# A transport formulation of moving fronts and application to dislocations dynamics

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June 27, 2007

**Abstract**

In this article, we consider hypersurfaces moving with normal velocity depending on the time-space coordinates and on the normal to the hypersurface. We naturally define a measure associated to this hypersurface. This measure is defined on a suitable space/unit normal/curvature configuration space. We show that, while the hypersurface stays smooth, then the measure is a solution to a linear transport equation, that we call a transport formulation. In the particular case of curves moving in the plane, we get a simple transport formulation. With this transport formulation in hands, it is then easy to complete the models of dislocations densities that were proposed in the 60's. As a consequence, we therefore propose a closed mean field model for the dynamics of dislocations densities.

**AMS Classification:** 35R35, 35F20.
**Keywords:** kinetic equation, Liouville equation, transport equation, fronts, normal velocity, curvature, dislocations dynamics, dislocations densities.

# 1 Introduction

## 1.1 Motivation

We are interested in the motion of a smooth bounded connected and oriented hypersurface $\Gamma_t \subset \mathbb{R}^N$ with the first order geometric motion given by the normal velocity

$$V = c(t, y, n(t, y)) \tag{1.1}$$

where $t \in [0, T)$ and $y \in \mathbb{R}^N$ denote respectively the time and space coordinates, and $n(t, y) \in \mathbb{S}^{N-1}$ denotes the unit normal to $\Gamma_t$ at the point $y$ (for a given choice of orientation). We denote by $K(t, y) \in \mathbb{R}^{N\times N}_{sym}$ the curvature of $\Gamma_t$ at the point $y$. This curvature is a symmetric $N \times N$ matrix, and will be defined precisely later. For a given $T > 0$, we define

$$\Gamma = \bigcup_{t\in[0,T)} \{t\} \times \Gamma_t \quad \subset \quad [0, T) \times \mathbb{R}^n \ .$$

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To avoid any problem of regularity in this article, we will assume that

$$
(1.2) \qquad \begin{cases} \Gamma \in C^3 \quad \text{and} \quad c \in C^3\left([0,T) \times \mathbb{R}^N \times \mathbb{S}^{N-1}\right) \\ \\ \forall t \in [0,T), \quad \Gamma_t \text{ is } C^3 \text{ bounded, oriented and connected .} \end{cases}
$$

It is well-known that we can not expect in general existence of smooth solutions $\Gamma_t$ for all time and that singularities may happen in finite time.
Our goal is to show that it is possible to provide a transport formulation of the motion of such fronts. Our motivation comes from the modelling of dislocations dynamics, i.e. in the dynamics of curves moving their slip planes in crystals. Physically, it is interesting to be able to sum the evolution of several lines to deduce statistical and mean properties of this dynamics. The challenge behind this question is the possibility to describe the dynamics of densities of such curves. We refer to the work of Sedláček, Kratochvil, Werner [27], which was a source of inspiration of the present article.
In this paper, we show that our first goal is achieved, at least while the solution $\Gamma_t$ stays smooth. Indeed we prove (see Theorem 4.1) that the "density"

$$
g(t,y,n,K) = \delta_{\Gamma_t}(y)\delta_0(n - n(t,y))\delta_0(K - K(t,y))
$$

which is a measure for $(t,y,n,K) \in [0,T) \times \mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}^{N\times N}_{sym}$, satisfies the following equation

$$
g_t + \text{div }(ag) + a_0 g = 0
$$

which is a linear transport equation for some function $a_0$ and some suitable vector field $a$ (which is related to characteristics of Hamilton-Jacobi equations). The precise meaning and the details of these expressions will be given later (see Subsection 1.3 and Theorem 4.1). Let us mention that the vector field $a$ has a quadratic growth at infinity, as a function of the curvature $K$. This naturally creates some mathematical difficulties (that will not be addressed in the present paper) to get long time existence of solutions. This is obviously related to the fact that geometrically, the curvature of $\Gamma_t$ can become infinite in finite time.

We were tempted to call the equation satisfied by $g$ a "kinetic formulation", but this terminology has already been used to denote a powerful approach to nonlinear hyperbolic equations (see Perthame [26] and the references therein). This approach to nonlinear hyperbolic equations allows to get existence and uniqueness results, even after the appearence of singularities for the solution. On the contrary, our transport formulation in this paper only deals with smooth evolutions (even if it would be interesting to extend it after the appearence of singularities). Let us also cite a related famous example of transport equation associated to nonlinear evolution: this is the hamiltonian formulation of Navier-Stokes equation by Oseledets [25].

