Large Momentum bounds from Flow Equations

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Abstract

We analyse the large momentum behaviour of 4-dimensional massive euclidean φ^4 theory using the flow equations of Wilson's renormalization group. The flow equations give access to a simple inductive proof of perturbative renormalizability. By sharpening the induction hypothesis we prove new and, as it seems, close to optimal bounds on the large momentum behaviour of the correlation functions. The bounds are related to what is generally called Weinberg's theorem.

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1 Introduction

The high energy or momentum behaviour of correlation functions in quantum field theory is of immediate physical interest. It is reflected in the high energy behaviour of measurable quantities such as interaction cross sections. It is also related to questions of theoretical consistency such as unitarity [1]. Four dimensional field theories of physical relevance to this day have been analysed rigourously in truncated form only, in particular in perturbation theory. The main reason for this is related to the fact that physical quantities calculated within these theories have to be renormalized, i.e. reparametrized, since when expressed in the original bare parameters of the theory they diverge. In the framework of perturbation theory renormalization can be carried out in full rigour. A particularly attractive tool for performing the renormalization proof is the flow equation of the Wilson renormalization group [2]. The proof is considerably simplified as compared to the traditional Feynman diagram based proofs, and at the same time the technical question of eliminating infinities is traced back to the physical problem of analysing the renormalization group flow of the theory. The statement of renormalizability of the theory then can be phrased as follows : On fixing the physical structure (i.e. the field and symmetry content) and on fixing a finite number of relevant parameters by physical renormalization conditions the perturbative correlation functions of the theory are finite.

From these remarks it is obvious that the analysis of the large momentum behaviour of the correlation functions cannot be performed rigorously before settling the renormalization issue. Historically Weinberg [3] performed his famous analysis of the high energy behaviour of euclidean Feynman amplitudes about ten years before the achievement of rigorous renormalization theory. His conviction that the renormalization procedure would not invalidate his results was confirmed in the 70ies, in particular through the work of Bergère, Lam and Zuber [4]. Their result is of the following form : For a given Feynman diagram with given euclidean external momenta $p_1, \ldots, p_n$ the associated Feynman amplitude $I(\lambda) := I(\lambda p_1, \ldots, \lambda p_n)$ for λ large, has the following asymptotic expansion

$$I(\lambda) = \sum_{r=r_{\max}}^{-\infty} \sum_{s=0}^{s_{\max}} a_{rs} \lambda^r (\ln \lambda)^s \ .$$

The powers of logarithms are related to the number of renormalization operations performed on the graph, whereas the leading Weinberg power $r_{\max}$ is the maximal scaling dimension of all subgraphs which are irrigated by the flow of large external momenta. Bergère, de Calan and Malbouisson [4] generalized the previous result to the situation where only a subset of momenta is scaled by λ. As regards the technique of proof, it is based on the

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Zimmermann forest formula in parametric space together with the Mellin transform.

Our results are related to those of Weinberg and followers. Since the flow equations do not require cutting up perturbative amplitudes into Feynman amplitudes, the result is stated for the full amplitude, and *it depends on the geometry of the set of external momenta only*. It is written directly in general form such that the bound can also be read off in situations where only subsets of momenta grow large.

We restrict our considerations to the simplest item of a renormalizable field theory in four dimensions. The flow equations have been used to prove renormalizability of most theories of physical interest, including theories with massless fields, and also nonabelian gauge theories [6]. The present considerations could then be extended to those theories to prove strict UV bounds. The method of proof is in accord with the standard flow equation inductive proofs. It uses sharpened induction hypotheses incorporating the improvement of UV behaviour when momentum derivatives are applied to the correlation functions. In closing we note that the flow equations have been used extensively in recent years beyond the field of mathematical physics, in theoretical physics and phenomenology. For a review see [7].

2   Renormalisation and large momentum bounds from the Flow Equations

2.1   The Flow equation framework

Renormalization theory based on the Wilson flow equation (FE) has been exposed quite often in the literature [5]. So we will introduce it rather shortly. The object studied is the regularized generating functional L^{Λ,Λ_0} of connected (free propagator) amputated Green functions (CAG). The upper indices Λ and Λ_0 enter through the regularized propagator

$$C^{\Lambda,\Lambda_0}(p) \;=\; \frac{1}{p^2+m^2}\{e^{-\frac{p^2+m^2}{\Lambda_0^2}} - e^{-\frac{p^2+m^2}{\Lambda^2}}\}$$

or its Fourier transform $\hat{C}^{\Lambda,\Lambda_0}(x) = \int_p C^{\Lambda,\Lambda_0}(p)\, e^{ipx}$, with $\int_p := \int_{\mathrm{I\!R}^4} \frac{d^4p}{(2\pi)^4}$. We assume $0 \leq \Lambda \leq \Lambda_0 \leq \infty$ so that the Wilson flow parameter Λ takes the role of an infrared (IR) cutoff[3], whereas Λ_0 is the ultraviolet (UV) regularization. The full propagator is recovered for $\Lambda = 0$ and $\Lambda_0 \to \infty$. For the "fields" and their Fourier transforms we write $\hat{\varphi}(x) = \int_p \varphi(p)\, e^{ipx}$,

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[3]Such a cutoff is of course not necessary in a massive theory. The IR behaviour is only modified for Λ above m.

