L. Lovász: "Graphs and Geometry". AMS, 2019, 444 pp[1]

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Graphs are an important subject in mathematics and can be found in various mathematical branches, e.g., combinatorics, probability, topology, etc. Graphs, as a way to model networks or interactions between entities, play also an important role in many other areas of science. They form one of the central objects of this book, and each chapter relies in a way or the other to them. The main purpose of the book is to illustrate how geometry can be used to explore their combinatorial properties, and how, in return, graphs can provide useful concepts to address geometric questions. It has been written by a major contributor of this fascinating topic, and is thus a unique opportunity to discover this latter.

The present review aims at conveying some of the ideas and results of this book. As we will see in Section 1, one idea is that traditional notions of matchings and vertex covers in graphs can be extended to vector-labeled graphs in a very fruitful way. This section, which actually presents only results from one chapter (among twenty), somehow illustrates the general spirit of the book: nice mathematical results, which are beneficial to other areas (here, to pure graph theory and to physics). A more central message of the book is that combinatorial properties of graphs can be found or studied via "graph representations," which are well-chosen labelings of the vertices with vectors. Section 2 gathers a few results and methods around this theme. Another interesting aspect of the book is the place of physics, and this will be already clear from the first two sections. Two extra applications to this discipline are described in Section 3. A few complementary comments are given in Section 4.

1. A first glimpse of the book: matchings and covers in frameworks

A (simple) graph G is a pair (V, E) of finite sets where the elements of E are pairs of elements of V. The elements of V are the *vertices* and the elements of E are the *edges*. The common way to visualize a graph is to identify each vertex with a distinct point of the plane and each edge with a line connecting its two vertices. In the sequel, n will always be the number of vertices of G.

Matchings and vertex covers are fundamental notions related to graphs. A *matching* is a set of pairwise disjoint edges. The maximum cardinality of a matching, the *matching number* of G, is denoted by $\nu(G)$. A *vertex cover* is a set of vertices meeting all edges. The minimum cardinality of a vertex cover, the *vertex cover number* of G, is denoted by $\tau(G)$. It is immediate that the inequality $\nu(G) \leqslant \tau(G)$ always holds. It is also clear that this inequality can be strict. Yet, there is an important family of graphs for which this inequality is actually an equality: bipartite graphs. A graph is *bipartite* when it is

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possible to partition the vertices into two parts such that every edge has a vertex in each part. The equality $\nu(G) = \tau(G)$ when G is bipartite is a theorem of Kőnig from the early 20th century and is a fundamental result in combinatorial optimization.

One of the main objects studied in the book is that of a framework. A *framework* is a pair $(G, \boldsymbol{u})$, where G is a graph and $\boldsymbol{u}$ is a collection $(\boldsymbol{u}_v)_{v \in V}$ of vectors of $\mathbb{R}^d$ labeling the vertices of G. A *matching* of a framework is a set of pairwise disjoint edges whose vertices are labeled by linearly independent vectors. We denote by $\nu(G, \boldsymbol{u})$ the maximum cardinality of a matching and by $\tau(G, \boldsymbol{u})$ the minimum rank of a vertex cover (the *rank* of a set $S \subseteq V$ being the rank of $\{\boldsymbol{u}_v \colon v \in S\}$). Note that when we label G with the standard unit vectors of $\mathbb{R}^n$, these quantities coincide with the usual matching and vertex cover numbers. The inequality between these latter extends to frameworks. The following extension of Kőnig's theorem is maybe more surprising.

Theorem 1 (Chapter 18, Theorem 18.14)**.** *If G is a bipartite graph, then $\nu(G, \boldsymbol{u}) = \tau(G, \boldsymbol{u})$.*

Further results about matchings and vertex covers in frameworks when the graph is not bipartite, generalizing classical results for non-labeled graphs, can also be established. A striking application is the following theorem found by Milić [9] about electrical networks. (A *gyrator* is a two-port electrical device that relates in an opposite way the current and the voltage of the two ports.)

