Spirals moving by mean curvature. Part I: a comparison principle

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Abstract

In this paper, we study the motion of spirals by mean curvature in a (two dimensional) plane. Our motivation comes from dislocation dynamics; in this context, spirals appear when a screw dislocation line attains the surface of a crystal. The main result of this paper is a comparison principle for the corresponding quasi-linear equation. As far as motion of spirals are concerned, the novelty and originality of our setting and results come from the fact that, first, the singularity generated by the fixed point of spirals is taken into account for the first time (to the best of our knowledge), and second, spirals are studied in the whole space. We also prove that the Cauchy problem is well-posed by using Perron's method.

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1 Introduction

In this paper we are interested in curves $(\Gamma_t)_{t>0}$ in $\mathbb{R}^2$ which are half lines attached at the origin. These lines are assumed to move with normal velocity

$$V_n = c + \kappa \tag{1.1}$$

where κ is the curvature of the line and $c > 0$ is a given constant.

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1.1 Motivations and known results

The question of defining the motion of spirals in a two dimensional space is motivated by the seminal paper of Burton, Cabrera and Frank [2] where the growth of crystals with the vapour is studied. When a screw dislocation line attains the boundary of the material, atoms are adsorbed on the surface in such a way that a spiral is generated; moreover, under appropriate physical assumptions, these authors prove that the geometric law governing the dynamics of the growth of the spiral is precisely given by (1.1) where $-c$ denotes a critical value of the curvature.

From a mathematical point of view, different methods have been used in order to define solutions of the geometric law (1.1): for instance, a phase-field approach is proposed in [10]; see also [11, 12]. In papers such as [9, 7], spirals are parametrized in polar coordinates; see also Subsection 1.2.

In [17, 13], the geometric flow is defined by using the level-set approach [14, 3, 5]. The spiral moves into an annulus $B(0,R) \setminus B(0,\rho)$, $R > \rho$; they reach perpendicularly the boundary of the annulus, $\partial B(0,R) \cup \partial B(0,\rho)$.

The aim of this paper is to study what happens when $R \to +\infty$ and $\rho \to 0$. In other words, we would like to handle the singularity generated at the origin and unbounded domains. We will see that this will lead us to study an equation which is quasi-linear, of evolution type and with unbounded data.

More precisely, we would like to get a comparison principle for the quasi-linear equation in order to study the large time behaviour of spirals; we would like to prove that the solution of the Cauchy problem converges toward a global solution of the form $\lambda t + \varphi(r)$; see [6] for more details and references.

We would like to conclude this paragraph by mentioning results about the coarsening of spirals [16].

1.2 The geometric formulation

In this section, we make precise the way spirals are defined. We will first define them as parametrized curve and then use the level-set approach.

Parametrization of spirals. We look for interfaces Γ which can be parametrized as follows

$$\Gamma = \{r(\theta)e^{i\theta} : \theta \in \mathbb{R}\}$$

for some increasing function r defined in $\mathbb{R}$ onto $[0,+\infty)$. We refer to such an interface as a *spiral*. This curve is oriented by choosing the normal vector field equal to $-(r'(\theta)+ir(\theta))e^{i\theta}$. Because of the monotonicity assumption on r, these curves can also be parametrized as follows

$$\Gamma = \{re^{i\theta} : r \geq 0, \theta = \Theta(r)\}$$

for some increasing function $\Theta : [0,+\infty) \to \mathbb{R}$. If now the interface evolves with a time variable $t > 0$, Θ also depends on the time variable $t > 0$. If now $\bar{u}(t,r)$ denotes $-\Theta(t,r)$, then the moving spiral is defined as follows

$$\Gamma_t = \{re^{i\theta} : r > 0, \theta \in \mathbb{R},\ \theta + \bar{u}(t,r) = 0\} \qquad (1.2)$$

and the geometric law (1.1) implies that $\bar{u}$ satisfies

$$r\bar{u}_t = c\sqrt{1+r^2\bar{u}_r^2} + \bar{u}_r\left(\frac{2+r^2\bar{u}_r^2}{1+r^2\bar{u}_r^2}\right) + \frac{r\bar{u}_{rr}}{1+r^2\bar{u}_r^2} \quad \text{for} \quad (t,r)\in(0,+\infty)\times(0,+\infty) \tag{1.3}$$

with the initial condition

$$\bar{u}(0,r) = \bar{u}_0(r) \quad \text{for} \quad r\in(0,+\infty)\,. \tag{1.4}$$

Eq. (1.3) is a quasi-linear evolution equation posed in the domain $\Omega = (0,+\infty)$; remark that *we do not impose any boundary condition for* $r=0$. This can be explained in the following way: remark that the solutions of (1.3) satisfies (at least formally) the following boundary condition at the origin

$$0 = c + 2\bar{u}_r \quad \text{for} \quad r = 0$$

which is a boundary condition compatible with the comparison principle. In other words, the boundary condition is embedded in the equation.

Rescaling time and space, we can assume that $c=1$. We will do so throughout the paper.

Link with the level-set approach. In view of (1.2), we see that our approach is closely related to the level-set one. We recall that the level-set approach was introduced in [14, 5, 3]; in particular, it permits to construct an interface moving by mean curvature, that is to say satisfying the geometric law (1.1). It consists in defining the interface Γ_t as the 0-level set of a function $\tilde{U}(t,\cdot)$ and in remarking that the geometric law is verified only if $\tilde{U}$ satisfies a non-linear evolution equation of parabolic type.

In an informal way, we can say that the quasi-linear evolution equation (1.3) is a "graph" equation associated with the classical mean curvature equation (MCE), but written in polar coordinates. More precisely, in Cartesian coordinates (MCE) writes

$$\tilde{U}_t = c|D_X\tilde{U}| + \widehat{D_X\tilde{U}}^{\perp}\cdot D^2_{XX}\tilde{U}\cdot\widehat{D_X\tilde{U}}^{\perp}$$

(where $\hat{p} = p/|p|$ and $p^\perp = (-p_2,p_1)$ for $p=(p_1,p_2)\in\mathbb{R}^2$) while, in polar coordinates, it writes

$$r\bar{U}_t = c\sqrt{\bar{U}_\theta^2 + r^2\bar{U}_r^2} + \bar{U}_r\left(\frac{2\bar{U}_\theta^2 + r^2\bar{U}_r^2}{\bar{U}_\theta^2 + r^2\bar{U}_r^2}\right) + \frac{r}{\bar{U}_\theta^2 + r^2\bar{U}_r^2}\left(\bar{U}_{rr}\bar{U}_\theta^2 + \bar{U}_{\theta\theta}\bar{U}_r^2 - 2\bar{U}_{r\theta}\bar{U}_r\bar{U}_\theta\right).$$

In order to investigate further the link with the level-approach, we can reformulate (1.2) by writing

$$\Gamma_t = \cup_{k\in\mathbb{Z}}\{X\in\mathbb{R}^2 : \theta(X) + \bar{u}(t,r(X)) = 2k\pi\}$$

where $(r(X),\theta(X))$ denotes the polar coordinates of a point X of the plan. We can even write

$$\Gamma_t = \{X\in\mathbb{R}^2 : \theta(X) + \bar{U}(t,r(X)) = 0\}$$

by considering multivalued $\bar{U}$'s and θ. Such an approach is systematically developed in [13].

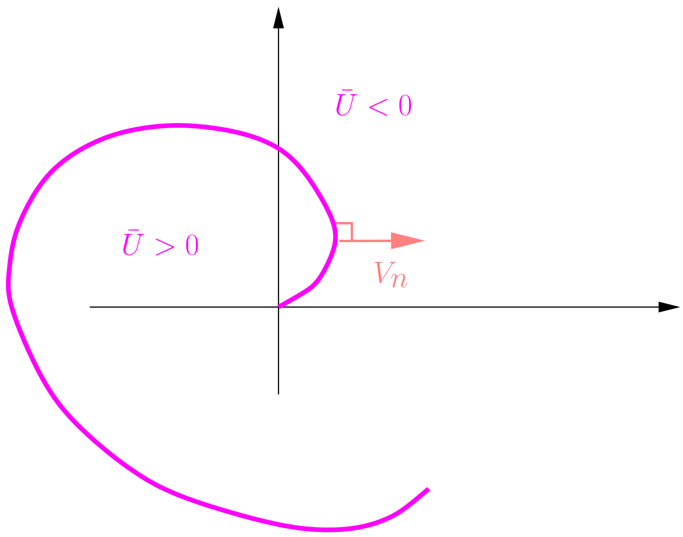

Figure 1: Motion of the spiral

1.3 Main results

Our first main result is a comparison principle for the Cauchy problem (1.3)-(1.4).

Theorem 1.1 (Comparison principle for (1.3))**.** *Assume that $\bar{u}_0$ is a Lipschitz continuous function. Consider a sub-solution $\bar{u}$ and a super-solution $\bar{v}$ of* (1.3)-(1.4) *such that there exist $C_1 > 0$ such that for all $t \in [0,T]$ and $r > 0$,*

$$|\bar{u}(t,r) - \bar{u}_0(r)| \le C_1 \quad \textit{and} \quad |\bar{v}(t,r) - \bar{u}_0(r)| \le C_1. \tag{1.5}$$

If $\bar{u}(0,r) \le \bar{u}_0(r) \le \bar{v}(0,r)$ for all $r \ge 0$, then $\bar{u} \le \bar{v}$ in $(0,T) \times (0,+\infty)$.