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$\frac{\delta}{\delta \hat\varphi(x)} = (2\pi)^4 \int_p \frac{\delta}{\delta \varphi(p)} e^{-ipx}$. For our purposes the fields $\hat\varphi(x)$ may be assumed to live in the Schwartz space $\mathbb{S}(\mathbb{R}^4)$. For finite Λ_0 and in finite volume the theory can be given rigorous meaning starting from the functional integral

$$e^{-(L^{\Lambda,\Lambda_0}(\hat\varphi)+I^{\Lambda,\Lambda_0})} = \int d\mu_{\Lambda,\Lambda_0}(\hat\phi)\, e^{-L^{\Lambda_0,\Lambda_0}(\hat\phi+\hat\varphi)}\,. \tag{1}$$

On the rhs of (1) $d\mu_{\Lambda,\Lambda_0}(\hat\phi)$ denotes the (translation invariant) Gaussian measure with covariance $\hat C^{\Lambda,\Lambda_0}(x)$. The functional $L^{\Lambda_0,\Lambda_0}(\hat\varphi)$ is the bare action including counterterms, viewed as a formal power series in the renormalized coupling g. Its general form for symmetric φ_4^4 theory is

$$L^{\Lambda_0,\Lambda_0}(\hat\varphi) = \frac{g}{4!} \int d^4x\, \hat\varphi^4(x) +$$

$$+ \int d^4x\, \{\frac{1}{2}a(\Lambda_0)\hat\varphi^2(x) + \frac{1}{2}b(\Lambda_0) \sum_{\mu=0}^{3}(\partial_\mu\hat\varphi)^2(x) + \frac{1}{4!}c(\Lambda_0)\hat\varphi^4(x)\}\,, \tag{2}$$

the parameters $a(\Lambda_0)$, $b(\Lambda_0)$, $c(\Lambda_0)$ fulfill

$$a(\Lambda_0) = O(g)\,, \quad b(\Lambda_0),\, c(\Lambda_0) = O(g^2)\,. \tag{3}$$

They are directly related to the standard mass, wave function and coupling constant counterterms. On the lhs of (1) there appears the normalization factor $e^{-I^{\Lambda,\Lambda_0}}$ which is due to vacuum contributions. It diverges in infinite volume so that we can take the infinite volume limit only when it has been eliminated. We do not make the finite volume explicit here since it plays no role in the sequel. For a more thorough discussion see [5] (in particular the last reference).

The FE is obtained from (1) on differentiating w.r.t. Λ. It is a differential equation for the functional L^{Λ,Λ_0} :

$$\partial_\Lambda(L^{\Lambda,\Lambda_0} + I^{\Lambda,\Lambda_0}) = \tag{4}$$

$$= \frac{1}{2}\langle\frac{\delta}{\delta\hat\varphi}, (\partial_\Lambda C^{\hat\Lambda,\Lambda_0})\frac{\delta}{\delta\hat\varphi}\rangle L^{\Lambda,\Lambda_0} - \frac{1}{2}\langle\frac{\delta}{\delta\hat\varphi}L^{\Lambda,\Lambda_0}, (\partial_\Lambda C^{\hat\Lambda,\Lambda_0})\frac{\delta}{\delta\hat\varphi}L^{\Lambda,\Lambda_0}\rangle\,.$$

By $\langle\,,\,\rangle$ we denote the standard scalar product in $L_2(\mathbb{R}^4, d^4x)$. Changing to momentum space and expanding in a formal powers series w.r.t. g we write (with slight abuse of notation)

$$L^{\Lambda,\Lambda_0}(\varphi) = \sum_{r=1}^{\infty} g^r\, L_r^{\Lambda,\Lambda_0}(\varphi)\,.$$

From $L_r^{\Lambda,\Lambda_0}(\varphi)$ we then obtain the CAG of order r in momentum space as

$$(2\pi)^{4(n-1)}\delta_{\varphi(p_1)}\ldots\delta_{\varphi(p_n)}L_r^{\Lambda,\Lambda_0}|_{\varphi\equiv 0} = \delta^{(4)}(p_1+\ldots+p_n)\,\mathcal{L}_{r,n}^{\Lambda,\Lambda_0}(p_1,\ldots,p_{n-1})\,, \tag{5}$$

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where we have written $\delta_{\varphi(p)} = \delta/\delta\varphi(p)$. Note that by our definitions the free two point function is not contained in $L_r^{\Lambda,\Lambda_0}(\varphi)$. This means that $\mathcal{L}_{0,2}^{\Lambda,\Lambda_0}$ vanishes. This is important for the set-up of the inductive scheme, from which we will prove renormalizability below. The FE (4) rewritten in terms of the CAG (5) takes the following form

$$\partial_\Lambda \partial^w \mathcal{L}_{r,n}^{\Lambda,\Lambda_0}(p_1, \ldots p_{n-1}) = \frac{1}{2} \int_k (\partial_\Lambda C^{\Lambda,\Lambda_0}(k)) \, \partial^w \mathcal{L}_{r,n+2}^{\Lambda,\Lambda_0}(k, -k, p_1, \ldots p_{n-1})$$

$$- \sum_{\substack{r_1 + r_2 = r, \\ w_1 + w_2 + w_3 = w \\ n_1 + n_2 = n+2}} \frac{1}{2} \Bigg[\partial^{w_1} \mathcal{L}_{r_1,n_1}^{\Lambda,\Lambda_0}(p_1, \ldots, p_{n_1-1}) \times \qquad (6)$$

$$\times (\partial^{w_3} \partial_\Lambda C^{\Lambda,\Lambda_0}(p')) \, \partial^{w_2} \mathcal{L}_{r_2,n_2}^{\Lambda,\Lambda_0}(p_{n_1}, \ldots, p_n) \Bigg]_{ssym} ,$$

$$\text{where} \quad p' = -p_1 - \ldots - p_{n_1-1} = p_{n_1} + \ldots + p_n .$$

Here we have written (6) directly in a form where also momentum derivatives of the CAG (5) are performed, and we used the shorthand notations

$$\partial^w := \prod_{i=1}^{n-1} \prod_{\mu=0}^{3} (\frac{\partial}{\partial p_{i,\mu}})^{w_{i,\mu}} \quad \text{with} \quad w = (w_{1,0}, \ldots, w_{n-1,3}),$$

$$|w_i| = \sum_\mu w_{i,\mu}, \; |w| = \sum |w_i|, \; w_{i,\mu} \in \mathbb{N}_0 .$$

