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1. Combinatorial Stokes formula and applications

1.1. **Combinatorial Stokes formula.** Let M be an d-dimensional pseudomanifold. Let $\lambda : V(\mathsf{M}) \rightarrow \{-1,+1,\ldots,-k,+k\}$ be a labeling of its vertices, where k is some positive integer. A ℓ-dimensional simplex σ of M is *negatively alternating* (resp. *positively alternating*) if $\lambda(V(\sigma))$ is of the form $\{-j_0,+j_1,\ldots,(-1)^{\ell-1}j_\ell\}$ (resp. of the form $\{+j_0,-j_1,\ldots,(-1)^{\ell}j_\ell\}$) with $1 \leq j_0 < j_1 < \cdots < j_\ell \leq k$. We denote by $\beta^-(\mathsf{M})$ (resp. $\beta^+(\mathsf{M})$) the number of negatively (resp. positively) alternating d-simplices of M.

The following proposition due to Fan [7] is a powerful tool for providing elementary proofs of fundamental topological results.

Proposition 1.1. *Suppose that $\lambda(\boldsymbol{u})+\lambda(\boldsymbol{v}) \neq 0$ whenever $\boldsymbol{u}$ and $\boldsymbol{v}$ are adjacent vertices. Then*

$$\beta^+(\mathsf{M})+\beta^-(\mathsf{M}) = \beta^-(\partial\mathsf{M}) \mod 2.$$

Proof. Consider the graph $G=(V,E)$ where the vertices are the d-simplices of M, plus an additional dummy vertex s. Two vertices distinct from s are connected by an edge if the corresponding simplices share a common facet that is negatively alternating. A vertex is connected to s by an edge if the corresponding simplex has a facet on $\partial\mathsf{M}$ that is negatively alternating.

The following facts can be easily checked. An *almost negatively alternating simplex* is a d-simplex that is not alternating while having a negatively alternating facet.

- A vertex corresponding to an alternating d-simplex is of degree 1.
- A vertex corresponding to an almost negatively alternating d-simplex is of degree 2.
- s is of degree $\beta^-(\partial\mathsf{M})$.
- Any other vertex is of degree 0.

The equality to be proved is then a consequence of the evenness of the number of odd degree vertices in G. ◻

1.2. **Tucker's lemma, the Borsuk-Ulam theorem, and beyond.**

Theorem 1.2 (Combinatorial Ky Fan's theorem [7])**.** *Let T be a centrally symmetric triangulation of S^d and let*

$$\lambda : V(\mathsf{T}) \rightarrow \{-1,+1,\ldots,-k,+k\}$$

be a labeling of its vertices such that $\lambda(-\boldsymbol{v}) = -\lambda(\boldsymbol{v})$ for every $\boldsymbol{v} \in V(\mathsf{T})$. If $\lambda(\boldsymbol{u})+\lambda(\boldsymbol{v}) \neq 0$ whenever $\boldsymbol{u}$ and $\boldsymbol{v}$ are adjacent vertices, then both $\beta^+(\mathsf{T})$ and $\beta^-(\mathsf{T})$ are odd.

We prove the statement when T is a centrally symmetric triangulation that refines the boundary of the crosspolytope: $\partial\Diamond^{d+1} = \{\boldsymbol{x} \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1}|x_i| = 1\}$. The statement is also true for any symmetric triangulations, but the proof is less easy. Anyway, for the applications of Theorem 1.2 that follow, this kind of triangulations is enough.

Proof of Theorem 1.2. The proof works by induction on d. For $d=0$, the statement is obvious. Suppose now that $d \geq 1$. Consider the triangulation T' induced by T on the equator of $\partial\Diamond^{d+1}$, which is $\{\boldsymbol{x} \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1}|x_i| = 1,\ x_{d+1}=0\}$. This triangulation satisfies the condition of the theorem. Thus, by induction $\beta^-(\mathsf{T}')$ is odd. Denote T^+ (resp. T^-) the triangulation induced by T on the positive (resp. negative) hemisphere $\{\boldsymbol{x} \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1} |x_i| = 1,\ x_{d+1} \geq 0\}$ (resp. $\{\boldsymbol{x} \in \mathbb{R}^{d+1} : \sum_{i=1}^{d+1} |x_i| = 1,\ x_{d+1} \leq 0\}$). According to Proposition 1.1, we have $\beta^+(\mathsf{T}^+) + \beta^-(\mathsf{T}^+)$ odd. Since λ is antipodal, we have $\beta^+(\mathsf{T}^-) = \beta^-(\mathsf{T}^+)$ and $\beta^+(\mathsf{T}^+) + \beta^+(\mathsf{T}^-) = \beta^+(\mathsf{T})$. Thus, $\beta^+(\mathsf{T})$ is odd. By antipodality, $\beta^-(\mathsf{T})$ is also odd. □

Corollary 1.3 (Tucker's lemma [20])**.** *Let T be a centrally symmetric triangulation of S^d and let*

$$\lambda : V(\mathsf{T}) \to \{-1, +1, \ldots, -k, +k\}$$

be a labeling of its vertices such that $\lambda(-\boldsymbol{v}) = -\lambda(\boldsymbol{v})$ for every $\boldsymbol{v} \in V(\mathsf{T})$. If $\lambda(\boldsymbol{u}) + \lambda(\boldsymbol{v}) \neq 0$ whenever $\boldsymbol{u}$ and $\boldsymbol{v}$ are adjacent vertices, then $k \geq d + 1$.

Theorem 1.4 (Borsuk-Ulam theorem)**.** *There is no continuous map $f : S^d \to S^{d-1}$ such that $f(-\boldsymbol{x}) = -f(\boldsymbol{x})$ for all $\boldsymbol{x} \in S^d$.*

To prove it, we follow the same scheme as on p.35 of [11].