After our work was achieved, we were aware of the developement of results by Hochrainer, Zaiser [13] (see also Hochrainer [12] and Zaiser, Hochrainer [31]) that seem similar in the special case of dimension $N = 2$ for velocities $c(t,y)$ independent on the normal $n$, these results beeing based on Lie-derivative of differential forms.
Our work focuses on transport formulations of hypersurfaces. In the Appendix, we only give some indications for the way to get other transport formulations associated to the wavefront of the evolution of submanifolds of codimension higher than 1, in particular for the case of

the transport of curves in $\mathbb{R}^N$.
In the particular case of curves moving in the plane, it is possible to use a simplified description. We can describe the normal $n = (\cos\theta, \sin\theta)$ by its angle $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ and choose a scalar curvature $\kappa \in \mathbb{R}$. In this framework a transport formulation (see Theorem 2.1) is proposed for the "density"

$$g(t, y, \theta, \kappa) = \delta_{\Gamma_t}(y)\delta_0(\theta - \theta(t, y))\delta_0(\kappa - \kappa(t, y)) \ .$$

Finally, let us mention that our analysis do not cover the case of velocities depending on the curvature itself. This would be an interesting extension in connection with random process (see for instance Buckdahn, Cardaliaguet, Quincampoix [3], Soner, Touzi [28, 29, 30]). We plan to study this problem in a future work.

## 1.2 Organization of the paper

In Section 2, we present our main result in dimension $N = 2$, namely Theorem 2.1 for the simplified description. We also propose as an application, a model for the dynamics of dislocations densities (see Subsection 2.2). In Section 3, we give the proof of Theorem 2.1 and of Proposition 2.2.
For sake of completeness, we state in Section 4 our main result in any dimensions $N \geq 2$, namely Theorem 4.1. The proof is basically similar to the one of Theorem 2.1, but technically more involved. This is the reason why we chosed to present the result in general dimension *after* the result in dimension $N = 2$. The proof of Theorem 4.1 is done in Section 5. In the Appendix, for sake of completeness, we give in Subsection 6.1 the proof of Lemma 5.1, in Subsection 6.2, we give some indications about a transport formulation of the motion of curves in $\mathbb{R}^N$ in the case of pure transport, and in Subsection 6.3, we propose an alternative transport equation for the wavefront of curves moving in the plane which is well-posed for long time existence of solutions.

## 1.3 Notation

For a smooth oriented hypersurface $\Gamma_t$ in $\mathbb{R}^N$, we denote by $n(t, y) \in \mathbb{S}^{N-1}$ the unit normal to $\Gamma_t$ at the point $y \in \Gamma_t$, and by $K(t, y) \in \mathbb{R}^{N\times N}_{sym}$ its curvature, where $\mathbb{R}^{N\times N}_{sym}$ is the set of symmetric $N \times N$ matrices. This matrix $K(t, y)$ is given by

$$K(t, y) = \sum_{i=1}^{N-1} K_i f_i \otimes f_i$$

where the $K_i$ are the principal curvatures and the $f_i$ are the principal directions of curvature of the surface $\Gamma_t$ at the point $y$. Recall that the $f_i, i = 1, ..., N - 1$ generate the tangent hyperplane to $\Gamma_t$ at the point $y$. We use here the convention that for a sphere, if the normal is pointing out of the ball, then the curvatures $K_i$ are negative.
In dimension $N = 2$, we set

$$n = (\cos\theta, \sin\theta), \quad \tau = (\sin\theta, -\cos\theta) \tag{1.3}$$

where $(\tau, n)$ is a direct orthonormal basis. Depending on the context (but without ambiguity) we will consider either general $\tau, n$ depending on the general variable $\theta \in \mathbb{R}/(2\pi\mathbb{Z})$ (and sometimes denoted by $\tau(\theta), n(\theta)$ to clearly specify the dependence on $\theta$), or depending on the particular value $\theta(t,y) \in \mathbb{R}/(2\pi\mathbb{Z})$ which is the angle associated to $n(t,y)$ for $y \in \Gamma_t$. We will also define the scalar curvature $\kappa(t,y)$ by

$$K(t,y) = \kappa(t,y)\tau \otimes \tau \ .$$

We denote by $\partial_\theta$ and $\partial_\kappa$ the derivatives respectively with respect to $\theta$ and to $\kappa$. With these notation, we have in particular