The symbol *ssym* means summation over those permutations of the momenta $p_1, \ldots, p_n$, which do not leave invariant the subsets $\{p_1, \ldots, p_{n_1-1}\}$ and $\{p_{n_1}, \ldots, p_n\}$. Note that the CAG are symmetric in their momentum arguments by definition. The simple inductive proof of the renormalizability of φ_4^4 theor [5] gives the following bounds, which serve at the same time as induction hypotheses :

A) Boundedness $\qquad |\partial^w \mathcal{L}_{r,n}^{\Lambda,\Lambda_0}(\vec{p})| \leq \kappa^{4-n-|w|} \, \mathcal{P}_1(log \frac{\kappa}{m}) \, \mathcal{P}_2(\frac{|\vec{p}|}{\kappa})$, $\qquad (7)$

B) Convergence $\quad |\partial_{\Lambda_0} \partial^w \mathcal{L}_{r,n}^{\Lambda,\Lambda_0}(\vec{p})| \leq \frac{1}{\Lambda_0^3} \, \kappa^{6-n-|w|} \, \mathcal{P}_3(log \frac{\Lambda_0}{m}) \, \mathcal{P}_4(\frac{|\vec{p}|}{\kappa})$. $\ (8)$

Here and in the following we set $\kappa = \Lambda + m$ and use the shorthand $\vec{p} = (p_1, \ldots, p_{n-1})$ and $|\vec{p}| = \sup\{|p_1|, \ldots, |p_n|\}$. The $\mathcal{P}_i$ denote polynomials with nonnegative coefficients, which depend on $r, n, |w|, m$, but not on $\vec{p}, \Lambda, \Lambda_0$. The degree of $\mathcal{P}_1$ can be shown to be bounded by $r + 1 - n/2$ for $n \geq 4$ and by $r - 1$ for $n = 2$. The statement (8) implies renormalizability, since it proves the limits $\lim_{\Lambda_0 \to \infty, \, \Lambda \to 0} \mathcal{L}_{r,n}^{\Lambda,\Lambda_0}(\vec{p})$ to exist to all orders r. But the statement (7) has to be obtained first to prove (8).

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2.2 Renormalisation together with large momentum bounds

The inductive scheme used to prove (7,8) will also be used to obtain the new bounds. What we need is a sharpened induction hypothesis, and better control of the high energy improvement generated by derivatives acting on the Green functions. We denote by $p_1, \ldots, p_n$ a set of external momenta with $p_1 + \ldots + p_n = 0$, and we introduce

$$\eta_{i,j}^{(n)}(p_1, \ldots, p_n) = \inf\left\{ |p_i + \sum_{k \in J} p_k| \,/\, J \subset (\{1, ..., n\} - \{i, j\}) \right\} . \quad (9)$$

Thus $\eta_{i,j}^{(n)}$ is the smallest subsum of external momenta which contains p_i and which does not contain p_j. Our new bounds are then given by

Proposition 1 : For $0 \le \Lambda \le \Lambda_0$, $\kappa = \Lambda + m$, and for $n \ge 4$

$$|\partial^w \mathcal{L}_{r,n}^{\Lambda,\Lambda_0}(\vec{p})| \le \kappa^{4-n} \prod_{\substack{i=1 \\ i \ne j}}^{n} \frac{1}{\left(\sup(\kappa, \eta_{i,j}^{(n)})\right)^{|w_i|}} \mathcal{P}_{r,n}^{|w|}\left(\log(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))\right) , \quad (10)$$

$$\text{for } n = 2 : |\partial^w \mathcal{L}_{r,2}^{\Lambda,\Lambda_0}(p)| \le \sup(|p|, \kappa)^{2-|w|} \mathcal{P}_{r,2}^{|w|}\left(\log(\sup(\frac{|p|}{\kappa}, \frac{\kappa}{m}))\right) . \quad (11)$$

Here $\mathcal{P}_{r,n}^{|w|}$ are (each time they appear possibly new) polynomials with non-negative coefficients which depend on $r, n, |w|, m$, but not on $\vec{p}, \Lambda, \Lambda_0$. They are of degree

$$\deg \mathcal{P}_{r,n}^{|w|} \le \begin{cases} |r - 1 - n/2| & \text{if } n = 2, |w| \ge 3 \\ |r - n/2| & \text{if } n = 2, |w| \le 2 \text{ or if } n = 4, |w| \ge 1 \\ |r + 1 - n/2| & \text{otherwise.} \end{cases}$$

Proof : We will use the standard inductive scheme which goes up in r and for given r descends in n, and for given r, n descends in $|w|$ starting from some arbitrary $|w|_{\max}$. The rhs of the FE is then prior the lhs in the inductive order, and the bounds can thus be verified for suitable boundary conditions on integrating the rhs of the FE over Λ, using the bounds of the proposition. To start the induction note that

$$\mathcal{L}_{r,n}^{\Lambda,\Lambda_0} \equiv 0 \quad \text{for} \quad n > 2r + 2$$

(as follows from the connectedness). Terms with $n + |w| \ge 5$ are integrated down from Λ_0 to Λ, since for those terms we have the boundary conditions at $\Lambda = \Lambda_0$ following from (2)

$$\partial^w \mathcal{L}_{r,n}^{\Lambda_0,\Lambda_0}(p_1, \ldots p_{n-1}) = 0 \quad \text{for} \quad n + |w| \ge 5 ,$$

whereas the terms with $n + |w| \le 4$ at the renormalization point - which we choose at zero momentum for simplicity - are integrated upwards from

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0 to Λ, since they are fixed at $\Lambda = 0$ by (Λ_0-independent) renormalization conditions, which define the relevant parameters of the theory. From symmetry considerations we deduce the absence of nonvanishing renormalization constants apart from those appearing in (3). The Schlömilch or integrated Taylor formula permits us to move away from the renormalization point, treating first $\mathcal{L}_{r,4}^{0,\Lambda_0}$ and then the momentum derivatives of $\mathcal{L}_{r,2}^{0,\Lambda_0}$, in descending order.