Remark 1.2. *The growth of the sub-solution u and the super-solution v is made precise by assuming Condition* (1.5)*. Such a condition is motivated by the large time asymptotic study carried out in [6]; indeed, we construct in [6] a global solution of the form $\lambda t + \bar{u}_0(r)$.*

The proof of Theorem 1.1 is based on the doubling of variable method, which consists in regularizing the sub- and super-solutions. Obviously, this is a difficulty here because the curve is attached at the origin and the doubling of variables at the origin is not well defined. To overcome this difficulty, we work with logarithmic coordinates $x = \ln r$ for r close to 0. But then the equation becomes

$$I_x[u] = u_t - e^{-x}\sqrt{1+u_x^2} \quad \text{with} \quad I_x[u] := e^{-2x}u_x + e^{-2x}\frac{u_{xx}}{1+u_x^2}$$

We then apply the doubling of variables in the x coordinates. There is a persistence of the difficulty, because we have now to bound terms like

$$A := e^{-x}\sqrt{1+u_x^2} - e^{-y}\sqrt{1+v_y^2}$$

that can blow-up as $x, y \to -\infty$. We are lucky enough to be able to show that A can be controlled by $I_x[u] - I_y[v]$, where the term $I_x[u] = (V_n - \kappa)\sqrt{1+u_x^2}$ is a natural nonlinear parabolic operator which contains in particular the curvature term. In view of the study from [6], $\bar{u}_0$ has to be chosen sub-linear in Cartesian coordinates and thus so are the sub- and super-solutions to be compared. The second difficulty arises when passing to logarithmic coordinates for large r's; indeed, the sub-solution and the super-solution then grow exponentially in $x = \ln r$ at infinity and we did not manage to adapt the previous reasoning in this setting. There is for instance a similar difficulty to deal with the Mean Curvature Motion equation. In this framework, for super-linear initial data, the uniqueness of the solution is not known in full generality (see [1]). In other words, the change of variables do not seem to work far from the origin. We thus have to stick to Cartesian coordinates for large r's and see the equation in different coordinates when r is either small or large (see Section 4).

Our second main result is a corollary of the comparison principle. It asserts the existence and the uniqueness of a viscosity solution to the Cauchy problem (1.3)-(1.4) and allows us to construct properly a geometric flow.

Theorem 1.3 (Existence and uniqueness for the Cauchy problem)**.** *Assume that $\bar{u}_0 \in W^{2,\infty}([0,+\infty))$ and that*

$$1 + 2(\bar{u}_0)_r(0) = 0. \tag{1.6}$$

Then there exists a unique viscosity solution $\bar{u}$ of (1.3)-(1.4) *such that there exists $\bar{C} > 0$ such that for all $t > 0$ and $r > 0$,*

$$|\bar{u}(t,r) - \bar{u}_0(r)| \leq \bar{C}t.$$

Remark 1.4. *We would like to mention that the viscosity solution constructed in Theorem 1.3 is expected to be smooth as soon as $t > 0$. However, we do not use here the classical parabolic theory to investigate regularity and we refer the interested reader to [6] for such a discussion.*

Organization of the article. In Section 2, we recall the definition of viscosity solutions for the quasi-linear evolution equation at stake in this paper. The change of variables that we will use in the proof of the comparison principle is also made precise. In Section 3, we give the proof of Theorem 1.1 in the bounded case. The proof in the general case is given in Section 4. In Section 5, we prove Theorem 1.3. Finally, proofs of technical lemmas are gathered in Appendix A.

Notation. If a is a real number, a_+ denotes $\max(0, a)$. If $p = (p_1, p_2) \in \mathbb{R}^2$, $p \neq 0$, then $\hat{p}$ denotes $p/|p|$ and $p^\perp$ denotes $(-p_2, p_1)$.

2 Preliminaries

2.1 Viscosity solutions for the main equation

In view of (1.3), it is convenient to introduce the following notation

$$\bar{F}(r, q, Y) = \sqrt{1 + r^2q^2} + q\left(\frac{2 + r^2q^2}{1 + r^2q^2}\right) + \frac{rY}{1 + r^2q^2}. \tag{2.7}$$

We first recall the notion of viscosity solution for an equation such as (1.3).

Definition 2.1 (Viscosity solutions for (1.3)-(1.4))**.**
A lower semi-continuous (resp. upper semi-continuous) function $u : [0, +\infty) \times (0, +\infty) \to \mathbb{R}$ *is a (viscosity)* super-solution *(resp.* sub-solution*) of* (1.3)-(1.4) *if for any* C^2 *test function* ϕ *such that* $u - \phi$ *attains a local minimum (resp. maximum) at* $(t, r) \in [0, +\infty) \times (0, +\infty)$*, we have*

(i) If $t \geq 0$*:*

$$r\phi_t \geq \bar{F}(r, \phi_r, \phi_{rr}) \quad \left(\textit{resp. } r\phi_t \leq \bar{F}(r, \phi_r, \phi_{rr}) \right).$$

(ii) If $t = 0$*:*

$$u(0, r) \geq \bar{u}_0(r) \quad \left(\text{resp. } u(0, r) \leq \bar{u}_0(r) \right).$$

A continuous function $u : [0, +\infty) \times (0, +\infty) \to \mathbb{R}$ *is a (viscosity)* solution *of* (1.3)-(1.4) *if it is both a super-solution and a sub-solution.*

Remark 2.2. *Remark that we do not impose any condition at* $r = 0$*; in other words, it is not necessary to impose a condition on the whole parabolic boundary of the domain. This is due to the "singularity" of our equation at* $r = 0$*.*

Remark 2.3. *Remark that the Compatibility Condition* (1.6) *satisfied by the initial datum reads*

$$\bar{F}(0, (\bar{u}_0)_r(0), (\bar{u}_0)_{rr}(0)) = 0\,. \tag{2.8}$$

Since we only deal with this weak notion of solution, (sub-/super-)solutions will always refer to (sub-/super-)solutions in the viscosity sense.

When constructing solutions by Perron's method, it is necessary to use the following classical discontinuous stability result. The reader is referred to [3] for a proof.

Proposition 2.4 (Discontinuous stability)**.** *Consider a family* $(u_\alpha)_{\alpha \in \mathcal{A}}$ *of sub-solutions of* (1.3)-(1.4) *which is uniformly bounded from above. Then the upper semi-continuous envelope of* $\sup_{\alpha \in \mathcal{A}} u_\alpha$ *is a sub-solution of* (1.3)-(1.4)*.*

2.2 A change of unknown function

We will make use of the following change of unknown function: $u(t, x) = \bar{u}(t, r)$ with $x = \ln r$ satisfies for all $t > 0$ and $x \in \mathbb{R}$

$$u_t = e^{-x}\sqrt{1 + u_x^2} + e^{-2x} u_x + e^{-2x} \frac{u_{xx}}{1 + u_x^2} \tag{2.9}$$

submitted to the initial condition: for all $x \in \mathbb{R}$,

$$u(0, x) = u_0(x) \tag{2.10}$$

where $u_0(x) = \bar{u}_0(e^x)$. Eq. (2.9) can be rewritten $u_t = F(x, u_x, u_{xx})$ with

$$F(x, p, X) = e^{-x}\sqrt{1 + p^2} + e^{-2x} p + e^{-2x} \frac{X}{1 + p^2}. \tag{2.11}$$

Remark that functions F and $\bar{F}$ are related by the following formula

$$F(x, u_x, u_{xx}) = \frac{1}{r}\bar{F}(r, \bar{u}_r, \bar{u}_{rr})\,. \tag{2.12}$$

Since the function ln is increasing and maps $(0, +\infty)$ onto $\mathbb{R}$, we have the following elementary lemma which will be used repeatedly throughout the paper.

Lemma 2.5 (Change of variables)**.** *A function $\bar{u}$ is a solution of* (1.3)-(1.4) *if and only if the corresponding function u is a solution of* (2.9)-(2.10) *with $u_0(x) = \bar{u}_0(e^x)$.*

The reader is referred to [4] (for instance) for a proof of such a result, where we use a definition of viscosity solution as it is given for instance in [4].

When proving the comparison principle in the general case, we will also have to use Cartesian coordinates. From a technical point of view, the following lemma is needed.

Lemma 2.6 (Coming back to the Cartesian coordinates)**.** *We consider a subsolution u (resp. supersolution v) of* (2.9)-(2.10) *and we define the function $\tilde{U}$ (resp. $\tilde{V}$)* $: (0, +\infty) \times \mathbb{R}^2 \to \mathbb{R}$ *defined by*

$$\tilde{U}(t, X) = \theta(X) + u(t, x(X)) \quad \text{(resp.} \quad \tilde{V}(t, Y) = \theta(Y) + u(t, x(Y))\text{)}$$

where $(\theta(Z), x(Z))$ is defined such that $Z = e^{x(Z)+i\theta(Z)} \neq 0$. Then $\tilde{U}$ (resp. $\tilde{V}$) is sub-solution (resp. super-solution) of

$$\begin{cases} w_t = |Dw| + \dfrac{Dw^\perp}{|Dw|} D^2 w \dfrac{Dw^\perp}{|Dw|} \\ \\ w(0, x) = \theta(X) + \bar{u}_0(x(X)). \end{cases} \tag{2.13}$$

Proof. Straightforward. □

Remark 2.7. *In Lemma 2.6, for $Z \neq 0$, the angle $\theta(Z)$ is only defined modulo 2π, but is locally uniquely defined by continuity. Then $D\theta, D^2\theta$ are always uniquely defined.*

3 A comparison principle for bounded solutions

As explained in the introduction, we first prove a comparison principle for (1.3) in the bounded case. Comparing with classical comparison results for geometric equations (see for instance [5, 3, 15, 8]), the difficulty is to handle the singularity at the origin ($r = 0$).