Theorem 2 (Chapter 18, Theorem 18.19)**.** *Let G be an electrical network of voltage and current sources, resistances, and gyrators. Suppose that the resistances and the parameters of the gyrators are algebraically independent over the rationals. Then there is a unique assignment of current and voltage differences to the edges if and only if G has a spanning tree which contains all voltage sources, no current source, and either both or none of the edges of each gyrator.*

Basic computations of multilinear algebra establish a formula leading to an efficient randomized algorithm for computing $\nu(G, \boldsymbol{u})$, which in turns can be used for deciding the existence of the spanning tree of Theorem 2.

Vertex covers in frameworks is a key tool for understanding a fundamental property of usual (non-labeled) graphs, namely the property of being cover-critical. A graph is *cover-critical* if removing any edge from it decreases the vertex cover number. A cover-critical graph is *basic* if every vertex has degree at least three. (The *degree* of a vertex v, denoted by $\deg(v)$, is the number of edges containing that vertex.) It is not too difficult to see that all cover-critical graphs can be described from the basic ones. The *covering defect* of a cover-critical graph is the quantity $2\tau(G) - n$. The following theorem is due to Lovász [6].

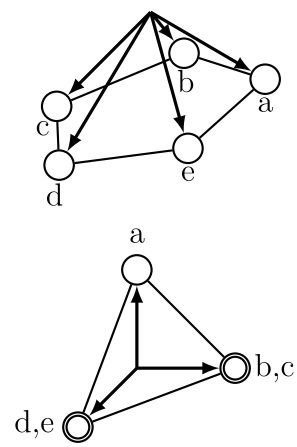
Theorem 3 (Chapter 18, Theorem 18.3)**.** *The number of basic cover-critical graphs with fixed covering defect is finite.*

The only known proof of this deep result relies on vertex cover numbers of frameworks. Roughly speaking, intermediary lemmas stating properties on vertex cover numbers of frameworks are established via inductive arguments that fail to work when restricted to usual vertex cover numbers.

2. Representations of graphs

Frameworks—defined as graphs whose vertices are labeled with vectors—form the main object discussed in the previous section. They can arise quite naturally from applications from physics. Theorem 2 is obtained this way. They can also come as an attempt of generalizing standard notions. Generalizations are sometimes fruitful, and Theorem 3 is a typical example. But there is another way to get frameworks, and this is via graph representations. A representation of a graph is a labeling of its vertices with vectors that captures some of its combinatorial properties. We present hereafter three examples from the book.

2.1. **Orthogonal representations.** An *orthogonal representation* of a graph is a labeling of its vertices with vectors of $\mathbb{R}^d$ such that non-adjacent vertices (i.e., pair of vertices not forming an edge) are labeled with orthogonal vectors. The adjacent figure show two orthogonal representations of the 5-cycle C_5 (graph that can be visualized as a pentagon).



Two orthogonal representations of C_5.

The graph C_5 is 2-connected, which means that removing a single vertex does not "disconnect" the graph; see the precise definition of k-connectivity below. The first representation in the figure is in general position (no three vectors are linearly dependent). This is an illustration of a more general phenomenon: quite surprisingly, there exists a relation found by Lovász, Saks, and Schrijver [8] between the dimension of such a representation (in general position) and the connectivity of the graph.

A graph is *connected* if for every pair of vertices v, v', there exists a sequence of edges such that consecutive edges share a vertex, the first edge contains the vertex v, and the last edge contains the vertex v'. A graph is *k-connected* if the removal of at most $k-1$ vertices cannot disconnect the graph.

Theorem 4 (Chapter 10, Theorem 10.9)**.** *A graph has an orthogonal representation in $\mathbb{R}^d$ in general position if and only if it is $(n-d)$-connected.*

The difficult part of the proof is the construction of an orthogonal representation in general position for $(n-d)$-connected graphs. The description of the construction itself is actually easy and provides an efficient randomized algorithm to compute such a representation. What is difficult is the fact that the representation is in general position. The proof uses clever arguments combining graphs, linear algebra, and probabilities.