Proof of Theorem 1.4. Suppose for a contradiction that such map f exists. We set $\varepsilon = \frac{1}{\sqrt{d}}$. With such an ε, for every $\boldsymbol{z} = (z_1, \ldots, z_d) \in S^{d-1}$, we always have $z_i > \varepsilon$ for some $i \in d$. Since f is uniformly continuous, there exists a $\delta > 0$ such that if the distance between $\boldsymbol{z}$ and $\boldsymbol{z}'$ is not larger than δ, then $||f(\boldsymbol{z}) - f(\boldsymbol{z}')||_\infty < 2\varepsilon$. We choose a triangulation T of S^d such that $\mathrm{diam}(\sigma) < \delta$ for every simplex $\sigma \in \mathsf{T}$. Now, let us define a labeling $\lambda : V(\mathsf{T}) \to \{-1, +1, \ldots, -d, +d\}$. To that purpose, let

$$k(\boldsymbol{v}) = \min\{i \in [d] :\ |f(\boldsymbol{v})_i| \geq \varepsilon\}.$$

We set then

$$\lambda(\boldsymbol{v}) = \begin{cases} +k(\boldsymbol{v}) & \text{if } f(\boldsymbol{v})_{k(\boldsymbol{v})} > 0, \\ -k(\boldsymbol{v}) & \text{if } f(\boldsymbol{v})_{k(\boldsymbol{v})} < 0. \end{cases}$$

The labeling λ satisfies the condition of Corollary 1.3 with $k = d$. Thus, there must be some edge $\boldsymbol{uv}$ with $\lambda(\boldsymbol{u}) = -\lambda(\boldsymbol{v}) > 0$. Setting $i = \lambda(\boldsymbol{u})$, we get $f(\boldsymbol{u})_i \geq \varepsilon$ and $f(\boldsymbol{v})_i \leq -\varepsilon$. Therefore, $||f(\boldsymbol{u}) - f(\boldsymbol{v})||_\infty \geq 2\varepsilon$; a contradiction. □

Theorem 1.5 (Ky Fan's theorem [7])**.** *Let $A_1, \ldots, A_k$ be k subsets of S^d satisfying the following conditions:*

- *They are all open or all closed.*
- *None of them contain antipodal points.*
- $\bigcup_{i=1}^k (A_i \cup (-A_i)) = S^d$.

Then there exist $d + 1$ integers $1 \leq j_0 < \cdots < j_d \leq k$ such that

$$A_{j_0} \cap (-A_{j_1}) \cap \cdots \cap ((-1)^d A_{j_d}) \neq \emptyset.$$

Proof. We first prove the case when all A_i are closed.

Let T be a centrally symmetric triangulation of S^d of arbitrary small mesh size. We define then a labeling λ of its vertices as follows. Let $\boldsymbol{v} \in V(\mathsf{T})$. We set $k(\boldsymbol{v}) = \min\{i \in [k] :\ \boldsymbol{v} \in A_i \cup (-A_i)$. We define then

$$\lambda(\boldsymbol{v}) = \begin{cases} k(\boldsymbol{v}) & \text{if } \boldsymbol{v} \in A_{k(\boldsymbol{v})} \\ -k(\boldsymbol{v}) & \text{if } \boldsymbol{v} \in -A_{k(\boldsymbol{v})} \end{cases}$$

The fact that none of the A_i contain antipodal points ensures that the sign of $\lambda(\boldsymbol{v})$ is well-defined and that no adjacent vertices have opposite labels. λ satisfies the condition of Theorem 1.2. There exists thus a positively alternating simplex. Considering a sequence of symmetric triangulations whose mesh size tends to zero and using the compactness of S^d, we get a sequence of positively alternating simplices converging toward a point and having all the same labels $j_0, -j_1, \ldots, (-1)^d j_d$.

The limit point is in the intersection $A_{j_0} \cap (-A_{j_1}) \cap \cdots \cap ((-1)^d A_{j_d})$, which gives the sought conclusion.

To get the result when all the A_i's are open, we proceed as follows. For each point $\boldsymbol{x}$ of the sphere, we choose an open neighborhood $V_{\boldsymbol{x}}$ whose closure is contained in some A_i or in some $-A_i$. We do it in such a way that $V_{-\boldsymbol{x}} = -V_{\boldsymbol{x}}$. By compactness, we can then build m closed subsets $A'_1, \ldots, A'_k$ such that $A'_i \subset A_i$ for $i = 1, \ldots, k$ and satisfying the condition of the theorem. The conclusion of the theorem for these sets provides the conclusion for the original sets. $\square$

Theorem 1.6 (Ky Fan's theorem [7] – alternate version)**.** *Let $A_1, \ldots, A_k$ be k subsets of S^d satisfying the following conditions:*

- *They are all open or all closed.*
- *None of them contain antipodal points.*
- *$\bigcup_{i=1}^k A_i = S^d$.*

Then there exist $d+2$ integers $1 \leq j_0 < \cdots < j_d \leq k$ such that

$$A_{j_0} \cap (-A_{j_1}) \cap \cdots \cap ((-1)^{d+1} A_{j_{d+1}}) \neq \emptyset.$$

Proof. The proof is similar as the one for Theorem 1.6, working first with closed sets, and with the same definition of $k(\boldsymbol{v})$. There is thus always a positively alternating simplex with labels $j_0, -j_1, \ldots, (-1)^d j_d$. Now, notice that the vertex with label $(-1)^d j_d$ is also in $(-1)^{d+1} A_\ell$ for some $\ell > j_d$. Indeed, $(-1)^{d+1} A_1, \ldots, (-1)^{d+1} A_k$ is a cover of S^d, and $\boldsymbol{v}$ is contained in none of the $(-1)^{d+1} A_i$ for $i \leq j_d$. By compactness, we get the result. We get the result for open sets similarly as for Theorem 1.6. $\square$

Ky Fan's theorem (Theorem 1.6) implies the following equivalent version of the Borsuk-Ulam theorem.

