SIMPLOTOPAL MAPS AND NECKLACE SPLITTING

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Abstract. We show how to prove combinatorially the Splitting Necklace Theorem by Alon for any number of thieves. Such a proof requires developing a combinatorial theory for abstract simplotopal complexes and simplotopal maps, which generalizes the theory of abstract simplicial complexes and abstract simplicial maps. Notions like orientation, subdivision, and chain maps are defined combinatorially, without using geometric embeddings or homology. This combinatorial proof requires also a $\mathbb{Z}_p$-simplotopal version of Tucker's Lemma.

1. Introduction

1.1. **Context and motivations.** The main motivation of this paper is the celebrated Splitting Necklace Theorem by Alon [1], which states that an open necklace, with t types of beads and with a number of beads of each type being a multiple of q, can always be fairly divided among q thieves using no more than $t(q-1)$ cuts. The original proof uses tools from algebraic topology and is not combinatorial. The section "Open problems" in the paper by Alon [2] starts with the following paragraph.

> "The first obvious problem is the problem of finding pure combinatorial proofs for the problems discussed in this paper [among them the Splitting Necklace Theorem]. After all, one would naturally expect that combinatorial objects should have combinatorial proofs. Such proofs are desirable, since they might shed more light on the problems. At the moment, there is no known combinatorial proof to any of the combinatorial applications of the Borsuk's Theorem mentioned in this paper."

Our objective is to find such a combinatorial proof for the Splitting Necklace Theorem. Two distinct combinatorial proofs for two thieves ($q = 2$) have been recently found. The first one, by the author [12], encodes the splittings as vertices of a cubical complex and the results of the splittings as labels taken in the vertex set of a cube, and applies a theorem by Ky Fan [7]. This theorem is actually a cubical version of Tucker's Lemma [14] (the combinatorial and simplicial counterpart of the Borsuk-Ulam Theorem). The second one, by Pálvölgyi [13], uses directly Tucker's Lemma, but in a way that does not seem easily adaptable.

A natural generalization of the first proof for the case with $q \geq 3$ thieves requires replacing the cube providing the labels by the Cartesian product of simplices with q vertices. It leads to the definition of *simplotopal maps*, which generalize both simplicial maps (between simplicial complexes) and cubical maps (between cubical complexes). Therefore, this paper has a second objective, namely to provide a combinatorial definition of simplotopal complexes and

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simplotopal maps and to provide combinatorial proofs of their properties. Regarding this second objective, there are several challenges, all having to be met in a combinatorial way:

- to define the notion of an oriented simplotope (without using relative homology or geometric embedding).
- to define the notion of a simplotopal map, using only finite sets (in the same vein as for abstract simplicial maps).
- to show how to derive a chain map from a given simplotopal map that keeps its essential properties.
- to describe a barycentric subdivision operator at the level of chains without any geometric embedding.
- to prove a simplotopal analogue of the cubical version of Tucker's Lemma by Ky Fan.

A paper by Ehrenborg and Hetyei [6] explains how to do the first three points above for cubical complexes.

1.2. **Results.** We show that it is possible to realize this program for simplotopal complexes and maps. We will see that this definition contains the notions of both simplicial and cubical maps (although our definition of a cubical map is then slightly more restrictive than that of Ehrenborg and Hetyei).

We use this new framework to give the first purely combinatorial proof of the Splitting Necklace Theorem, solving the open question by Alon stated above. According to Ziegler [15], a combinatorial proof in a topological context is a proof using no simplicial approximation, no homology, no continuous map. Actually, we get also a direct proof of the generalization found by Alon, Moshkovitz, and Safra [3] when there is not necessarily a multiple of q beads of each type. We take this opportunity to prove also a special case (when there are only two types) of a conjecture by Pálvölgyi stating that, in such a situation, we can always select, for each type, the thief who gets more beads of this type than the others.

1.3. **Plan.** The plan of the paper is the following.

In the first section (Section 2), we define abstract simplotopal complexes (Subsection 2.1), simplotopal maps (Subsection 2.2), the notion of oriented simplotope, and the notion of boundary operator for a simplotopal complex, inducing the notion of a chain complex (Subsection 2.3). It is possible to define the Cartesian product of two chains – a thing that is not possible for simplicial complexes (Subsection 2.4). Then we will see how a simplotopal map induces a chain map (Subsection 2.5). The combinatorial definition of the barycentric subdivision operator at the level of chains is provided in Subsection 2.6. In Subsection 2.7, an homotopy equivalence of simplotopal maps is proved. It is not required for the sequel but was done by Ehrenborg and Hetyei [6] for cubical maps.

With the tools of Section 2, we prove the Splitting Necklace Theorem combinatorially (Section 3), by explaining how to encode the splittings and the results for the thieves (Subsection 3.2) and how to generalize Ky Fan's cubical theorem (Subsection 3.3). We prove also a special case of Pálvölgyi's conjecture (Subsection 3.4).

In the last section (Section 4), open questions are stated.

2. Simplotopes, complexes, chains

2.1. **Abstract simplotopal complexes.**

Definition. Let $V_1, \ldots, V_m$ be m disjoint finite sets. Consider a family S of nonempty subsets of $V_1 \times \ldots \times V_m$ of the form $\sigma_1 \times \ldots \times \sigma_m$ with $\sigma_i \subseteq V_i$ for all i, such that if $\sigma_1 \times \ldots \times \sigma_m \in \mathsf{S}$ and if $\emptyset \neq \tau_i \subseteq \sigma_i$ for all i, then $\tau_1 \times \ldots \times \tau_m \in \mathsf{S}$. Such a family S is an *abstract simplotopal complex*. Each element σ of S is a *simplotope*.

When $m = 1$, the definition reduces to that of an ordinary abstract simplicial complex.

For $\sigma = \sigma_1 \times \ldots \times \sigma_m \in \mathsf{S}$, the *dimension* of σ is $\sum_{i=1}^m \dim \sigma_i$, where $\dim \sigma_i = |\sigma_i| - 1$. If $\sigma = \sigma_1 \times \ldots \times \sigma_m \in \mathsf{S}$ and $\tau = \tau_1 \times \ldots \times \tau_m \in \mathsf{S}$ with $\tau_i \subseteq \sigma_i$ for all $i \in \{1, \ldots, m\}$, then τ is a *face* of σ. Any face τ of σ such that $\dim \tau = \dim \sigma - 1$ is a *facet* of σ. The faces of dimension 0 are the *vertices* of σ. The set of all vertices of S is denoted $V(\mathsf{S})$. Note that $V(\mathsf{S}) \subseteq V_1 \times \ldots \times V_m$.

We state here a property that will later be useful.

Lemma 2.1. *Let σ be a d-dimensional simplotope $\sigma_1 \times \ldots \times \sigma_m$ with $d \geq 2$. The graph G whose vertices are the facets of σ and whose edges connect the facets sharing a common $(d-2)$-face is connected.*

Proof. Let $\tau = \tau_1 \times \ldots \times \tau_m$ be a facet of σ. We have $\tau_i = \sigma_i$ for each index i except exactly one, which we call j. Consider any other facet $\tau' = \tau'_1 \times \ldots \times \tau'_m$. Again, we have $\tau'_i = \sigma_i$ for all i except exactly one, which we call j'.

If $j \neq j'$, then define $\omega = \omega_1 \times \ldots \times \omega_m$ with $\omega_i = \sigma_i$ for all $i \notin \{j, j'\}$, with $\omega_j = \tau_j$, and with $\omega_{j'} = \tau_{j'}$. The simplotope ω is a common $(d-2)$-face of τ and τ', and thus τ and τ' are neighbors in G.

If $j = j'$ and $|\sigma_j| \geq 3$, then define $\omega = \omega_1 \times \ldots \times \omega_m$ with $\omega_i = \sigma_i$ for all $i \neq j$ and with $\omega_j = \tau_j \cap \tau'_j$. The simplotope ω is a common $(d-2)$-face of τ and τ', and thus again τ and τ' are neighbors in G.

Otherwise, $j = j'$, $|\sigma_j| = 2$, and there is a σ_k with $k \neq j$ such that $|\sigma_k| \geq 2$, because of the assumption on d. Pick any vertex v_k in σ_k and define $\rho = \rho_1 \times \ldots \times \rho_m$ with $\rho_i = \sigma_i$ for $i \neq k$ and with $\rho_k = \sigma_k \setminus \{v_k\}$. The simplotope ρ is a neighbor of both τ and τ' in G. ☐

The proof shows actually that the diameter of G is at most two.

