# A topological lower bound for the circular chromatic number of Schrijver graphs

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**Abstract**

In this paper, we prove that the Kneser graphs defined on a ground set of $n$ elements, where $n$ is even, have their circular chromatic numbers equal to their chromatic numbers.

# 1 Introduction

## 1.1 Kneser graphs and reduced Kneser graphs

Given an integer $n$, we will denote $\{1, \ldots, n\}$ by $[n]$, the collection of all subsets of $[n]$ will be denoted by $2^{[n]}$ and the collection of all $k$-subsets (subsets of cardinality $k$) of $[n]$ by $\binom{[n]}{k}$.

A subset $S$ of $[n]$ is called *stable* if $2 \leq |x - y| \leq n - 2$ for distinct elements $x$ and $y$ of $S$.

Let $\mathcal{F} \subseteq 2^{[n]}$ be a hypergraph. By $KG(\mathcal{F})$ we denote the graph whose vertex set is the edge set of $\mathcal{F}$, and $(A, B)$ is an edge of $KG(\mathcal{F})$ if and only if $A \cap B = \emptyset$. $KG(\mathcal{F})$ is called the *Kneser graph* associated to $\mathcal{F}$.

The *reduced Kneser graphs* $SG(\mathcal{F})$ is the subgraph of $KG(\mathcal{F})$ induced by all stable $k$-subsets.

If $n \geq 2k$, $KG(n, k)$ is $KG(\binom{[n]}{k})$ and $SG(n, k)$ is $SG(\binom{[n]}{k})$. Kneser [5] conjectured in 1955 that $\chi(KG(n, k)) = n - 2k + 2$. It was first proved by Lovász [6] in 1978 using tools of algebraic topology. Soon after the announcement of Lovász's breakthrough, Bárány [1] found a shorter proof; it is this latter that we will follow here.

This was proved by Schrijver [9] that $\chi(SG(n, k)) = \chi(KG(n, k))$.

## 1.2 Circular chromatic number

For two positive integers $p$ and $q$, $p \geq 2q$, a $(p, q)$*-coloring* of a graph $G$ is a mapping $\phi$ from the vertex set $V(G)$ into the set $[p]$ such that

$$(u, v) \in E(G) \Rightarrow q \leq |\phi(u) - \phi(v)| \leq p - q.$$

The *circular chromatic number* $\chi_c(G)$ is defined to be the infimum of $p/q$ such that $G$ admits a $(p, q)$-coloring. This concept was introduced by Vince [11] under the name *star chromatic number*. He proved that this number is in fact a minimum. It can be shown that

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G), \qquad (1)$$

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and thus $\lceil \chi_c(G) \rceil = \chi(G)$. The circular chromatic number has been studied intensively in recent years, for more see the survey [12], which presents a section about graphs with $\chi_c(G) = \chi(G)$. Johnson, Holroyd and Stahl [4] conjectured that among such graphs we can find the Kneser graphs, proving it for $n \leq 2k+2$, or $k = 2$. In 2003, Hajiabolhassan and Zhu [3] proved this conjecture for $n \geq 2k^2(k-1)$. The main result of this paper is the following theorem:

**Theorem 1** *For $n$ even, and $k$ such that $n \geq 2k$, we have $\chi_c(SG(n,k)) = \chi_c(KG(n,k)) = \chi(SG(n,k) = \chi(KG(n,k)) = n - 2k + 2$.*

This theorem states that the conjecture of Johnson, Holroyd and Stahl is true for $n$ even. Moreover, the equality between the circular chromatic numbers of Kneser graph and reduced Kneser graph enlightens a question of Lih and Liu [7]: they asked about the minimum $t(k)$ such that for any $n \geq t(k)$, $\chi_c(SG(n,k)) = \chi(KG(n,k))$.

# 2 Topological tools

## 2.1 Ky Fan's Theorem

The main theorem we will use is a theorem of Ky Fan [2] which generalizes the Lusternik-Schnirelmann theorem (which is the version of the Borsuk-Ulam theorem involving a cover of the $n$-sphere by $n+1$ sets, all open or all closed).

We give the theorem exactly as it is written in the original paper of Fan (without proof):

**Theorem 2 (Fan's theorem)** *Let $n$,$k$ be two arbitrary positive integers. If $k$ closed subsets $F_1$, $F_2$, ..., $F_k$ of the $n$-sphere $S^n$ cover $S^n$ and if no one of them contains a pair of antipodal points, then there exist $n+2$ indices $l_1$, $l_2$, ..., $l_{n+2}$, such that $1 \leq l_1 < l_2 < \ldots < l_{n+2} \leq k$ and*

$$F_{l_1} \cap -F_{l_2} \cap F_{l_3} \cap \ldots \cap (-1)^{n+1} F_{l_{n+2}} \neq \emptyset,$$

*where $-F_i$ denotes the antipodal set of $F_i$. In particular, $k$ is necessarily $\geq n+2$.*

Fan proved this theorem with a combinatorial lemma, which looks like Tucker's lemma. It holds for open sets too:

**Theorem 3** *Let $n$,$k$ be two arbitrary positive integers. If $k$ open subsets $U_1$, $U_2$, ..., $U_k$ of the $n$-sphere $S^n$ cover $S^n$ and if no one of them contains a pair of antipodal points, then there exist $n+2$ indices $l_1$, $l_2$, ..., $l_{n+2}$, such that $1 \leq l_1 < l_2 < \ldots < l_{n+2} \leq k$ and*

$$U_{l_1} \cap -U_{l_2} \cap U_{l_3} \cap \ldots \cap (-1)^{n+1} U_{l_{n+2}} \neq \emptyset,$$