Theorem 1.7 (Lyusternik-Shnirel'man theorem)**.** *Let $U_1, \ldots, U_{d+1}$ be a cover of the sphere S^d with $d+1$ open sets. Then there is at least one of them containing a pair of antipodal points.*

Proof. Suppose for a contradiction that none of the U_i contain antipodal points. We define $A_i = (-1)^{i-1} U_i$. These A_i's satisfy the condition of Theorem 1.6. There is thus a point $\boldsymbol{x}$ in $A_1 \cap (-A_2) \cap \cdots \cap ((-1)^d A_{d+1})$. In other words, $\boldsymbol{x} \in \bigcap_{i=1}^{d+1} U_i$. Since the U_i's form a cover of S^d, there is a j such that $-\boldsymbol{x} \in U_j$. We have then both $\boldsymbol{x}$ and $-\boldsymbol{x}$ in U_j, a contradiction. $\square$

1.3. **Homotopy and antipodality.**

Theorem 1.8. *There are no homotopic continuous maps $f, g : S^d \to S^d$ such that $f(-\boldsymbol{x}) = -f(\boldsymbol{x})$ and $g(-\boldsymbol{x}) = g(\boldsymbol{x})$ for all $\boldsymbol{x} \in S^d$.*

This theorem is a consequence of the following proposition.

Proposition 1.9. *Let T be a triangulation of $S^d \times [0,1]$ that is centrally symmetric on $S^d \times \{0\}$ and on $S^d \times \{1\}$. Let $\lambda : V(\mathsf{T}) \to \{-1, +1, \ldots, -k, +k\}$ be a labeling of its vertices. Suppose that $\lambda(-\boldsymbol{v}) = -\lambda(\boldsymbol{v})$ when $\boldsymbol{v}$ is a vertex on $S^d \times \{0\}$ and that $\lambda(-\boldsymbol{v}) = \lambda(\boldsymbol{v})$ when $\boldsymbol{v}$ is a vertex on $S^d \times \{1\}$. If $\lambda(\boldsymbol{u}) + \lambda(\boldsymbol{v}) \neq 0$ whenever $\boldsymbol{u}$ and $\boldsymbol{v}$ are adjacent vertices, then both $\beta^+(\mathsf{T})$ and $\beta^-(\mathsf{T})$ are odd.*

Proof. $\partial\mathsf{T}$ is the union of a centrally symmetric triangulation of $S^d \times \{0\}$, which we denote T_0, and a centrally symmetric triangulation of $S^d \times \{1\}$, which we denote T_1. Because of the condition on λ, we have $\beta^+(\mathsf{T}_1)$ and $\beta^-(\mathsf{T}_1)$ both even. According to Theorem 1.2, we have $\beta^+(\mathsf{T}_0)$ and $\beta^-(\mathsf{T}_0)$ both odd. Proposition 1.1 implies then that both $\beta^+(\mathsf{T})$ and $\beta^-(\mathsf{T})$ are odd. $\square$

To prove Theorem 1.8, we follow the same scheme as for proving the Borsuk-Ulam from Tucker's lemma. We simply replace Tucker's lemma by Proposition 1.9.

Proof of Theorem 1.8. Suppose for a contradiction that such maps f and g exist. We have then a continuous mapping $H : S^d \times [0,1] \to S^d$ such that $H(\boldsymbol{x}, 0) = f(\boldsymbol{x})$ and $H(\boldsymbol{x}, 1) = g(\boldsymbol{x})$ for every $\boldsymbol{x} \in S^d$.

We set $\varepsilon = \frac{1}{\sqrt{d+1}}$. With such an ε, for every $\boldsymbol{z} = (z_1, \ldots, z_{d+2}) \in S^d \times [0,1]$, we always have $z_i > \varepsilon$ for some $i \in [d+1]$. Since the mapping H is uniformly continuous, there exists a $\delta > 0$ such that if the distance between $\boldsymbol{z}$ and $\boldsymbol{z}'$ is not larger than δ, then $||H(\boldsymbol{z}) - H(\boldsymbol{z}')||_\infty < 2\varepsilon$. We choose a triangulation T of $S^d \times [0,1]$ such that $\mathrm{diam}(\sigma) < \delta$ for every simplex $\sigma \in \mathsf{T}$.

Now, let us define a labeling $\lambda : V(\mathsf{T}) \to \{-1, +1, \ldots, -(d+1), +(d+1)\}$. To that purpose, let

$$k(\boldsymbol{v}) = \min\{i \in [d+1] : \; |H(\boldsymbol{v})_i| \geq \varepsilon\}.$$

We set then

$$\lambda(\boldsymbol{v}) = \begin{cases} +k(\boldsymbol{v}) & \text{if } H(\boldsymbol{v})_{k(\boldsymbol{v})} > 0, \\ -k(\boldsymbol{v}) & \text{if } H(\boldsymbol{v})_{k(\boldsymbol{v})} < 0. \end{cases}$$

The labeling λ satisfies the condition of Proposition 1.9 with $k = d+1$. There are not enough labels to get any negatively or positively alternating d-simplex in T. Thus, both $\beta^+(\mathsf{T})$ and $\beta^-(\mathsf{T})$ are even, which implies that there must be some edge $\boldsymbol{uv}$ with $\lambda(\boldsymbol{u}) = -\lambda(\boldsymbol{v}) > 0$. Setting $i = \lambda(\boldsymbol{u})$, we get $H(\boldsymbol{u})_i \geq \varepsilon$ and $H(\boldsymbol{v})_i \leq -\varepsilon$. Therefore, $||H(\boldsymbol{u}) - H(\boldsymbol{v})||_\infty \geq 2\varepsilon$; a contradiction. $\square$

2. Combinatorial proof of the Lovász-Kneser theorem

2.1. **The Lovász-Kneser theorem.** Let m, ℓ be two integers such that $m \geq 2\ell$. The *Kneser graph* $\mathrm{KG}(m, \ell)$ is defined by

$$\begin{array}{rcl} V(\mathrm{KG}(m,\ell)) & = & \binom{[m]}{\ell} \\ E(\mathrm{KG}(m,\ell)) & = & \left\{AB : A, B \in \binom{[m]}{\ell},\; A \cap B = \emptyset\right\}. \end{array}$$

Theorem 2.1. *Let $m \geq 2\ell$. We always have $\chi(\mathrm{KG}(m,\ell)) = m - 2\ell + 2$.*

The inequality $\chi(\mathrm{KG}(m,\ell)) \leq m - 2\ell + 2$ is easy. It is a consequence of the explicit coloring

$$c : U \in \binom{[m]}{\ell} \longmapsto \min(\min(U), m - 2\ell + 2) \in [m - 2\ell + 2].$$

The original proof of the reverse inequality used the Borsuk-Ulam theorem. It is the celebrated proof by Lovász [10]. In this section, we explain Matoušek's purely combinatorial proof of this inequality [12].