$$\partial_\theta \tau = n, \quad \partial_\theta n = -\tau \ .$$

In any dimension $N$, considering a direct orthonormal basis $(e_1, ..., e_N)$, and for tensors $\mathcal{T} = \sum_{i_1,...,i_T=1,...,N} \mathcal{T}_{i_1,...,i_T} e_{i_1} \otimes ... \otimes e_{i_T}$, $\mathcal{S} = \sum_{j_1,...,j_T=1,...,N} \mathcal{S}_{j_1,...,j_S} e_{j_1} \otimes ... \otimes e_{j_S}$, we set the simple contraction of tensors

$$\mathcal{T} \cdot \mathcal{S} = \sum_{i_1,...,i_{T-1}=1,...,N} \quad \sum_{j_2,...,j_S=1,...,N} \left( \sum_{k=1,...,N} \mathcal{T}_{i_1,...,i_{T-1},k} \ \mathcal{S}_{k,j_2,...,j_S} \right) e_{i_1} \otimes ... \otimes e_{i_{T-1}} \otimes e_{j_2} \otimes ... e_{j_S}$$

and the double contraction of tensors

$$\mathcal{T} : \mathcal{S} = \sum_{i_1,...,i_{T-2}=1,...,N} \quad \sum_{j_3,...,j_S=1,...,N} \left( \sum_{k,l=1,...,N} \mathcal{T}_{i_1,...,i_{T-2},l,k} \ \mathcal{S}_{k,l,j_3,...,j_S} \right) e_{i_1} \otimes ... \otimes e_{i_{T-2}} \otimes e_{j_3} \otimes ... e_{j_S} \ .$$

Assuming that the tensor $\mathcal{T}$ depends on $K \in \mathbb{R}^{N\times N}_{sym}$, we define

$$\partial_K \mathcal{T} = \sum_{p,q=1,...,N} \frac{1}{2}(1+\delta_{pq}) \sum_{i_1,...,i_T=1,...,N} \left( \frac{\partial}{\partial K_{pq}} \mathcal{T}_{i_1,...,i_T} \right) \ e_p \otimes e_q \otimes e_{i_1} \otimes ... \otimes e_{i_T} \ .$$

With this definition, we have for instance $\mathcal{T} : \partial_K K = \mathcal{T}$ if $\mathcal{T}$ is symmetric on its two last indices. Similarly, for a tensor $\mathcal{T}$ depending on $y \in \mathbb{R}^N$, we define

$$\partial_y \mathcal{T} = \sum_{p=1,...,N} \quad \sum_{i_1,...,i_T=1,...,N} \left( \frac{\partial}{\partial y_p} \mathcal{T}_{i_1,...,i_T} \right) \ e_p \otimes e_{i_1} \otimes ... \otimes e_{i_T} \ .$$

Similarly, if the tensor $\mathcal{T}(n)$ depends on $n \in \mathbb{S}^{N-1}$ (among other possible variables), we consider $\tilde{n}$ in a neighborhood of $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$ and then define $\widetilde{\mathcal{T}}_{i_1,...,i_T}(\tilde{n}) = \mathcal{T}_{i_1,...,i_T}(\tilde{n}/|\tilde{n}|)$. We set

$$\partial_n \mathcal{T}(n) = \sum_{p=1,...,N} \quad \sum_{i_1,...,i_T=1,...,N} \left( \frac{\partial}{\partial \tilde{n}_p} \left( \widetilde{\mathcal{T}}_{i_1,...,i_T}(\tilde{n}) \right) \right)_{|\tilde{n}=n} \ e_p \otimes e_{i_1} \otimes ... \otimes e_{i_T} \ .$$

We have in particular $n \cdot \partial_n = 0$ and $\partial_n n = I - n \otimes n$ with the identity matrix $I = \sum_{i=1,...,N} e_i \otimes e_i$. We also define

$$\partial^2_{yy} \mathcal{T} = \partial_y \left( \partial_y \mathcal{T} \right), \quad \partial^2_{yn} \mathcal{T} = \partial_y \left( \partial_n \mathcal{T} \right), \quad \partial^2_{ny} \mathcal{T} = \partial_n \left( \partial_y \mathcal{T} \right), \quad \partial^2_{nn} \mathcal{T} = \partial_n \left( \partial_n \mathcal{T} \right) - n \otimes \partial_n \mathcal{T}$$

where we can check that $\partial^2_{nn}\mathcal{T}$ is symmetric in its two first indices. We also call $\partial_t \mathcal{T}$ the tensor whose components are time derivatives of the components of $\mathcal{T}$. We set

$$D_t = \partial_t + cn \cdot \partial_y, \quad D_\tau = -(I - n \otimes n) \cdot \partial_y + K \cdot \partial_n \ .$$

