it possible to do the division with less cuts? The answer is yes, as shown by the following theorem proved by Goldberg and West:

Theorem 1 *A fair division of the necklace with t types of beads between two thieves can be done with no more than t cuts.*

In 1987, Alon proved the following generalization, for a necklace having qa_i beads for each type i, a_i integer, using a generalization of the Borsuk-Ulam theorem [1]:

Theorem 2 *A fair division of the necklace with t types of beads between q thieves can be done with no more than $t(q-1)$ cuts.*

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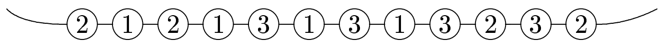

Figure 1: An open necklace, with 12 beads, of 3 different types, to be divided between Alice and Bob.

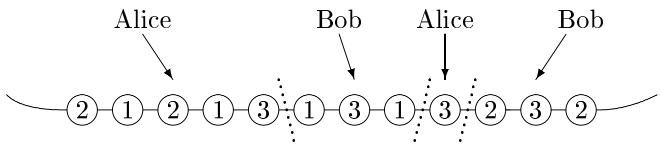

Figure 2: A fair division between Alice and Bob.

For $q = 2$, this last theorem coincides with Theorem 1. Furthermore, this bound is tight, even when $a_1 = a_2 = \ldots = a_t = 1$: consider the necklace for which all the beads of the first type come first, then all beads of the second type, then all beads of the third type, and so on.

Except for particular cases, all known proofs of this theorem are topological and use continuous tools. Section 2 is devoted to direct and non-topological proofs for particular cases of these theorems: one on them is the first proof of this type for the case $q = 2$ and $t = 3$.

In Section 3, we give a purely combinatorial[1] and constructive proof for the complete case $q = 2$ (although the topological inspiration remains notable).

It uses a cubical version of Tucker's lemma found by Ky Fan (Theorem 3), involving cubical complexes instead of simplicial complexes. We give a constructive proof of this theorem (such a proof was not known before the present paper; nevertheless, our proof is inspired by the constructive proof given by Prescott and Su for a combinatorial generalization of Tucker's lemma found by Ky Fan, [13]) and show that it suits to the necklace problem. It gives an efficient algorithm for the division of the necklace between two thieves: contrary to the known algorithm, our result needs no approximation or rounding method and no arbitrarily fine triangulation of the cross-polytope. Furthermore, the d-cubes of our cubical complex have a simple interpretation: each d-cube corresponds to the choice of d beads. We will also see that our method gives also an "almost-fair" division of the necklace between two thieves when the number of beads in any type is not necessarily even (2-splittings in the terminology of Alon, Moshkovitz and Safra [3]).

2 Direct proofs

2.1 A direct proof for $t \leq 2$ and any q

This proof was first given in the paper by Epping, Hochstättler and Oertel [7]. This paper deals with different aspects of a paint shop problem: one of the cases corresponds to the Necklace theorem presented as a conjecture (Conjecture 10) by the authors who did not

[1]According to Ziegler [15], a combinatorial proof in a topological context is a proof using no simplicial approximation, no homology, no continuous map.

know the results of Goldberg, West and Alon. For more about the paint shop problem and the necklace, see [11].

If $t = 1$, i.e., if the beads are only of one type, the solution is straightforward: split the necklace in q parts of the same size, which needs $q - 1$ cuts (and has complexity $O(q)$).

If $t = 2$: the proof works by induction on q. For $q = 1$, there is nothing to prove. Let us now suppose that $q \geq 2$. We (virtually) split the necklace into q disjoint sub-necklaces of $a_1 + a_2$ consecutive beads; the necklace is open and horizontal; the beads get the numbers 1 to n, from left to right.

If one of these q sub-necklaces contains a_1 beads of type 1 and a_2 beads of type 2, we remove this sub-necklace, and give it to one of the thieves, and by induction, we know that there is a fair splitting of the remaining in $2(q-2)$ cuts; all together, this gives $2+2(q-2) = 2(q-1)$ cuts.

Hence, we can assume that there is no such sub-necklace among the q sub-necklaces. Thus, there is at least one sub-necklace which contains strictly more than a_1 beads of type 1, and another which has strictly less than a_1 beads of type 1. Let's call the first sub-necklace M^+ and the second one M^-.

If we slide a window of size $a_1 + a_2$ over the necklace, starting at M^- until we reach M^+ using moves of exactly one bead, the window will contain a_1 beads of type 1 at least once. The $a_1 + a_2$ beads of the window can be given to one of the thieves, and then by induction, as before, $2 + 2(q-2) = 2(q-1)$ cuts are sufficient.

The time complexity can be computed as follows: each bead receives the number of beads of each of the two types preceding it on the necklace (the bead itself included), which can be done in $O(n)$ operations: so, when we consider a sub-necklace, we know immediately how many beads of each type are in this sub-necklace; a dichotomy allows us to find a sub-necklace having a_1 beads of type 1 and a_2 beads of type 2 in $O(\log n)$. The total complexity is thus: $O(n + q \log n)$.

2.2 A direct proof for the case $a_1 = a_2 = \ldots = a_t = 1$, any t and any q

We consider the beads one after the other; from left to right. We always have a "current thief". When considering the kth bead, we check if the current thief has a bead of this type or not.

If he does: we change the current thief and take a thief not having a bead of this type, we give the bead to this new current thief and go to the next bead.

If he does not: we give the bead to the current thief and go to the next bead.

This procedure ensures a division in at most $t(q-1)$ cuts: indeed, we never change the current thief when we meet the first bead of a given type. A change of the current thief is precisely a cut of the necklace. The number of cuts is thus less than or equal to n, the number of all beads, minus t, the number of first meetings of a type of beads: in total: $n - t = qt - t = t(q-1)$.

Moreover, this algorithm is greedy and has complexity $O(n)$.