2.2. **Another combinatorial Ky Fan's theorem.** The main tool for proving Lemma 3.2 is the following lemma. It uses "signed vectors", which are elements of $\{+,-,0\}^m$. We endow this set with a partial order $\preceq$ as follows. We have $\boldsymbol{x} \preceq \boldsymbol{y}$ if for every index $i \in [m]$ whenever $x_i \neq 0$, then $x_i = y_i$.

Let $\lambda : \{+,-,0\}^m \setminus \{\boldsymbol{0}\} \to \{-1, +1, \ldots, -k, +k\}$ for some positive integers k and m. A *positively alternating m-chain* is a sequence $\boldsymbol{x}^1 \preceq \boldsymbol{x}^2 \preceq \cdots \preceq \boldsymbol{x}^m$ such that $\lambda\left(\{\boldsymbol{x}^1, \ldots, \boldsymbol{x}^m\}\right)$ is of the form $\{+j_1, -j_2, \ldots, (-1)^{m-1} j_m\}$ with $1 \leq j_1 < j_2 < \cdots < j_m \leq k$.

Lemma 2.2 (Octahedral Ky Fan's lemma)**.** *Suppose that λ satisfies the following conditions*

- $\lambda(-\boldsymbol{x}) = -\lambda(\boldsymbol{x})$ *for every* $\boldsymbol{x}$.
- $\lambda(\boldsymbol{x}) + \lambda(\boldsymbol{y}) \neq 0$ *whenever* $\boldsymbol{x} \preceq \boldsymbol{y}$.

Then the number of positively alternating m-chains is odd. In particular, we have $k \geq m$.

This lemma without the statement on the number of positively alternating m-chains is usually known under the name "Octahedral Tucker's lemma".

Proof of Lemma 2.2. Let T be barycentric subdivision of the boundary of the unit cube:

$$\mathsf{T} = \mathrm{sd}\,\partial[-1,+1]^m.$$

Note that T is a triangulation of S^{m-1}. The signed vectors are precisely the vertices of T, once $+$ is replaced by 1 and $-$ replaced by -1, and there is a one-to-one correspondence between the chains for $\preceq$ and the simplices of T. In particular, an edge of the barycentric subdivision corresponds to two comparable signed vectors. Theorem 1.2 applies and it ensures that there is an odd number of positively alternating $(m-1)$-simplices in T, which are precisely the positively alternating m-chains. ☐

2.3. **The combinatorial proof.** For the proof, we introduce the following notation. For $\boldsymbol{x} \in \{+,-,0\}^m$, we define $|\boldsymbol{x}|$ as being the number of its nonzero components. Moreover we define

$$\boldsymbol{x}^+ = \{i \in [m] :\ x_i = +\} \quad \text{and} \quad \boldsymbol{x}^- = \{i \in [m] :\ x_i = -\}.$$

Proof of Theorem 2.1. The inequality $\chi(\mathrm{KG}(m,\ell)) \leq m - 2\ell + 2$ has already noted to be true. Let us prove the reverse inequality.

Let $c : \binom{[m]}{\ell} \to [t]$ be a proper coloring of $\mathrm{KG}(m,\ell)$ with t colors. For a subset A of $[m]$ of cardinality at least ℓ, we define

$$c(A) = \max\{c(U) :\ U \subseteq A,\ |U| = \ell\}.$$

Now, for $\boldsymbol{x} \in \{+,-,0\}^m \setminus \{\boldsymbol{0}\}$, we define

$$\lambda(\boldsymbol{x}) = \begin{cases} |\boldsymbol{x}| & \text{if } |\boldsymbol{x}| \leq 2\ell-2 \text{ and } \min(\boldsymbol{x}^+) < \min(\boldsymbol{x}^-), \\ -|\boldsymbol{x}| & \text{if } |\boldsymbol{x}| \leq 2\ell-2 \text{ and } \min(\boldsymbol{x}^-) < \min(\boldsymbol{x}^+), \\ c(\boldsymbol{x}^+) + 2\ell - 2 & \text{if } |\boldsymbol{x}| \geq 2\ell-1 \text{ and } c(\boldsymbol{x}^+) > c(\boldsymbol{x}^-), \\ -c(\boldsymbol{x}^-) - 2\ell + 2 & \text{if } |\boldsymbol{x}| \geq 2\ell-1 \text{ and } c(\boldsymbol{x}^-) > c(\boldsymbol{x}^+). \end{cases}$$

(We use the convention that the minimum (resp. maximum) of a function over an empty set is $+\infty$ (resp. $-\infty$)). The fact that c is proper coloring ensures that $c(\boldsymbol{x}^+) \neq c(\boldsymbol{x}^-)$ once $|\boldsymbol{x}| \geq 2\ell - 1$.

λ obviously satisfies $\lambda(-\boldsymbol{x}) = -\lambda(\boldsymbol{x})$ for every $\boldsymbol{x}$. Now, consider two nonzero signed vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ such that $\lambda(\boldsymbol{x}) = -\lambda(\boldsymbol{y})$. Without loss of generality, we assume that $\lambda(\boldsymbol{x}) > 0$. Suppose first that $|\boldsymbol{x}| \leq 2\ell - 2$, then $\boldsymbol{x}$ and $\boldsymbol{y}$ necessarily have the same support cardinality, but opposite value for their first nonzero component. Thus $\boldsymbol{x}$ and $\boldsymbol{y}$ are not comparable. Suppose now that $|\boldsymbol{x}| \geq 2\ell - 1$. There is a subset U of cardinality ℓ in $\boldsymbol{x}^+$ and a subset V of cardinality ℓ in $\boldsymbol{y}^-$ such that $c(U) = c(V)$. As c is proper coloring, these two sets intersect. It implies that again $\boldsymbol{x}$ and $\boldsymbol{y}$ are not comparable. λ satisfies therefore the condition of Lemma 2.2 with $k = t + 2\ell - 2$.