A *subcomplex* of an abstract simplotopal complex S is an abstract simplotopal complex whose simplotopes are simplotopes of S. Let $X \subseteq V(\mathsf{S})$. The set $\{\sigma \in \mathsf{S} : \sigma \subseteq X\}$ is the subcomplex of S *induced* by X.

Remark 1. Just with abstract simplicial complexes, we can define the *geometric realization* of an abstract simplotopal complex S. It is a set $\mathcal{S}$ of geometric simplotopes (product of geometric simplices) such that (i) each face of any simplotope is also a simplotope of the set, (ii) the intersection of two simplotopes is a face of both, and (iii) the vertex sets of the geometric simplotopes of $\mathcal{S}$ are exactly the simplotopes of S. Every abstract simplotopal complex S has a geometric realization (but we will not use it).

Remark 2. There are sets $\mathcal{S}$ of geometric simplotopes satisfying conditions (i) and (ii) in Remark 1 while not being the geometric realization of an abstract simplotopal complex as defined in the present paper. The fact that we require each V_i to be fixed for the whole complex is indeed restrictive. The example given in Figure 1 shows a 2-dimensional cubical

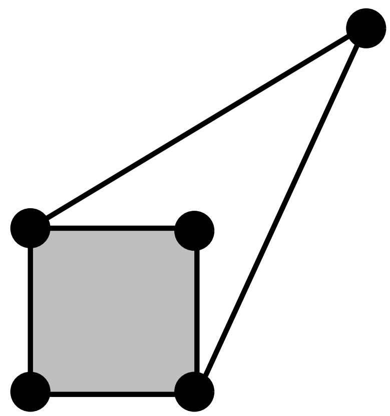

FIGURE 1. A 2-dimensional geometric cubical complex which is not the geometric realization of an abstract simplotopal complex as defined in the present paper

complex in the traditional terminology but does not satisfy the definition of an abstract simplotopal complex given above. However, this can be fixed with the definition of isomorphic simplotopal complexes, which is given in Subsection 2.2: two abstract simplotopal complexes S_1 and S_2 can be "glued" together by identifying a subcomplex of S_1 with an isomorphic subcomplex of S_2. This definition of an abstract simplotopal complex will not be useful for our purpose, and we will not explore further properties of these more general simplotopal complexes.

Product. When S and T are two abstract simplotopal complexes, we define $\mathsf{S} \times \mathsf{T}$ to be the abstract $(\dim \mathsf{S} + \dim \mathsf{T})$-simplotopal complex whose simplotopes are all $\sigma \times \tau$ such that $\sigma \in \mathsf{S}$ and $\tau \in \mathsf{T}$. For the Cartesian product of r copies of S, we use the notation S^r.

Order complex. Given an abstract simplotopal complex S, we denote the containment poset on its faces by $\mathcal{F}(\mathsf{S})$. This poset induces an abstract simplicial complex $\Delta(\mathcal{F}(\mathsf{S}))$, called the *order complex* of $\mathcal{F}(\mathsf{S})$, whose vertices are the simplotopes of S and whose simplices are the chains of the poset. Here, "chains" is understood in the poset terminology: a simplex F in $\Delta(\mathcal{F}(\mathsf{S}))$ is a collection $\{\sigma^{(0)}, \sigma^{(1)}, \ldots, \sigma^{(\ell)}\}$ of simplotopes of S such that $\sigma^{(0)} \subset \sigma^{(1)} \subset \cdots \subset \sigma^{(\ell)}$. With a slight abuse of notation, given a simplotope σ, we define similarly the abstract simplicial complexes $\Delta(\mathcal{F}(\sigma))$ whose vertices are the faces of σ and whose simplices are the chains of its face poset.

2.2. **Simplotopal maps.** Let S and T be two abstract simplotopal complexes. A map $\lambda : V(\mathsf{S}) \to V(\mathsf{T})$ is a *simplotopal map* if for every abstract d-dimensional simplotope σ of S, $\lambda(\sigma)$ is a subset of an abstract d'-dimensional simplotope of T with $d' \leq d$.

An alternative definition goes as follows. Given a subset $\tau \subseteq V_1 \times \ldots \times V_r$, let $\pi_i(\tau) \subseteq V_i$ denote the set of ith coordinates of τ. The map λ is a simplotopal map if and only if for any d-dimensional simplotope $\sigma \in \mathsf{S}$, the set $\pi_1(\lambda(\sigma)) \times \ldots \times \pi_r(\lambda(\sigma))$ is a simplotope whose dimension does not exceed the dimension of σ.

It is worth noting that this is **not** a combinatorial version of cellular maps for regular CW-complexes: $\lambda(\sigma)$ is not necessarily the whole vertex set of a simplotope.

If the simplotopal map is bijective and has an inverse that is also simplotopal, then it is an *isomorphism* and S and T are *isomorphic.*

When S and T are two abstract simplicial complexes, the definition of the simplotopal map reduces to the ordinary definition of a simplicial map.

When S and T are two cubical complexes, the classical definition of a cubical map is the following (see Fan [7] or Ehrenborg and Hetyei [6] for instance): a map $\lambda : V(\mathsf{S}) \to V(\mathsf{T})$ is a *cubical map* if the following conditions are both fulfilled:

(1) for every cube σ of S, $\lambda(\sigma)$ is a subset of the vertices of a cube of T
(2) λ takes adjacent vertices to adjacent vertices or to the same vertex.

Our simplotopal map when the abstract simplotopal complexes are cubical complexes is a cubical map in the sense of Fan or of Ehrenborg and Hetyei, but the converse is not true as the following example shows. Indeed, the following map from a 3-dimensional cube to a 4-dimensional cube is a cubical map in the sense of Fan-Ehrenborg-Hetyei but not in the sense of the present paper. The minimal face containing the image of the 3-cube is the 4-cube itself.

$$\begin{array}{ccc}
(0,0,0) & \to & (0,0,0,0) \\
(1,0,0) & \to & (1,0,0,0) \\
(0,1,0) & \to & (0,1,0,0) \\
(1,1,0) & \to & (0,0,0,0) \\
(0,0,1) & \to & (0,0,0,1) \\
(1,0,1) & \to & (0,0,0,0) \\
(0,1,1) & \to & (0,0,0,0) \\
(1,1,1) & \to & (0,0,1,0)
\end{array}$$

Whether it is possible to define a more general notion of a simplotopal map that would contain a cubical map in this more general sense is an open question (see Subection 4.1 for a complementary discussion).

2.3. **Oriented simplotopes.** Let $\sigma = \sigma_1 \times \ldots \times \sigma_m$ be a d-dimensional simplotope of an abstract simplotopal complex S. An ordering on $\sigma_1 \cup \ldots \cup \sigma_m$ is *σ-compatible* if the vertices of each simplex σ_i are consecutive in the ordering. Define two σ-compatible orderings on $\sigma_1 \cup \ldots \cup \sigma_m$ to be *equivalent* if they differ by an even permutation. If the dimension of σ exceeds 0, then we get two equivalence classes, each of them being an *orientation.* A simplotope σ together with an orientation is an *oriented simplotope.* For the $v_{j,i}$ being the vertices of σ_i for each i, we denote by $\big[v_{0,1}, \ldots, v_{p_1,1} | \ldots | v_{0,m}, \ldots, v_{p_m,m}\big]$ the equivalence class of the ordering $v_{0,1}, \ldots, v_{p_1,1}, \ldots, v_{0,m}, \ldots, v_{p_m,m}$. We denote the opposite orientation by

$$-\big[v_{0,1}, \ldots, v_{p_1,1} | \ldots | v_{0,m}, \ldots, v_{p_m,m}\big].$$

Again, in the special case of an abstract simplex, the definition of an orientation reduces to the ordinary notation of an orientation.

Let $\tau = \tau_1 \times \ldots \times \tau_m$ be a facet of $\sigma = \sigma_1 \times \ldots \times \sigma_m$, with i the only index such that $\tau_i \neq \sigma_i$. The simplex τ_i has the form $\{v_{0,i}, \ldots, v_{p_i,i}\} \setminus \{v_{j,i}\}$ for some index $j \in \{0, \ldots, p_i\}$. This latter set is also denoted $\{v_{0,i}, \ldots, \widehat{v_{j,i}}, \ldots, v_{p_i,i}\}$, where the hat is the deletion operator. This notation is used throughout the paper.

