Computing kernels in graphs with a clique-cutset

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In a directed graph, a kernel is a subset of the vertices that is both independent and absorbing. Not all directed graphs have a kernel, and finding classes of graphs having always a kernel or for which deciding the existence of a kernel is polynomial has been the topic of many works in graph theory. We formalize some techniques to build a kernel in a graph with a clique-cutset, knowing kernels in the pieces with respect to the clique-cutset. As a consequence, we obtain for instance that computing a kernel in a clique-acyclic orientation of a chordal graph can be done in polynomial time. We enlighten some consequences in the theory of hedonic games.

1 Introduction

A subset S of the vertices of a directed graph is *independent* if no two vertices in S are adjacent, and *absorbing* if for any vertex u not in S, there is a vertex $v \in S$ such that the arc (u, v) exists in the graph. A *kernel* is a subset of vertices that is both independent and absorbing. Kernels have been introduced in 1944 by Von Neumann and Morgenstern [15] as a tool for studying positional or Nim-type games. Since then, other applications in game theory have been found [3, Sections 7 and 8]. They also play a role in graph theory: they are for instance at the heart of Galvin's proof of Dinitz's conjecture on list coloring [8].

A directed graph such that every induced subgraph has a kernel is *kernel-perfect*. Identifying classes of kernel-perfect graphs has been the motivation of many works, see the bibliography in [3]. A *clique-cutset* of a directed graph $D = (V, A)$ is a subset $C \subseteq V$ such that C induces a clique in D and $D[V \setminus C]$ is disconnected. For each connected component B of $D[V \setminus C]$, the directed graph induced by $B \cup C$ is a *piece* of D with respect to C. Jacob [11] proved that if every piece with respect to some clique-cutset C is kernel-perfect, so is D. Jacob's theorem has been one of the main tools used by Maffray for proving his result about kernels in i-triangulated graphs [12]. We prove the following lemma, which strengthens Jacob's theorem and adds to it an algorithmic flavor.

Lemma 1. *Let C be a clique-cutset of a digraph $D = (V, A)$ and B, B' a bipartition of $V \setminus C$ such that $B \cup C$ is a piece of D with respect to C. Suppose that there exist subsets of vertices $K^{(i)}$ such that $K^{(i)}$ is a kernel of $D\left[B \cup \left(C \setminus \bigcup_{j=1}^{i-1} K^{(j)}\right)\right]$ for every $i \in \{1, \ldots, |C|+1\}$ and such that $D\left[B' \cup \left(C \cap \bigcup_{j=1}^{|C|} K^{(j)}\right)\right]$ has a kernel K. Then there exists $i \in \{1, \ldots, |C|+1\}$ such that $K \cup K^{(i)}$ is a kernel of D.*

This lemma implies in particular that, if $B \cup C$ and $B' \cup C$ induce kernel-perfect graphs, then D is kernel-perfect as well. We can then see that Lemma 1 implies Jacob's theorem by applying it recursively on the directed graph induced by $B' \cup C$. It is worth noting that the proof of our lemma is much shorter than the proof Jacob gave for his theorem. We provide the full proof at the end of this extended abstract.

In a graph class closed under taking induced subgraphs, the *atoms* are the graphs that have no clique-cutset. According to Jacob's theorem, if the atoms of such a class are kernel-perfect, so are all graphs in the class. The following theorem, proved almost directly from Lemma 1, ensures that if kernels are polynomially computable in the atoms of this class and in their induced subgraphs, then they are polynomially computable on the whole class. Deciding whether a directed graph has a kernel or is kernel-perfect are NP-complete problems [1, 4]. Moreover, not so many classes of kernel-perfect graphs with polynomial algorithms for computing kernels are known.

Theorem 2. *Consider a class of directed graphs closed under taking induced subgraphs. Suppose that there is a polynomial algorithm that, for any induced subgraph of an atom, computes a kernel when it exists. Then there is a polynomial algorithm that, given any digraph D of the class, returns either a kernel of D, or an induced subgraph of D with no kernel.*

If every directed graph in the class is kernel-perfect, then the algorithm computes in polynomial time a kernel. However, if there are directed graphs in the class that are not kernel-perfect, the algorithm may fail to find a kernel, even if one exists, but it outputs then a certificate of non kernel-perfectness.