Theorem 3.1 (Comparison principle for (2.9)-(2.10))**.** *Assume that u_0 is Lipschitz continuous. Consider a bounded sub-solution u and a bounded super-solution v of* (2.9)-(2.10)*. Then $u \le v$ in $(0, +\infty) \times \mathbb{R}$.*

Proof of Theorem 3.1. We classically fix $T > 0$ and argue by contradiction by assuming that

$$M = \sup_{0<t<T, x\in\mathbb{R}} (u(t, x) - v(t, x)) > 0.$$

This means that there exist $t^* > 0$ and $x^* \in \mathbb{R}$ such that

$$u(t^*, x^*) - v(t^*, x^*) \ge M/2 > 0.$$

We then consider the following approximation of M

$$M_{\epsilon,\alpha} = \sup_{0<t<T, x,y\in\mathbb{R}} \left\{ u(t,x) - v(t,y) - e^{Kt}\frac{(x-y)^2}{2\epsilon} - \frac{\eta}{T-t} - \alpha\frac{x^2}{2} \right\}$$

where ϵ, α, η are small parameters and $K > 0$ is a large constant which depends on parameters and which will be fixed later.

Penalization. Since u and v are bounded functions, this supremum is attained at a point (t,x,y). By optimality of (t,x,y), we have in particular

$$\begin{aligned} u(t,x) - v(t,y) - e^{Kt}\frac{(x-y)^2}{2\epsilon} - \frac{\eta}{T-t} - \alpha\frac{x^2}{2} & \\ \geq u(t^*,x^*) - v(t^*,x^*) - \frac{\eta}{T-t^*} - \alpha\frac{(x^*)^2}{2} \geq M/3 \end{aligned}$$

for α and η small enough (only depending on M). In particular,

$$\frac{(x-y)^2}{2\epsilon} + \alpha\frac{x^2}{2} \leq \|u\|_\infty + \|v\|_\infty\,.$$

Hence, there exists a constant C_0 only depending on $\|u\|_\infty$ and $\|v\|_\infty$ such that

$$|x-y| \leq C_0\sqrt{\epsilon} \quad \text{and} \quad \alpha|x| \leq C_0\sqrt{\alpha}\,. \tag{3.14}$$

In the remaining of the proof, ϵ is fixed (even if we will choose it small enough) and α goes to 0 (even if it is not necessary to pass to the limit).

Initial condition. Assume first that $t = 0$. In this case, we use the fact that u_0 is Lipschitz continuous and (3.14) in order to get

$$\frac{M}{3} \;\leq\; u_0(x) - u_0(y) \leq \|Du_0\|_\infty|x-y| \leq C_0\|Du_0\|_\infty\sqrt{\varepsilon}$$

which is absurd for ε small enough (depending only on M, C_0 and $\|Du_0\|_\infty$).

Viscosity inequalities. It is convenient to define

$$p = \frac{(x-y)}{\epsilon}e^{Kt}.$$

In view of the previous discussion, we can assume that, for ϵ small enough, we have $t > 0$ for all $\alpha > 0$ small enough (independent on ϵ). We thus can write two viscosity inequalities.

We now use the Jensen-Ishii Lemma [4] in order to get, for all $\gamma_1 > 0$, four real numbers a, b, A, B such that

$$a \;\leq\; e^{-x}\sqrt{1+(p+\alpha x)^2} + e^{-2x}(p+\alpha x) + e^{-2x}\frac{A+\alpha}{1+(p+\alpha x)^2} \tag{3.15}$$

$$b \;\geq\; e^{-y}\sqrt{1+p^2} + e^{-2y}p + e^{-2y}\frac{B}{1+p^2}\,. \tag{3.16}$$

Moreover a, b satisfy the following inequality

$$a - b \geq \frac{\eta}{(T-t)^2} + Ke^{Kt}\frac{(x-y)^2}{2\epsilon} \tag{3.17}$$

and for any $\gamma_1 > 0$ small enough, there exists two reals A, B satisfying the following matrix inequality

$$\begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix} \leq \frac{e^{Kt}}{\epsilon}(1+\gamma_1)\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

This matrix inequality implies

$$A\xi_1^2 \leq B\xi_2^2 + \frac{e^{Kt}}{\epsilon}(1+\gamma_1)(\xi_1 - \xi_2)^2 \tag{3.18}$$

for all $\xi_1, \xi_2 \in \mathbb{R}$. The remaining of the proof consists in proving the following lemma.

Lemma 3.2 (Getting a contradiction)**.** *Assume that u and v satisfy* (3.14)*,* (3.15)*,* (3.16)*,* (3.17) *and* (3.18)*. Then $\eta \leq 0$.*

This is the desired contradiction and the proof of the comparison principle in the bounded case is complete. □

It remains to prove Lemma 3.2.

Remark 3.3. *Inequality* (3.17) *is in fact an equality. However, we will use later Lemma 3.2 with an inequality.*

Proof of Lemma 3.2. We first combine (3.15) and (3.16) and we obtain

$$\begin{aligned} \frac{\eta}{T^2} + Ke^{Kt}\frac{(x-y)^2}{2\epsilon} &\leq e^{-x}\sqrt{1+(p+\alpha x)^2} - e^{-y}\sqrt{1+p^2} \\ &\quad + e^{-2x}\alpha x + (e^{-2x} - e^{-2y})p + e^{-2x}\frac{\alpha}{1+(p+\alpha x)^2} + A\xi_x^2 - B\xi_y^2 \end{aligned}$$

where $\xi_x^2 = e^{-2x}(1+(p+\alpha x)^2)^{-1}$ and $\xi_y^2 = e^{-2y}(1+p^2)^{-1}$. We next use (3.18) and elementary computations in order to get, after rearranging terms,

$$\begin{aligned} &\frac{\eta}{T^2} + Ke^{Kt}\frac{(x-y)^2}{2\epsilon} - e^{-2x+Kt}(1-e^{2(x-y)})\frac{x-y}{\epsilon} \\ &\qquad \leq e^{-x}|\alpha x| + e^{-2x}\alpha x + e^{-2x}\alpha + \sqrt{1+p^2}(e^{-x} - e^{-y}) + \frac{e^{Kt}}{\epsilon}(1+\gamma_1)(\xi_x - \xi_y)^2. \end{aligned}$$

Using (3.14), we can write

$$-(1-e^{2(x-y)})\frac{x-y}{\epsilon} = (2 + o_\epsilon(1))\frac{(x-y)^2}{\epsilon}$$

where $o_\epsilon(1)$ denotes a function H of ϵ which only depend on C_0 in (3.14) and such that $H(\epsilon) \to 0$ as $\epsilon \to 0$. We thus get from the previous inequality

$$\begin{aligned} &\frac{\eta}{T^2} + e^{Kt}(K + (4+o_\epsilon(1))e^{-2x})\frac{(x-y)^2}{2\epsilon} \\ &\qquad \leq e^{-x}|\alpha x| + e^{-2x}\alpha x + e^{-2x}\alpha \\ &\qquad\qquad + \sqrt{1+p^2}(e^{-x} - e^{-y}) + \frac{e^{Kt}}{\epsilon}(1+\gamma_1)(\xi_x - \xi_y)^2. \end{aligned} \tag{3.19}$$

The remaining of the proof consists in controlling terms from the right hand side by terms from the left hand side.

Let us explain in an informal way how we will proceed. Roughly speaking, the second non-negative term of the left hand side of (3.19) will permit us to control error terms in the right hand side generated from the difference $x-y$. We will thus be left with error terms generated from the penalization αx^2. For α small enough, they will be controlled by ηT^{-2}.

First estimate. We first study the difference $(\xi_x - \xi_y)^2$. It involves both the difference $x-y$ and the error term αx^2. We use (3.14) in order to get

$$\begin{aligned}
(\xi_x - \xi_y)^2 &= \left(\frac{e^{-x}}{\sqrt{1+(p+\alpha x)^2}} - \frac{e^{-y}}{\sqrt{1+p^2}} \right)^2 \\
&\le \left(\left[\frac{e^{-x}}{\sqrt{1+(p+\alpha x)^2}} - \frac{e^{-x}}{\sqrt{1+p^2}} \right] + \left[\frac{e^{-x}}{\sqrt{1+p^2}} - \frac{e^{-y}}{\sqrt{1+p^2}} \right] \right)^2 \\
&\le (1+\gamma_2^{-1})e^{-2x}|\alpha x|^2 + (1+\gamma_2)e^{-2x}(1-e^{x-y})^2 \\
&= (1+\gamma_2^{-1})e^{-2x}|\alpha x|^2 + (1+\gamma_2)e^{-2x}(1+o_\epsilon(1))(x-y)^2
\end{aligned}$$

for some $\gamma_2 > 0$ to be fixed later. Hence

$$\begin{aligned}
\frac{e^{Kt}}{\epsilon}(1+\gamma_1)(\xi_x-\xi_y)^2 \le{}& \frac{e^{Kt-2x}}{\epsilon}(1+\gamma_1)(1+\gamma_2^{-1})|\alpha x|^2 \\
&+ \frac{e^{Kt-2x}}{\epsilon}(1+o_\epsilon(1))(1+\gamma_1)(1+\gamma_2)(x-y)^2.
\end{aligned} \tag{3.20}$$

We now fix ϵ, γ_1 and γ_2 small enough so that

$$2(1+o_\epsilon(1))(1+\gamma_1)(1+\gamma_2) \le (4+o_\epsilon(1)) - 1. \tag{3.21}$$

Combining (3.19), (3.20) and (3.21), and using (3.14), we obtain

$$\begin{aligned}
&\frac{\eta}{T^2} + e^{Kt}(K+e^{-2x})\frac{(x-y)^2}{2\epsilon} \\
&\qquad \le e^{-x}|\alpha x| + e^{-2x}\alpha x + e^{-2x}\alpha + C_\gamma e^{-2x}\frac{\sqrt{\alpha}}{\epsilon}|\alpha x| \\
&\qquad\qquad + \sqrt{1+p^2}(e^{-x}-e^{-y})
\end{aligned} \tag{3.22}$$

where C_γ only depends on γ_1, γ_2, T and C_0.