Orthogonal representations are also useful for defining the following important graph parameter. The *Lovász number* of a graph is the quantity

$$\vartheta(G) = \min_{\boldsymbol{u},\boldsymbol{c}} \max_{v \in V} \frac{1}{(\boldsymbol{c} \cdot \boldsymbol{u}_v)^2},$$

where the minimum is taken over all orthonormal representations $\boldsymbol{u}$ of G (orthogonal representations with unit vectors) and all unit vectors $\boldsymbol{c}$. From duality theory in semidefinite optimization, we can get the following equivalent definition:

$$\vartheta(G) = \max_{\boldsymbol{u}',\boldsymbol{c}'} \sum_{v \in V} (\boldsymbol{c}' \cdot \boldsymbol{u}'_v)^2 \,, \tag{1}$$

where the maximum is taken over all orthonormal representations $\boldsymbol{u}'$ of the complementary graph $\overline{G}$ (graph in which non-adjacent vertices become adjacent, and vice versa), and all unit vectors $\boldsymbol{c}'$.

The Lovász number is sandwiched between two important graph parameters: the *stability number*, denoted by $\alpha(G)$, and the *clique cover number*, denoted by $\chi(\overline{G})$. The first is the maximum number of pairwise non-adjacent vertices; the second is the minimum number of cliques (complete subgraphs) needed to cover all vertices of G. They are NP-hard, which means that the existence of a polynomial time algorithm (i.e., an efficient algorithm) to compute any of them is very unlikely.

Theorem 5 (Chapter 11, Theorem 11.1)**.** *The inequalities $\alpha(G) \leqslant \vartheta(G) \leqslant \chi(\overline{G})$ hold for every graph G.*

The proof of these two inequalities is rather simple. The value of $\vartheta(G)$ is polynomial time computable—a non-trivial fact found by Grötschel, Lovász, and Schrijver [3]—which makes this parameter a useful tool for bounding the values of the stability number and the clique cover number.

The Lovász number is relevant in many other contexts. Its most celebrated application is maybe the determination of the Shannon capacity of C_5 by Lovász [7]. The Shannon capacity is a quantity that has been introduced by Shannon and is related to the asymptotic number of non-confusable messages of a noisy channel. Formally, it is the value

$$\Theta(G) = \lim_{k \to +\infty} \alpha(\underbrace{G \boxtimes \cdots \boxtimes G}_{k \text{ times}})^{1/k} \,,$$

where the operation $\boxtimes$ is defined as follows. Given two graphs G and H, we denote by $G \boxtimes H$ the graph whose vertices are the pairs (v, w) with v a vertex of G and w a vertex of H. Two vertices (v, w) and (v', w') form an edge if either vv' and ww' are edges of respectively G and H, or vv' is an edge of G and $w = w'$, or $v = v'$ and ww' is an edge of H.

Theorem 6 (Chapter 11, Corollary 11.32 (special case))**.** *The following equality holds:* $\Theta(C_5) = \sqrt{5}$.

2.2. **Rubber band representations.** Theorem 4 relates orthogonal representations and the connectivity of the graph. Another representation can be used to characterize the connectivity of a graph. Identify the vertices with points in $\mathbb{R}^d$. A subset S of the vertices is fixed (they are "nailed"); the other vertices are free. Now, replace each edge with a rubber band of unit stiffness and let the free vertices settle in equilibrium, which is unique (here, we only consider the forces exerted by the rubber bands). The labeling of the vertices with their equilibrium position is a *rubber band* representation of the graph.

Theorem 7 (Chapter 3, Corollary 3.8)**.** *A graph has a rubber band representation in* $\mathbb{R}^d$ *in general position with* S *nailed for every* $S \subseteq V$ *with* $|S| = d+1$ *if and only if it is* $(d+1)$*-connected.*

This theorem, found by Linial, Lovász, and Wigderson [5], is obviously relevant for its applications in physics. Yet, we have again an unexpected application the other way around since this theorem provides a very efficient randomized algorithm to compute the connectivity of a graph. Roughly speaking, it boils down to assign once and for all random weights to the edges, invert some $n \times n$ matrix built from the "Laplacian" of the graph, multiply this matrix with a polylogarithmic number of different $d \times n$ matrices (which all have a very simple description), and check the affine independence of the rows of all matrices obtained this way.