Finally in dimension $N = 2$, we keep the same notation for defining

$$D_\tau = \tau \cdot \partial_y + \kappa \partial_\theta$$

and define

$$\partial^2_{yy} \cdot = \partial_y(\partial_y \cdot), \quad \partial^2_{y\theta} \cdot = \partial_y(\partial_\theta \cdot), \quad \partial^2_{\theta y} \cdot = \partial_\theta(\partial_y \cdot), \quad \partial^2_{\theta\theta} \cdot = \partial_\theta(\partial_\theta \cdot) \ .$$

For a function $f$, we also set

$$\partial_y^\perp f = -\left(\frac{\partial f}{\partial y_2}\right) e_1 + \left(\frac{\partial f}{\partial y_1}\right) e_2 \ .$$

# 2 Result in dimension $N = 2$

## 2.1 Main result

In dimension $N = 2$, let us consider a closed connected and oriented curve $\Gamma_t$ for $t \in [0, T)$ for some fixed $T > 0$, with the normal pointing out the bounded set whose boundary is the curve. At a point $y$ of the curve, we recall that we can write the unit normal $n(t, y) = (\cos\theta(t, y), \sin\theta(t, y))$ with $\theta(t, y) \in \mathbb{R}/(2\pi\mathbb{Z})$, and call $\kappa(t, y) \in \mathbb{R}$ the curvature (negative for a circle).
We set

$$\overline{c}(t, y, \theta) = c(t, y, n(\theta))$$

that up to an abuse of notation, we will continue to call it $c(t, y, \theta)$.

Then for any function $\varphi \in C_c^\infty([0, T) \times \mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z})) \times \mathbb{R})$, we define the distribution $g_\Gamma(t, y, \theta, \kappa)$ by

$$< g_\Gamma, \varphi > = \int_0^T dt \ \int_{\Gamma_t} \varphi(t, y, \theta(t, y), \kappa(t, y)) \ . \tag{2.4}$$

Given any distribution $g$ (with compact support in the variable $\kappa \in \mathbb{R}$), we also define formally the distribution $\hat{g}(t, y, \theta)$ by

$$'' \ \hat{g} := \int_{\mathbb{R}} d\kappa \ D_\tau g \ '' \quad \text{with} \quad D_\tau = \tau \cdot \partial_y + \kappa \partial_\theta$$

where $\tau$ is defined in (1.3), i.e. rigorously, for any $\psi \in C_c^\infty([0, T) \times \mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z})))$

$$< \hat{g}, \psi > := < D_\tau g, \overline{\psi} > \quad \text{with} \quad \overline{\psi}(t, y, \theta, \kappa) = \psi(t, y, \theta) \ . \tag{2.5}$$

Then we have the following result

**Theorem 2.1 (Equivalence geometric motion/ linear transport, $N = 2$)**
*Under the regularity assumption (1.2), if $(\Gamma_t)_t$ solves equation (1.1) on the time interval $[0, T)$, then the distribution $g_\Gamma(t, y, \theta, \kappa)$ defined by (2.4) solves the following equation*

$$g_t + div \ (ag) + a_0 g = 0 \quad in \quad \mathcal{D}'\big((0, T) \times \mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z})) \times \mathbb{R}\big) \tag{2.6}$$

*with*

$$\text{(2.7)} \qquad div\ (ag) = \partial_y \cdot (a_y g) + \partial_\theta (a_\theta g) + \partial_\kappa (a_\kappa g) \qquad \textit{for} \qquad a = (a_y, a_\theta, a_\kappa)$$

*and*

$$\text{(2.8)} \qquad \left\{ \begin{array}{l} a_0 = \kappa(c + \partial^2_{\theta\theta} c) + \tau \cdot \partial^2_{y\theta} c, \quad a_y = cn - \tau \partial_\theta c, \quad a_\theta = \tau \cdot \partial_y c, \\ a_\kappa = \kappa^2 (c + \partial^2_{\theta\theta} c) + \kappa \ \left(n \cdot \partial_y c + 2\tau \cdot \partial^2_{y\theta} c\right) + \tau \otimes \tau : \partial^2_{yy} c \ . \end{array} \right.$$