Note that j in (10) is arbitrary, so the bound arrived at will be in fact

$$
|\partial^w \mathcal{L}_{r,n}^{\Lambda,\Lambda_0}(\vec{p})| \leq \kappa^{4-n} \inf_{j,1\leq j \leq n} \prod_{\substack{i=1 \\ i \neq j}}^{n} \frac{1}{\left(\sup(\kappa,\eta_{i,j}^{(n)})\right)^{|w_i|}} \mathcal{P}_{r,n}^{|w|}\left(\log(\sup(\frac{|\vec{p}|}{\kappa},\frac{\kappa}{m}))\right) .
$$

We will choose $j = n$ since the proof is independent of this choice.

A) $n + |w| \geq 5$:

A1) $n \geq 4$: Integrating the FE (4) w.r.t. the flow parameter κ' from κ to $\Lambda_0 + m$ gives the following bound for the *first* term on the rhs of the FE

$$
\int_{\kappa}^{\Lambda_0+m} d\kappa' \int d^4 p \; e^{-\frac{p^2+m^2}{\Lambda'^2}} \; \kappa'^{4-(n+2)-3} \prod_{i=1}^{n-1} \frac{1}{\left(\sup(\kappa',\eta_{i,n}^{(n+2)})\right)^{|w_i|}}
$$

$$
\times \mathcal{P}_{r,n+2}^{|w|}\left(\log(\sup(\frac{|\vec{p}|}{\kappa'},\frac{|p|}{\kappa'},\frac{\kappa'}{m}))\right) \qquad (\Lambda' = \kappa' - m)
$$

$$
\leq \int_{\kappa}^{\Lambda_0+m} d\kappa' \int d^4(\frac{p}{\kappa'}) \; e^{-\frac{p^2}{\Lambda'^2}} \; \kappa'^{3-n-|w|} \prod_{i=1}^{n-1} \frac{1}{\left(\sup(1,\frac{\eta_{i,n}^{(n+2)}}{\kappa'})\right)^{|w_i|}}
$$

$$
\times \mathcal{P}_{r,n+2}^{|w|}\left(\log(\sup(\frac{|\vec{p}|}{\kappa'},\frac{|p|}{\kappa'},\frac{\kappa'}{m}))\right)
$$

$$
\leq \kappa^{4-n} \prod_{i=1}^{n-1} \frac{1}{\left(\sup(\kappa,\eta_{i,n}^{(n)})\right)^{|w_i|}} \; \mathcal{P}_{r,n+2}^{|w|}\left(\log(\sup(\frac{|\vec{p}|}{\kappa},\frac{\kappa}{m}))\right) ,
$$

which satisfies the required bound. Here we used the important inequality:

$$
\int d^4 x \; e^{-x^2} \; \mathcal{P}(\log|x|) \prod_{i=1}^{k} \frac{1}{\sup(1,|x+a_i|)} \leq c(k) \prod_{i=1}^{k} \frac{1}{\sup(1,|a_i|)} \qquad (12)
$$

for suitable $c(k) > 0$. This inequality will again be used in the subsequent considerations. It is easily established using the rapid fall-off of e^{-x^2} .

The required bound on the *second* contribution from the rhs of the FE (6) is established when using the induction hypothesis for the terms $\partial^{w_1} \mathcal{L}_{r_1,n_1}^{\Lambda,\Lambda_0}$ and $\partial^{w_2} \mathcal{L}_{r_2,n_2}^{\Lambda,\Lambda_0}$. The only new ingredient needed is a bound for the derivatives of the regularization factor appearing in this second term:

$$
|\partial^w e^{-\frac{q^2+m^2}{\Lambda^2}}| \leq c(|w|) \; \kappa^{-|w|} e^{-\frac{q^2}{\Lambda^2}}
$$

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for suitable $c(|w|) > 0$. Note also that by the induction hypothesis

$$\deg \mathcal{P}^{|w_1|}_{r_1,n_1} + \deg \mathcal{P}^{|w_2|}_{r_2,n_2} \leq \begin{cases} |r+1-n/2| & \text{if } n = 4, \ w = 0 \text{ or if } n \geq 6 \\ |r-n/2| & \text{if } n = 4, \ |w| \geq 1 \end{cases}$$

in all cases (also if $|w| \geq 1$, and $w_1, w_2 = 0$).

A2) The case ($n = 2, w = 3$) which is simpler due to the appearance of one external momentum only is treated analogously.

B) $n + |w| \leq 4$:

For the relevant terms of dimension ≤ 4 the induction hypothesis is easily verified at zero momentum where it agrees with the results from [5] [4]. To extend it to general momenta we shall choose a suitable integration path from zero to the momentum configuration considered.