2.3. **Laplacian.** The *Laplacian* of a graph is defined as the $V \times V$ matrix L such that

$$L_{uv} = \begin{cases} \deg(u) & \text{if } u = v, \\ -1 & \text{if } uv \text{ is an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

The study of the Laplacian can reveal interesting properties of a graph. The next theorem is established via a careful analysis of equations involving the Laplacian. The *effective resistance* $R(s,t)$ of a graph between vertices s and t is the resistance of the graph when one sends current from s to t, where each edge has resistance 1. We denote by $F(s,t)$ the force pulling the nails when G is a rubber band structure as in Section 2.2 with s and t nailed. The expected time of a random walk starting at s to hit t and then return to s is the *commute time* between s and t and is denoted by $\mathrm{comm}(s,t)$ (each neighbor of a vertex is selected uniformly at random between all neighbors). The relation between the resistance and the commute time is due to Nash-Williams [10].

Theorem 8 (Chapter 4, Theorem 4.6)**.** *Let* G *be a connected graph and* $s, t \in V$*. The effective resistance* $R(s,t)$*, the force* $F(s,t)$*, and the commute time* $\mathrm{comm}(s,t)$ *are related by the equations*

$$R(s,t) = \frac{1}{F(s,t)} = \frac{\mathrm{comm}(s,t)}{2m}\,.$$

The notion of G-matrix generalizes the Laplacian and is used for a striking algebraic characterization of planar graphs. A graph is *planar* if it can be drawn in the plane so that the vertices are points and the edges are Jordan curves joining their vertices, without intersecting each other except maybe at their endpoints. A *G-matrix* is a symmetric $V \times V$-matrix such that $M_{uv} = 0$ for every pair of non-adjacent vertices u, v. It is *well-signed* if $M_{uv} < 0$ for every edge uv. The Laplacian is an example of a well-signed G-matrix.

The *Colin de Verdière number* $\mu(G)$ of the graph G is the maximum corank of a well-signed transversal G-matrix with exactly one negative eigenvalue. (We do not define transversal here: it is a condition of non-degeneracy.) The following theorem is due to Colin de Verdière [2].

Theorem 9 (Chapter 16, Theorem 16.15 (special case))**.** *A connected graph* G *is planar if and only if* $\mu(G) \leqslant 3$.

The theorem admits generalizations and variations. Let us mention that $\mu(G) \leqslant 4$ also means something about possible embeddings of G, and the version of this theorem given in the book treats this case as well.

3. Interplay between graphs, geometry, and physics

Graphs and geometry can be very useful to derive physical properties: Theorems 2 and 8 are illustrations. Sometimes, the physical motivation leads to interesting mathematical properties, and Theorem 7 is an example.

We give here two other such interactions between graphs, geometry, and physics presented in the book.

3.1. **Global rigidity.** A *linkage* is a pair (G, ℓ) where G is a graph and $\ell\colon E \to \mathbb{R}_+$ is a "prescription" of a distance between the vertices of every edge. A linkage can be interpreted as a graph in which each edge is a rigid bar of a given length. A *realization* of a linkage is a framework $(G, \boldsymbol{u})$ such that $\ell(vv') = \|\boldsymbol{u}_v - \boldsymbol{u}_{v'}\|_2$ for every edge vv'. The vector $\boldsymbol{u}_v$ can be thought as the position of the vertex v. A framework $(G, \boldsymbol{u})$ in $\mathbb{R}^d$ is *globally rigid* if every other labeling $\boldsymbol{u}'$ with vectors in $\mathbb{R}^d$ such that the framework $(G, \boldsymbol{u}')$ realizes the same linkage can be obtained from $\boldsymbol{u}$ via an isometric transformation of $\mathbb{R}^d$: A concrete "physical" realization of a globally rigid framework cannot be deformed.

The following theorem by Alfakih [1], proved with the help of Theorem 4, provides a characterization of globally rigid frameworks whose vector labeling is in affinely general position. A graph is *complete* if every pair of distinct vertices form an edge.

Theorem 10 (Chapter 15, Theorem 15.35)**.** *A framework in $\mathbb{R}^d$ with a vector labeling in affinely general position is globally rigid if and only if its graph is complete or $(d+1)$-connected.*

Global rigidity is also a relevant notion in the theory of sensor networks, as seen in the following example. Consider a number of sensors scattered in a room. Suppose that we are able to measure the distance between some pairs and we would like to reconstruct their geometric positions. A necessary condition is that the measured distances determine the distance between all pairs, i.e., that the framework with sensors as vertices, measured distances as edges, and positions in the room as the vectors in the labeling is globally rigid. This condition becomes sufficient as soon as we know the position of four affinely independent sensors.