The *induced orientation* of $\big[v_{0,1},\ldots,v_{p_1,1}|\ldots|v_{0,m},\ldots,v_{p_m,m}\big]$ on τ is

$$(-1)^{j+p_1+\ldots+p_{i-1}}\big[v_{0,1},\ldots,v_{p_1,1}|\ldots|v_{0,i},\ldots,\widehat{v_{j,i}},\ldots,v_{p_i,i}|\ldots|v_{0,m},\ldots,v_{p_m,m}\big].$$

We can now define the chain complex of an abstract simplotopal complex S. The group of formal sums of oriented d-dimensional simplotopes with coefficients in $\mathbb{Z}$ is denoted by $C_d(\mathsf{S})$. The boundary operator is defined on an oriented d-dimensional simplotope by

$$\partial\big[v_{0,1},\ldots,v_{p_1,1}|\ldots|v_{0,m},\ldots,v_{p_m,m}\big]=$$

$$\sum_{i\in\{1,\ldots,m\}:\,p_i\geq 1}\sum_{j=0}^{p_i}(-1)^{j+p_1+\ldots+p_{i-1}}\big[v_{0,1},\ldots,v_{p_1,1}|\ldots|v_{0,i},\ldots,\widehat{v_{j,i}},\ldots,v_{p_i,i}|\ldots|v_{0,m},\ldots,v_{p_m,m}\big].$$

The following lemma is obtained via a direct calculation.

Lemma 2.2. *$\partial\circ\partial=0$.*

Hence, the $C_d(\mathsf{S})$'s provide a chain complex $\mathcal{C}(\mathsf{S})$. Note that according to the definition, the boundary of a 0-chain is 0.

2.4. **Product of chains.** An interesting property of abstract simplotopal complexes is that the Cartesian product of two abstract simplotopal complexes is still an abstract simplotopal complex. It is possible to exploit this property at the level of chains identifying $C_d(\mathsf{S})\otimes C_{d'}(\mathsf{T})$ and $C_{d+d'}(\mathsf{S}\times\mathsf{T})$ by identifying

$$\big[v_{0,1},\ldots,v_{p_1,1}|\ldots|v_{0,m},\ldots,v_{p_m,m}\big]\otimes\big[v'_{0,1},\ldots,v'_{p'_1,1}|\ldots|v'_{0,m'},\ldots,v'_{p'_{m'},m'}\big]$$

and

$$\big[v_{0,1},\ldots,v_{p_1,1}|\ldots|v_{0,m},\ldots,v_{p_m,m}|v'_{0,1},\ldots,v'_{p'_1,1}|\ldots|v'_{0,m'},\ldots,v'_{p'_{m'},m'}\big].$$

This identification is used in Subsection 3.3.

Remark 3. The above identification is a combinatorial analogue of the following topological observation: the chain complex associated to a direct product is identifiable with the tensor product of the chain complexes associated to the factors in the direct product.

The following lemma is obtained by a direct calculation.

Lemma 2.3. *If $c\in C_d(\mathsf{S})$ and $c'\in C_{d'}(\mathsf{T})$, then*

$$\partial(c\otimes c')=\partial c\otimes c'+(-1)^d c\otimes\partial c'.$$

2.5. **Chain maps.** A simplotopal map induces a natural chain map of the corresponding chain complexes.

Theorem 2.4. *Given a simplotopal map $\lambda: V(\mathsf{S})\to V(\mathsf{T})$, there exists a unique chain map $\lambda_\#:\mathcal{C}(\mathsf{S})\to\mathcal{C}(\mathsf{T})$ satisfying the following properties.*

- *$\lambda_\#(v)=\lambda(v)$ for every $v\in V(\mathsf{S})$.*
- *For every oriented simplotope $\sigma\in\mathsf{S}$, we have $\lambda_\#(\sigma)=\alpha_\sigma\sigma'$ for some $\alpha_\sigma\in\mathbb{Z}$ and some orientation of σ', where $\sigma'\in\mathsf{T}$ is the simplotope of smallest dimension such that $\lambda(\sigma)\subseteq\sigma'$. In addition, if $\dim\sigma'<\dim\sigma$ or $\lambda(\sigma)\neq\sigma'$, then $\alpha_\sigma=0$.*

This construction is functorial: the identity map at the level of simplotopal complexes induces the identity map at the level of chain complexes, and $(\lambda\circ\mu)_\#=\lambda_\#\circ\mu_\#$.

Proof. The proof consists in building the map $\lambda_\#$ using induction on the dimension d of the simplotopes σ. Once the map is defined for all $(d-1)$-dimensional simplotopes, we show how to extend it for the d-dimensional simplotopes, and we prove that this extension is unique.

For $d = 0$, the map $\lambda_\#$ is fully defined: $\lambda_\#(v) = \lambda(v)$ for every $v \in V(\mathsf{S})$.

For $d = 1$, let σ be an oriented 1-dimensional simplotope and σ' be the oriented d'-dimensional simplotope of T such that $\lambda(\sigma) \subseteq \sigma'$ with d' minimal. For $d' = 0$, we define $\alpha_\sigma = 0$. For $d' = 1$, we proceed as follows. Let u and v be the two vertices of σ, and let u' and v' be the two vertices of σ'. Since $\lambda_\#$ is already defined for the vertices, we have $\lambda_\#(\partial\sigma) = \pm(\lambda(v) - \lambda(u))$, the sign being given by the orientation of σ. We have $\lambda(v) \neq \lambda(u)$ because $d' = 1$. We define α_σ to be -1 or $+1$, depending on whether $\lambda(v) - \lambda(u)$ is equal to $\partial\sigma'$ or $-\partial\sigma'$. The map $\lambda_\#$ defined in such a way satisfies the requirements, and there is no other way to define it.

Now, suppose that $\lambda_\#$ has been defined up to dimension $d-1$, with $d \geq 2$. Let σ be an oriented d-dimensional simplotope of S. Let σ' be the oriented d'-dimensional simplotope of T such that $\lambda(\sigma) \subseteq \sigma'$ with d' minimal.

A facet τ of σ is *vanishing* if $\lambda(\tau)$ is included in a face of σ' of dimension at most $d-2$. By the induction hypothesis, if τ is vanishing, then $\alpha_\tau = 0$. Note that if τ is non-vanishing, then there is a unique facet of σ' containing $\lambda(\tau)$ by definition of a simplotopal map. Denote by $f(\tau)$ this facet with the orientation induced by the orientation of σ'.

By the induction hypothesis, $\lambda_\#$ is defined for $\partial\sigma$. We have

$$\lambda_\#(\partial\sigma) = \sum_{\tau \text{ non-vanishing facet of } \sigma} \alpha_\tau f(\tau).$$

Regrouping those τ having same image, define $\beta_{\tau'} \in \mathbb{Z}$ associated to each facet τ' of σ', so that

$$\lambda_\#(\partial\sigma) = \sum_{\tau' \text{ facet of } \sigma'} \beta_{\tau'}\tau', \tag{1}$$

where τ' is endowed with the induced orientation of σ'.

By the induction hypothesis, one has also

$$(\partial \circ \lambda_\#)(\partial\sigma) = 0$$

since ∂ and $\lambda_\#$ commute when applied to a $(d-1)$-chain. We get that

$$\partial \sum_{\tau' \text{ facet of } \sigma'} \beta_{\tau'}\tau' = 0.$$

Any $(d-2)$-face of σ' is contained in exactly two facets. Hence, for any two facets τ'_1 and τ'_2 sharing a common $(d-2)$-face, we have $\beta_{\tau'_1} = \beta_{\tau'_2}$ since the induced orientations by τ'_1 and τ'_2 on the common $(d-2)$-face are opposite (corollary of Lemma 2.2). Therefore, with the help of Lemma 2.1, all the $\beta_{\tau'}$ are equal. Call this common value α_σ. Now define $\lambda_\#(\sigma)$ to be $\alpha_\sigma\sigma'$. There is no other possible choice for $\lambda_\#(\sigma)$ since we want to have $(\lambda_\# \circ \partial)(\sigma) = (\partial \circ \lambda_\#)(\sigma)$. This proves the existence and the uniqueness of the chain map as in the statement of the theorem.

Finally, the functoriality of $\lambda \mapsto \lambda_\#$ is easily checked. $\square$

When we work with simplicial maps or cubical maps, the $\lambda_\#$ reduces to the ordinary chain map.

2.6. **Barycentric subdivision.** The purpose of this subsection is to define the *barycentric subdivision operator* in a combinatorial way, acting at the level of chains, without passing through any geometric realization.

Given a simplotope $\sigma = \sigma_1 \times \ldots \times \sigma_m$, we consider the face poset $\mathcal{F}(\sigma)$ of its faces. A simplex F of the order complex $\Delta(\mathcal{F}(\sigma))$ is a set $\{\sigma^{(0)}, \sigma^{(1)}, \ldots, \sigma^{(\ell)}\}$ of faces of σ such that $\sigma^{(0)} \subset \sigma^{(1)} \subset \cdots \subset \sigma^{(\ell)} \subseteq \sigma$.