B1) For $n = 2$ we proceed in descending order of $|w|$ starting from $|w|_{\max}$. We use

$$\partial^w \mathcal{L}^{\Lambda,\Lambda_0}_{r,2}(p) = \partial^w \mathcal{L}^{\Lambda,\Lambda_0}_{r,2}(0) + |\sum_\mu p_\mu \int_0^1 d\lambda \, \partial_\mu \partial^w \mathcal{L}^{\Lambda,\Lambda_0}_{r,2}(\lambda p)|$$

and bound the second term with the aid of the induction hypothesis by

$$|\sum_\mu p_\mu \int_0^1 d\lambda \, \partial_\mu \partial^w \mathcal{L}^{\Lambda,\Lambda_0}_{r,2}(\lambda p)| \leq$$

$$|p| \int_0^1 \frac{d\lambda}{\left(\sup(\lambda|p|,\kappa)\right)^{|w|-1}} \mathcal{P}^{|w|+1}_{r,2}\left(\log(\sup(\frac{|p|}{\kappa}, \frac{\kappa}{m}))\right) \leq$$

$$|p| \left(\int_0^{\inf(1,\frac{\kappa}{|p|})} \frac{d\lambda}{\kappa^{|w|-1}} + \int_{\inf(1,\frac{\kappa}{|p|})}^1 \frac{d\lambda}{(\lambda|p|)^{|w|-1}} \right) \mathcal{P}^{|w|+1}_{r,2}\left(\log(\sup(\frac{|p|}{\kappa}, \frac{\kappa}{m}))\right)$$

$$\leq |p|^{2-|w|} \mathcal{P}^{|w|}_{r,2}\left(\log(\sup(\frac{|p|}{\kappa}, \frac{\kappa}{m}))\right) \ .$$

B2) To prove the proposition for ($n = 4, w = 0$) we will use repeatedly the
Lemma: For $\lambda \in [0,1]$ and $x, y \in \mathbb{R}^d$, if $|x + y| \geq |x|$ then $|\lambda x + y| \geq \lambda|x|$.
Proof: $|\lambda x + y| \geq |x + y| - |(1-\lambda)x| \geq |x| - (1-\lambda)|x| = \lambda|x|$. ∎

In fact, the case $n = 4, w = 0$ will be treated by distinguishing four different situations as regards the momentum configurations. We use the previously established bounds for the case $n = 4, w = 1$. These bounds are in terms of the functions $\eta^{(4)}_{i,j}$ from (9). Assuming (without loss of generality)

$$|p_4| \geq |p_1|, \ |p_2|, \ |p_3|$$

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[4] We note that when performing the integration over κ from m to $\Lambda + m$ for the terms with $n + |w| = 4$ there appears a logarithm, which is the origin of the polynomial $\mathcal{P}^{|w|}_{r,n}$, present also at zero momentum.

we realize that $\eta_{i,4}^{(4)}$ can always be realized by a sum of *at most two* momenta from the set $\{p_1,\, p_2,\, p_3\}$. It is then obvious that the subsequent cases ii) and iv) cover all possible situations. The cases i) and iii) correspond to exceptional configurations for which the bound has to be established before proceeding to the general ones. The four cases are

i) $\{p_1,\, p_2,\, p_3\} = \{0,\, q,\, v\}$

ii) $\{p_1,\, p_2,\, p_3\}$ such that $\inf_i \eta_{i,4}^{(4)} = \inf_i |p_i|$

iii) $\{p_1,\, p_2,\, p_3\} = \{p,\, -p,\, v\}$

iv) $\{p_1,\, p_2,\, p_3\}$ such that $\inf_i \eta_{i,4}^{(4)} = \inf_{j\neq k} |p_j + p_k|$.

i) To prove the proposition in this case we use an integrated Taylor formula :

$$|\mathcal{L}_{r,4}(0,q,v)| \leq$$

$$|\mathcal{L}_{r,4}(0,0,0)| + \sum_\mu \int_0^1 d\lambda \left(|q_\mu \partial_{q_\mu} \mathcal{L}_{r,4}(0,\lambda q, \lambda v)| + |v_\mu \partial_{v_\mu} \mathcal{L}_{r,4}(0,\lambda q, \lambda v)| \right).$$

The second term is bounded using the induction hypothesis:

$$\sum_{i=2,3} |p_i| \int_0^1 d\lambda \frac{1}{\sup(\kappa, \eta_{i,4}^{(4)}(\lambda))} \mathcal{P}_{r,4}^1 \left(\log(\sup(\frac{|p_4|}{\kappa}, \frac{\kappa}{m})) \right). \tag{13}$$

We have written $\eta(\lambda)$ for the η-parameter in terms of the scaled variables $p_2^\lambda = \lambda q$, $p_3^\lambda = \lambda v$. We directly find $\eta_{2,4}^{(4)}(\lambda) = \lambda|q|$, $\eta_{3,4}^{(4)}(\lambda) = \lambda|v|$ and thus obtain the following bound for (13)

$$|q| \left(\int_0^{\inf(1,\frac{\kappa}{|q|})} \frac{d\lambda}{\kappa} + \int_{\inf(1,\frac{\kappa}{|q|})}^1 \frac{d\lambda}{\lambda|q|} \right) \mathcal{P}_{r,4}^1 \left(\log(\sup(\frac{|p_4|}{\kappa}, \frac{\kappa}{m})) \right) + \left(q \to v \right)$$

$$\leq \left(\frac{|q|}{\kappa} \frac{\kappa}{|q|} + \log(\frac{|q|+\kappa}{\kappa}) + \frac{|v|}{\kappa} \frac{\kappa}{|v|} + \log(\frac{|v|+\kappa}{\kappa}) \right) \mathcal{P}_{r,4}^1 \left(\log(\sup(\frac{|p_4|}{\kappa}, \frac{\kappa}{m})) \right)$$

$$\leq \mathcal{P}_{r,4}^0 \left(\log(\sup(\frac{|p_4|}{\kappa}, \frac{\kappa}{m})) \right),$$

which ends the proof of case i).