*Moreover, if $g$ (with compact support in the variable $\kappa \in \mathbb{R}$) satisfies equation (2.6), then $\hat{g}$ defined by (2.5), satisfies the following equation*

$$\text{(2.9)} \qquad \hat{g}_t + div\ (a'\hat{g}) = 0 \quad in \quad \mathcal{D}'\left((0,T) \times \mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z}))\right)$$

*with*

$$\text{(2.10)} \qquad div\ (a'\hat{g}) = \partial_y \cdot (a_y \hat{g}) + \partial_\theta (a_\theta \hat{g}) \qquad \textit{for} \qquad a' = (a_y, a_\theta)\ .$$

*Finally, for $g_\Gamma$ defined by (2.4), $\hat{g_\Gamma}$ defined in (2.5) satisfies $\hat{g_\Gamma} = 0$.*

We note that a single planar curve is now represented as a measure on a space of dimension 4, and this measure satisfies the linear transport equation (2.6). Note also that for a curve we necessarily have $\hat{g_\Gamma} = 0$, which can be interpreted as a kind of compatibility condition. Moreover this compatibility condition is preserved by the equation on $g$, because $\hat{g}$ satisfies equation (2.9).
Here, general distributions $g$ can be interpreted as the density of curves in the generalized space of space/angle/curvature coordinates. We do not know, if in some sense, any distribution $g$ solving equation (2.6)) and satisfying $\hat{g} = 0$ , can be written as a linear combination of measures $g_\Gamma$ for a possibly infinite number of evolutions $\Gamma$.
We easily see by an integration in $\theta$, that $\hat{g} = 0$ implies (at least formally) that

$$\text{(2.11)} \qquad \partial_y \cdot \left( \int_{\mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R}} \tau(\theta) g(t, y, \theta, \kappa)\ d\theta d\kappa \right) = 0$$

which can be interpreted as a conservation equation, namely the conservation of the Burgers vector along the dislocatons lines, in the terminology of dislocations dynamics (see Lardner [19]). From a physical point of view (see [2]), the Burgers vector of a dislocation line is an invariant associated to the underlying lattice crystal. Mathematically this can be interpreted as the fact that the dislocation line has to be a closed loop.
More precisely, we have the following result

**Proposition 2.2 (Transport equation for the vectorial density)**
*If $g$ is a solution of (2.6) on the time interval $(0,T)$, then the vectorial distribution $\tau g$ satisfies in $\left(\mathcal{D}'\left((0,T) \times \mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z})) \times \mathbb{R}\right)\right)^2$*

$$\text{(2.12)} \qquad 0 = (\tau g)_t - \partial_y^\perp (cg) + (D_\tau g) a_y + \partial_\theta \left((D_\tau c)\tau g - c\kappa n g\right) + \partial_\kappa (a_\kappa \tau g)\ .$$

Equation (2.12) joint to assumption $\hat{g} = 0$, shows in particular by an integration in $\theta$ and $\kappa$, that we have (at least formally) the following evolution equation for the reduced vectorial density

$$\text{(2.13)} \quad \frac{\partial}{\partial t} \left( \int_{\mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R}} \tau(\theta) g(t, y, \theta, \kappa)\ d\theta d\kappa \right) = \partial_y^\perp \left( \int_{\mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R}} c(t, y, \theta) g(t, y, \theta, \kappa)\ d\theta d\kappa \right)$$

which clearly preserves the divergence free property (2.11).

Let us mention the mathematical difficulty due to the fact that the vector field $a$ has a quadratic growth in the curvature $\kappa$, which only allows to expect short time existence of solutions to the transport equation in general. See the Appendix (Subsection 6.3) for a different possible transport equation which overcomes this difficulty, and well-describes the wavefront solution.

**Remark 2.3** *After this work was completed, T. Hochrainer indicated to me the following important remark (see Hochrainer, Zaiser, Gumbsch [14]). For any distribution $g$ satisfying (2.6) with compatibility condition $\hat{g} = 0$, we can define:*

$$\left\{ \begin{array}{l} \overline{\overline{g}}(t,y,\theta) = \int_{\mathbb{R}} d\kappa \ g(t,y,\theta,\kappa), \\ \\ \overline{\overline{\kappa}}(t,y,\theta) = \int_{\mathbb{R}} d\kappa \ \kappa g(t,y,\theta,\kappa) \ . \end{array} \right.$$

*Then we get the compatibility condition*

$$\tau \cdot \partial_y \overline{\overline{g}} + \partial_\theta \overline{\overline{\kappa}} = 0$$