If $\ell = \dim \sigma$, then $\sigma^{(j+1)} = \sigma_1^{(j+1)} \times \ldots \times \sigma_m^{(j+1)}$ and $\sigma^{(j)} = \sigma_1^{(j)} \times \ldots \times \sigma_m^{(j)}$ differ only in one term: that is $\sigma_i^{(j)} = \sigma_i^{(j+1)}$ for each index i except exactly one, say $\iota(j+1)$, for which there is a vertex $v^{(j+1)} \in V_{\iota(j+1)}$ such that $\sigma^{(j+1)}_{\iota(j+1)} = \sigma^{(j)}_{\iota(j+1)} \cup \{v^{(j+1)}\}$. The simplotope $\sigma^{(0)}$ is 0-dimensional; we write it $\sigma^{(0)} = (v_1^{(0)}, v_2^{(0)}, \ldots, v_m^{(0)})$.

When $\ell = \dim \sigma$, we can thus associate to the simplex F a map $f_F : \bigcup_{i=1}^m \sigma_i \to \{0, 1, \ldots, \ell\}$ by $f_F(v^{(j)}) = j$ for $1 \leq j \leq \ell$ and $f_F(v_i^{(0)}) = 0$ for $1 \leq i \leq m$. Define g_F mapping $V(\sigma)$ to $V(\Delta(\mathcal{F}(\sigma)))$ by $g_F(v_1, \ldots, v_m) = \sigma^{(\max_{i=1,\ldots,m} f_F(v_i))}$.

Lemma 2.5. *g_F is a simplotopal map from $\mathcal{F}(\sigma)$ to $\Delta(\mathcal{F}(\sigma))$.*

Proof. Take a face $\tau = \tau_1 \times \ldots \times \tau_m$ of σ. The cardinality of $\{\max_{i=1,\ldots,m} f_F(v_i) : (v_1, \ldots, v_m) \in V(\tau)\}$ is at most $\sum_{j=1}^m |\tau_i| - (m-1)$. The $m-1$ comes from the fact that the $m-1$ smallest values (with multiplicity) of $\{f(v) : v \in \bigcup_{i=1}^m \tau_i\}$ do not contribute to the cardinality. Hence $\dim \tau \geq \dim g_F(\tau)$. $\square$

According to Lemma 2.5 and Theorem 2.4, we can define a chain map $g_{F\#}$. For any oriented d-dimensional simplotope σ, let

$$\mathrm{sd}_\#(\sigma) = \sum_{F \in \Delta(\mathcal{F}(\sigma)) :\, \dim F = d} g_{F\#}(\sigma).$$

We extend it by linearity and get the *barycentric subdivision operator*

$$\mathrm{sd}_\# : C_d(\mathsf{S}) \to C_d(\Delta(\mathcal{F}(\mathsf{S}))).$$

We are going to prove that this operator is a chain map. Before, we need to prove two technical lemmas.

Lemma 2.6. *Let F and F' be two d-simplices of $\Delta(\mathcal{F}(\sigma))$ and assume that F and F' share a common facet G. For a given orientation of σ, the contributions on G by $g_{F\#}(\partial\sigma)$ and by $g_{F'\#}(\partial\sigma)$ are opposite.*

Proof. Let us assume $d \geq 2$. For $d = 1$, the result is direct.

Let $F = \{\sigma^{(0)}, \ldots, \sigma^{(d)}\}$, where $\sigma^{(j)}$ is a j-dimensional face of σ. The facet G has the form $\{\sigma^{(0)}, \ldots, \widehat{\sigma^{(j)}}, \ldots, \sigma^{(d)}\}$ for some integer $j \neq d$ and F' has the form $\{\sigma^{(0)}, \ldots, \sigma'^{(j)}, \ldots, \sigma^{(d)}\}$. Let β and β' be integers such that

$$g_{F\#}(\sigma) = \beta[\sigma^{(0)}, \ldots, \sigma^{(d)}] \tag{2}$$

and

$$g_{F'\#}(\sigma) = \beta'[\sigma^{(0)}, \ldots, \sigma'^{(j)}, \ldots, \sigma^{(d)}]. \tag{3}$$

There is a unique simplotope τ in the support of $\partial\sigma$ whose image by $g_{F\#}$ has a nonzero component on G. Define similarly τ'. Orienting τ and τ' according to σ, we have actually

$$g_{F\#}(\tau) = \beta(-1)^j[\sigma^{(0)}, \ldots, \widehat{\sigma^{(j)}}, \ldots, \sigma^{(d)}]$$

according to (2) and

$$g_{F'\#}(\tau') = \beta'(-1)^j[\sigma^{(0)}, \ldots, \widehat{\sigma^{(j)}}, \ldots, \sigma^{(d)}]$$

according to (3). We want to prove that $\beta = -\beta'$.

To that end, note that the simplotope τ has the form $\tau_1 \times \ldots \times \tau_m$ with $\tau_i = \sigma_i$ for each index i, except for one index k. Furthermore, $\tau_k = \sigma_k \setminus \{v\}$ for some $v \in V_k$ such that $f_F(v) = j$. Define similarly k' and v' associated to F'. Then we have $f_F(v') = f_{F'}(v) = j+1$. Define also $s : \mathcal{F}(\sigma) \to \mathcal{F}(\sigma)$ by $s(\rho) = \rho$ for all faces ρ of σ, except $s(\sigma^{(j)}) = \sigma^{(j+1)}$ and $s(\sigma^{(j+1)}) = \sigma'^{(j)}$. Note that s is a simplicial map from $\Delta(\mathcal{F}(\sigma))$ into itself and that $g_{F'} = s \circ g_F$.

From (2) and the equality $f_F(v') = j+1$, we get

$$g_{F\#}(\tau') = (-1)^{j+1}\beta[\sigma^{(0)}, \ldots, \widehat{\sigma^{(j+1)}}, \ldots, \sigma^{(d)}].$$

Using $s_\#([\sigma^{(0)}, \ldots, \widehat{\sigma^{(j+1)}}, \ldots, \sigma^{(d)}]) = [\sigma^{(0)}, \ldots, \widehat{\sigma^{(j)}}, \ldots, \sigma^{(d)}]$, we get $\beta = -\beta'$, as required. □

Lemma 2.7. *Let F be a facet of $\Delta(\mathcal{F}(\sigma))$, let G be the facet of F whose vertices are proper faces of σ, and let τ be the facet of σ having all vertices of G as faces. Choose an arbitrary orientation for σ and orient τ with the induced orientation. For an arbitrary orientation of G, the coefficient of G in $\partial g_{F\#}(\sigma)$ is then the coefficient of G in $g_{G\#}(\tau)$.*

Proof. We have $\partial g_{F\#}(\sigma) = g_{F\#}(\partial\sigma)$. Take $\tau' \neq \tau$, another facet of σ. We have $g_F(\tau') \neq G$. Indeed, the vertex $v \in (\bigcup_{i=1}^m \sigma_i)\setminus(\bigcup_{i=1}^m \tau'_i)$ is such that $\sigma^{(f_F(v))} \in G \setminus g_F(\tau')$ (even if $f_F(v) = 0$, in which case, 0 is never the maximum of the $f(v_i)$'s for a $(v_1, \ldots, v_m)$ in $V(\tau')$).

The coefficient of G in $\partial g_{F\#}(\sigma)$ is therefore its coefficient in $g_{F\#}(\tau)$. Now, since f_F and f_G coincide on $\bigcup_{i=1}^m \tau_i$, which is $(\bigcup_{i=1}^m \sigma_i) \setminus \{v^{(\ell)}\}$, the map g_F coincides with g_G when restricted to τ. □

Proposition 2.8. *We have $\mathrm{sd}_\# \circ \partial = \partial \circ \mathrm{sd}_\#$.*

Proof. Let $\sigma = \sigma_1 \times \ldots \times \sigma_m$ be a d-dimensional simplotope with $d \geq 1$. We compute

$$\begin{aligned}(\partial \circ \mathrm{sd}_\#)(\sigma) &= \sum_{F \in \Delta(\mathcal{F}(\sigma)) :\, \dim F = d} (\partial \circ g_{F\#})(\sigma) &= \sum_{(\tau \text{ facet of } \sigma)} \sum_{(G \in \Delta(\mathcal{F}(\tau)) :\, \dim G = d-1)} g_{G\#}(\tau) \\ &= \sum_{\tau \text{ facet of } \sigma} \mathrm{sd}_\#(\tau) &= (\mathrm{sd}_\# \circ \partial)(\sigma),\end{aligned}$$

where each τ is oriented according to the orientation of σ. The second equality is obtained thank to Lemmas 2.6 and 2.7. □

Remark 4. Given a simplotope σ, we have $g_{F\#}(\sigma) = \pm F$ for some orientation of F. This can be easily proved with the help of Lemma 2.7 and using induction on the dimension of σ. This equality implies the following property: $\mathrm{sd}_\#$ maps any simplotope σ to the sum of all full-dimensional simplices of its (geometric) barycentric subdivision, each of these simplices getting the orientation induced by σ.