ii) We assume without loss of generality $\inf_i \eta_{i,4}^{(4)} = |p_1|$. We use again an integrated Taylor formula along the integration path $(p_1^\lambda, p_2^\lambda, p_3^\lambda) = (\lambda p_1, p_2, p_3 + (1-\lambda)p_1)$. By the Lemma we find $\eta_{1,4}^{(4)}(\lambda) = |p_1^\lambda| = \lambda|p_1|$, $\eta_{3,4}^{(4)}(\lambda) \geq \lambda|p_1|$. The boundary term for $\lambda = 0$ is bounded through i). For the second term we bound

$$|\sum_\mu \int_0^1 d\lambda \left(p_{1,\mu} \left(\partial_{p_{1,\mu}} - \partial_{p_{3,\mu}} \right) \mathcal{L}(p_1^\lambda, p_2^\lambda, p_3^\lambda) \right)|$$

$$\leq |p_1| \int_0^1 d\lambda \left(\frac{1}{\sup(\kappa, \eta_{1,4}^{(4)}(\lambda))} + \frac{1}{\sup(\kappa, \eta_{3,4}^{(4)}(\lambda))} \right) \mathcal{P}_{r,4}^1 \left(\log(\sup(\frac{|p_4|}{\kappa}, \frac{\kappa}{m})) \right)$$

$$\leq |p_1| \left(\int_0^{\inf(1,\frac{\kappa}{|p_1|})} \frac{d\lambda}{\kappa} + \int_{\inf(1,\frac{\kappa}{|p_1|})}^1 \frac{d\lambda}{\lambda|p_1|} \right) \mathcal{P}_{r,4}^1 \left(\log(\sup(\frac{|p_4|}{\kappa}, \frac{\kappa}{m})) \right) ,$$

which gives the required bound similarly as in i).

iii) We choose the integration path $(p_1^\lambda, p_2^\lambda, p_3^\lambda) = (\lambda p, -p, v)$. Here we assume without restriction that $|v| \leq |v - (1-\lambda)p|$, otherwise we interchange the role of v and $-v$. The boundary term leads again back to i). The integral $\int_0^1 d\lambda$ of the second term is cut into four pieces

$$\int_0^1 = \int_0^{\inf(1/2,\frac{\kappa}{|p_1|})} + \int_{\inf(1/2,\frac{\kappa}{|p_1|})}^{1/2} + \int_{1/2}^{\sup(1/2,1-\frac{\kappa}{|p_1|})} + \int_{\sup(1/2,1-\frac{\kappa}{|p_1|})}^1 .$$

They are bounded in analogy with ii) using $\eta_{1,4}^{(4)}(\lambda) = \lambda|p_1|$ for $\lambda \leq 1/2$, $\eta_{1,4}^{(4)}(\lambda) = (1-\lambda)|p_1|$ for $\lambda \geq 1/2$, relations easily established with the aid of the Lemma.

iv) We assume without loss of generality $\inf_i \eta_{i,4}^{(4)} = |p_1 + p_2|$ and integrate along $(p_1^\lambda, p_2^\lambda, p_3^\lambda) = (p_1, -p_1 + \lambda(p_1 + p_2), p_3)$. The boundary term has been bounded in iii). Using the Lemma again we find $\inf \eta_{2,4}^{(4)}(\lambda) = \lambda|p_1 + p_2|$, and the integration term is then bounded through

$$|\sum_\mu \int_0^1 d\lambda \left((p_{1,\mu} + p_{2,\mu}) \, \partial_{p_{2,\mu}} \mathcal{L}(p_1^\lambda, p_2^\lambda, p_3^\lambda) \right)| \leq$$

$$|p_1 + p_2| \left(\int_0^{\inf(1,\frac{\kappa}{|p_1+p_2|})} \frac{d\lambda}{\kappa} + \int_{\inf(1,\frac{\kappa}{|p_1+p_2|})}^1 \frac{d\lambda}{\lambda|p_1 + p_2|} \right) \mathcal{P}_{r,4}^1 \left(\log(\sup(\frac{|p_4^\lambda|}{\kappa}, \frac{\kappa}{m})) \right)$$

which gives the required bound as before. ∎

Bounds like those of Proposition 1 can also be proven using regularizations different from the one applied here. To analyse properties of Green functions in Minkowski space it is useful to have regulators which stay bounded for large momenta in the whole complex plane. An example is

$$C^{\Lambda,\Lambda_0}(p) = \frac{1}{p^2 + m^2} \left((\frac{\Lambda_0^2}{p^2 + m^2 + \Lambda_0^2})^k - (\frac{\Lambda^2}{p^2 + m^2 + \Lambda^2})^k \right) .$$

One realizes that an inequality analogous to (12) in this case requires that $2k > |w|_{\max} + 2$. Since $|w|_{\max}$ should be at least 3 (to be able perform the renormalization proof for the two point function), we need $k \geq 3$. Then the proof can be performed as before.

2.3 Weighted trees and large momentum fall-off

In this section we want to show that for $n \geq 6$ the n-point functions of symmetric massive φ_4^4 fall off for large external momenta. The following definitions are required for a precise formulation of these fall-off properties.

A 4-tree of order r is defined to be a connected graph without loops and with a set of $r \geq 1$ vertices of coordination number 4. The tree has n external lines with $n = 2r + 2$, which are assumed to be numbered, and it has a set $\mathfrak{I}$ of internal lines with $|\mathfrak{I}| = r - 1$. We then denote by $\mathfrak{T}^{4,n}$ the set of all 4-trees with n external lines. A weighted 4-tree is a 4-tree with a weight $\mu(I) = 2$ attached to each $I \in \mathfrak{I}$. We now define for $1 \leq k \leq n - 4$ *k-times reduced* (weighted) trees obtained from (weighted) 4-trees:

A 0-times reduced tree is a 4-tree.

