A COMBINATORIAL APPROACH TO COLOURFUL SIMPLICIAL DEPTH

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Abstract. The colourful simplicial depth conjecture states that any point in the convex hull of each of $d+1$ sets, or colours, of $d+1$ points in general position in $\mathbb{R}^d$ is contained in at least d^2+1 simplices with one vertex from each set. We verify the conjecture in dimension 4 and strengthen the known lower bounds in higher dimensions. These results are obtained using a combinatorial generalization of colourful point configurations called octahedral systems. We present properties of octahedral systems generalizing earlier results on colourful point configurations and exhibit an octahedral system which cannot arise from a colourful point configuration. The number of octahedral systems is also given.

1. Introduction

1.1. **Preliminaries.** An n-uniform hypergraph is said to be *n-partite* if its vertex set is the disjoint union of n sets $V_1, \ldots, V_n$ and each edge intersects each V_i at exactly one vertex. Such a hypergraph is an $(n+1)$-tuple $(V_1, \ldots, V_n, E)$ where E is the set of edges. An *octahedral system* Ω is a simple n-uniform n-partite hypergraph $(V_1, \ldots, V_n, E)$ with $|V_i| \geq 2$ for $i = 1, \ldots, n$ and satisfying the following *parity condition*: the number of edges of Ω induced by $X \subseteq \bigcup_{i=1}^n V_i$ is even if $|X \cap V_i| = 2$ for $i = 1, \ldots, n$. Simple means that there are no two edges with same vertex set.

A *colourful point configuration* in $\mathbb{R}^d$ is a collection of $d+1$ sets, or colours, $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$. A *colourful simplex* is defined as the convex hull of a subset S of $\bigcup_{i=1}^{d+1} \mathbf{S}_i$ with $|S \cap \mathbf{S}_i| = 1$ for $i = 1, \ldots, d+1$. The Octahedron Lemma [3, 6] states that, given a subset $X \subseteq \bigcup_{i=1}^{d+1} \mathbf{S}_i$ of points such that $|X \cap \mathbf{S}_i| = 2$ for $i = 1, \ldots, d+1$, there is an even number of colourful simplices generated by X and containing the origin $\mathbf{0}$. Therefore, the hypergraph $\Omega = (V_1, \ldots, V_{d+1}, E)$, with $V_i = \mathbf{S}_i$ for $i = 1, \ldots, d+1$ and where the edges in E correspond to the colourful simplices containing $\mathbf{0}$ forms an octahedral system. This property motivated Bárány to suggest octahedral systems as a combinatorial generalization of colourful point configurations, see [8].

Let $\mu(d)$ denote the minimum number of colourful simplices containing $\mathbf{0}$ over all colourful point configurations satisfying $\mathbf{0} \in \bigcap_{i=1}^{d+1} \mathrm{conv}(\mathbf{S}_i)$ and $|\mathbf{S}_i| = d+1$ for $i = 1, \ldots, d+1$. Bárány's colourful Carathéodory theorem [2] states that $\mu(d) \geq 1$. The quantity $\mu(d)$ was investigated in [6] where it is shown that $2d \leq \mu(d) \leq d^2+1$, that $\mu(d)$ is even for odd d, and that $\mu(2) = 5$. This paper also conjectures that $\mu(d) = d^2+1$ for all $d \geq 1$. Subsequently, Bárány and Matoušek [3] verified the conjecture for $d = 3$ and provided a lower bound of $\mu(d) \geq \max(3d, \left\lceil \frac{d(d+1)}{5} \right\rceil)$ for

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$d \geq 3$, while Stephen and Thomas [16] independently proved that $\mu(d) \geq \left\lfloor \frac{(d+2)^2}{4} \right\rfloor$, before Deza, Stephen, and Xie [8] showed that $\mu(d) \geq \left\lceil \frac{(d+1)^2}{2} \right\rceil$. The lower bound was slightly improved in dimension 4 to $\mu(4) \geq 14$ via a computational approach presented in [9].

An octahedral system arising from a colourful point configuration $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$, such that $\mathbf{0} \in \bigcap_{i=1}^{d+1} \mathrm{conv}(\mathbf{S}_i)$ and $|\mathbf{S}_i| = d+1$ for all i, is without isolated vertices; that is, each vertex belongs to at least one edge. Indeed, according to a strengthening of the colourful Carathéodory theorem [2], any point of such a colourful configuration is the vertex of at least one colourful simplex containing $\mathbf{0}$. Theorem 1, whose proof is given in §4, provides a lower bound for the number of edges of an octahedral system without isolated vertices.

Theorem 1. *An octahedral system without isolated vertices and with* $|V_1| = |V_2| = \ldots = |V_n| = m$ *has at least* $\frac{1}{2}m^2 + \frac{5}{2}m - 11$ *edges for* $4 \leq m \leq n$ *.*

Setting $m = n = d+1$ in Theorem 1 yields a lower bound for $\mu(d)$ given in Corollary 1.

Corollary 1. $\mu(d) \geq \frac{1}{2}d^2 + \frac{7}{2}d - 8$ *for* $d \geq 3$*.*

Corollary 1 improves the known lower bounds for $\mu(d)$ for all $d \geq 5$. Refining the combinatorial approach for small instances in §5, we show that $\mu(4) = 17$, i.e. the conjectured equality $\mu(d) = d^2 + 1$ holds in dimension 4, see Proposition 12. Properties of octahedral systems generalizing earlier results on colourful point configurations are presented in §2. We answer open questions raised in [5] in §3 by determining in Theorem 2 the number of distinct octahedral systems with given $|V_i|$'s, and by showing that the octahedral system given in Figure 3 cannot arise from a colourful point configuration.

Bárány's sufficient condition for the existence of a colourful simplex containing $\mathbf{0}$ has been recently generalized in [1, 11, 14]. The related algorithmic question of finding a colourful simplex containing $\mathbf{0}$ is presented and studied in [4, 7]. We refer to [10, 13] for a recent breakthrough for a monocolour version.

1.2. **Definitions.** Let $E[X]$ denote the set of edges induced by a subset X of the vertex set $\bigcup_{i=1}^n V_i$ of an octahedral system $\Omega = (V_1, \ldots, V_n, E)$. The degree of X, denoted by $\deg_\Omega(X)$, is the number of edges containing X. An octahedral system $\Omega = (V_1, \ldots, V_n, E)$ with $|V_i| = m_i$ for $i = 1, \ldots, n$ is called a $(m_1, \ldots, m_n)$-octahedral system. Given an octahedral system $\Omega = (V_1, \ldots, V_n, E)$, a subset $T \subseteq \bigcup_{j=1}^n V_j$ is a *transversal* of Ω if $|T| = n-1$ and $|T \cap V_j| \leq 1$ for $j = 1, \ldots, n$. The set T is called an $\hat{\imath}$*-transversal* if i is the unique index such that $|T \cap V_i| = 0$. Let $\nu(m_1, \ldots, m_n)$ denote the minimum number of edges over all $(m_1, \ldots, m_n)$-octahedral systems without isolated vertices. The minimum number of edges over all $(d+1, \ldots, d+1)$-octahedral systems has been considered by Deza et al. [5] where this quantity is denoted by $\nu(d)$. By a slight abuse of notation, we identify $\nu(d)$ with $\nu(\underbrace{d+1, \ldots, d+1}_{d+1 \text{ times}})$. We have $\mu(d) \geq \nu(d)$, and the inequality is conjectured to hold with equality.

Throughout the paper, given an octahedral system $\Omega = (V_1, \ldots, V_n, E)$, the *parity property* refers to the evenness of $|E[X]|$ if $|X \cap V_i| = 2$ for $i = 1, \ldots, n$. In a slightly weaker form, the parity property refers to the following observation: If e is an edge, T an $\hat{\imath}$-transversal disjoint from e, and x a vertex in $V_i \setminus e$, then there is an edge distinct from e in $e \cup T \cup \{x\}$. Indeed, $|(e \cup T \cup \{x\}) \cap V_j| = 2$ for $j = 1, \dots, n$ implies that the number of edges in $E[e \cup T \cup \{x\}]$ is even. An octahedral system being a simple hypergraph, there in an edge distinct from e in $e \cup T \cup \{x\}$.

Let $D(\Omega)$ be the directed graph (V, A) associated to $\Omega = (V_1, \dots, V_n, E)$ with vertex set $V := \bigcup_{i=1}^n V_i$ and where (u, v) is an arc in A if, whenever $v \in e \in E$, we have $u \in e$. In other words, (u, v) is an arc of $D(\Omega)$ if any edge containing v contains u as well.