Remark 5. The map $g_{F\#}$ provides a combinatorial way to define the induced orientation of a simplotope σ on a simplex F of its barycentric subdivision.

2.7. **Homotopy equivalence.** An important notion when dealing with induced homology maps is the notion of *homotopic maps*. We will not need such a notion for the remainder of the paper, but, in the spirit of finding simplotopal counterparts of results by Ehrenborg and Hetyei for cubical maps, we show how to do this here.

Two simplotopal maps $\lambda, \mu : \mathsf{S} \to \mathsf{T}$ are *homotopic* when there is a path $P_n = v_0 \ldots v_n$ (seen as a 1-dimensional cubical complex) and a simplotopal map $\phi : \mathsf{S} \times P_n \to \mathsf{T}$ such that for every $v \in V(\mathsf{S})$ we have $\lambda(v) = \phi(v, v_0)$ and $\mu(v) = \phi(v, v_n)$. If we can take P_n to be a path of length 1, then we call λ and μ *elementary homotopic* maps.

Consider Definition 30 on page 285, in the paper by Ehrenborg and Hetyei [6] to see that this notion contains the notion of homotopic cubical maps (with the restriction already underlined at the end of Subsection 2.2). It contains also the notion of homotopic simplicial maps as we explain now. Two simplicial maps λ and μ are *contiguous* if, for each simplex $v_0, \ldots, v_d$ of S, the points

$$\lambda(v_0), \ldots, \lambda(v_d), \mu(v_0), \ldots, \mu(v_d)$$

span a simplex of T. Two simplicial maps λ and μ are homotopic according to the traditional meaning if we can go from λ to μ by a sequence of contiguous maps. Starting with λ and substituting progressively the image by λ of the vertices of S by their image by μ, we see that two contiguous simplicial maps are homotopic simplotopal maps. Actually, the minimal length of the path P_n is the chromatic number of the 1-skeleton of S since, at each step, we can substitute the image of a stable set in this graph.

We prove now that homotopic simplotopal maps induce homotopic chain maps, that is, there is a morphism $D : C_i(\mathsf{S}) \to C_{i+1}(\mathsf{T})$ such that

$$D \circ \partial + \partial \circ D = \lambda_\# - \mu_\#.$$

Lemma 2.9. *If the simplotopal maps $\lambda, \mu : \mathsf{S} \to \mathsf{T}$ are homotopic, then the induced chain maps $\lambda_\#, \mu_\#$ are chain homotopic.*

Proof. By transitivity, it is enough to prove it for elementary homotopic simplotopal maps. We denote by v_0 and v_1 the two vertices of P_1, which we identify with the 1-dimensional oriented simplex $[v_0, v_1]$.

Let $s : \mathcal{C}(\mathsf{S}) \to \mathcal{C}(\mathsf{S} \times P_1)$ be defined by $s(\tau) = (-1)^{\dim \tau - 1} \tau \otimes [v_0, v_1]$ for any oriented simplotope τ in S.

Take now a simplotope σ in S. According to the definition of s, we have

$$(s \circ \partial)(\sigma) = (-1)^{\dim \sigma - 2} (\partial \sigma) \otimes [v_0, v_1].$$

On the other hand, with the help of Lemma 2.3, we compute

$$(\partial \circ s)(\sigma) = (-1)^{\dim \sigma - 1} (\partial \sigma) \otimes [v_0, v_1] - \sigma \otimes v_1 + \sigma \otimes v_0.$$

Hence,

$$(s \circ \partial + \partial \circ s)\sigma = \sigma \otimes v_0 - \sigma \otimes v_1.$$

Letting $D = \phi_\# \circ s$ gives the morphism required by the definition of homotopic chain maps. ∎

3. Splitting necklaces

3.1. **The Necklace Theorem.** We turn now to the Splitting Necklace Theorem. The first version of this theorem, for two thieves, was first proved by Goldberg and West [8], and later by Alon and West [4] with a shorter proof. The version with any number q of thieves (but still a multiple of q of beads of each type) was proved by Alon [1]. The version proved here is slightly more general and was proved by Alon, Moshkovitz, and Safra [3] (they proved first a continuous version, and then proved that a 'rounding-procedure' is possible with flows).

Suppose that the necklace has n beads, each of a certain type i, where $i \in \{1, \ldots, t\}$. Suppose there are A_i beads of type i for $i = 1, \ldots, t$, with $\sum_{i=1}^{t} A_i = n$, where A_i is not necessarily a multiple of q. A *q-splitting* of the necklace is a partition of it into q parts, each consisting of a finite number of non-overlapping subnecklaces of beads whose union captures either $\lfloor A_i/q \rfloor$ or $\lceil A_i/q \rceil$ beads of type i, for $i = 1, \ldots, t$.

Theorem 3.1. *Every necklace with A_i beads of type i for $1 \leq i \leq t$ has a q-splitting requiring at most $t(q-1)$ cuts.*

By a well-known trick (see [1, 10]), it is enough to prove Theorem 3.1 for prime q. In the next three subsections, the number of beads is therefore assumed to be prime and is denoted by p. The necklace is identified with the interval $[0, n)$ going from left to right. The kth bead uniformly occupies the interval $[k-1, k)$. A bead of type i is an *i-bead*.

3.2. **Encoding of the cuts and the resulting distribution to thieves.** Let R be the graph consisting of p paths of length n with a common endpoint. The vertex of degree p is o. The vertices along the rth path are $o, (1, r), \ldots, (n, r)$ with the first coordinate increasing along the path. We view o as the left end of the necklace and (k, r) as the location of a cut and a specification of a thief r.

We now view R as a 1-dimensional abstract simplicial complex and consider the abstract $(t(p-1)+1)$-dimensional simplotopal complex $\mathsf{R}^{t(p-1)+1}$ (which is actually an abstract cubical complex). Each vertex $\boldsymbol{v}$ of $\mathsf{R}^{t(p-1)+1}$ has the form $(v_1, \ldots, v_{t(p-1)+1})$. Define K as the subcomplex of $\mathsf{R}^{t(p-1)+1}$ induced by X, where

$$X = \{(v_1, \ldots, v_{t(p-1)+1}) \in V(\mathsf{R}^{t(p-1)+1}) : \, v_{\bar{j}} = (n, r) \text{ for some } \bar{j} \text{ and some } r\}.$$

A vertex $\boldsymbol{v}$ of K provides a splitting of the necklace with at most $t(p-1)$ cuts. Indeed, for each $j \in \{1, \ldots, t(p-1)+1\}$, we have $v_j = o$ or $v_j = (k, r)$ for some $k \in \{1, \ldots, n\}$ and some $r \in \{1, \ldots, p\}$, which gives a cut at a position $k \in \{0, \ldots, n\}$. At that stage, we do not care about the thief r if the v_j has the form (k, r). Since there is a $\bar{j}$ such that $v_{\bar{j}} = (n, r)$, one of these cuts is at position n and is therefore not a real cut. We have thus indeed at most $t(p-1)$ cuts.

We explain now how the assignment to the thieves is encoded by a vertex $\boldsymbol{v}$ of K. We consider the $t(p-1)+1$ components v_j of this vertex $\boldsymbol{v}$ and the resulting $t(p-1)+1$ subnecklaces. We describe to which thief the ℓth subnecklace (starting from the left) is assigned, for any $\ell \in \{1, \ldots, t(p-1)+1\}$. Denote by x the left endpoint of that subnecklace ℓ and by y its right endpoint. The subnecklace is the interval $[x, y) \subseteq [0, n)$. Note that $[x, x)$ is empty. Take $v_j = (k, r)$, with the smallest j such that $k \geq y$. The thief r is the one who gets the ℓth subnecklace.