*and the system (using an integration by parts)*

$$\left\{ \begin{array}{l} \overline{\overline{g}}_t + div \ \left(a' \overline{\overline{g}}\right) + (\tau \cdot \partial^2_{y\theta} c)\overline{\overline{g}} + (c + \partial^2_{\theta\theta} c)\overline{\overline{\kappa}} = 0 \\ \\ \overline{\overline{\kappa}}_t + div \ \left(a' \overline{\overline{\kappa}}\right) - (n \cdot \partial_y c + \tau \cdot \partial^2_{y\theta} c)\overline{\overline{\kappa}} - (\tau \otimes \tau : \partial^2_{yy} c)\overline{\overline{g}} = 0 \end{array} \right. \tag{2.14}$$

*with $a'$ defined in (2.10). Here, a remarkable property of the linear system (2.14) is that the coefficients are bounded, which ensures the existence of a solution $(\overline{\overline{g}}, \overline{\overline{\kappa}})$ for all time.*

## 2.2 Application to dislocations dynamics

As a matter of application, let us give a natural model for the dynamics of dislocations densities, using our transport formulation.
To simplify the presentation, let us consider only one slip system in a tridimensional crystal with orthonormal basis $(e_1, e_2, e_3)$, and with dislocations curves moving in planes perpendicular to $e_3$, and with Burgers vector $b \in \mathbb{R}^3$ (with $b \cdot e_3 = 0$ for mobile dislocations without climb). We assume that the density of such dislocations is represented by the quantity

$$g(t,x,\theta,\kappa) \quad \text{with} \quad x = (x_1,x_2,x_3) \in \mathbb{R}^3, \quad \theta \in \mathbb{R}/(2\pi\mathbb{Z}), \quad \kappa \in \mathbb{R} \ .$$

The strain $e(t,x) \in \mathbb{R}^{3\times 3}_{sym}$ solves on $\mathbb{R}^3$ (see for instance Alvarez et al. [2])

$$\left\{ \begin{array}{l} \mathrm{div}\,(\Lambda : e) = 0 \\ \\ \mathrm{inc}\ e = (\mathrm{curl}_{row}\,(b \otimes \beta))_{sym} \quad \text{with} \quad \beta(t,x) = \int_{\mathbb{R}/(2\pi\mathbb{Z})\times\mathbb{R}} d\theta d\kappa \ \tau g(t,x,\theta,\kappa) \end{array} \right. \tag{2.15}$$

where $\Lambda = (\Lambda_{ijkl})_{i,j,k,l=1,2,3}$ is the fourth order tensor of elastic coefficients, and the operator inc $e$ is obtained, taking first the curl of the column vectors of the matrix $e$, and then the curl of the row vectors of the new matrix. The $\mathrm{curl}_{row}$ is the curl of the row vectors of the matrix, and the index $(\quad)_{sym}$ means that we consider the symmetric part of the matrix. The quantity $b \otimes \beta$ is called the Nye tensor of dislocations densities. Here we set

$$\tau = (\sin\theta)e_1 - (\cos\theta)e_2, \quad n = (\cos\theta)e_1 + (\sin\theta)e_2 \ . \tag{2.16}$$

The normal velocity of the dislocations is proportional to the resolved Peach-Koehler force up to a drag coeffient. Even if is easy (using equation (2.6)) to write the model for an anisotropic drag coefficient, for simplicity let us restrict ourself to the case where this coefficient is equal to 1. Then the velocity is simply given by

$$c(t,x) = (b \otimes e_3) : e(t,x) \tag{2.17}$$

and for $x = (y, x_3)$ with $y = (x_1, x_2)$, for each $x_3 \in \mathbb{R}$, $g^{x_3}(t, y, \theta, \kappa) := g(t, y, x_3, \theta, \kappa)$ solves equation (2.6) with a normal velocity independent on $\theta$ defined by $c^{x_3}(t, y, \theta) = c(t, y, x_3)$, i.e.
(2.18)

$$0 = g_t^{x_3} + \kappa c^{x_3} g^{x_3} + n \cdot \partial_y (c^{x_3} g^{x_3}) + \partial_y c^{x_3} \cdot \partial_\theta (\tau g^{x_3}) + \partial_\kappa \left( g^{x_3} \left( c^{x_3} \kappa^2 + \kappa n \cdot \partial_y c^{x_3} + \tau \otimes \tau : \partial^2_{yy} c^{x_3} \right) \right)$$

i.e.