The book explores other versions of rigidity—local, universal, or infinitesimal— and various equilibrium properties of frameworks, which leads to several other unexpected connections between mechanical properties of systems and graph theory.

3.2. **Hidden variables in quantum physics.** The applications to physics we have seen so far rely on classical physics. Quantum physics is also addressed in the book (in Chapter 12).

A quantum system can be observed via a measurement. The output of this measurement is somehow random and the measurement changes the state of the system; these are now well-popularized facts from quantum physics. Consider such a quantum system on which n possible measure settings can be chosen and denote by e_i the event that may be observed by this measurement (the event e_i occurs or not).

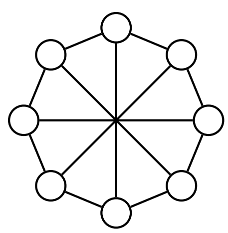
The theory of "hidden variables," proposed by Einstein, Podolsky, and Rosen in 1935, provides a classical interpretation of this system. In this theory, the state of the system is determined before the measurement is performed. Consider the "exclusivity graph" G whose vertices are the events e_i and whose edges connect *exclusive* events, i.e., events that cannot occur simultaneously (no state is such that both events occur). Proceed now with the following experiment: select i uniformly at random, and then check with the suitable measurement if e_i occurs. Denote by p the probability that one sees the occurrence of the randomly selected event. Whatever mechanism determines the state of the system, we will have

$$p \leqslant \frac{\alpha(G)}{n}\,.$$

According to the quantum (non-classical) interpretation, though, the state of the system is described as a unit vector $\boldsymbol{x}$ in a complex Hilbert space, and the probability that a given event e_i occurs is given by $|\boldsymbol{u}_i \cdot \boldsymbol{x}|^2$, where $\boldsymbol{u}_i$ is some unit vector determined by the event e_i. For two events e_i and e_j, being exclusive translate into $\boldsymbol{u}_i$ and $\boldsymbol{u}_j$ being orthogonal. According to quantum theory, we have thus $p = \frac{1}{n}\sum_{i=1}^{n} |\boldsymbol{u}_i \cdot \boldsymbol{x}|^2$ for the experiment described above. The book proposes as an exercise (Chapter 11, Exercise 11.9) to show that equation (1) remains correct when the complex Hilbert space is taken instead of the real Euclidean space. Hence, we get

$$p \leqslant \frac{\vartheta(G)}{n}\,. \tag{2}$$

Experimental settings have been designed to get W_8 as exclusivity graph; see a visualization of this graph on the figure below. We have $\alpha(W_8) = 3$ (easy), $\vartheta(W_8) = 2+\sqrt{2}$ (less easy), and $n = 8$. According to the theory of hidden variables, we should find $p \leqslant 0.375$. The experiment performed by Hensen et al. [4] provides $p \approx 0.401$ and can be considered as a disproof of the theory of the hidden variables (several older experiments, relying on other constructions, have lead to the same conclusion). The upper bound provided by the Lovász number is $(2+\sqrt{2})/8 \approx 0.427$, and is thus not in contradiction with quantum theory. Whether concrete experiments could lead to a probability p closer to the upper bound in inequality (2) is an open question.



The graph W_8, used in the experiment testing the theory of hidden variables.

4. Complementary comments on the book

4.1. **Other topics.** There are many other topics that are addressed in the book and that are not presented here, such as discrete harmonic functions, conformal invariance of scaling limits, square tilings, semi-definite optimization, or metric representations.

4.2. **Non-degeneracy and combinatorial properties.** A common theme in many results given in the book is that non-degeneracy conditions are often needed to characterize combinatorial properties with geometric or algebraic notions. In this review, this is illustrated by Theorems 2, 4, 7, 10, and the definition of the Colin de Verdière number. There is probably more to understand between non-degeneracy and combinatorics, and this is one of the concluding thoughts of the book (Chapter 20).

4.3. **Exercises.** The book proposes at the end of each chapter a series of exercises that not only help understand the main tools and concepts of the book, but also provides complementary results that are often interesting for their own sake. (No solution is proposed.)

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