An example is given in Figure 2. There are three thieves Alice, Bob, and Charlie, and four types of beads. The example illustrates how the cuts and the assignments are encoded

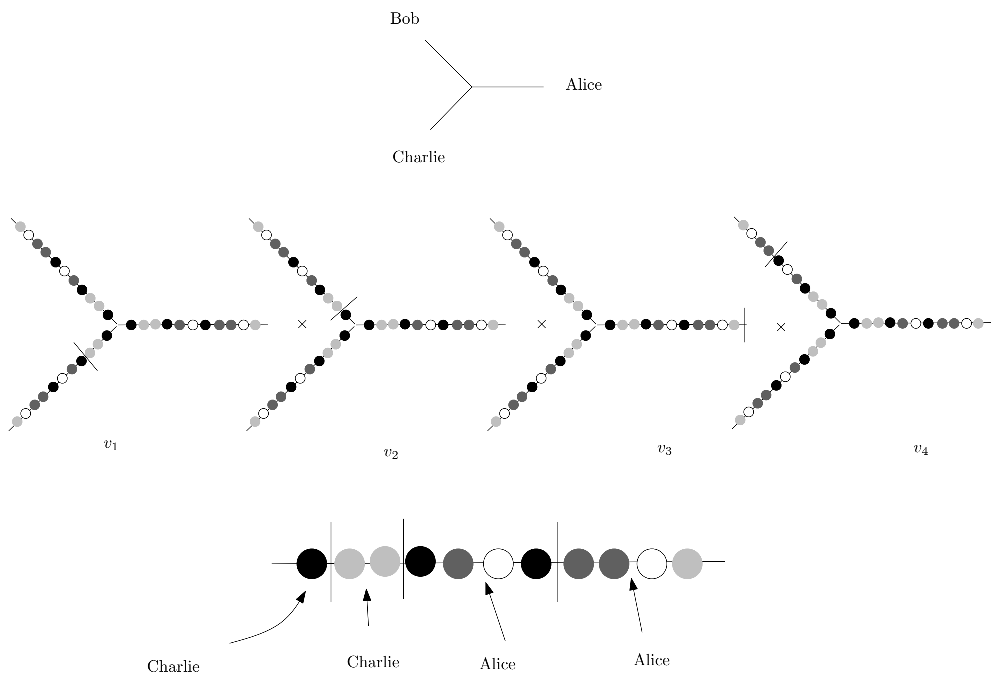

FIGURE 2. Encoding of a splitting as a vertex of K. Here the v_j of the form (n, r) is v_3 with $r =$ Alice.

by a vertex $\boldsymbol{v} = (v_1, v_2, v_3, v_4)$ with

$$v_1 = (3, \text{Charlie}),\ v_2 = (1, \text{Bob}),\ v_3 = (11, \text{Alice}),\ v_4 = (7, \text{Bob}).$$

The vertex $\boldsymbol{v}' = (v_1', v_2', v_3', v_4')$ with $v_1' = v_2$, $v_2' = v_1$, $v_3' = v_3$, and $v_4' = v_4$ gives the same cuts as $\boldsymbol{v}$ and the same assignments, except the first subnecklace, which is assigned to Bob in $\boldsymbol{v}'$, while it is assigned to Charlie in $\boldsymbol{v}$. This encoding was proposed in the paper [12] for the case with two thieves.

Consider now the abstract simplotopal complex $\mathsf{L} := (\Delta_{p-1})^t$, where Δ_{p-1} denotes the abstract $(p-1)$-dimensional simplex whose vertices are $1, \ldots, p$. Note that K and L both have dimension $t(p-1)$ and that L has a unique simplotope of maximal dimension.

For each vertex $\boldsymbol{v}$ of K, define $\lambda_i(\boldsymbol{v})$ as the thief who gets the largest amount of beads of type i when one splits the necklace according to $\boldsymbol{v}$. Using the positions of the beads on the necklace as a total order, one can avoid a tie: in case of equality, the thief with the i-bead at the rightmost position is considered as advantaged. This thief is called the *i-winner*.

3.3. **Simplotopal Ky Fan's Theorem and proof of the Necklace Theorem.** Let ν be the cyclic shift $r \mapsto r + 1$ modulo p. Note that it induces a free action on K. The map

$$\begin{array}{rccc} \lambda: & V(\mathsf{K}) & \to & V(\mathsf{L}) \\ & \boldsymbol{v} & \mapsto & (\lambda_1(\boldsymbol{v}), \ldots, \lambda_t(\boldsymbol{v})) \end{array}$$

is then an equivariant simplotopal map (see Lemma 3.2 below), that induces an equivariant chain map $\mathcal{C}(\mathsf{K}) \to \mathcal{C}(\mathsf{L})$ (we use here Theorem 2.4 to derive the construction of this induced chain map).

Lemma 3.2. *λ is an equivariant simplotopal map.*

Proof. The equivariance is straightforward. Let us check that λ is simplotopal, that is, that the image of a d-cube σ in K is contained in a simplotope of L having dimension at most d.

Take a d-cube σ. It is defined by d edges, each of them selected in a distinct copy of R in the product $\mathsf{R}^{t(p-1)+1}$. Each of these edges corresponds to a cut "sliding" along a bead. The other $t(p-1)-d$ cuts are fixed throughout the vertices of σ. We denote by d_i the number of these cuts sliding on i-beads. In particular, we have $\sum_{i=1}^t d_i = d$. We define W_i as the set of thieves that are i-winners for some vertex of σ. In other words, a thief is in W_i if he is a i-winner for some position of the d sliding cuts among the 2^d possible positions, which correspond to the 2^d vertices of σ. We are going to show that $|W_i| \leq d_i + 1$ for $i = 1, \ldots, t$. By the definition of λ, this will show that the image of σ is contained in a simplotope of L having dimension at most $\sum_{i=1}^t d_i = d$.

We introduce for each i a hypergraph H_i whose vertices are the thieves and whose edges correspond to the i-beads on which there is at least one sliding cut from σ: for such an i-bead, the set of thieves who can get this bead for some position of the sliding cuts is an edge of H_i. In other words, denoting by $b(F)$ the bead corresponding to an edge $F \in E(H_i)$, a thief is in F if there is a vertex of σ – that is a position of the sliding cuts – for which he gets the bead $b(F)$. Note that H_i has at most d_i edges and that some of them may have only one vertex.

We select now a special position for the d sliding cuts. For each i, we choose a position of the cuts sliding on i-beads that gives to the i-winner the smallest possible number of i-beads among the 2^{d_i} positions of these cuts. In case of equality, we choose a position such that the rightmost i-bead of the i-winner has the smallest possible position. Since two cuts sliding on distinct beads keep their relative positions, we can choose the positions of the sliding cuts independently for each i. We get in such a way our special position for the d cuts, which corresponds to a special vertex $\boldsymbol{v}$. We denote by w_i the thief who is the i-winner in $\boldsymbol{v}$ for $i = 1, \ldots, t$. In the special vertex $\boldsymbol{v}$, each w_i gets $a_i + b_i$ beads, where $a_i = \left\lfloor \frac{A_i}{p} \right\rfloor$, for some $b_i \geq 0$. We denote by $T_i^{(x)}$ the set of thieves in W_i who get $a_i + x$ beads of type i in this special vertex. Let $n_i^{(x)} = |T_i^{(x)}|$.

Recall that we want to bound $|W_i|$, which is exactly $\sum_{x=-a_i}^{b_i} n_i^{(x)}$. We will get this bound by studying the $T_i^{(x)}$ and H_i. We define the *head* of an edge F in H_i to be the thief who gets $b(F)$ in the special vertex $\boldsymbol{v}$.

CLAIM 1: Let $i \in \{1, \ldots, t\}$ and $x \in \{-a_i, \ldots, b_i\}$. For each thief $w \in T_i^{(x)}$, there are at least $b_i - x$ edges F of H_i such that w is in F while not being its head.

Indeed, this thief w is a i-winner for some position of the d cuts. By definition of w_i, the thief w gets at least $a_i + b_i$ beads of type i in the position making him a i-winner. There is a difference of at least $b_i - x$ beads with what he gets in the special position. There are therefore at least $b_i - x$ beads w does not get in the special position but he gets in the position making him a i-winner.

The following claim strengthens CLAIM 1 for $x = b_i$.

CLAIM 2: Let $i \in \{1, \ldots, t\}$. For each thief $w \in W_i \setminus \{w_i\}$, there is an edge F of H_i such that w is in F while not being its head.

Indeed, assume to the contrary that such a thief is the head of all edges of H_i containing him, or that he is in no edge at all. According to CLAIM 1, he gets $a_i + b_i$ beads of type i in the special vertex $\boldsymbol{v}$. Now, consider $\boldsymbol{v}'$ a vertex of σ making him a i-winner. Such a vertex exists since $w \in W_i$. In $\boldsymbol{v}'$, the thief w gets at least $a_i + b_i$ beads by definition of w_i. Because of the assumption, he gets no additional bead in $\boldsymbol{v}'$. Hence, in $\boldsymbol{v}'$, he gets exactly $a_i + b_i$ beads. Thus the set of i-beads he gets is the same in $\boldsymbol{v}$ and $\boldsymbol{v}'$. Since he is not the i-winner in $\boldsymbol{v}$, he has his rightmost i-bead at a smaller position than w_i. Now there is a contradiction: the special vertex should have been $\boldsymbol{v}'$ and not $\boldsymbol{v}$.