For an arc $(u, v) \in A$, v is an *outneighbour* of u, and u is an *inneighbour* of v. The set of all outneighbours of u is denoted by $N^+_{D(\Omega)}(u)$. Let $N^+_{D(\Omega)}(X) = \left(\bigcup_{u \in X} N^+_{D(\Omega)}(u)\right) \setminus X$; that is, the subset of vertices, not in X, being heads of arcs in A having tail in X. The *outneighbours* of a set X are the elements of $N^+_{D(\Omega)}(X)$. Note that $D(\Omega)$ is a transitive directed graph: if (u, v) and (v, w) with $w \neq u$ are arcs of $D(\Omega)$, then (u, w) is an arc of $D(\Omega)$. In particular, it implies that there is always a nonempty subset X of vertices without outneighbours inducing a complete subgraph in $D(\Omega)$. Moreover, a vertex of $D(\Omega)$ cannot have two distinct inneighbours in the same V_i.

2. Combinatorial properties of octahedral systems

This section presents properties of octahedral systems generalizing earlier results holding for $n = |V_1| = \ldots = |V_n| = d + 1$. While Proposition 1 and Proposition 2 deal with octahedral systems possibly with isolated vertices, Propositions 3, 4, 5, and 6 deal with octahedral systems without isolated vertices.

Proposition 1. *An octahedral system $\Omega = (V_1, \dots, V_n, E)$ with even $|V_i|$ for $i = 1, \dots, n$ has an even number of edges.*

This proposition provides an alternate definition for octahedral systems where the condition "$|X \cap V_i| = 2$" is replaced by "$|X \cap V_i|$ is even" for $i = 1, \dots, n$.

Proof. Let Ξ be the set $\{X \subseteq \bigcup_{i=1}^n V_i : |X \cap V_i| = 2\}$. Since Ω satisfies the parity property, $|E[X]|$ is even for any $X \in \Xi$, and $\sum_{X \in \Xi} |E[X]|$ is even. Each edge of Ω being counted $(|V_1| - 1)(|V_2| - 1) \dots (|V_n| - 1)$ times in the sum, we have $\sum_{X \in \Xi} |E[X]| = (|V_1| - 1) \dots (|V_n| - 1)|E|$. As $(|V_1| - 1) \dots (|V_n| - 1)$ is odd, the number $|E|$ of edges in Ω is even. □

Proposition 2. *Besides the trivial octahedral system without edges, an octahedral system has at least* $\min_i |V_i|$ *edges.*

Proof. Assume without loss of generality that V_1 has the smallest cardinality. If no vertex of V_1 is isolated, the octahedral system has at least $|V_1|$ edges. Otherwise, at least one vertex x of V_1 is isolated and the parity property applied to an edge, $(|V_1| - 1)$ disjoint $\hat{1}$-transversals, and x gives at least $|V_1|$ edges. The bound is tight as a $\hat{1}$-transversal forming an edge with each vertex of V_1 is an octahedral system with $|V_1|$ edges. □

Setting $n = |V_1| = \ldots = |V_n| = d + 1$ in Proposition 1 and Proposition 2 yields results given in [5].

Proposition 3. *An octahedral system without isolated vertices has at least* $\max_{i \neq j}(|V_i| + |V_j|) - 2$ *edges.*

The special case for octahedral systems arising from colourful point configurations, i.e. $\mu(d) \geq 2d$, has been proven in [6].

Proof. Assume without loss of generality that $2 \leq |V_1| \leq \ldots \leq |V_{n-1}| \leq |V_n|$. Let v^* be the vertex minimizing the degree in Ω over V_n. If $\deg(v^*) \geq 2$, then there are at least $2|V_n| \geq |V_n| + |V_{n-1}| - 2$ edges. Otherwise, $\deg(v^*) = 1$ and we note $e(v^*)$ the unique edge containing v^*. Pick w_i in $V_i \setminus e(v^*)$ for all $i < n$. Applying the octahedral property to the transversal $\{w_1, \ldots, w_{n-1}\}$, $e(v^*)$, and any $w \in V_n \setminus \{v^*\}$ yields at least $|V_n|$ edges not intersecting with $V_{n-1} \setminus (e(v^*) \cup \{w_{n-1}\})$. In addition, $|V_{n-1}| - 2$ edges are needed to cover the vertices in $V_{n-1} \setminus (e(v^*) \cup \{w_{n-1}\})$. In total we have at least $|V_n| + |V_{n-1}| - 2$ edges. $\square$

The remaining of the section deals with upper bounds for $\nu(m_1, \ldots, m_n)$.

Proposition 4. $\nu(m_1, \ldots, m_n) \leq 2 + \sum_{i=1}^{n}(m_i - 2)$.

Proof. For all $(m_1, \ldots, m_n)$, we construct an octahedral system $\Omega^{(m_1,\ldots,m_n)} = (V_1, \ldots, V_n, E^{(m_1,\ldots,m_n)})$ without isolated vertices and with $|V_i| = m_i$, such that

$$|E^{(m_1,\ldots,m_n)}| = 2 + \sum_{i=1}^{n}(m_i - 2).$$

Starting from $\Omega^{(m_1)}$, we inductively build $\Omega^{(m_1,\ldots,m_{n+1})}$ from $\Omega^{(m_1,\ldots,m_n)}$.

The unique octahedral system without isolated vertices with $n = 1$ and $|V_1| = m_1$ is $\Omega^{(m_1)} = (V_1, E^{(m_1)})$ where $E^{(m_1)} = \{\{v\} : v \in V_1\}$. Assuming that $\Omega^{(m_1,\ldots,m_n)} = (V_1, \ldots, V_n, E^{(m_1,\ldots,m_n)})$ with $|E^{(m_1,\ldots,m_n)}| = 2 + \sum_{i=1}^{n}(m_i - 2)$ has been built, we build the octahedral system $\Omega^{(m_1,\ldots,m_{n+1})} = (V_1, \ldots, V_n, V_{n+1}, E^{(m_1,\ldots,m_{n+1})})$ by picking an edge e_1 in $E^{(m_1,\ldots,m_n)}$ and setting

$$E^{(m_1,\ldots,m_{n+1})} = \{e_1 \cup \{u_i\} : i = 1, \ldots, m_{n+1} - 1\} \cup \{e \cup \{u_{m_{n+1}}\} : e \in E^{(m_1,\ldots,m_n)} \setminus \{e_1\}\}$$

where $u_1, \ldots, u_{m_{n+1}}$ are the vertices of V_{n+1}. Clearly, $|E^{(m_1,\ldots,m_{n+1})}| = m_{n+1} - 1 + |E^{(m_1,\ldots,m_n)}| - 1$; that is, $|E^{(m_1,\ldots,m_{n+1})}| = 2 + \sum_{i=1}^{n+1}(m_i - 2)$. Each vertex of $\Omega^{(m_1,\ldots,m_{n+1})}$ belongs to at least one edge by construction and we need to check the parity condition. Let $X \subseteq \bigcup_{i=1}^{n} V_i$ such that $|X \cap V_i| = 2$ for $i = 1, \ldots, n+1$ and consider the following four cases:

Case (a): $X \cap V_{n+1} = \{u_j, u_k\}$ with $j \neq m_{n+1}$ and $k \neq m_{n+1}$, and $e_1 \subseteq X$. Then, $e_1 \cup \{u_j\}$ and $e_1 \cup \{u_k\}$ are the only two edges induced by X in $\Omega^{(m_1,\ldots,m_{n+1})}$.

Case (b): $X \cap V_{n+1} = \{u_j, u_k\}$ with $j \neq m_{n+1}$ and $k \neq m_{n+1}$, and $e_1 \not\subseteq X$. Then, no edges are induced by X in $\Omega^{(m_1,\ldots,m_{n+1})}$.

Case (c): $X \cap V_{n+1} = \{u_j, u_{m_{n+1}}\}$ and $e_1 \subseteq X$. Then, the number of edges in $E^{(m_1,\ldots,m_n)} \setminus \{e_1\}$ induced by X in $\Omega^{(m_1,\ldots,m_n)}$ is odd by the parity property. Hence, the number of edges in $\{e \cup \{u_{m_{n+1}}\} : e \in E^{(m_1,\ldots,m_n))} \setminus \{e_1\}\}$ induced by X in $\Omega^{(m_1,\ldots,m_{n+1})}$ is odd as well. These edges, along with the edge $e_1 \cup \{u_j\}$, are the only edges induced by X in $\Omega^{(m_1,\ldots,m_{n+1})}$, i.e. the parity condition holds.

Case (d): $X \cap V_{n+1} = \{u_j, u_{m_{n+1}}\}$ and $e_1 \not\subseteq X$. Then, the number of edges in $E^{(m_1,\ldots,m_n)} \setminus \{e_1\}$ induced by X in $\Omega^{(m_1,\ldots,m_n)}$ is even by the parity property.