Hence, $t + 2\ell - 2 \geq m$ and the conclusion follows. ☐

3. Circular chromatic number of Kneser graphs

3.1. **Context and main results.** Let $G = (V,E)$ be a graph. For two integers $p \geq q \geq 1$, a (p,q)-*coloring* of G is a mapping $c : V \to [p]$ such that $q \leq |c(u) - c(v)| \leq p - q$ for every edge uv of G. The *circular chromatic number* of G is

$$\chi_c(G) = \inf\{p/q :\ G \text{ admits a } (p,q)\text{-coloring}\}.$$

It is known that $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ and that the infimum in the definition is actually a minimum, i.e. $\chi_c(G)$ is attained for some (p,q)-coloring (and thus the circular chromatic number is always rational), see [22] for details. The question of determining which graphs are such that $\chi_c(G) = \chi(G)$ has received a considerable attention. The following theorem has been conjectured by Johnson, Holroyd, and Stahl [9]. It has been proved independently for even m by M. [13] and Simonyi and Tardos [17]. The general case has been proved by Chen [5].

Theorem 3.1. *Let $m \geq 2\ell$. We always have $\chi_c(\mathrm{KG}(m,\ell)) = \chi(\mathrm{KG}(m,\ell))$.*

This theorem is actually a consequence of the following more general result, also proved by Chen in the same paper. We denote by $K^*_{q,q}$ the bipartite complete graph $K_{q,q}$ from which a perfect matching has been removed.

Lemma 3.2. *Any proper coloring of* $\mathrm{KG}(m,\ell)$ *with* $m-2\ell+2$ *colors contains a colorful copy of* $K^*_{m-2\ell+2,m-2\ell+2}$.

It is quite easy to prove Theorem 3.1 from this lemma.

Proof of Theorem 3.1. Let c be a (p,q)-coloring of $\mathrm{KG}(m,\ell)$ with $p \leq q(m-2\ell+2)$. Define a new coloring of the vertices by $\hat{c}(v) = \lceil c(v)/q \rceil$. It is proper coloring of the vertices with at most $m-2\ell+2$ colors. According to Lemma 3.2, there is a colorful copy of $K^*_{m-2\ell+2,m-2\ell+2}$ in $\mathrm{KG}(m,\ell)$ for the coloring $\hat{c}$. If $m=1$ or $m=2$, the statement we have to prove is obvious. Let us thus assume that $m \geq 3$.

If m is even, we have thus a cycle $v_1,\ldots,v_{m-2\ell+2},v_1$ of length $m-2\ell+2$, with $\hat{c}(v_i)=i$. Hence, we have $c(v_i) \leq c(v_{i+1})$ for every $i \in [m-2\ell+1]$. Since c is a (p,q)-coloring, we have $c(v_1)+iq \leq c(v_{i+1})$ for every $i \in [m-2\ell+1]$. In particular, $c(v_1)+(m-2\ell+1)q \leq c(v_{m-2\ell+2})$. Moreover, $v_{m-2\ell+2}$ and v_1 are adjacent on the cycle. Therefore, $c(v_{m-2\ell+2})-c(v_1) \leq p-q$, which implies $(m-2\ell+1)q \leq p-q$, and we get the conclusion.

If m is odd, we have thus a cycle $v_1,\ldots,v_{m-2\ell+2},u_1,\ldots,u_{m-2\ell+2},v_1$ of length $2m-4\ell+4$, with the colors $1,2,\ldots,m-2\ell+2,1,2,\ldots,m-2\ell+2$ appearing in this order on the cycle: $\hat{c}(v_i) = \hat{c}(u_i) = i$. Without loss of generality, we can assume that $c(u_1) \leq c(v_1)$. We have $c(v_i) \leq c(v_{i+1})$ for every $i \in [m-2\ell+1]$, and hence we have $c(v_1)+iq \leq c(v_{i+1})$ for every $i \in [m-2\ell+1]$. In particular, $c(v_1)+(m-2\ell+1)q \leq c(v_{m-2\ell+2})$. Moreover, $v_{m-2\ell+2}$ and u_1 are adjacent on the cycle. Therefore, $c(v_{m-2\ell+2})-c(u_1) \leq p-q$, which implies $c(v_{m-2\ell+2}) \leq p-q+c(v_1)$, and we get $\geq (m-2\ell+1)q \leq p-q$, as required. $\square$

The remaining of the section is devoted to the proof of Lemma 3.2. Chang, Liu, and Zhu [4] simplified Chen's proof. We propose here a further simplification.

3.2. **Proof of Lemma 3.2.**

Proof of Lemma 3.2. Let $c : \binom{[m]}{\ell} \to [m-2\ell+2]$ be a proper coloring of $\mathrm{KG}(m,\ell)$ with $m-2\ell+2$ colors. For a subset A of $[m]$ of cardinality at least ℓ, we define

$$c(A) = \max\{c(U) : \ U \subseteq A, \ |U| = \ell\}.$$

Now, for $\boldsymbol{x} \in \{+,-,0\}^m \setminus \{\boldsymbol{0}\}$, we define as in the proof of Theorem 2.1

$$\lambda(\boldsymbol{x}) = \begin{cases} |\boldsymbol{x}| & \text{if } |\boldsymbol{x}| \leq 2\ell-2 \text{ and } \min(\boldsymbol{x}^+) < \min(\boldsymbol{x}^-), \\ -|\boldsymbol{x}| & \text{if } |\boldsymbol{x}| \leq 2\ell-2 \text{ and } \min(\boldsymbol{x}^-) < \min(\boldsymbol{x}^+), \\ c(\boldsymbol{x}^+)+2\ell-2 & \text{if } |\boldsymbol{x}| \geq 2\ell-1 \text{ and } c(\boldsymbol{x}^+) > c(\boldsymbol{x}^-), \\ -c(\boldsymbol{x}^-)-2\ell+2 & \text{if } |\boldsymbol{x}| \geq 2\ell-1 \text{ and } c(\boldsymbol{x}^-) > c(\boldsymbol{x}^+). \end{cases}$$