$$0 = g_t + \kappa c g + n \cdot \partial_x (cg) + \partial_x c \cdot \partial_\theta (\tau g) + \partial_\kappa \left( g \left( c\kappa^2 + \kappa n \cdot \partial_x c + \tau \otimes \tau : \partial^2_{xx} c \right) \right) \ . \tag{2.19}$$

The complete system of equations satisfied by $g$ is then (2.15)-(2.16)-(2.17)-(2.19), with a choice of the initial data satisfying the compatibility condition $\hat{g} = 0$ with the notation (2.5), i.e.

$$\int_{\mathbb{R}} d\kappa \ \{\tau \cdot \partial_x g + \kappa \partial_\theta g\} = 0 \ .$$

This system is a generalization of the model of Groma, Balogh [10, 11] that was restricted to the motion of straight lines dislocations with curvature $\kappa = 0$ and only two possible angles $\theta = 0$ or $\pi$. See for instance El Hajj, Forcadel [8] for a mathematical analysis of the Groma, Balogh model in a particular geometry. In equation (2.19), the term $\kappa c g$ can be interpreted as a source term created by the curvatures of the dislocations. Our model is also a natural generalization of the model of Sedláček, Kratochvil, Werner [27] whose transport equation was written for $\beta(t, y, x_3)$, namely

$$\beta_t = \partial_y^\perp \left( c|\beta| \right) \ .$$

This equation has to be compared to our equation (2.13) which containes more degrees of freedom, or even to (2.12) or (2.19). We also underline that equation (2.19) is a natural transport equation that was missing for instance in the theory of Kröner [17, 18] or that was under investigation in the theories of Mura [23] or Kosevich [16].

In the case where there are several slip systems, the contribution of each slip system must be summed on the right hand side of the equation giving the inc $e$, and each density solves an evolution equation similar to (2.6) in its own slip plane direction with its corresponding velocity. The complete system will be studied in a future work.

Let us remark that our model (2.15)-(2.16)-(2.17)-(2.19) only describes pure transport of dislocations lines in a quite rough mean field model. For instance we do not treat self-annihilation of dislocation lines, contrarily to the eikonal equation. We really describe only a kind of wavefront propagation (like in Osher et al. [24], see also the Appendix of the present article). Moreover our mean field model is really a zero-order approximation. A more realistic model would also contain some short distances corrections similar to the homogeneization problem studied in Imbert, Monneau [15]. In a more realistic model, other source or collisions terms should be added to describe Frank-Read sources, annihilations of dislocations, cross-slip, etc. See for instance the proposition of El-Azab [7, 6].

# 3 Proofs in dimension $N = 2$

We start with the following result

**Lemma 3.1** *Let $\psi \in C_c^1((0,T) \times \mathbb{R}^2)$ and $(\Gamma_t)_t$ be a smooth evolution with normal velocity $c(t,y)$. Then we have*

$$\frac{d}{dt}\left(\int_{\Gamma_t} \psi\right) = \int_{\Gamma_t} D_t\psi - c\kappa\psi \qquad \textit{for} \qquad D_t\psi := \psi_t + cn \cdot \partial_y \psi \tag{3.20}$$

*where $\psi, n, \kappa$ and $c$ are evaluated at the current point $(t,y)$.*

**Proof of Lemma 3.1**
We fix the time $t$, and consider a parametrization $\gamma$ by the curvilinear abscissa $s$ of the connected curve $\Gamma_t$, and set $\gamma(s)$ the point associated to $s \in \mathbb{R}/(L\mathbb{Z})$, with $L$ the length of $\Gamma_t$ and $\dfrac{d\gamma}{ds} = \tau$. Then we can parametrize the curve $\Gamma_{t+h}$ by $\gamma_{t+h}$ for $h$ small (even if the parametrization is not yet by curvilinear abscissa)

$$\gamma_{t+h}(s) = \gamma(s) + r(h,s)n(s)$$

where $n(s)$ is the normal to $\Gamma_t$ at the point $\gamma(s)$. Moreover $r(0,s) = 0$ and $r_h(0,s) = c(t,\gamma(s))$ (where $r_h$ stands for $\dfrac{\partial r}{\partial h}$ and $r_s$ stands for $\dfrac{\partial r}{\partial s}$). We compute

$$\begin{aligned}
\int_{\Gamma_{t+h}} \psi(t+h,\cdot) &= \int_{\mathbb{R}/(L\mathbb{Z})} ds\ \psi(t+h,\gamma_{t+h}(s)) \left|\frac{d}{ds}\gamma_{t+h}(s)\right| \\
&= \int_{\mathbb{R}/(L\mathbb{Z})} ds\ \psi(t+h,\gamma(s)+r(h,s)n) \left|(1-r(h,s)\ \kappa)\tau + r_s(h,s)\ n\right| .
\end{aligned}$$