2.3 A direct proof for $t = 3$ and $q = 2$

2.3.1 Main steps

The necklace is identified with the interval $]0, n[\subset \mathbb{R}$. The beads are numbered with the integers $1, 2, \ldots, n$, from left to right. The kth bead occupies uniformly the interval $]k-1, k[$.

For $S \subseteq]0, n[$, we denote by $\phi_i(S)$ the quantity of beads of type i in positions included in S. More precisely, we denote by $\mathbf{1}_i(u)$ the mapping which indicates if there is a bead of type i at point u of $]0, n[$:

$$\mathbf{1}_i(u) = \begin{cases} 1 \text{ if the } \lceil u \rceil \text{th bead is of type } i \\ 0 \text{ if not.} \end{cases}$$

We have then

$$\phi_i(S) := \int_S \mathbf{1}_i(u) du.$$

If there is a window of $n/2$ beads inducing a fair division, there is nothing to prove. Thus, we can assume that there is no x such that $\phi_i(]x, x + n/2[) = a_i$ for all $i = 1, 2, 3$. Moreover, we can also assume that $\phi_1(]0, n/2[) > a_1$. These two assumptions are called *the starting assumptions.*

First, we unite the types 2 and 3 in a single type, called type 2'. Hence, we define $\phi_{2'} := \phi_2 + \phi_3$.

Then, we build a graph G, whose vertices are triples (c_1, c_2, c_3), such that

- c_1, c_2 and c_3 are integers,
- $0 \leq c_1 \leq c_2 \leq c_3 \leq n$,
- $\phi_1(]0, c_1[\cup]c_2, c_3[) = a_1$ and $\phi_{2'}(]0, c_1[\cup]c_2, c_3[) = a_2 + a_3$.

Distinct triples (c_1, c_2, c_3) and (d_1, d_2, d_3) are neighbors in G if $|c_i - d_i| \leq 1$ for all $i = 1, 2, 3$.

Vertices of G correspond thus to cuts of the necklace giving the same number of beads to each thief, and dividing fairly the beads of type 1. Without loss of generality, we decide that Alice gets the part $]0, c_1[\cup]c_2, c_3[$ and Bob the remaining.

Remark that the starting assumption implies that there is no vertex of the form $(0, c_2, n)$. In the next paragraph, the following lemma is proved:

Lemma 2.1 *The following holds under the starting assumptions:*

(i) *The vertices (c_1, c_2, c_3) of odd degree have either $c_1 = 0$ or $c_3 = n$.*

(ii) *The number of vertices of odd degree such that $c_1 = 0$ id odd and equals the number of vertices of odd degree such that $c_3 = n$.*

The end of the proof is then straightforward. We split the set of vertices of odd degrees of G into two disjoint subsets A and B: A is the set of vertices of odd degrees such that $\phi_3(]0, c_1[\cup]c_2, c_3[) < a_3$ and B the set of vertices of odd degrees such that $\phi_3(]0, c_1[\cup]c_2, c_3[) > a_3$ (according to the starting assumptions, there is no vertex (c_1, c_2, c_3) of odd degree such that $\phi_3(]0, c_1[\cup]c_2, c_3[) = a_3$). Hence $|A| = |B|$ and this number is odd. This implies that there is a path in G that links a vertex in A to a vertex in B. ϕ_1 and $\phi_{2'} = \phi_2 + \phi_3$ are constant on G. Moreover, the value of ϕ_3 for two adjacent vertices differs by at most one. Hence, we necessarily have a vertex on this path such that $\phi_1 = a_1$, $\phi_2 = a_2$ and $\phi_3 = a_3$.

2.3.2 Proof of Lemma 2.1 ($t = 3$ and $q = 2$)

In the proof, we distinguish the vertices (c_1, c_2, c_3) according to the type of beads adjacent to the point where the necklace is cut. Below, the first | represents c_1, the second one c_2 and the third one c_3. The numbers on both sides of the jth line | show the type of the bead which is on the left side of point c_j and the type on the right side of point c_j. For instance:

$$\underbrace{\ldots 1}_{\text{Alice}} | \underbrace{2' \ldots 2'}_{\text{Bob}} | \underbrace{1 \ldots 1}_{\text{Alice}} | \underbrace{2' \ldots}_{\text{Bob}}.$$

This vertex is of degree 0, because no move of a $|$ can be compensated by another move in order to maintain the number of beads of types 1 and 2' received by each thief. Another example is the following (where $c_1 = 0$):

$$|1\ldots1|1\ldots1|2'\ldots.$$

The degree of such a vertex equals 3. Indeed, there are 3 neighbors:

$$|1\ldots|11\ldots|12'\ldots,$$

$$1|\ldots11|\ldots1|2'\ldots$$

and

$$1|\ldots1|1\ldots|12'\ldots$$

In any of these three cases, the move of a cut is compensated by another move.

Let us compute the degree of a vertex (c_1, c_2, c_3) in general. Three cases have to be distinguished