Second estimate. Thanks to (3.14), we first write

$$e^{-x} - e^{-y} = e^{-x}(1-e^{x-y}) \le 2e^{-x}|x-y|.$$

We next obtain for some small $\gamma_3 > 0$ (to be chosen later)

$$\begin{aligned}
\sqrt{1+p^2}(e^{-x}-e^{-y}) &\le 2e^{-x}|x-y|\frac{e^{Kt}}{\epsilon}\sqrt{\epsilon^2 e^{-2Kt} + |x-y|^2} \\
&\le \frac{e^{Kt}}{\epsilon}\left(\gamma_3 e^{-2x}(\epsilon^2 e^{-2Kt} + |x-y|^2) + \gamma_3^{-1}|x-y|^2\right) \\
&\le e^{-2x}(\gamma_3\epsilon) + \gamma_3 e^{Kt-2x}\frac{|x-y|^2}{\epsilon} + \frac{1}{\gamma_3}e^{Kt}\frac{|x-y|^2}{\epsilon}.
\end{aligned}$$

We choose $K \geq 2\max(2, \epsilon\alpha^{-1})$ and γ_3 in the interval $(\frac{2}{K}, \min(\frac{1}{2}, \epsilon^{-1}\alpha))$ in order to get from the previous inequality the following estimate

$$\sqrt{1+p^2}(e^{-x} - e^{-y}) \leq e^{-2x}\alpha + e^{Kt}(K + e^{-2x})\frac{|x-y|^2}{2\epsilon}. \tag{3.23}$$

Conclusion. We combine (3.22) and (3.23) in order to get

$$\frac{\eta}{T^2} \leq e^{-x}|\alpha x| + e^{-2x}\left(\alpha x + 2\alpha + C_\gamma \frac{\sqrt{\alpha}}{\epsilon}|\alpha x|\right).$$

Pick now $a > 0$ such that for $x \leq -a$,

$$e^x|x| - \frac{|x|}{2} + 2 \leq 0.$$

In order to get a contradiction, we finally distinguish two cases.
Case 1: $\mathbf{x_n \leq -a}$ for some $\boldsymbol{\alpha_n \to 0}$. We choose n large enough so that $C_\gamma \frac{\sqrt{\alpha_n}}{\epsilon} \leq \frac{1}{2}$ and we get

$$\frac{\eta}{2T^2} \leq e^{-2x}(e^x|\alpha x| + \alpha x + 2\alpha + \frac{1}{2}|\alpha x|) \leq 0$$

which implies $\eta \leq 0$.
Case 2: $\mathbf{x \geq -a}$ for α small enough. We use (3.14) and get

$$\frac{\eta}{2T^2} \leq e^{2a}((2 + C_\gamma \frac{\sqrt{\alpha}}{\epsilon})|\alpha x| + 2\alpha) \leq e^{2a}((2 + C_\gamma \frac{\sqrt{\alpha}}{\epsilon})C_0\sqrt{\alpha} + 2\alpha)$$

and we let $\alpha \to 0$ to get $\eta \leq 0$ in this case too. The proof of Lemma 3.2 is now complete. □

4 Comparison principle for sub-linear solutions

Proof of Theorem 1.1. Thanks to the change of unknown function described in Subsection 2.2, we can consider the functions u and v defined on $(0, +\infty) \times \mathbb{R}$ which are sub- and super-solutions of (2.9). We can either prove that $\bar{u} \leq \bar{v}$ in $(0, +\infty) \times (0, +\infty)$ or that $u \leq v$ in $(0, +\infty) \times \mathbb{R}$.

For $\theta \in \mathbb{R}$, we define

$$U(t, x, \theta) = \theta + u(t, x) \quad \text{and} \quad V(t, x, \theta) = \theta + v(t, x).$$

Note that U and V are respectively sub and super-solution of

$$W_t(t, x, \theta) = e^{-x}|DW| + e^{-2x}DW \cdot e_1 + e^{-2x}\frac{DW^\perp}{|DW|}D^2W\frac{DW^\perp}{|DW|}. \tag{4.24}$$

We fix $T > 0$ and we argue by contradiction by assuming that

$$M = \sup_{t \in [0,T], x, \theta \in \mathbb{R}} \{U(t, x, \theta) - V(t, x, \theta)\} > 0.$$

In order to use the doubling variable technique, we need a smooth interpolation function Ψ between polar coordinates for small r's and Cartesian coordinates for large r's. Precisely, we choose Ψ as follows.

Lemma 4.1 (Interpolation between logarithmic and Cartesian coordinates)**.** *There exists a smooth (C^∞) function $\psi : \mathbb{R}^2 \mapsto \mathbb{R}3$ such that*

$$\left\{\begin{array}{rll} \psi(x,\theta+2\pi) & = \psi(x,\theta) & \\ \psi(x,\theta) & = (x, e^{i\theta}) & \text{if} \quad x \le 0 \\ \psi(x,\theta) & = (0, e^x e^{i\theta}) & \text{if} \quad x \ge 1 \end{array}\right. \tag{4.25}$$

and such that there exists two constants $\delta_0 > 0$ and $m_\psi > 0$ such that for $x \le 1$ and $\theta \in \mathbb{R}$,

$$\psi(x,\theta) = (a,b) \text{ with } |b| \le e \tag{4.26}$$

and such that for all x, y, σ, θ, if $|\psi(x,\theta) - \psi(y,\sigma)| \le \delta_0$ and $|\theta - \sigma| \le \frac{\pi}{2}$, then

$$|\psi(x,\theta) - \psi(y,\sigma)| \ \ge \ m_\psi |(x,\theta) - (y,\sigma)|, \tag{4.27}$$

$$\left|D\psi(x,\theta)^T \otimes (\psi(x,\theta) - \psi(y,\sigma))\right| \ \ge \ m_\psi |(x,\theta) - (y,\sigma)| \tag{4.28}$$

where $\odot$ is defined for a p tensor $A = (A_{i_1,\dots,i_p})$ and a q tensor $B = (B_{j_1,\dots,j_q})$ by

$$(A \odot B)_{i_1,\dots,i_{p-1},j_2,\dots,j_q} = \sum_k A_{i_1,\dots,i_{p-1},k} B_{k,j_2,\dots,j_q}.$$

The proof of this lemma is given in Appendix A.

Penalization. We consider the following approximation of M

$$M_{\varepsilon,\alpha} = \sup_{t\in[0,T],x,\theta,y,\sigma\in\mathbb{R}} \Bigg\{ U(t,x,\theta) - V(t,y,\sigma) - e^{Kt} \frac{|\psi(x,\theta)-\psi(y,\sigma)|^2}{2\varepsilon} - \frac{1}{\varepsilon}\left(|\theta-\sigma| - \frac{\pi}{3}\right)_+^2 - \frac{\alpha}{2}(\psi(x,\theta))^2 - \frac{\eta}{T-t} \Bigg\} \tag{4.29}$$

where $\varepsilon, \alpha, \eta$ are small parameters and $K \ge 0$ is a large constant to be fixed later. For α and η small enough we remark that $M_{\varepsilon,\alpha} \ge \frac{M}{2} > 0$. In order to prove that the maximum $M_{\varepsilon,\alpha}$ is attained, we need the following lemma whose proof is postponed until Appendix A.

Lemma 4.2 (A priori estimates)**.** *There exists a constant $C_2 > 0$ such that the following estimate holds true for any $x, y, \theta, \sigma \in \mathbb{R}$*

$$\begin{aligned} |u_0(x) - u_0(y)| &\le C_2 + e^{Kt}\frac{|\psi(x,\theta)-\psi(y,\sigma)|^2}{4\varepsilon} \\ |\theta-\sigma| &\le C_2 + \frac{1}{2\varepsilon}\left(|\theta-\sigma| - \frac{\pi}{3}\right)_+^2. \end{aligned}$$

Using this lemma, we then deduce that

$$\begin{aligned} & U(t,x,\theta) - V(t,y,\sigma) - e^{Kt}\frac{|\psi(x,\theta)-\psi(y,\sigma)|^2}{2\varepsilon} - \frac{1}{\varepsilon}\left(|\theta-\sigma| - \frac{\pi}{3}\right)_+^2 \\ & \le |u(t,x) - u_0(x)| + |v(t,y) - u_0(y)| + |\theta - \sigma| + |u_0(x) - u_0(y)| \\ & \quad - e^{Kt}\frac{|\psi(x,\theta)-\psi(y,\sigma)|^2}{2\varepsilon} - \frac{1}{\varepsilon}\left(|\theta-\sigma| - \frac{\pi}{3}\right)_+^2 \\ & \le 2C_1 + 2C_2 - e^{Kt}\frac{|\psi(x,\theta)-\psi(y,\sigma)|^2}{4\varepsilon} - \frac{1}{2\varepsilon}\left(|\theta-\sigma| - \frac{\pi}{3}\right)_+^2. \end{aligned} \tag{4.30}$$

Using the 2π-periodicity of ψ, the maximum is achieved for $\theta \in [0, 2\pi]$. Then, using the previous estimate and the fact that $-\alpha(\psi(x,\theta))^2 \to -\infty$ as $|x| \to \infty$, we deduce that the maximum is reached at some point that we still denote $(t, x, \theta, y, \sigma)$.