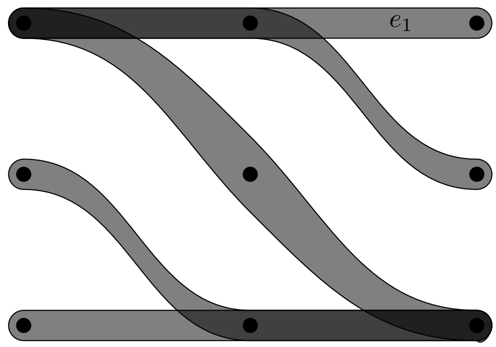

FIGURE 1. $\Omega^{(3,3)}$: a $(3,3,3)$-octahedral system matching the upper bound given in Proposition 4

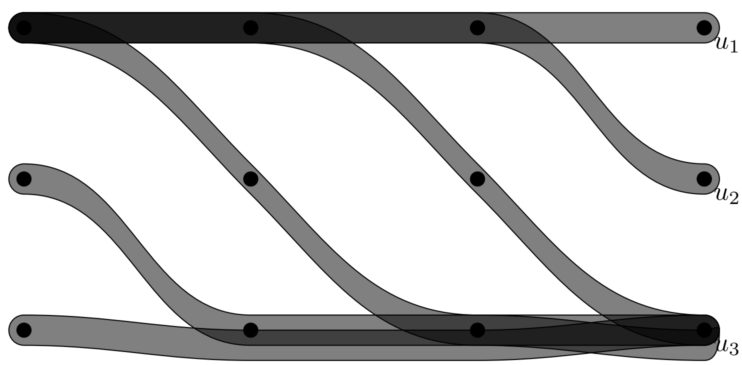

FIGURE 2. $\Omega^{(3,4)}$: a $(3,3,3,3)$-octahedral system matching the upper bound given in Proposition 4

Hence, the number of edges in $\{e \cup \{u_{m_{n+1}}\} : e \in E^{(m_1,\dots,m_n)} \setminus \{e_1\}\}$ induced by X in $\Omega^{(m_1,\dots,m_{n+1})}$ is even as well. These edges are the only edges induced by X in $\Omega^{(m_1,\dots,m_{n+1})}$, i.e. the parity condition holds. □

Figures 1 and 2 illustrate the construction in the proof of Proposition 4 for $n = m_1 = m_2 = m_3 = 3$, and for $n-1 = m_1 = m_2 = m_3 = m_4 = 3$.

Proposition 4 combined with Proposition 3 directly implies Proposition 5.

Proposition 5. $\nu(2,\dots,2,m_{n-1},m_n) = m_{n-1} + m_n - 2$ *for* $m_{n-1}, m_n \geq 2$.

When all m_i are equal, the bound given in Proposition 4 can be improved as follows.

Proposition 6. $\nu(\overbrace{m,\dots,m}^{n \text{ times}}) \leq \min(m^2, n(m-2)+2)$ *for all* $m, n \geq 1$.

Proof. We construct an $(m,\dots,m)$-octahedral system without isolated vertices and with m^2 edges. Consider m disjoint $\hat{n}$-transversals, and form m edges from each of these $\hat{n}$-transversals by adding a distinct vertex of V_n. We obtain an octahedral system without isolated vertices with m^2 edges. The other inequality is a corollary of Proposition 4. □

Propositions 4 and 6 can be seen as combinatorial counterparts and generalizations of $\mu(d) \leq d^2 + 1$ proven in [6].

An approach similar to the one developed in Section 5 shows that

$$\nu(\overbrace{2,\ldots,2}^{z \text{ times}}, \overbrace{3,\ldots,3}^{4-z \text{ times}}, 4) = 8 - z \text{ and } \nu(\overbrace{3,\ldots,3}^{z \text{ times}}, \overbrace{4,\ldots,4}^{5-z \text{ times}}) = 12 - z \text{ for } z = 0,\ldots,4.$$

In other words, the inequality given in Proposition 4 holds with equality for small m_i's and n at most 5. While this inequality also holds with equality for any n when $m_1 = \ldots = m_{n-2} = 2$ by Proposition 5, the inequality can be strict as, for example, $\nu(3,\ldots,3) < 2 + n$ for $n \geq 8$ by Proposition 6.

3. Additional results

This section provides answers to open questions raised in [5] by determining the number of distinct octahedral systems and by showing that some octahedral systems cannot arise from a colourful point configuration. We first remark that the symmetric difference of two octahedral systems forms an octahedral system.

Proposition 7. *Let $\Omega_1 = (V_1,\ldots,V_n,E_1)$ and $\Omega_2 = (V_1,\ldots,V_n,E_2)$ be two octahedral systems on the same sets of vertices, the symmetric difference $\Omega_1 \triangle \Omega_2 = (V_1,\ldots,V_n,E_1 \triangle E_2)$ is an octahedral system.*

Proof. Consider $X \subseteq \bigcup_{i=1}^n V_i$ such that $|X \cap V_i| = 2$ for $i = 1,\ldots,n$. We have $|(E_1 \triangle E_2)[X]| = |E_1[X]| + |E_2[X]| - 2|(E_1 \cap E_2)[X]|$, and therefore the parity condition holds for $\Omega_1 \triangle \Omega_2$. □

Proposition 7 can be used to build octahedral systems or to prove the non-existence of others. For instance, Proposition 7 implies that there is a $(3,3,3)$-octahedral system without isolated vertices with exactly 22 edges by setting Ω_1 to be the complete $(3,3,3)$-octahedral system with 27 edges, and Ω_2 to be the $(3,3,3)$-octahedral system with exactly 5 edges given in Figure 1. The octahedral system $\Omega_1 \triangle \Omega_2$ is without isolated vertices since each vertex in Ω_1 is of degree 9. Similarly, Proposition 7 shows that no $(3,3,3)$-octahedral system with exactly 25 or 26 edges exists. Otherwise a $(3,3,3)$-octahedral system with exactly 1 or 2 edges would exist, contradicting Proposition 2.

Proposition 7 shows that the set of all octahedral systems defined on the same V_i's equipped with the symmetric difference as addition is an $\mathbb{F}_2$ vector space. We further specify the structure of this $\mathbb{F}_2$ vector space by giving a generating set. Let F_i denote the binary vector space $\mathbb{F}_2^{V_i}$ and $\mathcal{H}$ denote the tensor product $F_1 \otimes \ldots \otimes F_n$. There is a one to one mapping between the elements of $\mathcal{H}$ and the simple n-uniform n-partite hypergraphs on vertex sets $V_1,\ldots,V_n$. Each edge $\{v_1,\ldots,v_n\}$ of such a hypergraph H with $v_i \in V_i$ for all i is identified with the vector $x_1 \otimes \ldots \otimes x_n$ where x_i is the unit vector of F_i having a 1 at position v_i and 0 elsewhere.

Proposition 8. *The subspace of $\mathcal{H}$ generated by the vectors of the form $x_1 \otimes \ldots \otimes x_{j-1} \otimes e \otimes x_{j+1} \otimes \ldots \otimes x_n$, with $j \in \{1,\ldots,n\}$ and $e = (1,\ldots,1) \in \mathbb{F}_2^{V_j}$, forms precisely the set of all octahedral systems.*

Proof. Each of these vectors is an octahedral system, and so are the linear combinations of these vectors. Conversely, any octahedral system is a linear sum of such vectors. Indeed, given an octahedral system and one of its vertices v of nonzero degree, we can add vectors of the above form in order to make v isolated. Repeating this argument for each V_i, we get an octahedral system with an isolated vertex in each V_i. Such an octahedral system is empty; that is, is the zero vector of the space of octahedral systems. □

Karasev [12] noted that the set of all colourful simplices in a colourful point configuration forms a d-dimensional coboundary of the join $\mathbf{S}_1 * \ldots * \mathbf{S}_{d+1}$ with mod 2 coefficients, see [15] for precise definitions of joins and coboundaries. With the help of Proposition 8, we further note that the octahedral systems form precisely the $(n-1)$-coboundaries of the join $V_1 * \ldots * V_n$ with mod 2 coefficients. Indeed, the vectors of the form $x_1 \otimes \ldots \otimes x_{j-1} \otimes \hat{x}_j \otimes x_{j+1} \otimes \ldots \otimes x_n$, with $j \in \{1, \ldots, n\}$, generate the $(n-2)$-cochains of $V_1 * \ldots * V_n$, and the coboundary of a vector $x_1 \otimes \ldots \otimes x_{j-1} \otimes \hat{x}_j \otimes x_{j+1} \otimes \ldots \otimes x_n$ is $x_1 \otimes \ldots \otimes x_{j-1} \otimes e \otimes x_{j+1} \otimes \ldots \otimes x_n$ with $e = (1, \ldots, 1) \in \mathbb{F}_2^{V_j}$.