1. $c_1 = 0$: the vertex is of the form: $|a\ldots b|c\ldots d|e\ldots$ with $a, b, c, d, e \in \{1, 2'\}$. Four subcases have to be distinguished, the checking being easy:
   - If $b \neq e$ and $c = d$, the degree is 1 if $a \neq c$ and 3 if not.
   - If $b \neq e$ and $c \neq d$, the degree is 1 if $b = c$ and 3 if not.
   - If $b = e$ and $c = d$, the degree is 4 if $a = b = c$, 2 if $a \neq b$, and 0 if $a = b$ and $b \neq c$.
   - If $b = e$ and $c \neq d$, the degree is 2.
2. $c_3 = n$: this case can be treated exactly as the case before.
3. $c_1 > 0$ and $c_3 < n$: we enumerate subcases while avoiding subcases which can be deduced from the others by symmetries or the exchange of 1 and 2': we give only a few subcases, for the checking is always straightforward:
   - Subcase $\ldots1|1\ldots1|1\ldots1|1\ldots$: the degree equals 6.
   - Subcase $\ldots1|1\ldots1|1\ldots1|2'\ldots$: the degree equals 4.
   - Subcase $\ldots1|1\ldots1|1\ldots2'|1\ldots$: the degree equals 4.
   - Subcase $\ldots1|1\ldots1|1\ldots2'|2'\ldots$: the degree equals 2. Indeed, there are exactly two neighbors: $\ldots11|\ldots11|\ldots2'|2'\ldots$ and $\ldots|11\ldots|11\ldots2'|2'\ldots$.
   - Subcase $\ldots1|1\ldots1|2'\ldots2'|1\ldots$: the degree equals 2.
   - Subcase $\ldots1|1\ldots2'|1\ldots1|2'\ldots$: the degree equals 2.
   - Subcase $\ldots1|1\ldots1|2'\ldots2'|2'\ldots$: the degree equals 2. Indeed, there are exactly two neighbors: $\ldots1|1\ldots12'|\ldots2'2'|\ldots$ et $\ldots|11\ldots|12'\ldots2'|2'\ldots$.
   - Subcase $\ldots1|1\ldots2'|1\ldots2'|2'\ldots$: the degree equals 2.
   - Subcase $\ldots1|2'\ldots2'|1\ldots1|2'\ldots$: the degree equals 0.
   - Subcase $\ldots1|2'\ldots1|2'\ldots2'|1\ldots$: the degree equals 4.

   All degrees are even.

It follows that a vertex of odd degree is necessarily such that $c_1 = 0$ or $c_3 = n$. This proves point (i) of Lemma 2.1. Note that a vertex has an odd degree if and only if $b \neq e$ (with the notations defined in the first case above).

We prove now that: *There is an odd number of vertices of odd degree such that* $c_1 = 0$.

Let $\epsilon > 0$ be a very small real. We first show that a vertex has an odd degree if the following holds: one of the intervals $]c_2 - 1, c_3 - 1 + \epsilon[$ and $]c_2, c_3 + \epsilon[$ has a proportion of beads of type 1 strictly smaller than $\frac{a_1}{a_1+a_2+a_3}$ and the other has a proportion of beads of type 1 strictly greater than $\frac{a_1}{a_1+a_2+a_3}$. Formally, we are going to show that the odd vertices $(0, c_2, c_3)$ are such that

$$\frac{\phi_1(]c_2 - 1, c_3 - 1 + \epsilon[)}{a_1 + a_2 + a_3 + \epsilon} - \frac{a_1}{a_1 + a_2 + a_3}$$

and

$$\frac{\phi_1(]c_2, c_3 + \epsilon[)}{a_1 + a_2 + a_3 + \epsilon} - \frac{a_1}{a_1 + a_2 + a_3}$$

have opposite signs.

This two quantities are respectively equal to

$$\frac{a_1 + \mathbf{1}_{\{b=1\}} - (1 - \epsilon)\mathbf{1}_{\{d=1\}}}{a_1 + a_2 + a_3 + \epsilon} - \frac{a_1}{a_1 + a_2 + a_3}$$

and

$$\frac{a_1 + \epsilon\mathbf{1}_{\{e=1\}}}{a_1 + a_2 + a_3 + \epsilon} - \frac{a_1}{a_1 + a_2 + a_3},$$

where b, d, and e are beads in positions specified in the first case above, and $\mathbf{1}_P$ takes the value 1 (resp. 0) if P is true (resp. false). It is easy to see that these two quantities have opposite signs if and only if $b \neq e$. Hence they have opposite signs if and only if the vertex has odd degree.

We show now that there is an odd number of such vertices. With the same kind of arguments as in Subsection 2.1, i.e., with the interval $]c_2, c_3 + \epsilon[$ moving from left to right, we see that there is an odd number of such situations, for the sign of

$$\frac{\phi_1(]u, u + a_1 + a_2 + a_3 + \epsilon[)}{a_1 + a_2 + a_3 + \epsilon} - \frac{a_1}{a_1 + a_2 + a_3}$$

changes strictly when the interval $]u, u + a_1 + a_2 + a_3 + \epsilon[$ moves from $]0, a_1 + a_2 + a_3 + \epsilon[$ to $]a_1 + a_2 + a_3, n + \epsilon[$ (this is a true for $\epsilon = 0$, according to the starting assumptions, and thus for all $\epsilon > 0$ small enough). Hence, there is an odd number of vertices of odd degree such that $c_1 = 0$.

By symmetry, there is also an odd number of vertices of odd degree having $c_3 = n$. Furthermore, we have the following equivalence, since the condition $b \neq e$ is the same for the vertices $(0, c_2, c_3)$ and (c_2, c_3, n):

$$\deg(0, c_2, c_3) \text{ odd} \Leftrightarrow \deg(c_2, c_3, n) \text{ odd}.$$

The lemma is proved. ∎

2.3.3 Final comments

Let's give an illustration of this proof for the example of Figure 1.