Orientations of chordal graphs form a graph class satisfying the condition of the theorem (see Section 2). It does not seem to have been known that a kernel can be found in polynomial time in kernel-perfect orientations of chordal graphs. We emphasize that, while the proof given by Jacob for his theorem is constructive and combines kernels of the pieces, we have not been able to adapt it to get a polynomial algorithm for chordal graphs or any graph class satisfying Theorem 2. Yet, there are families of interval graphs on which the straightforward application of the implicit algorithm in the latter proof is exponential.

2 Clique-acyclic orientations of perfect graphs

An arc (u, v) in a directed graph is *reversible* if the arc (v, u) exists, and *irreversible* otherwise. An *orientation* of an undirected graph G is a directed graph obtained by orienting each edge either in one direction or both ways. A *suborientation* is an orientation in which every edge in oriented in only one direction (there are no reversible arcs in a suborientation). An orientation is *clique-acyclic* if in every clique the subgraph of irreversible arcs is acyclic, or, equivalently, if every clique has a vertex absorbing all other vertices in the clique.

One of the main results on kernel-perfect graphs is a theorem by Boros and Gurvich [2], originally conjectured by Berge and Duchet in 1980, stating that every clique-acyclic orientation of a perfect graph has a kernel. An intriguing feature of their proof and of the subsequent proofs is that none of them provides an efficient method to compute a kernel. The complexity of this problem is an open question [9]. There are only few subclasses of perfect graphs for which the problem is known to be polynomial, e.g., bipartite graphs or line-graphs of bipartite graphs (via the Gale-Shapley algorithm for stable marriages [7, 13]). Theorem 2 allows to add to this list chordal and DE graphs. *DE graphs*, or "directed edge path graphs", introduced by Monma and Wei [14], are defined as the intersection graphs of families of directed paths in directed trees (where two paths intersect if they share an arc). The atoms of chordal graphs (resp. DE graphs) are cliques (resp. line-graphs of bipartite graphs) and they satisfy thus the condition of Theorem 2.

Corollary 3. *The problem of computing a kernel in a clique-acyclic orientation of a chordal graph is polynomial.*

Corollary 4. *The problem of computing a kernel in a clique-acyclic orientation of a DE graph is polynomial.*

The complexity of kernel computation seems to be simpler when limited to suborientations. For instance, the polynomiality of finding a kernel is an easy exercise in the case of clique-acyclic suborientations of chordal graphs and was already known for clique-acyclic suborientations of DE graphs [5]. The following result also goes in that direction. A *circular-arc graph* is the intersection graph of intervals on a circle.

Proposition 5. *There is a polynomial algorithm that decides if a clique-acyclic suborientation of a circular-arc graph has a kernel and computes such a kernel when it exists.*

3 An application to hedonic games

A *hedonic game with graph structure* [10] is a triple $(N, (\succeq_i)_{i\in N}, L)$ where N is a finite set of *players*, each $\succeq_i$ is a complete and transitive *preference relation* over the subsets including i, and L is a set of pairs of players. A subset of players is a *feasible coalition* if they induce a connected subgraph of (N, L), seen as a graph. A partition π of N into feasible coalitions is *core stable* if, for every feasible coalition S, there exists a player $i \in S$ who weakly prefers his current coalition $\pi(i)$ to S, i.e., $\pi(i) \succeq_i S$.

One can associate core stable partitions with kernels as follows. Let $\mathcal{F}_L$ be the set of all feasible coalitions. Consider the directed graph $(\mathcal{F}_L, A)$ where $(S, T) \in A$ if there exists a player $i \in S \cap T$ with $T \succeq_i S$. The core stable partitions of a hedonic game are precisely the kernels of $(\mathcal{F}_L, A)$. Demange [6, Theorem 2, p.767] proved that finding a core stable partition when (N, L) is a tree can be done in polynomial time in the number of feasible coalitions. Since the intersection graphs of subtrees of a tree (where intersecting here means sharing a common vertex) are exactly the chordal graphs, Corollary 3 strengthens Demange's result to the more general hedonic game where $\mathcal{F}_L$ is restricted to some predetermined subfamily $\mathcal{F}$ of subsets of players.