Let $\alpha(\boldsymbol{x},\lambda)$ be the number of positively alternating m-chains containing $\boldsymbol{x}$. According to Lemma 2.2, we have that $\sum_{\boldsymbol{x}:|\boldsymbol{x}|=2\ell-2}\alpha(\boldsymbol{x},\lambda)$ is odd. There exists a positively alternating m-chain $\boldsymbol{x}^1 \preceq \boldsymbol{x}^2 \preceq \cdots \preceq \boldsymbol{x}^m$ such that $\alpha(\boldsymbol{x}^{2\ell-2},\lambda)$ is odd.

The fact that λ is increasing and the alternation of signs implies we may denote $[m] = S \cup T \cup \{a_1,\ldots,a_{m-2\ell+2}\}$, where

$$\boldsymbol{x}^{(2\ell-2+i)+} = S \cup \{a_1,a_3,\ldots,a_i\} \quad \text{and} \quad \boldsymbol{x}^{(2\ell-2+i)-} = T \cup \{a_2,a_4,\ldots,a_{i-1}\} \quad \text{if } i \text{ is odd},$$

$$\boldsymbol{x}^{(2\ell-2+i)+} = S \cup \{a_1,a_3,\ldots,a_{i-1}\} \quad \text{and} \quad \boldsymbol{x}^{(2\ell-2+i)-} = T \cup \{a_2,a_4,\ldots,a_i\} \quad \text{if } i \text{ is even},$$

and S and T are two disjoint $(\ell-1)$-subsets of $[m]$. Actually, $S = \boldsymbol{x}^{(2\ell-2)+}$ and $T = \boldsymbol{x}^{(2\ell-2)-}$. (We have even that $\lambda(\boldsymbol{x}^i) = (-1)^{i-1}i$.) Define now a new labeling μ by

$$\mu(\boldsymbol{x}) = \begin{cases} -\lambda(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \{\boldsymbol{x}^{2\ell-2}, -\boldsymbol{x}^{2\ell-2}\} \\ \lambda(\boldsymbol{x}) & \text{otherwise.} \end{cases}$$

Since $\sum_{\boldsymbol{x}:|\boldsymbol{x}|=2\ell-2} \alpha(\boldsymbol{x},\lambda)$ and $\sum_{\boldsymbol{x}:|\boldsymbol{x}|=2\ell-2} \alpha(\boldsymbol{x},\mu)$ are both odd by Lemma 2.2 and since $\alpha(\boldsymbol{x},\lambda) = \alpha(\boldsymbol{x},\mu)$ when $|\boldsymbol{x}| = 2\ell-2$ except when $\boldsymbol{x} \in \{\boldsymbol{x}^{2\ell-2}, -\boldsymbol{x}^{2\ell-2}\}$, we get

$$\alpha(\boldsymbol{x}^{2\ell-2},\lambda) + \alpha(-\boldsymbol{x}^{2\ell-2},\lambda) = \alpha(\boldsymbol{x}^{2\ell-2},\mu) + \alpha(-\boldsymbol{x}^{2\ell-2},\mu) \mod 2.$$

Note that $\alpha(-\boldsymbol{x}^{2\ell-2},\lambda) = 0$ and that $\alpha(\boldsymbol{x}^{2\ell-2},\mu) = 0$. Since $\alpha(\boldsymbol{x}^{2\ell-2},\lambda)$ is odd, we get that $\alpha(-\boldsymbol{x}^{2\ell-2},\mu)$ is also odd. There exists thus a positively alternating m-chain $\boldsymbol{y}^1 \preceq \boldsymbol{y}^2 \preceq \cdots \preceq \boldsymbol{y}^m$ for the labeling μ, with $\boldsymbol{y}^{2\ell-2} = -\boldsymbol{x}^{2\ell-2}$. For the same reasons as for λ, we get that we may denote $[m] = S \cup T \cup \{b_1,\ldots,b_{m-2\ell+2}\}$, where

$$\boldsymbol{y}^{(2\ell-2+i)+} = T \cup \{b_1,b_3,\ldots,b_i\} \quad \text{and} \quad \boldsymbol{y}^{(2\ell-2+i)-} = S \cup \{b_2,b_4,\ldots,b_{i-1}\} \quad \text{if } i \text{ is odd,}$$

and

$$\boldsymbol{y}^{(2\ell-2+i)+} = T \cup \{b_1,b_3,\ldots,b_{i-1}\} \quad \text{and} \quad \boldsymbol{y}^{(2\ell-2+i)-} = S \cup \{b_2,b_4,\ldots,b_i\} \quad \text{if } i \text{ is even.}$$

We have thus

$$c(S \cup \{a_1,a_3,\ldots,a_i\}) = c(T \cup \{b_1,b_3,\ldots,b_i\} = i \quad \text{for odd } i$$

and

$$c(T \cup \{a_2,a_4,\ldots,a_i\}) = c(S \cup \{b_2,b_4,\ldots,b_i\}) = i \quad \text{for even } i.$$

We show now that $a_i = b_i$ and that $c(S\cup\{a_i\}) = c(T\cup\{a_i\}) = i$ for all $i \in [m-2\ell+2]$. Once this is done, the proof will be complete: the subgraph of $\mathrm{KG}(m,\ell)$ induced by the vertices $\{S\cup\{a_i\}, T\cup\{a_i\} : i \in [m-2\ell+2]\}$ is a colorful copy of $K^*_{m-2\ell+2,m-2\ell+2}$.