The corresponding graph G is illustrated in Figure 3. There is a path (actually, many) linking vertex $(0, 4, 10)$, which gives more beads of type 3 to Alice, to vertex $(4, 10, 12)$,

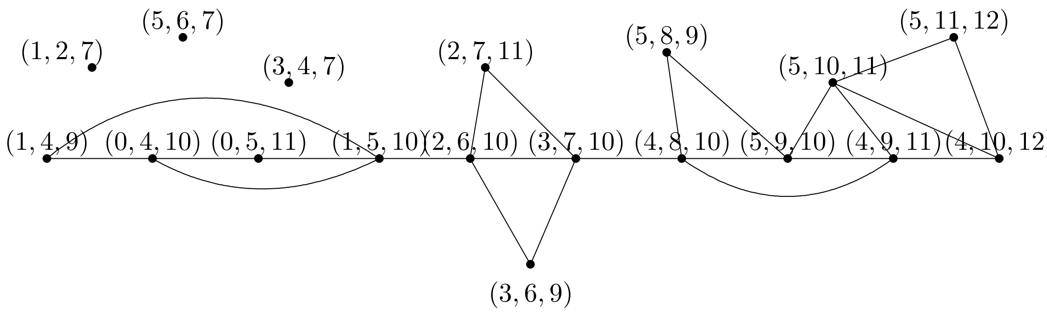

Figure 3: Graph G corresponding to the example (with $t = 3$ and $q = 2$) of Figure 1.

which gives more beads of type 3 to Bob. The vertices $(1, 5, 10)$ and $(2, 6, 10)$ lie on this path and provide fair divisions.

On the other hand, it is not clear whether it is possible to derive from this proof an efficient algorithm that would have a better complexity than $O(n^3)$ (G might be not explicitly computed).

Furthermore, the method is tedious. Is it possible to find some simplification?

3 Ky Fan's cubical theorem and constructive proof for two thieves

3.1 Preliminaries

We give now a constructive and combinatorial proof of Theorem 1. We could think that such a proof could be easily deduced from Tucker's lemma. Indeed, Theorem 1 is a direct application of Borsuk-Ulam's theorem, which has a constructive proof through Tucker's lemma. This is done for instance by Simmons and Su in [14], but the result is not completely satisfactory because this method requires a triangulation T of the $(t+1)$-dimensional cross-polytope such that the diameter of any simplex in T is a $O(1/(nt))$, and then it requires a rounding method.

Our method does not present those difficulties. It bases itself on a theorem (Theorem 3) found by Ky Fan in 1960 which is a cubical version of Tucker's lemma [8]. We give below a constructive proof of this theorem. The cubical complex we need is completely natural.

3.2 Notation and definitions

Let C be a cube, ∂C the boundary of C, and $V(C)$ the set of vertices of C.Two facets of a d-cube are said to be *adjacent* if their intersection is a $(d-2)$-cube. For a given facet σ of C, there is only one non-adjacent facet. Two non-adjacent facets are *opposite*.

A *cubical complex* is a collection C of cubes embedded in an euclidian space such that

- if $\tau \in \mathsf{C}$ and if σ is a face of τ then $\sigma \in \mathsf{C}$ and
- if σ and σ' are in C, then $\sigma \cap \sigma'$ is a face of both σ and σ'.

A cubical complex is said to be *pure* if all maximal cubes with respect to inclusion have same dimension.

$V(\mathsf{C})$ is the set of vertices of the cubical complex C.

A *d-dimensional cubical pseudo-manifold* M is a pure d-dimensional cubical complex in which each $(d-1)$-dimensional cube is contained in at most two d-dimensional cubes. The *boundary* of M, denoted by $\partial\mathsf{M}$, is the set of all $(d-1)$-cubes σ such that σ belongs to exactly one d-dimensional cube of M.

Let us denote by C^d the cubical complex which is the union of the cube $\square^d := [-1,1]^d$ and all its faces (see Figure 6).

Let $k \geq 0$ be an integer. Given k distinct integers $i_1, i_2, \ldots, i_k$, each of them in $\{1, 2, \ldots, d\}$, and k numbers $\epsilon_1, \epsilon_2, \ldots, \epsilon_k$, each of them belonging to $\{-1, +1\}$, we denote

$$F\begin{pmatrix} i_1 & i_2 & \ldots & i_k \\ \epsilon_1 & \epsilon_2 & \ldots & \epsilon_k \end{pmatrix} := \{(x_1, x_2, \ldots, x_d) \in \square^d : x_{i_j} = \epsilon_j \text{ for } j = 1, 2, \ldots, k\},$$

which is a $(d-k)$-face of $\square^d$. For $k = 0$, this face is $\square^d$ itself.

We endow $\partial\square^d$ with *hemispheres.* These hemispheres are denoted $+H_0$, $-H_0$, $+H_1$, $-H_1$, …, $+H_{d-1}$, $-H_{d-1}$. $+H_i$ (resp. $-H_i$) is the set of points $(x_1, x_2, \ldots, x_d)$ of $\partial\square^d$ such that $x_{i+1} \geq 0$ (resp. $x_{i+1} \leq 0$), and such that, if $i \leq d-2$, then $x_j = 0$ for $j > i+1$ (in Figure 4, the hemispheres of $\partial\square^3$ are highlighted).

The *carrier hemisphere* of a cube is the hemisphere of smallest possible dimension containing it.

Observation 2.1 *For $i \in \{0, 1, \ldots, d-1\}$, H_i is homeomorphic to the ball B^i, $(+H_i) \cup (-H_i)$ is homeomorphic to the i-dimensional sphere S^i, if $i \geq 1$, $\partial(+H_i) = \partial(-H_i) = (+H_{i-1}) \cup (-H_{i-1})$ and $H_{d-1} \cup -H_{d-1} = \partial\square^d$.*

Let C and D be cubical complexes, and let $\lambda : V(\mathsf{C}) \to V(\mathsf{D})$ be a map. λ is called a *cubical map* if λ satisfies the following conditions:

1. for every $\sigma \in \mathsf{C}$, there is a $\tau \in \mathsf{D}$ such that $\lambda(V(\sigma)) \subseteq V(\tau)$
2. λ takes adjacent vertices to adjacent vertices or both to the same vertex.

By a slight abuse of notation, we write $\lambda : \mathsf{C} \to \mathsf{D}$. We emphasize the fact that $\lambda(V(\sigma))$ in the definition above is not necessarily the vertex set of a cube.