A k-times reduced tree $T^{(k)}$ is obtained from a $(k-1)$-times reduced tree $T^{(k-1)}$ through the following process:

i) by suppressing *one* external line of $T^{(k-1)}$,

ii) by diminishing by *one* unit the weight of one among those internal lines of $T^{(k-1)}$, which are adjacent to the vertex where the external line was suppressed (there are at least 1 and at most 3 lines of this type),

iii) by suppressing any internal line I from the tree if it has acquired $\mu(I) = 0$ through this process, and fusing the two adjacent vertices into one,

iv) by suppressing the vertex from which the external line has been removed, in case this vertex has acquired coordination number 2 through this removal. If two internal lines have been attached to this vertex, they are fused into a single one and their weights are added. If one internal line had been attached to this vertex, it had necessarily weight 0 and was removed through iii).

It is then easy to realize that a k-times reduced tree $T^{(k)}$ with n external lines has the following properties :

a) It is a tree.

b) Its vertices have coordination numbers 3 or 4.

c) The weight $\mu(I)$ attached to each internal line $I \in \mathfrak{I}$ of $T^{(k)}$ satisfies

i) $\mu(I) \in \{1, 2\}$,   ii) $\sum_{I \in \mathfrak{I}} \mu(I) = n - 4$.

The set of weighted reduced trees with n external lines is denoted by $\mathfrak{T}^{n,\mu}$. We will use these trees to bound the lhs of the FE in terms of the rhs.

To the external lines of a tree $T^{n,\mu} \in \mathfrak{T}^{n,\mu}$ we associate n external incoming momenta $\vec{p} = (p_1, \ldots, p_n)$ and write $T^{n,\mu}(\vec{p})$ for the thus assigned tree. Let then $p(I)$ be the (uniquely fixed, by momentum conservation) momentum flowing through the internal line $I \in \mathfrak{I}$. For given κ the weight factor of an (assigned weighted) tree $T^{n,\mu}(\vec{p})$ (shortly T) is defined as

$$g^{\kappa}(T) = \prod_{I \in \mathfrak{I}(T)} \frac{1}{(\sup(\kappa, p(I))^{\mu(I)}} \, .$$

Our statement on the fall-off of the n-point-functions is then the following

Proposition 2: For $n \geq 4$ (and with $\kappa = \Lambda + m$)

$$|\mathcal{L}_{r,n}^{\Lambda,\Lambda_0}(\vec{p})| \leq \sup_{T \in \mathfrak{T}^{n,\mu}(\vec{p})} g^{\kappa}(T) \, \mathcal{P}_{r,n}\big(\log(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m}))\big) \, , \tag{14}$$

where $\deg \mathcal{P}_{r,n} \leq r + 1 - n/2$.

Remark: We could prove without hardly any change a slightly sharper version of Proposition 2, by restricting the sup in (14) to $2k$-times reduced trees with $k = r + 1 - n/2$. For k sufficiently large, both sets of trees become equal, however.

Proof: We again apply the standard inductive scheme. In starting we note that the $\mathcal{L}_{r,n}$ vanish for $n > 2r+2$ and are given by a sum over 4-tree graphs for $n = 2r + 2$, which obviously satisfy the bounds of the proposition. We also note that for $n = 4$ Proposition 2 follows from Proposition 1. Thus we assume $n \geq 6$.

i) We bound the *first* term on the rhs of the FE (6), integrated over κ:

$$\int_{\kappa}^{\Lambda_0+m} \frac{d\kappa'}{\kappa'^3} \int d^4p \, e^{-\frac{p^2+m^2}{\Lambda'^2}} \sup_{T \in \mathcal{T}^{n+2,\mu}(\vec{p},p,-p)} g^{\kappa'}(T) \, \mathcal{P}_{r,n+2}\big(\log(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{|p|}{\kappa'}, \frac{\kappa'}{m})))\big)$$

$$\leq \int_{\kappa}^{\Lambda_0+m} d\kappa' (\kappa')^{4-n-1} \int d^4(\frac{p}{\kappa'}) \, e^{-\frac{p^2}{\kappa'^2}} \prod_{I \in \mathcal{I}(T_{\max}^{\kappa'})} \frac{1}{\big(\sup(1, \frac{|p(I)|}{\kappa'})\big)^{\mu(I)}}$$

$$\times \mathcal{P}_{r,n+2}\big(\log(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{|p|}{\kappa'}, \frac{\kappa'}{m})))\big)$$

$$= \int_{\kappa}^{\Lambda_0+m} \frac{d\kappa'}{\kappa'} \prod_{i=1,2} \frac{1}{\big(\sup(1, \frac{|\hat{p}(I_i)|}{\kappa'})\big)} \, g^{\kappa'}(T_2^{\kappa'}) \, \mathcal{P}_{r,n+2}\big(\log(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m})))\big)$$

$$\leq \int_{\kappa}^{\Lambda_0+m} \frac{d\kappa'}{\kappa'} \, g^{\kappa'}(T_2^{\kappa'}) \, \mathcal{P}_{r,n+2}\big(\log(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m})))\big) \tag{15}$$

with the following explanations: The integral over p/κ' was bounded with the aid of the inequality (12). By $T_{\max}^{\kappa'}$ we denote a tree $T \in \mathcal{T}^{n+2,\mu}(\vec{p},p,-p)$ of maximal weight for given κ'. Then we denote by $T_2^{\kappa'}(\vec{p})$ or shortly $T_2^{\kappa'}$ a twice reduced tree of $T_{\max}^{\kappa'}$, obtained by suppressing the two external lines from $T_{\max}^{\kappa'}$, which carried the momenta p, $-p$, and by diminishing the weight of two internal lines I_1, I_2, adjacent to the respective vertices by one unit (it may happen that the two vertices and/or lines are identical). And we set $\hat{p}(I_i) := p(I_i)|_{p,-p=0}$. Now we note that