Penalization estimates. Using Estimate (4.30) and the fact that $M_{\varepsilon,\alpha} \geq 0$, we deduce that there exists a constant $C_0 = 4(C_1 + C_2)$ such that

$$\alpha(\psi(x,\theta))^2 + \frac{1}{\varepsilon}\left(|\theta-\sigma| - \frac{\pi}{3}\right)_+^2 + e^{Kt}\frac{|\psi(x,\theta)-\psi(y,\sigma)|^2}{2\varepsilon} \leq C_0. \tag{4.31}$$

On the one hand, an immediate consequence of this estimate is that

$$|\theta-\sigma| \leq \frac{\pi}{2}$$

for ε small enough. On the other hand, we deduce from (4.31) and (4.27)

$$m_\psi \frac{|\theta-\sigma|^2 + |x-y|^2}{2\varepsilon} \leq C_0.$$

Hence, we have $|\theta - \sigma| \leq \frac{\pi}{4}$ for ε small enough so that the constraint $|\theta-\sigma| \leq \frac{\pi}{3}$ is not saturated. We can also choose ε small enough so that

$$|x-y| \leq \frac{1}{2}.$$

In the sequel of the proof, we will also need a better estimate on the term $\alpha(\psi(x))^2$; precisely, we need to know that $\alpha(\psi(x))^2 \to 0$ as $\alpha \to 0$. Even if such a result is classical (see [4]), we give details for the reader's convenience. To prove this, we introduce

$$M_{\varepsilon,0} = \sup_{t\in[0,T],x,\theta,y,\sigma\in\mathbb{R}} \left\{ U(t,x,\theta) - V(t,y,\sigma) - e^{Kt}\frac{|\psi(x,\theta)-\psi(y,\sigma)|^2}{2\varepsilon} \right.$$
$$\left. - \frac{1}{\varepsilon}\left(|\theta-\sigma| - \frac{\pi}{3}\right)_+^2 - \frac{\eta}{T-t} \right\}$$

which is finite thanks to (4.30).

We remark that $M_{\varepsilon,\alpha} \leq M_{\varepsilon,0}$ and that $M_{\varepsilon,\alpha}$ is non-decreasing when α decreases to zero. We then deduce that there exists L such that $M_{\varepsilon,\alpha} \to L$ as $\alpha \to 0$. A simple computation then gives that

$$\frac{\alpha}{4}(\psi(x,\theta))^2 \leq M_{\varepsilon,\frac{\alpha}{2}} - M_{\varepsilon,\alpha} \to 0 \quad \text{as} \quad \alpha \to 0$$

and then

$$\frac{\alpha}{2}(\psi(x,\theta))^2 \to 0 \quad \text{as} \quad \alpha \to 0. \tag{4.32}$$

Initial condition. We now prove the following lemma.

Lemma 4.3 (Avoiding $t=0$)**.** *For ε small enough, we have $t > 0$ for all $\alpha > 0$ small enough.*

Proof. We argue by contradiction. Assume that $t = 0$. We then distinguish two cases.

If the corresponding x and y are small ($x \leq 2$ and $y \leq 2$) then, since u_0 is Lipschitz continuous and (4.27) holds true, there exists a constant $L_0 > 0$ such that

$$\begin{aligned} 0 < \frac{M}{2} \leq M_{\varepsilon,\alpha} &\leq U(0,x,\theta) - V(0,y,\sigma) - \frac{|\psi(x,\theta)-\psi(y,\sigma)|^2}{2\varepsilon} \\ &\leq L_0|(x,\theta)-(y,\sigma)| - m_\psi\frac{|(x,\theta)-(y,\sigma)|^2}{2\varepsilon} \\ &\leq \frac{L_0^2}{2m_\psi}\varepsilon \end{aligned}$$

which is absurd for ε small enough.

The other case corresponds to large x and y ($x \geq 1$ and $y \geq 1$). In this case, since $\bar{u}_0$ is Lipschitz continuous, we know that there exists a constant $L_1 > 0$ such that

$$0 \leq \frac{M}{2} \leq M_{\varepsilon,\alpha} \leq |U(0,x,\theta) - V(0,y,\sigma)| \leq |\theta - \sigma| + L_1|e^x - e^y|.$$

Using the fact

$$|\theta - \sigma| + L_1|e^x - e^y| \leq \left(\frac{1}{m_\psi} + L_1\right)|\psi(x,\theta) - \psi(y,\sigma)| \leq \left(\frac{1}{m_\psi} + L_1\right)\sqrt{2C_0}\sqrt{\varepsilon}$$

we get a contradiction for ε small enough. $\square$

Thanks to Lemma 4.3, we will now write two viscosity inequalities, combine them and exhibit a contradiction. We recall that we have to distinguish cases in order to determine properly in which coordinates viscosity inequalities must be written (see the Introduction).

Case 1: There exists $\alpha_n \to 0$ such that $x \geq \frac{3}{2}$ and $y \geq \frac{3}{2}$. We set $X = e^{x+i\theta}$ and $Y = e^{y+i\sigma}$. Consider $\tilde{U}$ and $\tilde{V}$ defined in Lemma 2.6. Remark that, even if $\theta(X)$ is defined modulo 2π, the quantity $\theta(X) - \theta(Y)$ is well defined (for $|X|, |Y| \geq e$ and $|X - Y| \leq \frac{1}{2}$) and thus so is $\tilde{U}(t,X) - \tilde{V}(t,Y)$. Recall also that $\tilde{U}, \tilde{V}$ are respectively sub and supersolutions of the following equation

$$w_t = |Dw| + \widehat{Dw}^{\perp} \cdot D^2 w \cdot \widehat{Dw}^{\perp}$$

Moreover, using the explicit form of ψ, we get that

$$M_{\varepsilon,\alpha} = \sup_{t\in[0,T], X,Y\in\mathbb{R}^2\setminus B_1(0)} \left\{ \tilde{U}(t,X) - \tilde{V}(t,Y) - \frac{e^{Kt}}{2\varepsilon}|X-Y|^2 - \frac{\alpha}{2}|X|^2 - \frac{\eta}{T-t} \right\}.$$

Moreover, $-|D_X\tilde{U}| \leq -\frac{1}{|X|}$ (in the viscosity sense). We set

$$p = \frac{X-Y}{\varepsilon}e^{Kt}.$$

We now use the Jensen-Ishii Lemma [4] in order to get four real number a, b, A, B such that

$$\begin{aligned} a &\leq |p + \alpha X| + \frac{(p+\alpha X)^{\perp}}{|p+\alpha X|}(A + \alpha I)\frac{(p+\alpha X)^{\perp}}{|p+\alpha X|}, \\ b &\geq |p| + \frac{p^{\perp}}{|p|}B\frac{p^{\perp}}{|p|}. \end{aligned}$$

Moreover, p satisfies the following estimate

$$|p + \alpha X| \geq \frac{1}{|X|}, \quad |p| \geq \frac{1}{|Y|}, \tag{4.33}$$

a, b satisfy the following equality

$$a - b = \frac{\eta}{(T-t)^2} + Ke^{Kt}\frac{|X-Y|^2}{2\varepsilon}$$

and A, B satisfy the following matrix inequality

$$\begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix} \leq \frac{2e^{Kt}}{\varepsilon} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}.$$

This matrix inequality implies

$$A\xi_1^2 \leq B\xi_2^2 + \frac{2e^{Kt}}{\varepsilon}|\xi_1 - \xi_2|^2 \tag{4.34}$$

for all $\xi_1, \xi_2 \in \mathbb{R}^2$. Subtracting the two viscosity inequalities, we then get

$$\begin{aligned}
\frac{\eta}{T^2} \leq & |p + \alpha X| - |p| + \alpha + \frac{(p+\alpha X)^\perp}{|p+\alpha X|} A \frac{(p+\alpha X)^\perp}{|p+\alpha X|} - \frac{p^\perp}{|p|} B \frac{p^\perp}{|p|} \\
\leq & \alpha|X| + \alpha + \frac{2e^{Kt}}{\varepsilon}\left(\frac{p+\alpha X}{|p+\alpha X|} - \frac{p}{|p|}\right)^2 \\
\leq & \sqrt{C_0}\sqrt{\alpha} + \alpha + \frac{2e^{Kt}}{\varepsilon}\left(2\left(\frac{\alpha X}{\frac{1}{|X|}}\right)^2 + 2\left(\frac{p}{|p|}\frac{\alpha X}{|p + |\alpha X||}\right)^2\right) \\
\leq & \sqrt{C_0}\sqrt{\alpha} + \alpha + \frac{8e^{Kt}}{\varepsilon}\left(\alpha|X|^2\right)^2
\end{aligned}$$

where we have used successively (4.34), (4.31) and (4.33). Recalling, by (4.32) that $\alpha|X|^2 = o_\alpha(1)$, we get a contradiction for α small enough.