According to Lemma 18 of the paper of Ehrenborg and Hetyei [6], if $\lambda : V(\square^n) \to V(\square^n)$ is a non-injective cubical map and if ρ is a facet of $\square^n$, then there are 0, 2 or 4 facets σ of $\square^n$ such that $\lambda(V(\sigma)) = V(\rho)$. It implies the following lemma:

Lemma 2.2 *Let $\lambda : V(\square^d) \to V(\square^m)$ be a cubical map (from C^d into C^m), with $m \geq d$, and let ρ be a $(d-1)$-face of $\square^m$. If λ is not injective, there are 0, 2 or 4 facets σ of $\square^d$ such that $\lambda(V(\sigma)) = V(\rho)$. Moreover, if there are 4 such facets and if σ is one of them, then the facet opposite to σ is also one of these 4 facets.*

Proof: Let $\lambda : V(\square^d) \to V(\square^m)$ be a non-injective cubical map and let ρ be a $(d-1)$-face of $\square^m$. If there is no facet σ such that $\lambda(V(\sigma)) = V(\rho)$, there is nothing to prove. So let us assume that there is a facet σ of $\square^d$ such that $\lambda(V(\sigma)) = V(\rho)$. We want to apply the result of Lemma 18 of [6], which is valid if $\lambda : V(\square^d) \to V(\square^d)$. Let us prove that we can make this assumption, that is, that there is a d-face of $\square^m$ which contains the image of λ.

- If $\lambda(V(\square^d)) = V(\rho)$, any d-face containing ρ contains the image of λ.
- If not, there is $v \in V(\square^d) \setminus V(\sigma)$ such that $\lambda(v) \notin V(\rho)$. Let w be the vertex of σ which is adjacent to v, and let w' be the single vertex of σ such that $d(w, w') = d-1$, where $d(\cdot, \cdot)$ is the distance in the 1-skeleton of a cube, counted in number of edges. It

is straightforward to check that $d(\lambda(w'),\lambda(v)) = d$ (take the minimal face containing both ρ and $\lambda(v)$). Let v' be any vertex of $\square^d$, we have the following chain of inequalities:

$$d = d(v,w') = d(v',v)+d(v',w') \geq d(\lambda(v'),\lambda(v))+d(\lambda(v'),\lambda(w')) \geq d(\lambda(v),\lambda(w')) = d.$$

Hence, $d(v',v) = d(\lambda(v'),\lambda(v))$ and $d(v',w') = d(\lambda(v'),\lambda(w'))$. It implies that $\lambda(v')$, for any v' in $\square^d$, is in the d-cube of $\square^m$ containing ρ and $\lambda(v)$.

We can thus assume that $\lambda : V(\square^d) \to V(\square^d)$.

If there are 4 facets such that $\lambda(V(\sigma)) = V(\rho)$ (in this case one has necessarily $d \geq 2$), at least two of them, say σ and σ', are adjacent. Let σ'' be another facet adjacent to σ and σ'. Let $x \in V(\sigma' \cap \sigma'') \setminus V(\sigma)$, and let y be the vertex adjacent to x in σ. We have $y \in \sigma \cap \sigma' \cap \sigma''$, and $\lambda(x)$ adjacent to $\lambda(y)$. Moreover, let z be the vertex of σ such that $\lambda(z) = \lambda(x)$ (which exists by injectivity). z is adjacent to y in σ and cannot be in σ'. Hence z is the only vertex of σ which is at distance 1 of y and not in σ'. It implies that z is in σ''. As $\lambda(z) = \lambda(x)$ and as z and x are two different vertices of σ'', $\lambda(V(\sigma'')) \neq V(\rho)$. A facet σ'' adjacent to σ and to σ' cannot satisfy $\lambda(V(\sigma'')) = V(\rho)$. ∎

3.3 Ky Fan's cubical formula

Suppose that we have a cubical map λ of M, a t-dimensional cubical pseudo-manifold, into C^m, the cubical complex made by $\square^m$ and its faces.

We denote by $\alpha_\lambda\begin{pmatrix} i_1 & i_2 & \dots & i_{m-t} \\ \epsilon_1 & \epsilon_2 & \dots & \epsilon_{m-t}\end{pmatrix}$ the number of those t-cubes σ of M such that the vertices of σ are mapped under λ in a one-to-one way onto the vertices of the t-face $F\begin{pmatrix} i_1 & i_2 & \dots & i_{m-t} \\ \epsilon_1 & \epsilon_2 & \dots & \epsilon_{m-t}\end{pmatrix}$ (if $m = t$, this face is $\square^m$ itself).

We will later make use of the following theorem (for the case $m = t$), whose constructive proof is given below.

Theorem 3 *Let M be the cubical pseudo-manifold without boundary obtained by subdividing $\partial\mathsf{C}^{t+1}$ by means of the hyperplanes $x_i = \frac{j}{n}$ where $i \in \{1,2,\dots,t+1\}$ and $j \in \{-n+1,-n+2,\dots,-1,0,1,\dots,n-2,n-1\}$. Let λ be a cubical map from M into C^m, with $m \geq t$. If λ is antipodal (i.e. if $\lambda(-x) = -\lambda(x)$), then we have*

- *if $m > t$,*

$$\sum_{\epsilon_2,\dots,\epsilon_{m-t}=-1,+1} \alpha_\lambda\begin{pmatrix} t+1 & t+2 & \dots & m \\ +1 & \epsilon_2 & \dots & \epsilon_{m-t}\end{pmatrix} = 1 \bmod 2.$$

- *if $m = t$, there is a t-cube σ of M such that $\lambda(V(\sigma)) = V(\mathsf{C}^m)$.*

For $t = 2$ and $n = 2$, M is illustrated in Figure 4.

3.4 Constructive proof of Theorem 3

We follow roughly the method found by Prescott and Su [13] for proving a similar result concerning simplicial complexes.