$$\int_{\kappa}^{\Lambda_0+m} \frac{d\kappa'}{\kappa'} \frac{g^{\kappa'}(T^{\kappa'})}{g^{\kappa}(T^{\kappa})} \, \mathcal{P}_{r,n+2}\big(\log(\sup(\frac{|\vec{p}|}{\kappa'}, \frac{\kappa'}{m})))\big) \leq \tag{16}$$

$$\leq \log(\frac{|\vec{p}|}{\kappa}) \, \tilde{\mathcal{P}}_{r,n+2}\big(\log(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})))\big)$$

and thus obtain finally the required bound for (15)

$$(15) \leq \sup_{T \in \mathcal{T}^{n,\mu}(\vec{p})} g^{\kappa}(T) \, \mathcal{P}_{r,n}\big(\log(\sup(\frac{|\vec{p}|}{\kappa}, \frac{\kappa}{m})))\big) \, .$$

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ii) To bound the *second* term on the rhs of (6) we use the inequality $\kappa^{-3}\,\exp(\frac{-p'^2}{\Lambda^2}) \le \big(\sup(\kappa,|p'|)\big)^{-2}\,\kappa^{-1}$ to obtain straightforwardly the following bound for any given term (with n_1, $n_2 \ge 4$)[5] in the sum appearing on the rhs of (6):

$$\frac{\kappa^{-1}}{\big(\sup(\kappa,|p'|)\big)^2}\,\sup_{T_1\in\mathcal{T}^{n_1,\mu_1}(\vec{p}_1)}\,g^\kappa(T_1)\,\mathcal{P}_{r_1,n_1}\big(\log(\sup(\frac{|\vec{p_1}|}{\kappa},\frac{\kappa}{m})))\big) \qquad (17)$$

$$\times \sup_{T_2\in\mathcal{T}^{n_2,\mu_2}(\vec{p}_2)}\,g^\kappa(T_2)\,\mathcal{P}_{r_2,n_2}\big(\log(\sup(\frac{|\vec{p_2}|}{\kappa},\frac{\kappa}{m})))\big)\ ,$$

where we used the notations of (6) and $\vec{p}_1 := (p_1,\ldots,p_{n_1-1},p')$, $\vec{p}_2 := (-p',p_{n_1},\ldots,p_n)$. We pick two trees $T^\kappa_{1,\max}$ and $T^\kappa_{2,\max}$, which realize the sup's in (17) and define the tree T^κ to be given by $T^\kappa_{1,\max}\cup T^\kappa_{2,\max}\cup\ell'$, where ℓ' is the internal line of the new tree T joining $T^\kappa_{1,\max}$ and $T^\kappa_{2,\max}$. This line carries the momentum $-p'$ (cf. (6)). We attach the weight 2 to ℓ'. We obviously have $T^\kappa \in \mathcal{T}^{n,\mu}$. Therefore integrating (17) from κ to $\Lambda_0 + m$ (using again (16)) the result is bounded by

$$g^\kappa(T^\kappa)\,\mathcal{P}_{r_1,n_1}\big(\log(\sup(\frac{|\vec{p_1}|}{\kappa},\frac{\kappa}{m})))\big)\,\mathcal{P}_{r_2,n_2}\big(\log(\sup(\frac{|\vec{p_2}|}{\kappa},\frac{\kappa}{m})))\big)\,\log(\frac{|\vec{p}|}{\kappa})$$

$$\le \sup_{T\in\mathcal{T}^{n,\mu}(\vec{p})}\,g^\kappa(T)\,\mathcal{P}_{r,n}\big(\log(\sup(\frac{|\vec{p}|}{\kappa},\frac{\kappa}{m})))\big)\ . \qquad (18)$$

∎

The present bounds seem close to optimal. They show for example that the high energy behaviour is not deteriorated if only one single external momentum becomes small, since our trees do not contain vertices of coordination number 2. In particular for n small (6,8,...) the number of weighted trees to be considered and thus the bound is easily explicited. For $n = 6$ we find three different trees[6], up to permutations of the external momenta. Their weight factors $g^m(T)$ are $\big(\sup(m,|p_1+p_2|)\,\sup(m,|p_1+p_2+p_3|)\big)^{-1}$, $\big(\sup(m,|p_1+p_2|)\,\sup(m,|p_3+p_4|)\big)^{-1}$, $\big(\sup(m,|p_1+p_2+p_3|)\big)^{-2}$. From them, from the geometry of the external momenta and from Proposition 2, we read off the bound on the six point function.

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[5]If e.g. $n_1 = 2$ for the first term, we use the bound (11) and then

$$\frac{\kappa^{-1}}{\big(\sup(\kappa,|p|)\big)^2}\,\big(\sup(|p|,\kappa)\big)^2\,\mathcal{P}_{r,2}\big(\log(\sup(\frac{|p|}{\kappa},\frac{\kappa}{m})))\big) \le \kappa^{-1}\,\mathcal{P}_{r,2}\big(\log(\sup(\frac{|p|}{\kappa},\frac{\kappa}{m})))\big)\ ,$$

and retain the contribution of the second term to verify the bound as in (17,18).

[6]When taking into account the Remark after Proposition 2, one finds that for $n = 6$, $r = 2$ only the last of the 3 weight factors above appears, a fact in accord with (trivial) direct calculation. For $r \ge 3$ we again obtain all 3 types of trees.

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