CLAIM 3: For each $1 \leq i \leq t$
$$\sum_{F \in E(H_i)} (|F| - 1) \leq d_i.$$

By definition of the assignment of the beads to the thieves, if l distinct thieves can get the same bead among the vertices of σ, then there are at least $l - 1$ cuts sliding on this bead. The fact that there are exactly d_i cuts sliding on beads of type i ends the proof of CLAIM 3.

Define $\delta_{r,i}^F \in \{0, 1\}$ so that $\delta_{r,i}^F = 1$ precisely when, in H_i, the thief r is in F while not being the head of the edge F. For a given i
$$\sum_{F \in E(H_i)} \sum_{r \in W_i} \delta_{r,i}^F = \sum_{F \in E(H_i)} (|F| - 1).$$

We also have
$$\sum_{r \in W_i} \sum_{F \in E(H_i)} \delta_{r,i}^F = \sum_{x=-a_i}^{b_i} \sum_{r \in T_i^{(x)}} \sum_{F \in E(H_i)} \delta_{r,i}^F \geq n_i^{(b_i)} - 1 + \sum_{x=-a_i}^{b_i - 1} (b_i - x) n_i^{(x)}.$$

The inequality is a consequence of CLAIM 1 and CLAIM 2. With the help of CLAIM 3, we get finally $|W_i| \leq d_i + 1$ for all i, as required. □

Remark 6. The map λ defined in the present section is a very particular simplotopal map, since the image of any cube is a simplotope (and not simply **contained** in a simplotope).

The following theorem can be interpreted as a $\mathbb{Z}_p$-simplotopal version of Tucker's Lemma (and a simplotopal version of Ky Fan's Theorem), generalizing the one used for the case with two thieves.

Theorem 3.3. *If $\mu : \mathsf{K} \to \mathsf{L}$ is an equivariant simplotopal map, then K has a $t(p-1)$-dimensional simplotope whose image under μ is the $t(p-1)$-dimensional simplotope of L.*

We postpone the proof of this theorem. Once this theorem is proved, we are done. Indeed, we can apply this theorem to $\mu = \lambda$, since the λ built in the present section satisfies its condition, and the following lemma leads then to the desired conclusion.

Lemma 3.4. *If K has a $t(p-1)$-dimensional simplotope σ whose image under λ is the $t(p-1)$-dimensional simplotope of L, then σ has at least one vertex corresponding to a p-splitting.*

Proof. We consider the special vertex $\boldsymbol{v}$ of such a simplotope σ and the same notations as in the proof of Lemma 3.2. For each type i, we have

$$p = |W_i| \leq n_i^{(b_i)} + \sum_{x=-a_i}^{b_i-1} (b_i - x) n_i^{(x)} \leq d_i + 1. \tag{4}$$

The first equality is a consequence of the full-dimensionality of $\lambda(\sigma)$. The first inequality comes from $\sum_x n_i^{(x)} = |W_i|$. The second inequality is proved within the proof of Lemma 3.2.

Since $\sum_{i=1}^t d_i = t(p-1)$, we have $d_i = p-1$ for each type i, and all inequalities are equalities in (4), which implies $n_i^{(x)} = 0$ for $x < b_i - 1$. Each thief gets $a_i + b_i - 1$ or $a_i + b_i$ beads of type i for $i = 1, \ldots, t$. Since there are A_i beads of type i in total, we get that $b_i \in \{0, 1\}$ and that each thief gets a_i or $a_i + 1$ beads of type i for $i = 1, \ldots, t$. It is a p-splitting. □

It remains only to prove Theorem 3.3. If we were allowed to use homology and contradiction, the proof would be a direct consequence of the Hopf-Lefschetz Formula. Indeed, K is a free cubical $\mathbb{Z}_p$-complex of dimension $t(p-1)$ and of connectedness $t(p-1)-1$. Suppose to the contrary that Theorem 3.3 does not hold. We have then an equivariant simplotopal map $\mu : \mathsf{K} \to \partial\mathsf{L}$, where $\partial\mathsf{L}$ is the free $\mathbb{Z}_p$-simplotopal complex of dimension $t(p-1)-1$ obtained by removing the top simplotope of L. Theorem 2.4 ensures that there is an equivariant chain map $\mu_\# : \mathcal{C}(\mathsf{K}) \to \mathcal{C}(\partial\mathsf{L})$. We can then write the Hopf-Lefchetz Formula and derive a contradiction in a similar way as for the proof of Theorem 6.2.5 (A "Borsuk-Ulam Theorem for G-space) on pages 139–141 in the book by Matoušek [10].

The following proof does not use homology and is a direct proof.

Proof of Theorem 3.3. Our proof uses three objects (all chains have coefficients in $\mathbb{Z}_p$):

(i) a sequence of d-chains (h_d) for $d = 0, \ldots, t(p-1)$ in $\mathcal{C}(\mathsf{K})$ such that

$$h_0 = (o, \ldots, o, (n, 1)), \quad \partial h_{2l} = (\nu - \mathrm{id}) h_{2l-1}, \quad \text{and} \quad \partial h_{2l+1} = \sum_{r=1}^{p} \nu^r h_{2l},$$

where id is the identity map at the level of chains.

(ii) an equivariant chain map $\eta_\# : \mathcal{C}(\partial\mathsf{L}) \to \mathcal{C}\left(\mathbb{Z}_p^{*(t(p-1)+1)}\right)$, where $*$ is the join operation and $\partial\mathsf{L}$ is L minus its unique face of maximal dimension.

(iii) a sequence of chain maps $\phi_{d\#} : C_d\left(\mathbb{Z}_p^{*(t(p-1)+1)}\right) \to \mathbb{Z}_p$ such that $\phi_{0\#}$ is equal to 1 for a unique vertex in the first copy of $\mathbb{Z}_p$ in $\mathbb{Z}_p^{*(t(p-1)+1)}$ and 0 elsewhere, and such that for all $l = 0, \ldots, \lceil t(p-1)/2 \rceil - 1$

$$\phi_{(2l+1)\#} \circ (\mathrm{id} - \nu) = \phi_{(2l)\#} \circ \partial \quad \text{and} \quad \phi_{(2l+2)\#} \circ \left(\sum_{r=1}^{p} \nu^r \right) = \phi_{(2l+1)\#} \circ \partial.$$

We explain now how to build these three objects.

(i) In each copy of R, let P_r be the rth path of length n leaving o, oriented from o to the endpoint (n, r). Define for $l \geq 1$, with the notation introduced in Subsection 2.4

$$\tilde{h}_1 = \sum_{r=1}^{p} \nu^r P_1 \tag{5}$$

$$\tilde{h}_{2l} = (\nu - \mathrm{id}) \left(P_1 \otimes \tilde{h}_{2l-1} \right) \tag{6}$$

$$\tilde{h}_{2l+1} = \sum_{r=1}^{p} \nu^r \left(P_1 \otimes \tilde{h}_{2l} \right) \tag{7}$$

Note that $\tilde{h}_d \in C_d(\mathsf{R}^d)$. Define

$h_0 := (o, \ldots, o, (n, 1))$ and $h_d := \partial(o \otimes \ldots \otimes o \otimes P_1 \otimes \tilde{h}_d) + o \otimes \ldots \otimes o \otimes \tilde{h}_d$ for $d = 1, \ldots, t(p-1)$. Using the fact that $(\nu - \mathrm{id}) \circ (\sum_{r=1}^{p} \nu^r) = 0$, checking that (h_d) satisfies the required relation is straightforward. The fact that $h_d \in \mathcal{C}(\mathsf{K})$ is proved as follows. For $d = 1, \ldots, t(p-1)$

$$h_d = -o \otimes \ldots \otimes o \otimes P_1 \otimes \partial \tilde{h}_d + o \otimes \ldots \otimes o \otimes (n, 1) \otimes \tilde{h}_d.$$

By a direct induction using the fact that coefficients are taken in $\mathbb{Z}_p$ and that $(\nu - \mathrm{id})\tilde{h}_{2l-1} = \sum_{r=1}^{p} \nu^r \tilde{h}_{2l} = 0$, we have that each vertex $(v_1, \ldots, v_d)$ of a simplotope in the support of $\partial \tilde{h}_d$ is such that one of the v_j's is of the form (n, r).