- A d-cube of M (with $0 \leq d \leq t$) is said to be *full* if its image under λ is of the form

$$F\begin{pmatrix} d+1 & d+2 & \dots & m \\ \epsilon_1 & \epsilon_2 & \dots & \epsilon_{m-d}\end{pmatrix}.$$

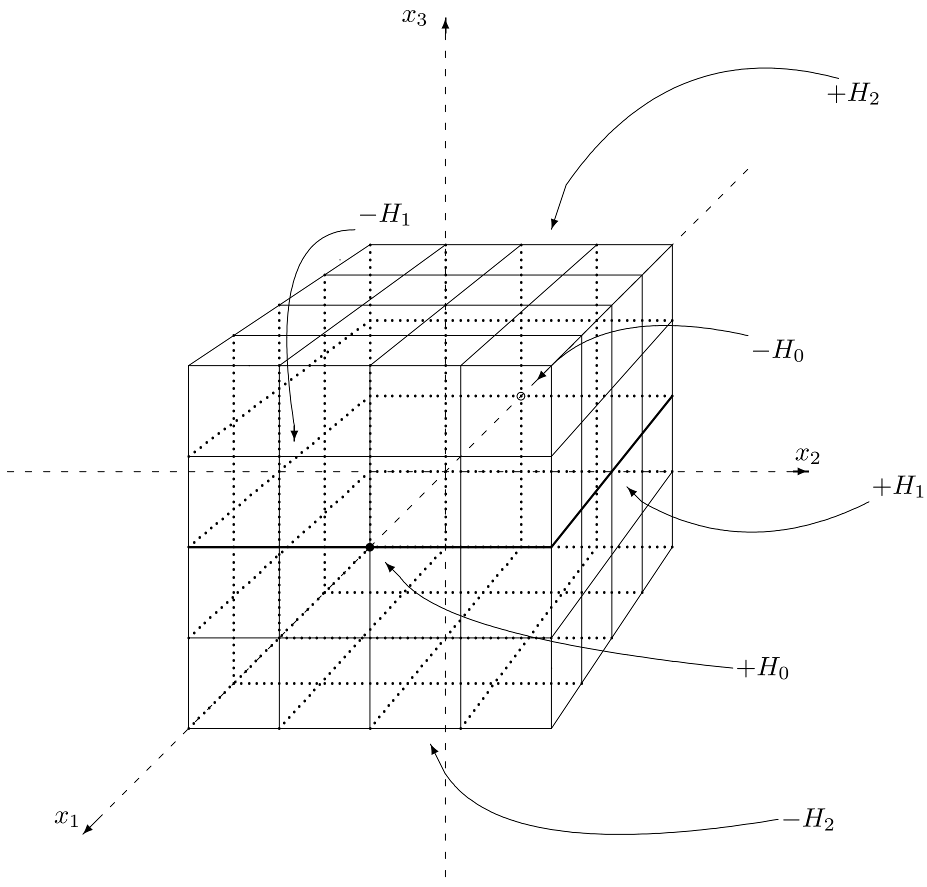

Figure 4: An illustration of M for $n = 2$ and $t = 2$.

- A d-cube of M is said to be *almost-full* if it is not full but one of its facets is full.

Except in the case when $d = m = t$, one can define the *sign* of a full d-cube as the value of ϵ_1 in the expressions above.

Lemma 3.1 *For an almost-full d-cube τ, there are only three possibilities:*

1. *either its vertices are mapped under λ in a one-to-one way onto the vertices of a d-face (with the entries j and ϵ_{j-d+1} deleted)*

$$F \begin{pmatrix} d & \dots & \hat{j} & \dots & m \\ \epsilon_1 & \dots & \hat{\epsilon}_{j-d+1} & \dots & \epsilon_{m-d+1} \end{pmatrix},$$

2. *or λ is not injective on the vertex set of τ and τ has exactly two full facets, each of them having the same sign,*

3. *or λ is not injective on the vertex set of τ and τ has exactly four full facets, each of them having the same sign. Moreover, in this case, if $\sigma \subseteq \tau$ is a full facet, then the opposite of σ is also a full facet.*

Proof: First remark that if λ is injective, the image of τ is a d-face (this easy assertion can be proved by induction). Since τ is not full, this image has necessarily the form given in point 1.

Hence, one can assume that λ is not injective. Take two full facets of τ, say σ and σ'. Since they are full, λ is injective on each of them. $\lambda(V(\sigma))$ and $\lambda(V(\sigma'))$ are the vertex sets of $(d-1)$-faces. They have a non-empty intersection, otherwise λ would have been injective on τ. Moreover, according to the definition of a full cube, the image of two full cubes of the same dimension are parallel. But, two parallel faces of a cube having same dimension and a nonempty intersection are equal. Thus, $\lambda(V(\sigma)) = \lambda(V(\sigma'))$, and there is a $(d-1)$-face ρ such that for all full facets σ of τ, one has $\lambda(V(\sigma)) = V(\rho)$. This settles the statement about the sign. Lemma 2.2 allows to conclude. ∎

We can then extend the notion of sign to almost-full cubes: the *sign* of an almost-full d-cube is the sign of any of its full facets.

A cube is said to be *agreeable* if its sign matches the sign of its carrier hemisphere.

We define the following graph G. A cube σ with carrier hemisphere $\pm H_d$ is a node of G if it is one of the following:

(1) an agreeable full $(d-1)$-cube,
(2) an agreeable almost-full d-cube, or
(3) a full d-cube.

Two nodes σ and τ are adjacent in G if all the following hold:

(a) σ is a facet of τ,
(b) σ is full,
(c) τ has the dimension of its carrier hemisphere, and
(d) the sign of the carrier hemisphere of τ matches the sign of σ.

We claim that G is a graph in which every vertex has degree 1, 2, or 4. Furthermore, a vertex has degree 1 if and only if its carrier hemisphere is $\pm H_0$, or if it is a full t-cube. Using antipodality, we will then show that the point H_0 is necessarily linked by a path in G to a full t-cube.