Theorem 2. *Given n disjoint finite vertex sets $V_1, \ldots, V_n$, the number of octahedral systems on $V_1, \ldots, V_n$ is $2^{\Pi_{i=1}^n |V_i| - \Pi_{i=1}^n (|V_i|-1)}$.*

Proof. We denote by G_i the subspace of F_i whose vectors have an even number of 1's. Let $\mathcal{X}$ be the tensor product $G_1 \otimes \ldots \otimes G_n$. Define now ψ as follows.

$$\begin{array}{cccc} \psi: & \mathcal{H} & \rightarrow & \mathcal{X}^* \\ & H & \mapsto & \langle H, \cdot \rangle \end{array}$$

By the above identification between $\mathcal{H}$ and the hypergraphs and according to the alternate definition of an octahedral system given by Proposition 1, the subspace $\ker \psi$ of $\mathcal{H}$ is the set of all octahedral systems on vertex sets $V_1, \ldots, V_n$. Note that by definition ψ is surjective. Therefore, we have $\dim \ker \psi + \dim \mathcal{X}^* = \dim \mathcal{H}$ which implies $\dim \ker \psi = \dim \mathcal{H} - \dim \mathcal{X}$ using the isomorphism between a vector space and its dual. The dimension of $\mathcal{H}$ is $\Pi_{i=1}^n |V_i|$ and the dimension of $\mathcal{X}$ is $\Pi_{i=1}^n (|V_i| - 1)$. This leads to the desired conclusion. □

Two isomorphic octahedral systems, that is, identical up to a permutation of the V_i's, or of the vertices in one of the V_i's, are considered distinct in Theorem 2, which means that we are counting *labelled* octahedral systems. A natural question is whether there is a non-labelled version of Theorem 2, that is whether it is possible to compute, or to bound, the number of non-isomorphic octahedral systems. Answering this question would fully answer Question 7 of [5].

Finally, Question 6 of [5] asks whether any octahedral system $\Omega = (V_1, \ldots, V_n, E)$ with $n = |V_1| = \ldots = |V_n| = d+1$ can arise from a colourful point configuration $\mathbf{S}_1, \ldots, \mathbf{S}_{d+1}$ in $\mathbb{R}^d$? That is, are all octahedral systems *realisable*? We give a negative answer to this question in Proposition 9.

Proposition 9. *Not all octahedral systems are realisable.*

Proposition 9 also holds for octahedral systems without isolated vertices.

Proof. We provide an example of a non-realisable octahedral system without isolated vertices in Figure 3. Indeed, suppose by contradiction that this octahedral system can be realized as a colourful point configuration $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$. Without loss of generality, we can assume that all the points lie on a circle centred at $\mathbf{0}$. Take $x_3 \in \mathbf{S}_3$, and consider the line ℓ going through x_3 and $\mathbf{0}$. There are at least two points x_1 and x_1' of $\mathbf{S}_1$ on the same side of ℓ. There is a point $x_2 \in \mathbf{S}_2$, respectively $x_2' \in \mathbf{S}_2$, on the other side of the line ℓ such that $\mathbf{0} \in \operatorname{conv}(x_1, x_2, x_3)$, respectively

$\mathbf{0} \in \text{conv}(x_1', x_2', x_3)$. Assume without loss of generality that x_2' is further away from x_3 than x_2. Then, $\text{conv}(x_1, x_2', x_3)$ contains $\mathbf{0}$ as well, contradicting the definition of the octahedral system given in Figure 3. □

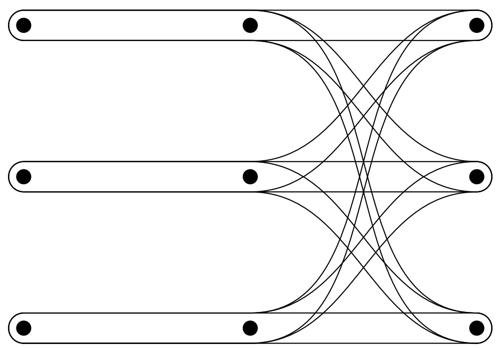

FIGURE 3. A non realisable $(3, 3, 3)$-octahedral system with 9 edges

We conclude the section with a question to which the intuitive answer is yes but we are unable to settle.

Question 1. *Is $\nu(m_1, \ldots, m_n)$ non-decreasing with each of the m_i?*

4. Proof of the main result

4.1. **Technical lemmas.** While Lemma 1 allows induction within octahedral systems, Lemmas 2, 3, and 4 are used in the subsequent sections to bound the number of edges of an octahedral system without isolated vertices.

If a subset X of the vertex set $\bigcup_{i=1}^n V_i$ of an octahedral system satisfies $|V_i \setminus X| \geq 2$ for all $i = 1, \ldots, n$, then the subhypergraph induced by $(\bigcup_{i=1}^n V_i) \setminus X$ is an octahedral system as well. Indeed, the parity property is clearly satisfied for this subhypergraph.

Lemma 1. *Consider an octahedral system Ω without isolated vertices. Let X be a subset of $\bigcup_{i=1}^n V_i$ inducing a complete subgraph in $D(\Omega)$, such that $|V_i \setminus X| \geq 2$ for all $i = 1, \ldots, n$. Let Ω' be the octahedral system induced by $(\bigcup_{i=1}^n V_i) \setminus X$. If $N^+_{D(\Omega)}(X) = \emptyset$, then Ω' is without isolated vertices.*

Proof. Each vertex v of Ω' is contained in at least one edge. Since X induces a complete subgraph, any edge of Ω intersecting X contains the whole subset X. Thus, since $v \notin N^+_{D(\Omega)}(X)$, the vertex v is in an edge of Ω disjoint from X. □

Lemma 2. *For $n \geq 4$, consider a $(\overbrace{k-1, \ldots, k-1}^{z \text{ times}}, \overbrace{k, \ldots, k}^{k-z \text{ times}}, m_{k+1}, \ldots, m_n)$-octahedral system $\Omega = (V_1, \ldots, V_n, E)$ without isolated vertices, with $3 \leq k \leq m_{k+1} \leq \ldots \leq m_n$ and $0 \leq z < k \leq n$. If there is a subset $X \subseteq \bigcup_{i=z+1}^n V_i$ of cardinality at least 2 inducing in $D(\Omega)$ a complete subgraph, then Ω has at least $(k-1)^2 + 2$ edges, unless Ω is a $(2, 2, 3, 3)$-octahedral system. Under the same condition on X, a $(2, 2, 3, 3)$-octahedral system has at least 5 edges.*

Proof. Any edge intersecting X contains X since X induces a complete subgraph in $D(\Omega)$, implying $\deg_\Omega(X) \geq 1$. Moreover, we have $|X \cap V_i| \leq 1$ for $i = 1, \ldots, n$.

Case (a): $\deg_\Omega(X) \geq 2$. Choose i^* such that $|X \cap V_{i^*}| \neq 0$. We first note that the degree of each w in $V_{i^*} \setminus X$ is at least $k-1$.

Indeed, take an edge e containing w and a $\widehat{i^*}$-transversal T disjoint from e and X. Note that e does not contain any vertex of X as underlined in the first sentence of the proof. Apply the weak form of the parity property to e, T, and the unique vertex x in $X \cap V_{i^*}$. There is an edge distinct from e in $e \cup T \cup \{x\}$. Note that this edge contains w, otherwise it would contain x and any other vertex in X. It also contains at least one vertex in T. For a fixed e, we can actually choose $k-2$ disjoint $\widehat{i^*}$-transversals T of that kind and apply the weak form of the parity property to each of them. Thus, there are $k-2$ distinct edges containing w in addition to e.

Therefore, we have in total at least $(k-1)^2$ edges, in addition to $\deg_\Omega(X) \geq 2$ edges.

Case (b): $\deg_\Omega(X) = 1$. Let $e(X)$ denote the unique edge containing X. For each i such that $|X \cap V_i| = 0$, pick a vertex w_i in $V_i \setminus e(X)$. Applying the weak form of the parity property to $e(X)$, the w_i's, and any colourful selection of $u_i \in V_i \setminus X$ when i is such that $|X \cap V_i| \neq 0$ shows that there is at least one additional edge containing all u_i's. We can actually choose $(k-1)^{|X|}$ distinct colourful selections of u_i's. With $e(X)$, there are in total $(k-1)^{|X|} + 1$ edges.