Case 2: There exists $\alpha_n \to 0$ such that $x \leq -\frac{1}{2}$ and $y \leq -\frac{1}{2}$. Using the explicit form of ψ and the fact that $U(t,x,\theta) = \theta + u(t,x)$ and $V(t,y,\sigma) = \sigma + v(t,y)$ with u and v respectively sub and super-solution of (2.9), we remark that

$$M_{\varepsilon,\alpha} = \sup_{t',x',y'}\{u(t',x') - v(t',y') - e^{Kt'}\frac{|\psi(x',\theta) - \psi(y',\sigma)|^2}{2\varepsilon} - \frac{\alpha}{2}|x'|^2 - \frac{\eta}{T-t'} + \theta - \sigma - \frac{\alpha}{2}\}.$$

Moreover, the maximum is reached at (t,x,y), where we recall that (t,x,θ,y,σ) is the point of maximum in (4.29). Using the Jensen-Ishii Lemma [4], we then deduce the existence, for all $\gamma_1 > 0$, of four real numbers a, b, A, B such that

$$\begin{aligned}
a &\leq e^{-x}\sqrt{1+(p+\alpha x)^2} + e^{-2x}(p+\alpha x) + e^{-2x}\frac{A+\alpha}{1+(p+\alpha x)^2} \\
b &\geq e^{-y}\sqrt{1+p^2} + e^{-2y}p + e^{-2y}\frac{B}{1+p^2}
\end{aligned}$$

where

$$p = \frac{x-y}{\varepsilon}e^{Kt}.$$

These inequalities are exactly (3.15) and (3.16). Moreover a, b satisfy the following equality

$$a - b = \frac{\eta}{(T-t)^2} + Ke^{Kt}\frac{|\psi(x,\theta) - \psi(y,\sigma)|^2}{2\epsilon} \geq \frac{\eta}{(T-t)^2} + Ke^{Kt}\frac{|x-y|^2}{2\epsilon},$$

which implies (3.17); moreover, A, B satisfy the following matrix inequality

$$\begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix} \le \frac{e^{Kt}}{\epsilon}(1+\gamma_1) \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

This inequality implies (3.18). Eventually, (4.31), the fact that $x \le 0$, $y \le 0$ and Lemma 4.1 imply (3.14) (with a different constant). We thus can apply Lemma 3.2 and deduce the desired contradiction.

Case 3: There exists $\alpha_n \to 0$ such that $-1 \le x, y \le 2$. Since $\theta, \sigma \in [-\pi, 3\pi]$, there then exists $M_\psi > 0$ (only depending on the function ψ) such that for all $x \in [-1, 2]$ and $\theta \in [-\pi, 3\pi]$,

$$|\psi(x,\theta)| + |D\psi(x,\theta)| + |D^2\psi(x,\theta)| + |D^3\psi(x,\theta)| \le M_\psi. \tag{4.35}$$

For simplicity of notation, we denote (x, θ) by $\bar{x}$ and (y, σ) by $\bar{y}$. We next define

$$p_{\bar{x}} = \frac{e^{Kt}}{\varepsilon} D\psi(\bar{x})^T \odot (\psi(\bar{x}) - \psi(\bar{y})) \quad \text{and} \quad p_{\bar{y}} = \frac{e^{Kt}}{\varepsilon} D\psi(\bar{y})^T \odot (\psi(\bar{x}) - \psi(\bar{y})).$$

We have $p_{\bar{x}}, p_{\bar{y}} \in \mathbb{R}^2$ and we set (e_1, e_2) a basis of $\mathbb{R}^2$.

Lemma 4.4 (Combining viscosity inequalities for $\alpha = 0$)**.** *We have for $\alpha = 0$*

$$\frac{\eta}{T^2} + K m_\psi^2 e^{Kt} \frac{|\bar{x}-\bar{y}|^2}{2\varepsilon} \le e^{-x}|p_{\bar{x}}| - e^{-y}|p_{\bar{y}}| + e^{-2x} p_{\bar{x}} \cdot e_1 - e^{-2y} p_{\bar{y}} \cdot e_1 + \frac{2e^{Kt}}{\varepsilon}(\mathcal{I}_1 + \mathcal{I}_2) \tag{4.36}$$

where

$$\begin{aligned} \mathcal{I}_1 &= (\psi(\bar{x}) - \psi(\bar{y})) \odot \left(D^2\psi(\bar{x}) e^{-x}\widehat{p_{\bar{x}}^\perp} \cdot e^{-x}\widehat{p_{\bar{x}}^\perp} - D^2\psi(\bar{y}) e^{-y}\widehat{p_{\bar{y}}^\perp} \cdot e^{-y}\widehat{p_{\bar{y}}^\perp} \right) \\ \mathcal{I}_2 &= \left| D\psi(\bar{x}) e^{-x}\widehat{p_{\bar{x}}^\perp} - D\psi(\bar{y}) e^{-y}\widehat{p_{\bar{y}}^\perp} \right|^2. \end{aligned}$$

Proof. Recall that U and V are respectively sub and super-solution of (4.24) and use the Jensen-Ishii Lemma [4] in order to deduce that there exist two real numbers a, b and two 2×2 real matrices A, B such that

$$\begin{aligned} a &\le e^{-x}|\tilde{p}_{\bar{x}}| + e^{-2x}\tilde{p}_{\bar{x}} \cdot e_1 \\ &\quad + e^{-2x} \frac{\tilde{p}_{\bar{x}}^\perp}{|\tilde{p}_{\bar{x}}|} \left(A + \alpha(\psi(\bar{x}) \odot D^2\psi(\bar{x}) + D\psi(\bar{x})^T \odot D\psi(\bar{x})) \right) \frac{\tilde{p}_{\bar{x}}^\perp}{|\tilde{p}_{\bar{x}}|} \\ b &\ge e^{-y}|p_{\bar{y}}| + e^{-2y} p_{\bar{y}} \cdot e_1 + e^{-2y} \frac{p_{\bar{y}}^\perp}{|p_{\bar{y}}|} B \frac{p_{\bar{y}}^\perp}{|p_{\bar{y}}|} \end{aligned}$$

where

$$\tilde{p}_{\bar{x}} = p_{\bar{x}} + \alpha D\psi(\bar{x})^T \odot \psi(\bar{x}).$$

Remark that , since $D_\theta U = D_\theta V = 1$, there exists $\delta_0 > 0$ such that

$$\tilde{p}_{\bar{x}} \ge \delta_0 > 0 \quad \text{and} \quad p_{\bar{y}} \ge \delta_0 > 0.$$

Moreover a, b satisfy the following equality

$$a - b = \frac{\eta}{(T-t)^2} + Ke^{Kt}\frac{|\psi(\bar{x}) - \psi(\bar{y})|^2}{2\epsilon}$$

and A, B satisfy the following matrix inequality

$$\begin{bmatrix} A & 0 \\ 0 & -B \end{bmatrix} \leq \frac{2e^{Kt}}{\epsilon}\Bigg\{ \begin{bmatrix} (\psi(\bar{x}) - \psi(\bar{y})) \odot D^2\psi(\bar{x}) & 0 \\ 0 & -(\psi(\bar{x}) - \psi(\bar{y})) \odot D^2\psi(\bar{y}) \end{bmatrix} + \begin{bmatrix} D\psi(\bar{x})^T \odot D\psi(\bar{x}) & -D\psi(\bar{y})^T \odot D\psi(\bar{x}) \\ -D\psi(\bar{y})^T \odot D\psi(\bar{x}) & D\psi(\bar{y})^T \odot D\psi(\bar{y}) \end{bmatrix} \Bigg\}.$$

This implies

$$A\xi \cdot \xi \leq B\zeta \cdot \zeta + \frac{2e^{Kt}}{\varepsilon}\Big\{ (\psi(\bar{x}) - \psi(\bar{y})) \odot D^2\psi(\bar{x})\xi \cdot \xi - (\psi(\bar{x}) - \psi(\bar{y})) \odot D^2\psi(\bar{y})\zeta \cdot \zeta + |D\psi(\bar{x})\xi - D\psi(\bar{y})\zeta|^2 \Big\}$$

for all $\xi, \zeta \in \mathbb{R}^2$. Combining the two viscosity inequalities and using the fact that $|\psi(\bar{x}) - \psi(\bar{y})| \geq m_\psi |\bar{x} - \bar{y}|$, we obtain

$$\begin{aligned} \frac{\eta}{T^2} + Km_\psi^2 e^{Kt}\frac{|\bar{x} - \bar{y}|^2}{2\varepsilon} \leq & e^{-x}|\tilde{p}_{\bar{x}}| - e^{-y}|p_{\bar{y}}| + e^{-2x}\tilde{p}_{\bar{x}} \cdot e_1 - e^{-2y}p_{\bar{y}} \cdot e_1 \\ & + \alpha e^{-2x}.\widehat{\tilde{p}_{\bar{x}}^{\perp}} \left(\psi(\bar{x}) \odot D^2\psi(\bar{x}) + D\psi(\bar{x})^T D\psi(\bar{x})\right) \widehat{\tilde{p}_{\bar{x}}^{\perp}} \\ & + \frac{2e^{Kt}}{\varepsilon}(\tilde{\mathcal{I}}_1 + \tilde{\mathcal{I}}_2) \end{aligned}$$

where $\tilde{\mathcal{I}}_1$ and $\tilde{\mathcal{I}}_2$ are defined respectively as $\mathcal{I}_1$ and $\mathcal{I}_2$ with $p_{\bar{x}}$ replaced by $\tilde{p}_{\bar{x}}$. Remarking that there exists a constant $C > 0$ such that