To see why, let's consider the three kinds of nodes in G:

(1) An agreeable full $(d-1)$-cube σ, with carrier hemisphere $\pm H_d$, is a facet of exactly two d-cubes, each of which must be a full or an agreeable almost-full cube in the same carrier.

This satisfies the adjacency conditions (a)–(d) with σ, hence σ has degree 2 in G. The cube σ cannot be adjacent in G to any of its facets, otherwise it would contradict condition (c).

(2) An agreeable almost-full d-cube σ, whose carrier hemisphere is $\pm H_d$, is adjacent in G to its two or four facets that are full $(d-1)$-cubes – see Lemma 3.1. A full facet of σ is automatically a node of G, since if it is not agreeable, then it is carried by $\mp H_{d-1}$ – the sign of the carrier of this full facet is then the opposite of the sign of σ. (Adjacency condition (d) is satisfied because σ is agreeable and because an agreeable almost-full d-cube has always the same sign as its full facets).

(3) Except if $d = 0$, a full d-cube σ with carrier $\pm H_d$ has exactly one facet τ that is full and whose sign is the same as the carrier hemisphere of σ. Hence σ is adjacent to τ in G.

On the other hand, σ is the facet of exactly two $(d+1)$-cubes in $+H_{d+1} \cup -H_{d+1}$: one in $+H_{d+1}$ and one in $-H_{d+1}$, but σ is adjacent in G to exactly one of them: which one is determined by the sign of σ, since adjacency condition (d) must be satisfied.

Finally, σ is of degree 2 or 4 in G unless $d = 0$ or σ is a full t-cube: if $d = 0$, σ is the point $\pm H_0$ and it has no face, hence σ has degree 1; and if σ is a full t-cube, then σ is not the facet of any other cube, and is therefore of degree 1.

Every node in G has degree 2 or 4, except for the points $\pm H_0$ and for the full t-cubes.

We transform G into a new graph G': we replace each node v of degree 4 in G by two nodes v' and v'', and we replace the four edges, by four new edges: two linking v' and a pair of opposite full facets, and two linking v'' and the other pair of opposite full facets.

In G', all vertices are of degree 1 or 2, hence G' is the union of disjoint paths and cycles.

Note that the antipode of a path in G' is also a path in G'. No path can have antipodal endpoints, otherwise it should be its own antipode and have a node or an edge which is antipodal to itself. Hence the number of endpoints is a multiple of four. Since exactly two endpoints are H_0 and $-H_0$, there are twice an odd number of endpoints which are full t-cubes. If $m > t$, half of them have a positive sign (if $m = t$, a full t-cube has no sign). ∎

A path of G' starting in H_0 can not terminate in $-H_0$, but in a full t-cube. Thus we have an algorithm which allows to find a full t-cube. It is illustrated in Figure 5.

3.5 Splitting the necklace between Alice and Bob

We can easily deduce Theorem 1 from Theorem 3. Indeed let $m = t$, and let $x := (x_1, x_2, \ldots, x_{t+1})$ be a vertex of M. We order the x_i in the increasing order of their absolute value. When $|x_i| = |x_{i'}|$, we put x_i before $x_{i'}$ if $i < i'$. This gives a permutation π such that

$$|x_{\pi(1)}| \leq |x_{\pi(2)}| \leq \ldots \leq |x_{\pi(t+1)}| = 1.$$

Then, we define $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_t) : \mathsf{M} \to \mathsf{C}^t$ as follows: we cut the necklace at points $n|x_i|$ (at most t are different from 1). By convention, $x_{\pi(0)} := 0$; we define:

$$\lambda_i(x) := \begin{cases} +1 \text{ if } \sum_{j=0}^{t} \operatorname{sign}(x_{\epsilon(j)})\phi_i\left(\left]n|x_{\pi(j)}|, n|x_{\pi(j+1)}|\right[\right) > 0 \\ -1 \text{ if not,} \end{cases}$$

for $i = 1, 2, \ldots, t$, where $\epsilon(j)$ is the smallest integer k such that $|x_k| \geq |x_{\pi(j+1)}|$, and with the convention $\operatorname{sign}(0) := 0$.

This formula can be understood as follows: λ_i equals $+1$ (resp. -1) if the first thief gets strictly more (resp. strictly less) beads of type i than the second one, when we cut at points $n|x_j|$ of the necklace, and when the sign of x_k shows the thief who gets the $(j+1)$th sub-necklace, where k is the smallest integer such that $n|x_k|$ is greater than or equal to the coordinate of any point of the sub-necklace.

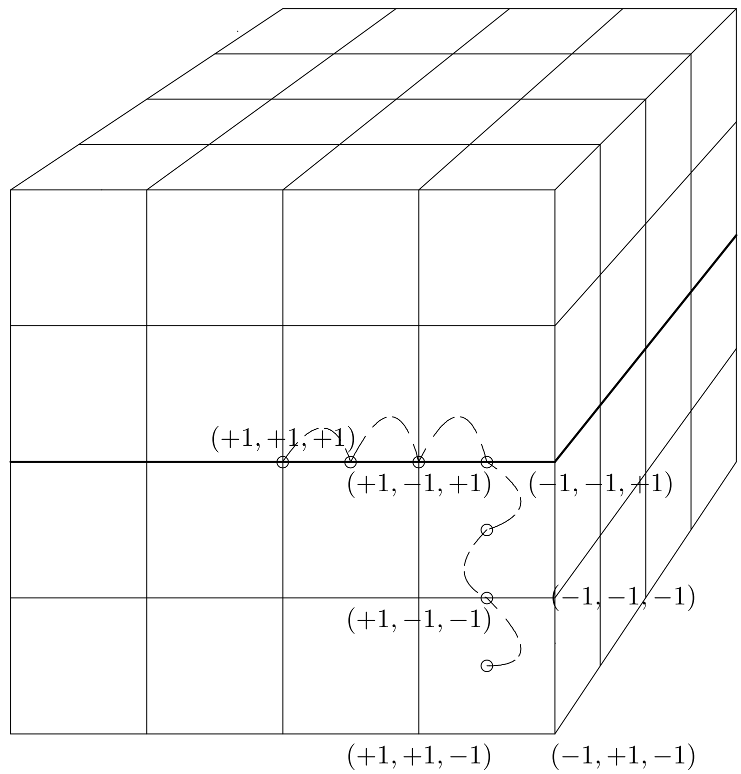

Figure 5: Illustration of the constructive proof of Theorem 3. The triples indicate the corresponding label on the cube image of the Figure 6.