If $|X| \geq 3$, then $(k-1)^{|X|} + 1 \geq (k-1)^2 + 2$. If $|X| = 2$, there exists $j \geq n-2$ such that $|X \cap V_j| = 0$. If $|V_j| \geq 3$, then at least $|V_j| - 2 \geq 1$ edges are needed to cover the vertices of V_j not belonging to these $(k-1)^{|X|} + 1$ edges. Otherwise, $|V_j| = 2$ and we have $j \leq z$ and $k = 3$. In this case, we have thus $k - 1 \geq z \geq n - 2$, i.e. $n = 4$ and $z = 2$. Ω is then a $(2,2,3,3)$-octahedral system and $(k-1)^{|X|} + 1 = 5$. $\square$

While Lemma 3 is similar to Lemma 2, we were not able to find a common generalization.

Lemma 3. *Consider a* $(\overbrace{k-1, \ldots, k-1}^{z \text{ times}}, \overbrace{k, \ldots, k}^{k-z \text{ times}}, m_{k+1}, \ldots, m_n)$*-octahedral system* $\Omega = (V_1, \ldots, V_n, E)$ *without isolated vertices, with* $3 \leq k \leq m_{k+1} \leq \ldots \leq m_n$ *and* $0 \leq z < k \leq n$. *If there is a subset* $X \subseteq \bigcup_{i=z+1}^n V_i$ *of cardinality at least 2 inducing in* $D(\Omega)$ *a complete subgraph without outneighbours, then* Ω *has at least* $(k-1)^2 + |V_{n-1}| + |V_n| - 2k + 1$ *edges.*

Proof. Choose i^* such that $X \cap V_{i^*} \neq \emptyset$. Choose $W_{i^*} \subseteq V_{i^*} \setminus X$ of cardinality $k-1$. For each vertex $w \in W_{i^*}$, choose an edge $e(w)$ containing w. Let v^* be the vertex v^* minimizing the degree in Ω over $V_n \setminus X$. Since X induces a complete subgraph without outneighbours, there is at least one edge disjoint from X containing v^*. We can therefore assume that there is a vertex $w^* \in W_{i^*}$ such that $e(w^*)$ contains v^*. Choose $W_i \subseteq V_i$ for $i \neq i^*$ such that $|W_i| = k-1$ and

$$\bigcup_{w \in W_{i^*}} e(w) \subseteq W = \bigcup_{i=1}^{n} W_i.$$

Case (a): the degree of v^* in Ω is at most $k-2$. For all $w \in W_{i^*}$, applying the parity property to $e(w)$, the unique vertex of $X \cap V_{i^*}$, and $k-2$ disjoint $\widehat{i^*}$-transversals in W yields $(k-1)^2$ distinct edges, in a similar way as in Case (a) of the proof of Lemma 2. Applying the weak form of the parity property to $e(w^*)$, any $\hat{n}$-transversal in W not intersecting the neighbourhood of v^* in Ω, and each vertex in $V_n \setminus W_n$ gives $|V_n|-k+1$ additional edges not intersecting $V_{n-1} \setminus W_{n-1}$. In addition, $|V_{n-1}|-k+1$ edges are needed to cover the vertices of $V_{n-1} \setminus W_{n-1}$. In total we have at least $(k-1)^2+|V_n|+|V_{n-1}|-2(k-1)$ edges.

Case (b): the degree of v^* in Ω is at least $k-1$. We have then at least $(k-1)(|V_n|-1)+1=(k-1)^2+(k-1)(|V_n|-k)+1 \geq (k-1)^2+|V_{n-1}|+|V_n|-2k+1$ edges. □

Lemma 4. *Consider a* $(\overbrace{k-1,\ldots,k-1}^{z \text{ times}}, \overbrace{k,\ldots,k}^{k-z \text{ times}}, m_{k+1},\ldots,m_n)$*-octahedral system* $\Omega=(V_1,\ldots,V_n,E)$ *without isolated vertices, with* $3 \leq k \leq m_{k+1} \leq \ldots \leq m_n$ *and* $0 \leq z < k \leq n$. *If there are at least two vertices of* V_n *having outneighbours in* $D(\Omega)$ *in the same* V_{i^*} *with* $i^* < k$*, then the octahedral system has at least* $|V_{i^*}|(k-1)+|V_{n-1}|+|V_n|-2k$ *edges.*

Proof. Let v and v' be the two vertices of V_n having outneighbours in V_{i^*}. Let u and u' be the two vertices in V_{i^*} with (v,u) and (v',u') forming arcs in $D(\Omega)$. Note that according to the basic properties of $D(\Omega)$, we have $u \neq u'$. For each vertex $w \in V_{i^*}$, choose an edge $e(w)$ containing w. We can assume that there is a vertex $w^* \in V_{i^*}$ such that $e(w^*)$ contains a vertex v^* in V_n of minimal degree in Ω.

Case (a): $|V_{i^*}|=k$. Choose $W_i \subseteq V_i$ such that $|W_i|=k-1$ for $i=1,\ldots,z$, $|W_i|=k$ for $i=z+1,\ldots,n$, and

$$\bigcup_{w \in V_{i^*}} e(w) \subseteq W = \bigcup_{i=1}^{n} W_i.$$

We first show that the degree of any vertex in V_{i^*} is at least $k-1$ in the hypergraph induced by W. Pick $w \in V_{i^*}$ and consider $e(w)$. If $v \in e(w)$, take $k-2$ disjoint $\widehat{i^*}$-transversals in W not containing v' and not intersecting with $e(w)$. In this case, we necessarily have $w \neq u'$ since $v' \notin e(w)$. Applying the weak form of the parity property to $e(w)$, u', and each of those $\widehat{i^*}$-transversals yields, in addition to $e(w)$, at least $k-2$ edges containing w. Otherwise, take $k-2$ disjoint $\widehat{i^*}$-transversals in W not containing v and not intersecting with $e(w)$, and apply the weak form of the the parity property to $e(w)$, u, and each of those $\widehat{i^*}$-transversals. Therefore, in both cases, the degree of w in the hypergraph induced by W is at least $k-1$.

Then, we add edges not contained in W. If the degree of v^* in Ω is at least 2, there are at least $2(|V_n|-k)$ distinct edges intersecting $V_n \setminus W_n$. Otherwise, the weak form of the parity property applied to $e(w^*)$, any $\hat{n}$-transversal in W, and each vertex in $V_n \setminus W_n$ provides $|V_n|-k$ additional edges not intersecting $V_{n-1} \setminus W_{n-1}$. Therefore, $|V_{n-1}|-k$ additional edges are needed to cover these vertices of $V_{n-1} \setminus W_{n-1}$.

In total, we have at least $k(k-1)+|V_{n-1}|+|V_n|-2k$ edges.

Case (b): $|V_{i^*}| = k-1$. Choose $W_i \subseteq V_i$ such that $|W_i| = k-1$ for $i = 1, \ldots, n-1$, $|W_n| = k$, and

$$\bigcup_{w \in V_{i^*}} e(w) \subseteq W = \bigcup_{i=1}^{n} W_i.$$

Similarly, we show that the degree of any vertex in V_{i^*} is at least $k-1$ in the hypergraph induced by W. Pick $w \in V_{i^*}$ and consider $e(w)$. If $v \in e(w)$, take $k-2$ disjoint $\widehat{i^*}$-transversals in W not containing v' and not intersecting with $e(w)$. Applying the weak form of the parity property to $e(w)$, u', and each of those $\widehat{i^*}$-transversals yields, in addition to $e(w)$, at least $k-2$ edges containing w. Otherwise, take $k-2$ disjoint $\widehat{i^*}$-transversals in W not containing v and not intersecting with $e(w)$, and apply the weak form of the the parity property to $e(w)$, u, and each of those $\widehat{i^*}$-transversals. Therefore, in both cases, the degree of w in the hypergraph induced by W is at least $k-1$.

Then, we add edges not contained in W. If the degree of v^* in Ω is at least 2, there are at least $2(|V_n|-k)$ distinct edges intersecting $V_n \setminus W_n$. Otherwise, the weak form of the parity property applied to $e(w^*)$, any $\hat{n}$-transversal in W, and each vertex in $V_n \setminus W_n$ provides $|V_n| - k$ additional edges not intersecting $V_{n-1} \setminus W_{n-1}$. Therefore, $|V_{n-1}|-k+1$ additional edges are needed to cover these vertices of $V_{n-1} \setminus W_{n-1}$.