We proceed by induction. We have $c(S\cup\{a_1\}) = c(T\cup\{b_1\}) = 1$. The vertices $S\cup\{a_1\}$ and $T\cup\{b_1\}$ are thus nonadjacent and necessarily $a_1 = b_1$. Assume now that we have proved $a_j = b_j$ and $c(S\cup\{a_j\}) = c(T\cup\{a_j\}) = j$ for all $j < i$. On the one hand, for $j \neq i$, we have $c(S\cup\{a_i\}) \neq c(T\cup\{a_j\})$, since $S\cup\{a_i\} \cap T\cup\{a_j\} = \emptyset$. Thus $c(S\cup\{a_i\}) \geq i$. On the other hand, $c(S\cup\{a_i\}) \leq c(S\cup\{a_i,a_{i-2},\ldots\}) = i$. Hence, $c(S\cup\{a_i\}) = i$. Similarly, $c(T\cup\{b_i\}) = i$. Finally, since $S\cup\{a_i\}$ and $T\cup\{b_i\}$ have same color, they are nonadjacent and thus $a_i = b_i$. □

4. Kneser hypergraphs

Let m,ℓ,r be three integers such that $m \geq r\ell$. The *Kneser hypergraph* $\mathrm{KG}^r(m,\ell)$ is defined by

$$\begin{array}{rcl} V(\mathrm{KG}^r(m,\ell)) & = & \binom{[m]}{\ell} \\ E(\mathrm{KG}^r(m,\ell)) & = & \left\{\{A_1,\ldots,A_r\} : A_i \in \binom{[m]}{\ell},\ A_i \cap A_j = \emptyset \text{ for } i \neq j\right\}. \end{array}$$

Theorem 4.1 (Alon-Frankl-Lovász theorem [2])**.**

$$\chi(\mathrm{KG}^r(m,\ell)) = \left\lceil \frac{m-r(\ell-1)}{r-1} \right\rceil.$$

Theorem 4.2 ([15])**.** *Let p be a prime number. Any proper coloring c of $\mathrm{KG}^p(m,\ell)$ with t colors contains a complete p-uniform p-partite hypergraph with parts $U_1,\ldots,U_p$ satisfying the following properties.*

- *It has $m-p(\ell-1)$ vertices.*
- *The values of $|U_j|$ differ by at most one.*
- *The vertices of U_j get distinct colors.*

Let $\mathcal{H} = (V, E)$ be a uniform hypergraph. For $X \subseteq V$, we define

$$\mathcal{N}(X) = \{v : \exists e \in E \text{ s.t. } e \setminus X = \{v\}\}$$

and $\mathcal{N}[X] := X \cup \mathcal{N}(X)$. The *local chromatic number* of $\mathcal{H}$ is then defined as

$$\psi(\mathcal{H}) = \min_{c} \max_{e \in E,\, v \in e} |c(\mathcal{N}[e \setminus \{v\}])|,$$

where the minimum is taken over all proper colorings c.

An easy consequence of the Zig-zag theorem for Kneser hypergraphs is the following theorem:

Theorem 4.3.

$$\psi(\mathrm{KG}^p(m, \ell)) \geq \min\left(\left\lceil \frac{m - p(\ell - 1)}{p} \right\rceil + 1, \left\lceil \frac{m - p(\ell - 1)}{p - 1} \right\rceil\right)$$

for any prime number p.

5. Open questions

5.1. Kneser graphs.

Question 5.1 ([17])**.** What is the local chromatic number of Kneser graphs?

For Schrijver graphs, the answer is almost known, see [17, 18].

Conjecture 5.2 ([19])**.** There exists a graph homomorphism $\mathrm{KG}(m, \ell) \to \mathrm{KG}(m', \ell')$ if and only if $m' \geq qm - 2k$, where $\ell' = q\ell - k$.

The existence of a graph homomorphism $\mathrm{KG}(m, \ell) \to \mathrm{KG}(m - 2, \ell - 1)$ has been proved by Stahl [19]. The case $m = 2\ell + 1$ and $m' = 2\ell' + 1$ has also been proved by Stahl (1996).

5.2. Zig-zag theorem for Kneser hypergraphs.

Question 5.3. Are Theorems 4.2 and 4.3 valid for nonprime p?

5.3. **Stable Kneser hypergraphs.** A subset A of $[m]$ is *s-stable* if $s \leq |u - v| \leq m - s$ for any distinct u and v taken in A. The *s-stable r-uniform Kneser hypergraph* $\mathrm{KG}^r(m, \ell, s)$ is the hypergraph defined by

$$\begin{array}{rcl} V(\mathrm{KG}^r(m, \ell, s)) & = & \left\{A \in \binom{[m]}{\ell} :\ A \text{ is } s\text{-stable}\right\} \\ E(\mathrm{KG}^r(m, \ell, s)) & = & \left\{\{A_1, \ldots, A_r\} :\ A_i \in V(\mathrm{KG}^r(m, \ell, s)),\ A_i \cap A_j = \emptyset \text{ for } i \neq j\right\}. \end{array}$$

Conjecture 5.4 ([14])**.** If $m \geq \max(s, r)\ell$

$$\chi(\mathrm{KG}^r(m, \ell, s)) = \left\lceil \frac{m - \max(s, r)(\ell - 1)}{r - 1} \right\rceil.$$

The case $s = r$ is the Alon-Drewnowski-Łuczak-Ziegler conjecture [1, 23]: it states that the chromatic number of a Kneser hypergraphs of rank r does not change when we restrict its vertex set to the r-stable ℓ-subsets of $[m]$ (see Section 4). It has been proved for r a power of 2 [1].

Otherwise, some cases have been proved (especially when $r = 2$ and s even [6]).

6. Exercices

6.1. **Sperner's lemma as a consequence of the combinatorial Ky Fan's theorem.** Prove that Sperner's lemma is a consequence of the combinatorial Ky Fan theorem (Theorem 1.2).