*where $-U_i$ denotes the antipodal set of $U_i$. In particular, $k$ is necessarily $\geq n+2$.*

**Proof:** The proof goes as for Borsuk (see [8]) using the compactness of the $n$-sphere: because of Theorem 2, it suffices to prove that there are $k$ closed sets $F_i \subset U_i$ which cover the $n$-sphere. This can be done as follows: For each $x \in U_i$, take $V_x$, an open neighborhood whose closure is in $U_i$. By compactness, there is a finite family of the $\{V_x\}_{x \in S^n}$ which covers $S^n$ and we define $F_i$ to be the union of the closures of the $V_x$ of this finite family which are strictly included in $U_i$. ∎

### 2.2 Gale's lemma - Ziegler's version

In our proof, we will need another topological result: Gale's lemma.

In Matousek's book [8] (a collection of beautiful topological methods in combinatorics), one can find a version of Gale's lemma strenghtening the original version of Gale, with a short proof using the moment curve. This new version was found by Ziegler (we state the lemma here without proof), in order to simplify the proof of Schrijver's theorem:

**Lemma 1 (Ziegler's version of Gale's lemma)** *For every $d \geq 0$ and every $k \geq 1$, there exists a $(2k+d)$-point set $X \subset S^d$ such that under a suitable identification of $X$ with $[2k+d]$, every open hemisphere contains a stable $k$-tuple.*

An open hemisphere of a sphere is determined by its "center" $x$ on this sphere: if we note $H(x)$ the hemisphere, we have formally: $H(x) = \{y \in S^d \colon \langle x, y\rangle > 0\}$.

## 3 Proof of Theorem 1

Let $n$ be an even positive integer, and $k$ a positive integer such that $n \geq 2k$. It is sufficient to prove that $\chi_c(SG(n,k)) \geq \chi(KG(n,k))$, since the reverse inequality is straightforward using the well-known $\chi_c(SG(n,k)) \leq \chi(KG(n,k)) \leq n-2k+2$.

Let $\phi$ be a $(p,q)$-coloring of $SG(n,k)$. Let us recall that the vertices of $SG(n,k)$ are the stable $k$-subsets of $[n]$. We identify these $n$ integers with $n$ points on the $(n-2k)$-sphere such that every open hemisphere contains a stable $k$-set (see lemma 1).

We define $p$ open subsets of the $(n-2k)$-sphere $U_1$, $U_2$, ..., $U_p$ as follows: $x$ is in $U_i$ if and only of $H(x)$ (the open hemisphere whose center is $x$) contains a stable $k$-set whose color is $i$.

It is easy to see that these subsets are all open and cover the $(n-2k)$-sphere by construction. For $i \in [p]$, $U_i$ cannot contain two antipodal points: otherwise, we would have two disjoint hemispheres, each of them containing a $k$-tuple of color $i$.

So, we can apply Fan's theorem: there exist integers $l_1$, $l_2$, ..., $l_{n-2k+2}$ such that $1 \leq l_1 < l_2 < \ldots < l_{n-2k+2} \leq p$ and $U_{l_1} \cap -U_{l_2} \cap U_{l_3} \cap \ldots \cap (-1)^{n-2k+1} U_{l_{n-2k+2}} \neq \emptyset$.

Let $i$ be in $[n-2k+1]$. $U_{l_i} \cap -U_{l_{i+1}} \neq \emptyset$. Since for every $x$ on the $(n-2k)$-sphere $H(x) \cap H(-x) = \emptyset$, there are two disjoint stable $k$-tuples, one of color $l_i$, the other of color $l_{i+1}$. Since $\phi$ is a $(p,q)$-coloring, we have $l_i + q \leq l_{i+1}$.

Moreover, $n$ is even. Thus $n-2k+1$ is odd, and $U_{l_1} \cap -U_{l_{n-2k+2}} \neq \emptyset$. For the same reasons as above, $l_1$ and $l_{n-2k+2}$ are colors of two disjoint stable $k$-tuples. Then we have: $l_{n-2k+2} - l_1 \leq p - q$.

We can write: $0 = (l_{n-2k+2} - l_1) + (l_1 - l_2) + (l_2 - l_3) + \ldots + (l_{n-2k+1} - l_{n-2k+2}) \leq (p-q) + (n-2k+1)(-q)$.

Hence: $0 \leq p - (n-2k+2)q$. Or, more clearly:

$$n - 2k + 2 \leq \frac{p}{q}.$$

∎

Theorem 1 follows.

**Remark 1** In the proof, we exploit the fact that $n$ is even: if $n$ is odd, we can not conclude that $l_1$ and $l_{n-2k+2}$ are colors of two disjoint stable $k$-tuples. Moreever, we have, for $k > 1$, $\chi_c(SG(2k+1,k)) = 2 + \frac{1}{k} < 3 = \chi(SG(2k+1,k))$. But no counterexample is known with $n$ odd for the equality $\chi(KG(n,k)) = n-2k+2$ to hold.

**Remark 2** It is interesting to note that the proof does not use the full power of Fan's result, but only the fact that, among the $n - 2k + 2$ selected sets $U_i$, each intersects the antipodal set of the next one.

**Remark 3** Theorem 1 was recently independently obtained by Simonyi and Tardos [10].

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