4 Proof of Lemma 1

To ease the notation, let us define $X^{(i)} := C \cap \bigcup_{j=1}^{i-1} K^{(j)}$. The sequence $(X^{(i)})_{i=1,2,\ldots}$ of subsets of C is nondecreasing (for inclusion). There is thus an index $k \in \{1, \ldots, |C| + 1\}$ such that $X^{(k)} = X^{(k+1)}$. Since $K^{(k)}$ is a kernel of $D[B \cup (C \setminus X^{(k)})]$, we have $K^{(k)} \cap C \subseteq C \setminus X^{(k)}$. The equality $X^{(k)} = X^{(k+1)}$ implies that $K^{(k)} \cap C \subseteq X^{(k)}$. Hence, $K^{(k)} \cap C = \varnothing$.

Assume that $D[B' \cup X^{(|C|+1)}]$ has a kernel K. Suppose first that K and C have an empty intersection. Since C is a clique-cutset, the set $K \cup K^{(k)}$ is independent. The vertices in $B \cup (C \setminus X^{(k)})$ being absorbed by $K^{(k)}$ and those of $B' \cup X^{(k)}$ being absorbed by K, we get that $K \cup K^{(k)}$ is a kernel of D.

Suppose then that K and C have a nonempty intersection. Denote by v a vertex in $K \cap X^{(|C|+1)}$. By definition of $X^{(|C|+1)}$, there is an index $\ell \in \{1, \ldots, |C|+1\}$ such that $v \in K^{(\ell)}$. Since C is a clique-cutset, the set $K \cup K^{(\ell)}$ is independent in D. The vertices in $B \cup (C \setminus X^{(\ell)})$ being absorbed by $K^{(\ell)}$ and those of $B' \cup X^{(\ell)}$ by K, we get that $K \cup K^{(\ell)}$ is a kernel of D. $\square$

References

[1] S.D. Andres and W. Hochstättler. Perfect digraphs. *Journal of Graph Theory*, 79:21–29, 2014.

[2] E. Boros and V. Gurvich. Perfect graphs are kernel solvable. *Discrete Mathematics*, 159:35–55, 1996.

[3] E. Boros and V. Gurvich. Perfects graphs, kernels, and cores of cooperative games. *Discrete Mathematics*, 306:2336–2354, 2006.

[4] V. Chvátal. On the computational complexity of finding a kernel. Technical Report CRM300, Centre de Recherches Mathématiques, Université de Montréal, 1973.

[5] O. Durand de Gevigney, F. Meunier, C. Popa, J. Reygner, and A. Romero. Solving coloring, minimum clique cover and kernel problems on arc intersection graphs of directed paths on a tree. *4OR*, 9:175–88, 2011.

[6] G. Demange. On group stability in hierarchies and networks. *Journal of Political Economy*, 112(4):754–778, 2004.

[7] D. Gale and L.S. Shapley. College Admissions and the Stability of Marriage. *The American Mathematical Monthly*, 69:9–15, 1962.

[8] F. Galvin. The list chromatic index of a bipartite multigraph. *Journal of Combinatorial Theory, Series B*, 63:153–158, 1995.

[9] Egerváry Research Group. Egres open, finding kernels in special digraphs. `http://lemon.cs.elte.hu/egres/open/Findingkernelsinspecialdigraphs`.

[10] A. Igarashi and E. Elkind. Hedonic games with graph-restricted communication. In *Proceedings of the 15th International Conference on Autonomous Agents and Multiagent Systems*, AAMAS 2016, pages 242–250, 2016.

[11] H. Jacob. Kernels in graphs with a clique-cutset. *Discrete Mathematics*, 156:265–267, 1996.

[12] F. Maffray. On kernels in i-triangulated graphs. *Discrete Mathematics*, 61:247–251, 1986.

[13] F. Maffray. Kernels in perfect line-graphs. *Journal of Combinatorial Theory, Series B*, 55:1–8, 1992.

[14] C.L. Monma and V.K. Wei. Intersection graphs of paths in a tree. *Journal of Combinatorial Theory, Series B*, 41:141–181, 1986.

[15] O. Morgenstern and J. von Neumann. *Theory of Games and Economic Behavior*. Princeton University Press, 1944.