(ii) Define a chain map $g_\# : \mathcal{C}(\Delta(\mathcal{F}(\partial \mathsf{L}))) \to \mathcal{C}\left(\mathbb{Z}_p^{*(t(p-1)+1)}\right)$ by taking in each orbit of $\Delta(\mathcal{F}(\partial \mathsf{L}))$ a face σ of $\partial \mathsf{L}$ (recall that $\mathbb{Z}_p$ acts on $\mathcal{F}(\partial \mathsf{L})$) and by defining $g(\sigma)$ to be any vertex in the $(\dim \sigma + 1)$-copy of $\mathbb{Z}_p$ in $\mathbb{Z}_p^{*(t(p-1)+1)}$. The map g is then an equivariant simplicial map.

Now, take the barycentric subdivision operator $\mathrm{sd}_\# : \mathcal{C}(\partial \mathsf{L}) \to \mathcal{C}(\Delta(\mathcal{F}(\partial \mathsf{L})))$ as defined in Subsection 2.6. Here, the construction described in that subsection makes $\mathrm{sd}_\#$ an equivariant chain map.

Finally define $\eta_\# = g_\# \circ \mathrm{sd}_\#$. Note that $\eta_\#$ applied on a vertex of L provides a vertex in the first copy of $\mathbb{Z}_p$; this remark will be useful below.

(iii) This sequence is constructed in [9]: $\phi_d = u \circ f_d$ where u is defined page 413 and f_d is defined page 411. An alternative proof would replace $\nu - \mathrm{id}$ in the formulas above by $\nu - \nu^{-1}$ and use $\phi_d = e_d$ where (e_d) is the sequence of cochains in $\mathrm{Hom}\left(C_d\left(\mathbb{Z}_p^{*(t(p-1)+1)}\right), \mathbb{Z}_p\right)$ defined in [11].

We show by induction that

$$(\phi_{(2l)\#} \circ \eta_\# \circ \mu_\#) \left(\left(\sum_{r=1}^{p} \nu^r \right) h_{2l} \right) = (-1)^l \bmod p$$

$$(\phi_{(2l+1)\#} \circ \eta_\# \circ \mu_\#) ((\nu - \mathrm{id}) h_{2l+1}) = (-1)^{l+1} \bmod p,$$

for $l = 0, \ldots, \lfloor (t(p-1) - 1)/2 \rfloor$. Start with $l = 0$. We have $(\phi_{0\#} \circ \eta_\# \circ \mu_\#)(\sum_{r=1}^{p} \nu^r h_0) = \phi_{0\#}(\sum_{r=1}^{p} \nu^r \eta_\#(1, \ldots, 1))$, where $(1, \ldots, 1) \in V(\mathsf{L})$. Hence, $(\phi_{0\#} \circ \eta_\# \circ \mu_\#)(\sum_{r=1}^{p} \nu^r h_0) = 1$ (here, we use the fact that $\eta_\#$ applied on a vertex L provides a vertex in the first copy of $\mathbb{Z}_p$). After that, the formulas above are proved by a straightforward induction.

Now, if $t(p-1)-1$ is even, we have $(\phi_{(t(p-1)-1)\#} \circ \eta_{\#} \circ \mu_{\#}) \left(\left(\sum_{r=1}^{p} \nu^r \right) h_{t(p-1)-1} \right) \neq 0$. It can be rewritten $(\phi_{(t(p-1)-1)\#} \circ \eta_{\#} \circ \mu_{\#})(\partial h_{t(p-1)}) \neq 0$, or equivalently $(\phi_{(t(p-1)-1)\#} \circ \circ \eta_{\#})\partial \mu_{\#}(h_{t(p-1)}) \neq 0$, which shows that $\mu_{\#}(h_{t(p-1)}) \neq 0$ and hence that there is an oriented $t(p-1)$-simplotope σ of K whose image by $\mu_{\#}$ is nonzero. The same holds if $t(p-1)-1$ is odd. □

3.4. **Some remarks about the Necklace Theorem.** Pálvölgyi conjectured in [13] that Theorem 3.1 can be strengthened as follows.

Conjecture 3.5. *If q thieves want to split a necklace with t types of beads such that the jth thief gets $a_i^{(j)}$ of the ith type where $a_i^{(j)} = \lfloor A_i/q \rfloor$ or $\lceil A_i/q \rceil$ and $\sum_j a_i^{(j)} = A_i$ for each i, then they can do it using at most $(q-1)t$ cuts.*

In [12], this conjecture was noted to be true for $q = 2$: for each type i, we can decide which thief gets $\lfloor A_i/2 \rfloor$ and which one gets $\lceil A_i/2 \rceil$ beads of this type.

Actually, our combinatorial proof of Theorem 3.1 suggests that the following weaker version of Conjecture 3.5 may be true. Note that both conjectures coincide for $q = 2$.

Conjecture 3.6. *If q thieves want to split a necklace with t types of beads such that each thief gets at least $\lfloor A_i/q \rfloor$ of the ith type for each i and such that they choose for each type i a thief getting at least as many beads of this type as any other thief, then they can do it using at most $(q-1)t$ cuts.*

We prove another special case of Conjecture 3.5.

Proposition 3.7. *Conjecture 3.5 is true for $t = 2$.*

Proof. By induction on q. If $q = 1$, there is nothing to prove. Assume now that $q \geq 2$.

Consider an arbitrary thief j. He wants to get $a_1^{(j)}$ beads of type 1 and $a_2^{(j)}$ beads of type 2. We identify the necklace with the interval $[0, A_1 + A_2)$. We slide continuously a "window" of length $(A_1 + A_2)/q$ all over the necklace. The left endpoint of the window is denoted by x. Let $f_1(x)$ denote the amount of beads of type 1 in the window. We have $\sum_{k=0}^{q-1} f_1(k(A_1 + A_2)/q) = A_1$, since each bead is contained in exactly one window with x of the form $k(A_1 + A_2)/q$. By continuity, there is necessarily an x such that $f_1(x) = A_1/q$. Now, if A_1/q is fractional, at least one of the endpoint of the window at position x is in the interior of a bead of type 1, and the same holds for type 2. It is then easy to see that we can move the endpoints in order to get $a_1^{(j)}$ beads of type 1 and $a_2^{(j)}$ beads of type 2.

We give this subnecklace to thief j, and we use the induction hypothesis to split the remainder of the necklace among the remaining thieves. □

4. Discussion

4.1. **Cubical maps.** We have outlined the fact that our definition of a simplotopal map, when specialized for cubical complexes, does not lead to the cubical map defined in its full generality (see Subsection 2.2). It remains open whether a more general version of a simplotopal map is possible that would correctly generalize the notion of a cubical map.

Just dropping the dimension requirement in the definition quickly leads to serious difficulties. For instance, consider the map assigning each vertex of a square to a distinct vertex of a tetrahedron. If we want to keep a combinatorial definition, in which the simplotopes are identified by their vertex sets, then we get a map whose image of a 2-dimensional simplotope is a 3-dimensional simplotope. It is not clear how to deal with such maps.

An application of a more general version of a simplotopal map might be a Sperner Lemma for a simplotopal complex that generalizes simultaneously the classical Sperner's Lemma and the cubical Sperner's Lemma found by Ky Fan [7].

4.2. **Algorithmic proof?** From a purely logical point of view, our proof is constructive since we use neither the Axiom of Choice nor contradiction. We can derive an algorithm from our proof, but this algorithm turns out to be a brute-force enumeration-based one: the algorithm scans all simplotopes of the simplotopal complex K, which is easy to build, and compute λ for each vertex of these simplotopes, until it finds a simplotope whose image by λ is the maximal simplotope of L (see Section 3).

We can ask whether there is a more efficient algorithm. A similar question was raised by De Longueville and Živaljević [5] about a theorem that is a $\mathbb{Z}_p$-generalization of Tucker's Lemma. They ask for a constructive proof of this theorem. At the end of Section 2, they explain what they understand by "constructive".

> "The proof [...] is based on the construction of a particular graph of degree at most two. Following a path in this graph starting at a known vertex of degree one, the inclined mathematician will end up in a vertex corresponding to the desired *(object)*. In order to do this one actually will only need to construct the graph along this path. In general this will be much quicker than to search all *(objects)* [...]."

Moreover, we can notice that the graph itself does not have to be kept in the memory of the computer: a neighbor of a vertex can be computed when needed. With some work, it is possible to build a similar graph for the proof of Theorem 3.3. Unfortunately, all attempts of the author have lead to graphs with vertex degrees taking at best values in $\{1, 2, p\}$. An algorithm starting at some vertex of degree 1 in order to find another degree one vertex requires to store the set of visited vertices and is somehow an enumeration-based algorithm.

An algorithmic proof of the Splitting Necklace Theorem without any enumeration remains to be designed.

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