In total, we have at least $(k-1)^2 + |V_{n-1}| + |V_n| - 2k$ edges. □

4.2. **Proof of the main result.** Theorem 1 is obtained by setting $(k, z) = (m, 0)$ in Proposition 10. This proposition is proven by induction on the cardinality of octahedral systems of the form illustrated in Figure 4. Either the deletion of a vertex results in an octahedral system satisfying the condition of Proposition 10 and we can apply induction, or we apply Lemma 3 or Lemma 4 to bound the number of edges of the system. Lemma 1 is a key tool to determine if the deletion of a vertex results in an octahedral system satisfying the condition of Proposition 10.

Proposition 10. *A* $(\overbrace{k-1, \ldots, k-1}^{z \text{ times}}, \overbrace{k, \ldots, k}^{k-z \text{ times}}, m_{k+1}, \ldots, m_n)$*-octahedral system* $\Omega = (V_1, \ldots, V_n, E)$ *without isolated vertices, with* $2 \le k \le m_{k+1} \le \ldots \le m_n$ *and* $0 \le z < k \le n$*, has at least*

$$\begin{aligned} \tfrac{1}{2}k^2 + \tfrac{1}{2}k - 8 + |V_{n-1}| + |V_n| - z &\quad \text{edges if } k \le n-2, \\ \tfrac{1}{2}n^2 + \tfrac{1}{2}n - 10 + |V_n| - z &\quad \text{edges if } k = n-1, \\ \tfrac{1}{2}n^2 + \tfrac{5}{2}n - 11 - z &\quad \text{edges if } k = n. \end{aligned}$$

Proof. The proof works by induction on $\sum_{i=1}^{n} |V_i|$. The base case is $\sum_{i=1}^{n} |V_i| = 2n$, which implies $z = 0$ and $k = |V_{n-1}| = |V_n| = 2$. The three inequalities trivially hold in this case.

Suppose that $\sum_{i=1}^{n} |V_i| > 2n$. We choose a pair (k, z) compatible with Ω. Note that (k, z) is not necessarily unique. If $k = 2$, Proposition 3 proves the inequality. We can thus assume that $k \ge 3$. We consider the two possible cases for the associated $D(\Omega)$.

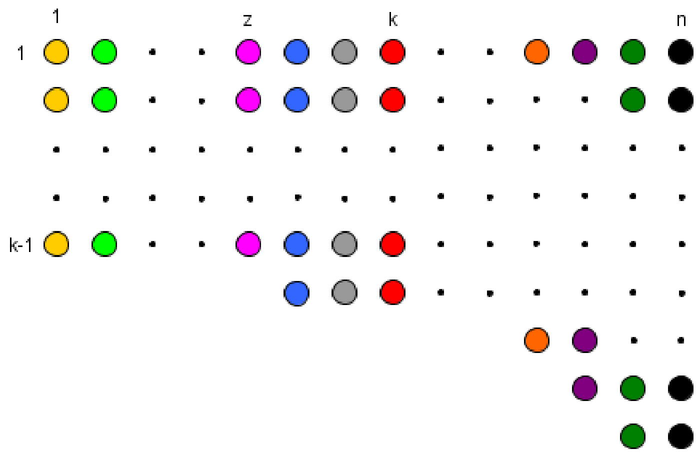

FIGURE 4. The vertex set of the $(k-1,\dots,k-1,k,\dots,k,m_{k+1},\dots,m_n)$-octahedral system $\Omega=(V_1,\dots,V_n,E)$ used for the proof of Proposition 10

If there are at least two vertices of V_n having an outneighbour in the same V_{i^*}, with $i^*<k$, we can apply Lemma 4. If $k\leq n-2$, the inequality follows by a straightforward computation, using that $z\geq 1$ when $|V_{i^*}|=k-1$; if $k=n-1$, we use the fact that $|V_{n-1}|=n-1$; and if $k=n$, we use the fact that $|V_{n-1}|\geq n-1$ and $|V_n|=n$.

Otherwise, for each $i<k$, there is at most one vertex of V_n having an outneighbour in V_i. Since $k-1<|V_n|$, there is a vertex x of V_n having no outneighbours in $\bigcup_{i=1}^{k-1}V_i$. Starting from x in $D(\Omega)$, we follow outneighbours until we reach a set X inducing a complete subgraph of $D(\Omega)$ without outneighbours. Since $D(\Omega)$ is transitive, we have $X\subseteq\bigcup_{i=k}^{n}V_i$. If $|X|\geq 2$, we apply Lemma 3. Thus, we can assume that $|X|=1$.

The subhypergraph Ω' of Ω induced by $(\bigcup_{i=1}^{n}V_i)\setminus X$ is an octahedral system without isolated vertices since X is a single vertex without outneighbours in $D(\Omega)$, see Lemma 1. Recall that the vertex in X belongs to $\bigcup_{i=k}^{n}V_i$. Let (k',z') be possible parameters associated to Ω' determined hereafter. Let i_0 be such that $X\subseteq V_{i_0}$. The induction argument is applied to the different values of $|V_{i_0}|$. It provides a lower bound on the number of edges in Ω'; adding 1 to this lower bound, we get a lower bound on the number of edges in Ω since there is at least one edge containing X.

If $|V_{i_0}|\geq k+1$, we have $(k',z')=(k,z)$ and we can apply the induction hypothesis with $|V_{n-1}|+|V_n|$ decreasing by at most one (in case $i_0=n-1$ or n) which is compensated by the edge containing X.

If $|V_{i_0}|=k$, $z\leq k-2$, and $k\leq n-1$, we have $(k',z')=(k,z+1)$ and we can apply the induction hypothesis with same $|V_{n-1}|$ and $|V_n|$ since $z\leq n-3$, while z'

replacing z takes away 1 which is compensated by the edge containing X.

If $|V_{i_0}| = k$, $z = k-1$, and $k \leq n-2$, we have $(k', z') = (k-1, 0)$ and we can apply the induction hypothesis with same $|V_{n-1}| + |V_n|$ since $z \leq n-3$. We get therefore $\frac{1}{2}(k-1)^2 + \frac{1}{2}(k-1) - 8 + |V_{n-1}| + |V_n|$ edges in Ω', plus at least one containing X. In total, we have $\frac{1}{2}k^2 + \frac{1}{2}k - 8 + |V_{n-1}| + |V_n| - k + 1$ edges in Ω, as required.

If $|V_{i_0}| = k$, $z = k-1$, and $k = n-1$, we have $(k', z') = (n-2, 0)$ and we can apply the induction hypothesis with $|V_{n-1}| + |V_n|$ decreasing by at most one. We get therefore $\frac{1}{2}(n-2)^2 + \frac{1}{2}(n-2) - 8 + |V_{n-1}| + |V_n| - 1$ edges in Ω', plus at least one containing X. Since $|V_{n-1}| = n-1$, we have in total $\frac{1}{2}n^2 + \frac{1}{2}n - 10 + |V_n| - (n-2)$ edges in Ω, as required.

If $|V_{i_0}| = k$, $z = k-1$, and $k = n$, we have $i_0 = n$ and $(k', z') = (n-1, 0)$. We can apply the induction hypothesis and get therefore $\frac{1}{2}n^2 + \frac{1}{2}n - 10 + (n-1)$ edges in Ω', plus at least one containing X. In total, we have $\frac{1}{2}n^2 + \frac{5}{2}n - 11 - (n-1)$ edges in Ω, as required.

If $|V_{i_0}| = k$, $z \leq k-2$, and $k = n$, we have $i_0 = n$. For Ω', the pair $(k', z') = (n, z+1)$ provides possible parameters. Note that in this case, the colors must be renumbered to keep them with non-decreasing sizes from 1 to n for Ω'. We can then apply the induction hypothesis and get therefore $\frac{1}{2}n^2 + \frac{5}{2}n - 11 - z - 1$ edges in Ω', plus at least one containing X. In total, we have $\frac{1}{2}n^2 + \frac{5}{2}n - 11 - z$ edges in Ω, as required. $\square$

Remark 1. A similar analysis, with $|V_i| = n$ for all i as a base case, shows that an octahedral system without isolated vertices and with $|V_1| = |V_2| = \ldots = |V_n| = m$ has at least $nm - \frac{1}{2}n^2 + \frac{5}{2}n - 11$ edges for $4 \leq n \leq m$.

5. Small instances and $\mu(4) = 17$

This section focuses on octahedral systems with m_i's and n at most 5.