6.2. **Combinatorial proof of the Zig-zag theorem.** The Zig-zag theorem of Simonyi and Tardos states in particular that any proper coloring of $\mathrm{KG}(m,\ell)$ with t colors contains a complete bipartite graph $K_{\lfloor (m-2\ell+2)/2\rfloor,\lceil (m-2\ell+2)/2\rceil}$ that is colorful, with the colors alternating on the two side when ordered in the increasing order.

Provide an alternate combinatorial proof of this statement with the help of the octahedral Ky Fan theorem (Lemma 2.2). Hint: adapt the combinatorial proof of the Lovász-Kneser theorem given in Section 2.3.

6.3. **A combinatorial proof of the splitting necklace theorem.** Consider an open necklace with m beads. The number of bead types is t. There are a_i beads of type i (and thus $\sum_{i=1}^t a_i = m$). Assuming that every a_i is even, the splitting necklace theorem due to Alon, Golberg, and West [3, 8] states that there exists a fair splitting of the necklace between two thieves with at most t cuts ("fair" means that each thief gets the same number of beads of each type). This theorem is a classic application of the Borsuk-Ulam theorem.

For $\boldsymbol{x} \in \{+,-,0\}^m$, denote by $\mathrm{alt}(\boldsymbol{x})$ the number of sign changes in $\boldsymbol{x}$ when reading from left to right (0 does not count). Define $h(\boldsymbol{x}) = \max\{\mathrm{alt}(\boldsymbol{y}) :\ \boldsymbol{y} \succeq \boldsymbol{x}\}$ and $s(\boldsymbol{x})$ to be y_1 for a $\boldsymbol{y}$ realizing the maximum in the definition of h (check that $s(\boldsymbol{x})$ is well-defined). Define moreover $\lambda(\boldsymbol{x})$ to be $s(\boldsymbol{x})h(\boldsymbol{x})$ when $h(\boldsymbol{x}) > t$.

Show that it is possible to extend the definition of λ for all $\boldsymbol{x} \in \{+,-,0\}^m \setminus \{\boldsymbol{0}\}$ in such a way that Lemma 2.2 implies the existence of a fair splitting, giving this way an alternate combinatorial proof of the splitting necklace theorem (this proof is due to Pálvölgyi [16]).

6.4. **The circular chromatic number is rational.** Let $G = (V, E)$ be a graph. Let $c' : V \to [0,1]$. The map c' is an *r-circular coloring* if any adjacent vertices u and v are such that

$$\frac{1}{r} \leq |c'(u) - c'(v)| \leq 1 - \frac{1}{r}.$$

1. Prove that $\inf\{r :\ \text{there exists an } r\text{-circular coloring}\}$ is actually a minimum (it is attained for some r).

Let $\overline{r} = \min\{r :\ \text{there exists an } r\text{-circular coloring}\}$ and let c' be an $\overline{r}$-circular coloring. Build the graph $D = (V, A)$ where (u, v) is an arc of D if $c'(v) = c'(u) + 1/r$ or $c'(u) = c'(v) + 1 - 1/r$.

2. Show that D has a circuit.

3. Deduce that $\overline{r}$ is a rational number.

4. Explain how to build a p/q-circular coloring from a (p, q)-coloring.

5. Explain how to build a (p, q)-coloring from a p/q-circular coloring.

6. Conclude: the circular chromatic number is rational.

6.5. **A direct proof of the validity of Hedetniemi's conjecture for Kneser graphs.** Hedetniemi's conjecture states that $\chi(G \times H) = \min(\chi(G), \chi(H))$, where $G \times H$ stands for the *categorical product* of G and H. This latter has vertex set $V(G) \times V(H)$ and edge set $\{(u,v)(u',v') :\ uu' \in E(G),\ vv' \in E(H)\}$.

The inequality $\chi(G \times H) \leq \min(\chi(G), \chi(H))$ is straightforward. The purpose of this exercice is to prove that Kneser graphs satisfy Hedetniemi's conjecture. There are various proofs of this result. Here we propose an elementary one (but using the Lovász-Kneser theorem – Theorem 2.1). It is due to Valencia-Pabon and Vrecia [21].

1. Show that there exists a graph homomorphism $\mathrm{KG}(m,\ell) \to \mathrm{KG}(m,\ell) \times \mathrm{KG}(m',\ell)$ for any integer $m' \geq m$.

2. Show that this implies that Hedetniemi's conjecture is true for Kneser graphs $\mathrm{KG}(m,\ell)$ and $\mathrm{KG}(m',\ell')$ when $\ell=\ell'$.

3. Use the existence of a graph homomorphism $\mathrm{KG}(m,\ell)\rightarrow\mathrm{KG}(m-2,\ell-1)$ (proved by Stahl [19]) to conclude.

There is also a circular version of Hedetniemi's conjecture: $\chi_c(G\times H)=\min(\chi_c(G),\chi_c(H))$.

4. Explain why the same proof actually shows that Kneser graphs satisfy this latter conjecture.

6.6. **Kneser graphs of matroids.** For M a matroid, we denote by $\mathrm{KG}(\mathsf{M})$ the graphs whose vertices are the bases of M and whose edges connect disjoint bases.

6.6.1. *F_7.* Prove that $\chi(\mathrm{KG}(F_7))=3$.

6.6.2. *Rank 1 and rank 2 matroids.*

6.6.3. *An upper bound for rank 3 matroids.* Prove that if M is a rank 3 matroid and has at least 5 elements, then

$$\chi(\mathrm{KG}(\mathsf{M}))\leq\min(n+1-\max_{L\in\mathcal{L}}|L|,n-4),$$

where $\mathcal{L}$ is the set of the hyperplanes of M.

6.7. **Explicit coloring of Kneser hypergraphs.** Describe an explicit proper coloring of $\mathrm{KG}^r(m,\ell)$ with $\left\lceil\frac{m-r(\ell-1)}{r-1}\right\rceil$ colors (see Section 4).

6.8. **Local chromatic number of Kneser hypergraphs.** Prove Theorem 4.3 from Theorem 4.2.

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