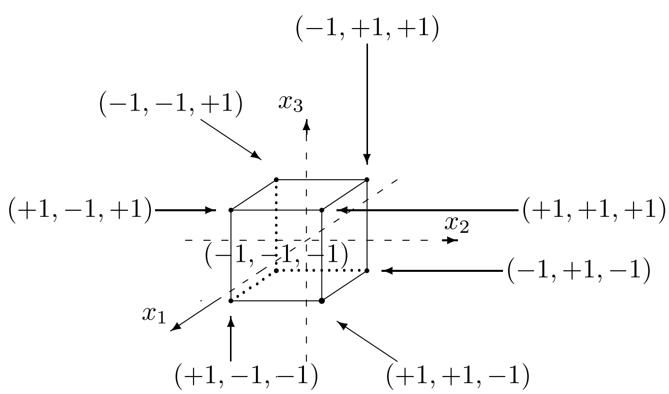

Figure 6: An illustration of C^m for $m = 3$.

The only problem with this definition is that λ may not be antipodal: if

$$\sum_{j=0}^{t} \operatorname{sign}(x_{\epsilon(j)})\phi_i\big(\big]n|x_{\pi(j)}|, n|x_{\pi(j+1)}|\big[\big) = 0,$$

then $\lambda_i(x) = \lambda_i(-x) = -1$. This can be avoided by a slight generic perturbation of the value of all the beads: a division using cuts between beads can then not be fair with respect to any type of bead[2].

λ is a cubical map. Indeed, the first condition defining cubical maps is trivially satisfied and the second one follows from the fact the images by λ of two adjacent vertices of M have at most one coordinate that differs – even in the case when there is j such that $\epsilon(j)$ differs. Theorem 3 shows that there is a t-cube σ whose image by λ is $\square^t$. One of the vertices of this t-cube corresponds to a fair division. Indeed, for any vertex x of such a σ, moving to each of the t adjacent vertices of x changes the value of each of the t components of $\lambda(x)$.

The constructive proof given above for Theorem 3 suits to the necklace problem. The cubical complex is completely natural in this context: a k-cube ($k \leq t$) corresponds precisely to the choice of k beads. Moreover, it is straightforward to check if a k-cube is full in terms of beads. We have thus a constructive proof of Theorem 1.

3.6 General discrete necklace splitting theorem

In [3], Alon, Moshkovitz and Safra prove the following extension of Theorem 2 (with flows).

Suppose that the necklace has n beads, each of a certain type i, where $1 \leq i \leq t$. Suppose there are A_i beads of type i, $1 \leq i \leq t$, $\sum_{i=1}^{t} A_i = n$, where A_i is not necessarily a multiple of q. A *q-splitting* of the necklace is a partition of it into q parts, each consisting of a finite number of non-overlapping sub-necklaces of beads whose union captures either $\lfloor A_i/q \rfloor$ or $\lceil A_i/q \rceil$ beads of type i, for every $1 \leq i \leq t$.

Theorem 4 *Every necklace with A_i beads of type i, $1 \leq i \leq t$, has a q-splitting requiring at most $t(q-1)$ cuts.*

For $q = 2$, the proof of the preceding section shows something more. The proof is still valid, even if the A_i's are not even. If A_i is odd, the $(t-1)$-facet of $\square^t$ whose ith coordinate is equal to 1 (resp. -1). correspond to division giving $\lceil A_i/2 \rceil$ beads of type i to Alice (resp. Bob) and $\lfloor A_i/2 \rfloor$ beads of type i to Bob (resp. Alice). We have thus the following theorem, also with a constructive proof:

Theorem 5 *Every necklace with A_i beads of type i, $1 \leq i \leq t$, has a 2-splitting requiring at most t cuts. Moreover, for each type i of beads such that A_i is odd, we can choose which thief gets $\lceil A_i/2 \rceil$ beads of type i.*

Note that this theorem can also be derived from the proof of Alon, Moshkovitz and Safra.

4 Open questions

There are a lot of questions that remain open.

1. Is it possible to extend the direct proofs of Section 2.1 to other cases?

[2] If one does not want to use a perturbation, there is another way for avoiding 0: choose a *particular* bead of each type and define $\lambda_i(x) := +1$ (resp. -1) if the first thief gets strictly more (resp. strictly less) beads of type i than the second thief, or if the first thief and the second one get the same number of beads of type i and the first one gets (resp. does not get) the particular bead of type i.

2. Is there a combinatorial or a constructive proof of Theorem 2? This question appears for instance in the paper of Alon [2].

3. What is the complexity of the following problem?

   INPUT: A necklace, with t beads, qa_i beads of each type (a_i is an integer), and q thieves,

   TASK: Find a fair division of the necklace using no more than $t(q-1)$ cuts,

   This question is still open (for more about problems of this type, see Papadimitriou [12]). The corresponding optimization problem, namely finding the minimum number of cuts providing a fair division, is NP-hard, even if $q = 2$ and $a_1 = a_2 = \ldots = a_t = 1$ (see [5], [11]).

4. Is there an analogue of Theorem 5 for any number q of thieves?

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