Proposition 11. *$\nu(3,3,3,3) = 6$.*

Proof. We first prove that $\nu(2,3,3,3) = 5$. Let $\Omega = (V_1, V_2, V_3, V_4, E)$ be a $(2,3,3,3)$-octahedral system. In $D(\Omega)$ there is at most one vertex of V_4 having an outneighbour in V_1, otherwise one vertex of V_4 would be isolated. Thus, there is a subset $X \subseteq V_2 \cup V_3 \cup V_4$ inducing in $D(\Omega)$ a complete subgraph without outneighbours. If $|X| \geq 2$, applying Lemma 2 with $(k, z) = (3, 1)$ gives at least 5 edges in that case. If $|X| = 1$, deleting X yields a $(2,2,3,3)$-octahedral system without isolated vertices since X has no outneighbours in $D(\Omega)$. As $\nu(2,2,3,3) = 4$ by Proposition 5, we have at least $4 + 1 = 5$ edges. Thus, the equality holds since $\nu(2,3,3,3) \leq 5$ by Proposition 4.

We then prove that $\nu(3,3,3,3) = 6$. Let $\Omega = (V_1, V_2, V_3, V_4, E)$ be a $(3,3,3,3)$-octahedral system. There is a subset X inducing in $D(\Omega)$ a complete subgraph without outneighbours. If $|X| \geq 2$, apply Lemma 2 with $(k, z) = (3, 0)$ gives at least 6 edges in that case. If $|X| = 1$, deleting X yields a $(2,3,3,3)$-octahedral system without isolated vertices since X has no outneighbours in $D(\Omega)$. As $\nu(2,3,3,3) = 5$, we have at least $5+1=6$ edges. Thus, the equality holds since $\nu(3,3,3,3) \leq 6$ by Proposition 4. $\square$

The main result this section, namely $\nu(5,5,5,5,5)=17$, is proven via a series of claims dealing with octahedral systems of increasing sizes. We first determine the values of $\nu(2,2,3,3,3)$, $\nu(2,3,3,3,3)$, and $\nu(3,3,3,3,3)$ in Claims 1, 2, and 3. To complete the proof of $\nu(5,5,5,5,5)=17$, we sequentially show $\nu(3,3,3,3,4) \geq 7$, $\nu(4,4,4,4,4)=12$, and finally $\nu(5,5,5,5,5)=17$. A key step consists in proving $\nu(4,4,4,4,4) \geq 11$ by induction using $\nu(3,3,3,3,4) \geq 7$ as a base case. We obtain then $\nu(4,4,4,4,4)=12$ by Propositions 1 and 4. The equality $\nu(5,5,5,5,5)=17$ is obtained by induction using $\nu(4,4,4,4,4)=12$ as a base case.

Claim 1. $\nu(2,2,3,3,3)=5$.

Proof. For $i=1$ and 2, there is at most one vertex of V_5 having an outneighbour in V_i as otherwise one vertex of V_5 would be isolated. Since $|V_5|=3$, there is a vertex of V_5 having no outneighbours in $V_1 \cup V_2$. Thus, there is a subset $X \subseteq V_3 \cup V_4 \cup V_5$ of cardinality 1, 2, or 3 inducing a complete subgraph in $D(\Omega)$ without outneighbours. If $|X| \geq 2$, applying Lemma 2 with $(k,z)=(3,2)$ gives at least 5 edges. If $|X|=1$, deleting X yields a $(2,2,2,3,3)$-octahedral system without isolated vertices since X has no outneighbours in $D(\Omega)$. As $\nu(2,2,2,3,3)=4$ by Proposition 5, we have at least $4+1=5$ edges. Thus, the equality holds since $\nu(2,2,3,3,3) \leq 5$ by Proposition 4. $\square$

Claim 2. $\nu(2,3,3,3,3)=6$.

Proof. We consider the two possible cases for the associated $D(\Omega)$.

Case (a): there are at least two vertices v and v' of V_5 having outneighbours in the same V_{i^*} in $D(\Omega)$ with $i^*=1$ or 2. Note that actually $i^*=2$ since otherwise $V_5 \setminus \{v,v'\}$ would be isolated. Applying Lemma 4 with $(k,z)=(3,1)$ gives at least $3 \times 2 + |V_4| + |V_5| - 6 = 6$ edges.

Case (b): there is at most one vertex of V_5 having an outneighbour in V_i for $i=1$ and 2 in $D(\Omega)$. Since $|V_5|=3$, there is a vertex of V_5 having no outneighbours in $V_1 \cup V_2$. Thus, there is a subset $X \subseteq V_3 \cup V_4 \cup V_5$ inducing in $D(\Omega)$ a complete subgraph without outneighbours. If $|X| \geq 2$, applying Lemma 2 with $(k,z)=(3,1)$ and $j=2$ gives at least 6 edges. If $|X|=1$, deleting X yields a $(2,2,3,3,3)$-octahedral system without isolated vertices since X has no outneighbours in $D(\Omega)$. As $\nu(2,2,3,3,3)=5$ by Claim 1, we have at least $5+1=6$ edges.

Thus, the equality holds since $\nu(2,3,3,3,3) \leq 6$ by Proposition 4. $\square$

Claim 3. $\nu(3,3,3,3,3)=7$.

Proof. There is a subset X inducing a complete subgraph in $D(\Omega)$ without outneighbours. Choose such an X of maximal cardinality. Without loss of generality, we assume that the indices i such that $|X \cap V_i| \neq 0$ are $n-|X|+1, n-|X|+2, \ldots, n$. Consider the different values for $|X|$.

- If $|X|=1$, deleting X yields a $(2,3,3,3,3)$-octahedral system without isolated vertices since X has no outneighbours in $D(\Omega)$. As $\nu(2,3,3,3,3)=6$ by Claim 2, we have at least $6+1=7$ edges.

- If $|X| = 2$ and $\deg_\Omega(X) \geq 2$, deleting X yields a $(2,2,3,3,3)$-octahedral system without isolated vertices. As $\nu(2,2,3,3,3) = 5$ by Claim 1, we have at least $5 + 2 = 7$ edges.
- If $|X| = 2$ and $\deg_\Omega(X) = 1$, denote $e(X)$ the unique edge containing X. For $i = 1, 2$, and 3, pick a vertex w_i in $V_i \setminus e(X)$. Applying the parity property to $e(X)$, w_1, w_2, w_3, and any $u_4 \in V_4 \setminus e(X)$, $u_5 \in V_5 \setminus e(X)$ yields at least 5 edges in $e(X) \cup \{w_1, w_2, w_3\} \cup V_4 \cup V_5$. At least 2 additional edges are needed to cover the 3 remaining vertices of V_1, V_2, and V_3 since a unique edge containing them would contradict the maximality of X. Thus, we have at least 7 edges.
- If $|X| = 3$ and $\deg_\Omega(X) \geq 3$, deleting X yields a $(2,2,2,3,3)$-octahedral system without isolated vertices. As $\nu(2,2,2,3,3) = 4$ by Proposition 5, we have at least $4 + 3 = 7$ edges.
- If $|X| = 3$ and $\deg_\Omega(X) \leq 2$, let $e(X)$ be an edge containing X. Pick $w_1 \in V_1 \backslash N_\Omega(X)$ and $w_2 \in V_2 \backslash N_\Omega(X)$ where $N_\Omega(X)$ denotes the vertices not in X contained in the edges intersecting X. Applying the parity property to $e(X)$, w_1, w_2, and any $u_i \in V_i \setminus e(X)$ for $i = 3, 4$, and 5 yields at least 9 edges in $e(X) \cup \{w_1, w_2\} \cup V_3 \cup V_4 \cup V_5$.
- If $|X| = 4$ and $\deg_\Omega(X) \geq 3$, take any vertex v in $V_2 \setminus X$. Applying the parity property to an edge $e(v)$ containing v, $V_2 \cap X$, and any $\hat{2}$-transversal disjoint from $e(v)$ and X shows that v is of degree at least 2. Since there are 2 vertices in $V_2 \setminus X$, we get, with 3 edges containing X, at least 7 edges.
- If $|X| = 4$ and $\deg_\Omega(X) \leq 2$, let $e(X)$ be an edge containing X. Pick $w_1 \in V_1 \setminus N_\Omega(X)$. Applying the parity property to $e(X)$, w_1, and any $u_i \in V_i \setminus e(X)$ for $i = 2, 3, 4$, and 5 yields at least 17 edges in $e(X) \cup \{w_1\} \cup V_2 \cup V_3 \cup V_4 \cup V_5$.
- If $|X| = 5$, the parity property applied to the edge $e(X)$ containing X, and any $u_i \in V_i \setminus e(X)$ for $i = 1, 2, 3, 4$, and 5 yields at least 33 edges.

Thus, the equality holds since $\nu(3,3,3,3,3) \leq 7$ by Proposition 4. □

Claim 4. $\nu(3,3,3,3,4) \geq 7$.

Proof. We first prove $\nu(2,3,3,3,4) \geq 6$ which in turn leads to $\nu(3,3,3,3,4) \geq 7$. The proof of these two inequalities are quite similar with the main difference being that, while the first inequality relies partially on Proposition 3, the second inequality relies on the first one.