$$\begin{aligned} e^{-x}|\tilde{p}_{\bar{x}}| \quad + \quad & e^{-2x}\tilde{p}_{\bar{x}} \cdot e_1 + \left| \alpha e^{-2x}\frac{\tilde{p}_{\bar{x}}^{\perp}}{|\tilde{p}_{\bar{x}}|} \left(\psi(\bar{x}) \odot D^2\psi(\bar{x}) + D\psi(\bar{x})^T D\psi(\bar{x})\right) \frac{\tilde{p}_{\bar{x}}^{\perp}}{|\tilde{p}_{\bar{x}}|} \right| \\ & \leq e^{-x}|p_{\bar{x}}| + e^{-2x}p_{\bar{x}} \cdot e_1 + C\alpha\left(|D^2\psi(\bar{x})|^2 + |D\psi(\bar{x})|^2 + |\psi(\bar{x})|^2\right) \\ & \leq e^{-x}|p_{\bar{x}}| + e^{-2x}p_{\bar{x}} \cdot e_1 + 3M_\psi^2 C\alpha \end{aligned}$$

and

$$\begin{aligned} |\tilde{\mathcal{I}}_1 - \mathcal{I}_1| + |\tilde{\mathcal{I}}_2 - \mathcal{I}_2| \leq & C\left|\widehat{\tilde{p}_{\bar{x}}^{\perp}} - \widehat{p_{\bar{x}}^{\perp}}\right| \\ \leq & C\left|\frac{\tilde{p}_{\bar{x}} - p_{\bar{x}}}{|\tilde{p}_{\bar{x}}|}\right| + |p_{\bar{x}}|\left|\frac{1}{|\tilde{p}_{\bar{x}}|} - \frac{1}{|p_{\bar{x}}|}\right| \\ \leq & C\left|\frac{\tilde{p}_{\bar{x}} - p_{\bar{x}}}{\delta_0}\right| + \left|\frac{|p_{\bar{x}}| - |\tilde{p}_{\bar{x}}|}{\delta_0}\right| \\ \leq & 2C\left|\frac{\tilde{p}_{\bar{x}} - p_{\bar{x}}}{\delta_0}\right| \\ \leq & \frac{2C^2\alpha}{\delta_0} \end{aligned}$$

and sending $\alpha \to 0$ (recall that $\bar{x}, \bar{y}$ ly in a compact domain), we get (4.36). □

Lemma 4.5 (Estimate on $\mathcal{I}_1$)**.** *There exists a constant $\overline{C}_1$ such that*

$$|\mathcal{I}_1| \leq \overline{C}_1 |x-y|^2 \tag{4.37}$$

Proof. In order to prove (4.37), we write

$$\begin{aligned}\frac{|\mathcal{I}_1|}{|\psi(\bar{x})-\psi(\bar{y}))|} \leq & |(D^2\psi(\bar{x}) - D^2\psi(\bar{y}))e^{-x}\widehat{p_{\bar{x}}^{\perp}} \cdot e^{-x}\widehat{p_{\bar{x}}^{\perp}}| \\ & + |D^2\psi(\bar{y})(e^{-x}-e^{-y})\widehat{p_{\bar{x}}^{\perp}} \cdot e^{-x}\widehat{p_{\bar{x}}^{\perp}}| \\ & + |D^2\psi(\bar{y})e^{-y}\left(\widehat{p_{\bar{x}}^{\perp}} - \widehat{p_{y}^{\perp}}\right) \cdot e^{-x}\widehat{p_{\bar{x}}^{\perp}}| \\ & + |D^2\psi(\bar{y})e^{-y}\widehat{p_{y}^{\perp}} \cdot (e^{-x}-e^{-y})\widehat{p_{\bar{x}}^{\perp}} \\ & + |D^2\psi(\bar{y})e^{-y}\widehat{p_{\bar{y}}^{\perp}} \cdot e^{-y}\left(\widehat{p_{\bar{x}}^{\perp}} - \widehat{p_{y}^{\perp}}\right)|.\end{aligned}$$

Thanks to (4.35) and $\max(|x|,|y|) \leq 2$, we have

$$\begin{aligned}|D^2\psi(\bar{x}) - D^2\psi(\bar{y})| &\leq M_\psi |\bar{x}-\bar{y}|, \\ |e^{-x}-e^{-y}| &\leq e^2|\bar{x}-\bar{y}|.\end{aligned}$$

We also have the following important estimate

$$\begin{aligned}\left|\widehat{p_{\bar{x}}^{\perp}} - \widehat{p_{\bar{y}}^{\perp}}\right| \leq & \left|\frac{p_{\bar{x}}-p_{\bar{y}}}{|p_{\bar{x}}|}\right| + |p_{\bar{y}}|\left|\frac{1}{|p_{\bar{x}}|} - \frac{1}{|p_{\bar{y}}|}\right| \\ \leq & \left|\frac{p_{\bar{x}}-p_{\bar{y}}}{|p_{\bar{x}}|}\right| + \left|\frac{|p_{\bar{y}}|-|p_{\bar{x}}|}{|p_{\bar{x}}|}\right| \\ \leq & 2\left|\frac{p_{\bar{x}}-p_{\bar{y}}}{|p_{\bar{x}}|}\right| \\ \leq & 2\frac{\frac{e^{Kt}}{\varepsilon}|D\psi(\bar{x})-D\psi(\bar{y})||\psi(\bar{x})-\psi(\bar{y})|}{\frac{e^{Kt}}{\varepsilon}m_\psi|\bar{x}-\bar{y}|} \\ \leq & \frac{2M_\psi^2}{m_\psi}|\bar{x}-\bar{y}|\end{aligned}$$

where we have used the fact that $|p_{\bar{x}}| \geq \frac{e^{Kt}}{\varepsilon}m_\psi|\bar{x}-\bar{y}|$ (see (4.28)). This finally gives that there exists a constant $\overline{C}_1$ (depending on m_ψ and M_ψ) such that (4.37) holds true. $\square$

Using the fact that $|p_{\bar{x}}|, |p_{\bar{y}}| \leq C\frac{e^{Kt}}{\varepsilon}|\bar{x}-\bar{y}|$, we can prove in a similar way the following lemma.

Lemma 4.6 (Remaining estimates)**.** *There exist three positive constants $\overline{C}_2, \overline{C}_3$ and $\overline{C}_4$ such that*

$$\begin{aligned}|\mathcal{I}_2| &\leq \overline{C}_2|\bar{x}-\bar{y}|^2, \\ e^{-x}|p_{\bar{x}}| - e^{-y}|p_{\bar{y}}| &\leq \overline{C}_3\frac{e^{Kt}}{\varepsilon}|\bar{x}-\bar{y}|^2, \\ e^{-2x}p_{\bar{x}}\cdot e_1 - e^{-2y}p_{\bar{y}}\cdot e_1 &\leq \overline{C}_4\frac{e^{Kt}}{\varepsilon}|\bar{x}-\bar{y}|^2.\end{aligned}$$

Use now Lemmas 4.5 and 4.6 in order to derive from (4.36) the following inequality

$$\frac{\eta}{T^2} + K m_\psi e^{Kt} \frac{|\bar{x} - \bar{y}|^2}{2\varepsilon} \leq \overline{C} \frac{e^{Kt}}{\varepsilon} |\bar{x} - \bar{y}|^2$$

with $\overline{C} = \overline{C}_1 + \overline{C}_2 + \overline{C}_3 + \overline{C}_4$. Choosing $K \geq \frac{2\overline{C}}{m_\psi}$, we get a contradiction. □

5 Barriers and Perron's method

Before constructing solutions of (1.3) submitted to the initial condition (1.4), we first construct appropriate barrier functions.

Proposition 5.1 (Barriers for the Cauchy problem)**.** *Assume* $\bar{u}_0 \in W^{3,\infty}([0,+\infty))$ *with* $\bar{u}_0$ *satisfying* (1.6)*. There exists a constant* $C > 0$ *such that* $\bar{u}^\pm(t,r) = \bar{u}_0(r) \pm Ct$ *are respectively a super- and a sub-solution of* (1.3)-(1.4)*.*

Proof. It is enough to prove that the following quantity is finite

$$\bar{C} = \sup_{r \geq 0} \frac{1}{r} \left| \bar{F}(r, (\bar{u}_0)_r(r), (\bar{u}_0)_{rr}(r)) \right|.$$

Use (2.8) to obtain

$$\bar{C} = \max(\bar{C}_1, \bar{C}_2)$$

with

$$\begin{aligned} \bar{C}_1 &= \sup_{r \in [0,1]} \frac{|\bar{F}(r, (\bar{u}_0)_r(r), (\bar{u}_0)_{rr}(r)) - \bar{F}(0, (\bar{u}_0)_r(0), (\bar{u}_0)_{rr}(0))|}{r}, \\ \bar{C}_2 &= \sup_{r \in [1,+\infty)} \frac{|\bar{F}(r, (\bar{u}_0)_r(r)(x), (\bar{u}_0)_{rr}(r))|}{r}. \end{aligned}$$

By using the regularity of F and $\bar{u}_0$, it is now easy to see that both quantities are finite and only depend on $\bar{u}_0$. □

We now turn to the proof of Theorem 1.3 which is very classical with the results we have in hand, namely the strong comparison principle and the existence of barriers. However, we give details for the sake of completeness.

Proof of Theorem 1.3. In view of Lemma 2.5, it is enough to construct a solution u of (2.9) satisfying (2.10) with $u_0(x) = \bar{u}_0(\exp x)$.