Let $\Omega = (V_1, \ldots, V_5, E)$ be a $(2,3,3,3,4)$- or a $(3,3,3,3,4)$-octahedral system. We consider the three possible cases for the associated $D(\Omega)$.

Case (a): there is a vertex of V_5 having no outneighbours. Deleting this vertex yields a $(2,3,3,3,3)$- or a $(3,3,3,3,3)$-octahedral system without isolated vertices. In both cases, we have at least 7 edges since $\nu(2,3,\ldots,3) = 6$ by Claim 2, and $\nu(3,3,3,3,3) = 7$ by Claim 3.

Case (b): each vertex of V_5 has an outneighbour and there are at least two vertices v and v' of V_5 having outneighbours in the same V_{i^*} in $D(\Omega)$ with $i^* = 1, 2$, or 3. Note that $|V_{i^*}| = 3$ since otherwise $V_5 \setminus \{v, v'\}$ would be isolated. Applying Lemma 4 with either $(k,z) = (3,1)$ or $(k,z) = (3,0)$ gives at least $3 \times 2 + |V_4| + |V_5| - 6 = 7$

edges.

Case (c): each vertex of V_5 has an outneighbour and there is at most one vertex of V_5 having an outneighbour in V_i for $i = 1, 2$, and 3. Since $|V_5| = 4$, there is a subset $X \subseteq V_4 \cup V_5$ inducing in $D(\Omega)$ a complete subgraph of cardinality 1 or 2 without outneighbours.

- If $|X| = 1$, we have $X \subseteq V_4$ since each vertex of V_5 has an outneighbour. Deleting X yields a $(2, 2, 3, 3, 4)$- or a $(2, 3, 3, 3, 4)$-octahedral system without isolated vertices. We obtain $\nu(2, 3, 3, 3, 4) \geq 6$ since $\nu(2, 2, 3, 3, 4) \geq 5$ by Proposition 3, and then $\nu(3, 3, 3, 3, 4) \geq 7$ since $\nu(2, 3, 3, 3, 4) \geq 6$.
- If $|X| = 2$, deleting X yields a $(2, 2, 3, 3, 3)$- or a $(2, 3, 3, 3, 3)$-octahedral system without isolated vertices. Since one additional edge is needed to cover X, we obtain $\nu(2, 3, 3, 3, 4) \geq 6$ since $\nu(2, 2, 3, 3, 3) = 5$ by Claim 1, and $\nu(3, 3, 3, 3, 4) \geq 7$ since $\nu(2, 3, 3, 3, 3) = 6$ by Claim 2.

$\square$

Claim 5. $\nu(\underbrace{3, \ldots, 3}_{z \text{ times}}, \underbrace{4, \ldots, 4}_{5-z \text{ times}}) \geq 11 - z$ *for* $z = 1, 2, 3$.

Proof. The proof works by a top-down induction on z using the inequality $\nu(3, 3, 3, 3, 4) \geq 7$ which holds by Claim 4. We consider the two possible cases for the associated $D(\Omega)$.

Case (a): there are at least two vertices v and v' of V_5 having outneighbours in the same V_{i^*} with $i^* \leq z$. Let u and u' be the two vertices in V_{i^*} with (v, u) and (v', u') forming arcs in $D(\Omega)$. For each vertex $w \in V_{i^*}$, choose an edge $e(w)$ containing w. Choose $W_i \subseteq V_i$ such that $|W_i| = 3$ for $i = 1, \ldots, 4$, $|W_5| = 4$, and

$$\bigcup_{w \in V_{i^*}} e(w) \subseteq W = \bigcup_{i=1}^{5} W_i.$$

Pick $w \in V_{i^*}$ and consider $e(w)$. If $v \in e(w)$, take 2 disjoint $\widehat{i^*}$-transversals in W not containing v' and not intersecting with $e(w)$. Applying the parity property to $e(w), u'$, and each of those $\widehat{i^*}$-transversals yields, in addition to $e(w)$, at least 2 edges containing w. Otherwise, take 2 disjoint $\widehat{i^*}$-transversals in W not containing v and not intersecting with $e(w)$, and apply the parity property to $e(w), u$, and each of those $\widehat{i^*}$-transversals. In both cases, the degree of w in the hypergraph induced by W is at least 3. Then, we add edges not contained in W. Since $V_4 \setminus W \neq \emptyset$, there is at least one additional edge. In total, we have at least $10 \geq 11 - z$ edges.

Case (b): there is at most one vertex of V_5 having an outneighbour in V_i for $i \leq z$. Since $|V_5| = 4$, there is at least one vertex of V_5 having no outneighbours in $\bigcup_{i=1}^{z} V_i$. Thus, there is a subset $X \subseteq \bigcup_{i=z+1}^{5} V_i$ inducing in $D(\Omega)$ a complete subgraph without outneighbours. If $|X| = 1$, deleting X yields a $(\underbrace{3, \ldots, 3}_{z+1 \text{ times}}, \underbrace{4, \ldots, 4}_{4-z \text{ times}})$-octahedral system without isolated vertices. As $\nu(\underbrace{3, \ldots, 3}_{z+1 \text{ times}}, \underbrace{4, \ldots, 4}_{4-z \text{ times}}) \geq 11 - (z + 1)$ we obtain $11 - z$ edges. If $|X| \geq 2$, we have at least $9 + 2 = 11$ edges by Lemma 2 with $(k, z) = (4, z)$. $\square$

Claim 6. $\nu(4,4,4,4,4)=12$.

Proof. There is a subset X inducing a complete subgraph in $D(\Omega)$ without outneighbours. If $|X|=1$, deleting X yields a $(3,4,\dots,4)$-octahedral system without isolated vertices. As $\nu(3,4,\dots,4)\geq 10$, we obtain 11 edges. If $|X|\geq 2$, we have at least 11 edges by Lemma 2 with $(k,z)=(4,0)$. Thus, $\nu(4,4,4,4,4)\geq 12$ by Proposition 1, and then $\nu(4,4,4,4,4)=12$ by Proposition 4. □

Claim 7. $\nu(\underbrace{4,\dots,4}_{z \text{ times}},\underbrace{5,\dots,5}_{5-z \text{ times}})=17-z$ *for* $z=1,2,3,4$.

Proof. The proof works by a top-down induction on z using the inequality $\nu(4,4,4,4,4)\geq 12$ which holds by Claim 6. We consider the two possible cases for the associated $D(\Omega)$.

Case (a)*:* there are at least two vertices v and v' of V_5 having outneighbours in the same V_{i^*} with $i^*\leq z$. We can apply Lemma 4 with $(k,z)=(5,z)$, we have at least $4\times 4+|V_4|+|V_5|-10\geq 17-z$ edges.

Case (b)*:* there is at most one vertex of V_5 having an outneighbour in V_i for $1\leq i\leq z$. Since $|V_5|=5$, there is a vertex of V_5 having no outneighbours in $\bigcup_{i=1}^{z} V_i$. Thus, there is a subset $X\subseteq\bigcup_{i=z+1}^{5} V_i$ inducing in $D(\Omega)$ a complete subgraph without outneighbours. If $|X|=1$, deleting X yields a $(\underbrace{4,\dots,4}_{z+1 \text{ times}},\underbrace{5,\dots,5}_{4-z \text{ times}})$-octahedral system. As $\nu(\underbrace{4,\dots,4}_{z+1 \text{ times}},\underbrace{5,\dots,5}_{4-z \text{ times}})\geq 16-z$, we obtain at least $17-z$ edges. If $|X|\geq 2$, we have at least 18 edges by Lemma 2.

Thus, the equality holds since $\nu(\underbrace{4,\dots,4}_{z \text{ times}},\underbrace{5,\dots,5}_{5-z \text{ times}})\leq 17-z$ by Proposition 4. □

Claim 8. $\nu(5,5,5,5,5)=17$.

Proof. There is a subset X inducing a complete subgraph in $D(\Omega)$ without outneighbours. If $|X|=1$, deleting X yields a $(4,5,5,5,5)$-octahedral system without isolated vertices. As $\nu(4,5,5,5,5)\geq 16$, we have at least 17 edges. If $|X|\geq 2$, we can apply Lemma 2, and we have at least 18 edges.

Thus, the equality holds since $\nu(5,5,5,5,5)\leq 17$ by Proposition 4. □

As $\nu(5,5,5,5,5)=\nu(4)$, Claim 8 and the relation $\mu(4)\geq\nu(4)$ directly imply that the conjectured equality $\mu(d)=d^2+1$ holds for $d=4$.

Proposition 12. $\mu(4)=17$.

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