Consider the set

$$\mathcal{S} = \{v : (0,+\infty) \times \mathbb{R} \to \mathbb{R}, \text{ sub-solution of (2.9) s.t. } v \leq u^+\}.$$

Remark that it is not empty since $u^- \in \mathcal{S}$ (where $u^\pm(t,x) = \bar{u}^\pm(t,r)$ with $x = \ln r$). We now consider the upper envelope u of $(t,r) \mapsto \sup_{v \in \mathcal{S}} v(t,r)$. By Proposition 2.4, it is a sub-solution of (2.9). The following lemma derives from the general theory of viscosity solutions as presented in [4] for instance.

Lemma 5.2. *The lower envelope* u_* *of* u *is a super-solution of* (2.9)*.*

We recall that the proof of this lemma proceeds by contradiction and consists in constructing a so-called bump function around the point the function u_* is not a super-solution of the equation. The contradiction comes from the maximality of u in $\mathcal{S}$.

Since for all $v \in \mathcal{S}$,

$$u_0(x) - Ct \leq v \leq u_0(x) + Ct,$$

we conclude that

$$u_0(x) = u_*(0, x) = u(0, x)\,.$$

We now use the comparison principle and get $u \leq u_*$ in $(0, T) \times \mathbb{R}$ for all $T > 0$. Since $u_* \leq u$ by construction, we deduce that $u = u_*$ is a solution of (2.9). The comparison principle also ensures that the solution we constructed is unique. The proof of Theorem 1.3 is now complete. □

A Appendix: proofs of technical lemmas

In this section, we prove Lemmas 4.1 and 4.2.

Proof of Lemma 4.1. We look for ψ under the following form: for $x, \theta \in \mathbb{R}$,

$$\psi(x, \theta) = (1 - \iota(x))(x, e^{i\theta}) + \iota(x)(0, e^{x+i\theta})$$

where $\iota : \mathbb{R} \to \mathbb{R}$ is increasing, smooth (C^∞) and such that $\iota(x) = 0$ if $x \leq 0$ and $\iota(x) = 1$ if $x \geq 1$. Remark that (4.25) and (4.26) are readily satisfied.

It remains to prove (4.27) and (4.28). Let us first find $\epsilon > 0$ and $m_\psi > 0$ such that for all x, y, θ, σ such that $|(x, \theta) - (y, \sigma)| \leq \epsilon$, we have (4.27) and (4.27).

Study of (4.27)**.** It is convenient to use the following notation: $\psi(x, \theta) = (\phi_1(x), \phi_2(x)e^{i\theta})$. We first write (4.27) in terms of ϕ_i:

$$|\phi_1(x) - \phi_1(y)| + |\phi_2(x) - \phi_2(y)\cos(\theta - \sigma)| + \phi_2(y)|\sin(\theta - \sigma)| \geq m_\psi(|x - y| + |\theta - \sigma|)$$

(we used a different norm in $\mathbb{R}^3$ and m_ψ is changed accordingly). It is enough to prove

$$\begin{aligned}|\phi_1(x) - \phi_1(y)| + |\phi_2(x) - \phi_2(y)| + \phi_2(y)(|\sin| - 1 + \cos)(\theta - \sigma)&\\ \geq m_\psi(|x - y| + |\theta - \sigma|).&\end{aligned}$$

We choose $\epsilon \leq 1$ and we remark that such an inequality is clear if $x \leq -1$ or $x \geq 2$. Through a Taylor expansion and using the fact that $\phi_2(y) \geq 1$, this reduces to check that

$$\min(\inf_{x \in (-1,2)} (|\phi_1'(x)| + |\phi_2'(x)|), 1) \geq 2m_\psi$$

which reduces to

$$\inf_{x \in (0,1)} \{|\phi_1'(x)| + |\phi_2'(x)|\} > 0.$$

For x far from 0, a simple computation shows that $\phi_2'(x) \geq \iota(x)e^x$ and this permits us to conclude. For x in a neighborhood of 0, $\phi_1'(x) = 1 + o(1)$ and $\phi_2'(x) = O(x)$ and we can conclude in this case too.

Study of (4.28)**.** We next write (4.28) in terms of ϕ_i

$$\text{(A.1)} \quad |\Phi(x,y) + \phi_2'(x)\phi_2(y)(1-\cos(\theta-\sigma))| + |\phi_2(x)\phi_2(y)||\sin(\theta-\sigma)| \geq \mu_\psi(|x-y|+|\theta-\sigma|)$$

where

$$\Phi(x,y) = \phi_1'(x)(\phi_1(x)-\phi_1(y)) + \phi_2'(x)(\phi_2(x)-\phi_2(y)).$$

Once again, such the previous inequality is true for $x \notin (-1,2)$ and for $x \in (-1,2)$, we choose m_ψ such that

$$\inf_{x\in(0,1)} \{\phi_1'(x))2 + (\phi_2'(x))2\} \geq 2m_\psi.$$

The same reasoning as above applies here too.

Reduction to the case: $|(x,\theta)-(y,\sigma)| \leq \epsilon$**.** It remains to prove that for $\epsilon > 0$ given, we can find $\delta_0 > 0$ such that, as soon as $|\psi(x,\theta)-\psi(y,\sigma)| \leq \delta_0$ and $|\theta-\sigma| \leq \frac{\pi}{2}$, then $|(x,\theta)-(y,\sigma)| \leq \epsilon$. We argue by contradiction by assuming that there exists $\epsilon_0 > 0$ and two sequences (x_n,θ_n) and (y_n,σ_n) such that

$$\begin{aligned} |\theta_n-\sigma_n| &\leq \frac{\pi}{2} \\ |x_n-y_n| + |\theta_n-\sigma_n| &\geq \epsilon_0 \\ \phi_1(x_n)-\phi_1(y_n) &\to 0 \\ \cos(\theta_n-\sigma_n)\phi_2(x_n)-\phi_2(y_n) &\to 0 \\ \phi_2(x_n)\sin(\theta_n-\sigma_n) &\to 0 \end{aligned}$$

as $n\to\infty$. Since ϕ_2 is bounded from below by 1, we deduce that $\sin(\theta_n-\sigma_n)\to 0$. Up to a subsequence, we can assume that $\theta_n-\sigma_n\to\delta$ and we thus deduce that $\delta=0$. Hence, $|x_n-y_n| \geq \frac{\epsilon_0}{2}$ for large n's. Thanks to a Taylor expansion in $\theta_n-\sigma_n$, we can also get that $\phi_2(x_n)-\phi_2(y_n)\to 0$. Because $|x_n-y_n| \geq \frac{\epsilon_0}{2}$, we then get that x_n and y_n remain in a bounded interval. We can thus assume that $x_n\to x_*$ and $y_n\to y_*$. Finally, we have $\phi_i(x_*)=\phi_i(y_*)$ for $i=1,2$ and $|x_*-y_*| >\geq \frac{\epsilon_0}{2}$ which is impossible. The proof of the lemma is now complete. □

Proof of Lemma 4.2. The second estimate is satisfied if C_2 is chosen such that

$$C_2 \geq \sup_{r>0}\left(r - \left(r-\frac{\pi}{3}\right)_+^2\right).$$

We now prove the first estimate. We distinguish three cases:

Case 1: $x\leq 1$ **and** $y\leq 1$**.** In this case, e^x and e^y are bounded and the definition of u_0 in terms of the Lipschitz continuous function $\bar{u}_0$ implies

$$|u_0(x)-u_0(y)| \leq C$$

for some constant $C>0$.

Case 2: ($x \le 1$ and $y \ge 1$) or ($x \ge 1$ and $y \le 1$). The two cases can be treated similarly and we assume here that $x \le 1$ and $y \ge 1$. In that case $\psi(x,\theta) = (a,b)$ with $a \in \mathbb{R}$ and $b \in \mathbb{C}$ with $|b| \le e$ (see (4.26)) and $\psi(y,\sigma) = (0, e^{y+i\sigma})$. Moreover, there exists a constant C such that

$$|u_0(x) - u_0(y)| \le C(1+e^y).$$

We also have

$$\begin{aligned}|\psi(x,\theta) - \psi(y,\sigma)| =& \sqrt{a^2 + |e^{y+i\sigma} - b|^2}\\ \ge& |e^{y+i\sigma} - b|\\ \ge& e^y - |b|\\ \ge& e^y - e.\end{aligned}$$

Hence,

$$\begin{aligned}|u_0(x) - u_0(y)| &\le C(1+e) + C(e^y - e)\\ &\le C(1+e) + C^2\epsilon e^{-Kt} + \frac{e^{Kt}}{4\epsilon}(e^y - e)^2\\ &\le C(1+e+C) + \frac{e^{Kt}}{4\varepsilon}|\psi(x,\theta) - \psi(y,\sigma)|^2\end{aligned}$$

which gives the desired estimate.

Case 3: $x \ge 1$ and $y \ge 1$. In this case,

$$|\psi(x,\theta) - \psi(y,\sigma)| = |e^{x+i\theta} - e^{y+i\sigma}| \ge |e^x - e^y|$$

and

$$|u_0(x) - u_0(y)| \le L_{u_0}|e^x - e^y|,$$

where L_{u_0} is the Lipschitz constant of $\bar{u}_0$. Hence, C_2 is chosen such that

$$C_2 \ge \sup_{r>0} L_{u_0} r - \frac{1}{4\epsilon} r^2.$$

The proof is now complete. $\square$

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