Therefore

$$\begin{aligned}
\frac{d}{dh}\left(\int_{\Gamma_{t+h}} \psi(t+h,\cdot)\right)_{|h=0} &= \int_{\mathbb{R}/(L\mathbb{Z})} ds\ \left\{\psi_t + (r_h(0,s))\ n\cdot\partial_y\psi + \psi\ (-r_h(0,s)\kappa)\right\} \\
&= \int_{\Gamma_t} \psi_t + cn\cdot\partial_y\psi - c\kappa\psi .
\end{aligned}$$

This ends the proof of the Lemma.

**Lemma 3.2** *For any $\varphi \in C_c^\infty((0,T)\times\mathbb{R}^2\times(\mathbb{R}/(2\pi\mathbb{Z}))\times\mathbb{R})$, we have for $g=g_\Gamma$*

$$< g_t, \varphi > = < g, -c\kappa\varphi + cn\cdot\partial_y\varphi + (D_t\tilde{\theta})\ \partial_\theta\varphi + (D_t\tilde{\kappa})\ \partial_\kappa\varphi > \tag{3.21}$$

*for any $C^1$ extension $\tilde{\theta}$ (resp. $\tilde{\kappa}$), which restricted to $\Gamma$ is equal to $\theta$ (resp. $\kappa$).*

**Proof of Lemma 3.2**
We have

$$\begin{aligned} < g_t, \varphi > &= - < g, \varphi_t > \\ &= -\int_0^T dt \int_{\Gamma_t} \varphi_t \\ &= \int_0^T dt \left\{ \frac{d}{dt}\left(\int_{\Gamma_t}\varphi\right) - \int_{\Gamma_t}\varphi_t \right\}. \end{aligned}$$

We now compute using (3.20) with $\psi(t,y) = \varphi(t,y,\tilde{\theta}(t,y),\tilde{\kappa}(t,y))$, and the velocity $c(t,y,\tilde{\theta}(t,y))$

$$\begin{aligned} \frac{d}{dt}\left(\int_{\Gamma_t}\varphi(t,y,\theta(t,y),\kappa(t,y))\right) &= \int_{\Gamma_t}\varphi_t + \tilde{\theta}_t\partial_\theta\varphi + \tilde{\kappa}_t\partial_\kappa\varphi - c\kappa\varphi \\ &\quad + \int_{\Gamma_t} cn\cdot\left\{\partial_y\varphi + (\partial_y\tilde{\theta})\partial_\theta\varphi + (\partial_y\tilde{\kappa})\partial_\kappa\varphi\right\}. \end{aligned}$$

And then

$$< g_t, \varphi > = \int_0^T dt \int_{\Gamma_t} -c\kappa\varphi + cn\cdot\partial_y\varphi + \left\{\tilde{\theta}_t + cn\cdot\partial_y\tilde{\theta}\right\}\partial_\theta\varphi + \left\{\tilde{\kappa}_t + cn\cdot\partial_y\tilde{\kappa}\right\}\partial_\kappa\varphi$$

which gives the result.

**Lemma 3.3** *With the notation of Lemma 3.2, we have on $\Gamma$*

$$\left\{ \begin{aligned} D_t\tilde{\theta} &= D_\tau c \\ D_t\tilde{\kappa} &= a_\kappa + (\partial_\theta c)\ (\tau\cdot\partial_y\tilde{\kappa}) \end{aligned} \right.$$

*with $a_\kappa$ is given in (2.8).*

**Proof of Lemma 3.3**
We remark that the vector field $D_t$ is tangent to the hypersurface $\Gamma$. This means that $D_t\tilde{\theta}$ and $D_t\tilde{\kappa}$ are intrinsec quantities only depending respectively on the values of $\theta$ and $\kappa$ on the hypersurface $\Gamma$. For this reason, it is possible to compute these quantities, considering a particular parametrization of the hypersurface $\Gamma$.
We consider $(t_0,y_0)\in\Gamma$. Up to a translation and rotations of the coordinates, we can assume that $t_0=0=y_0$ and consider a local representation of $\Gamma$ as

$$z = u(t,x) \quad \text{for} \quad y=(x,z) \quad \text{with} \quad u_x(0,0)=0$$

where $u_x$ denotes $\dfrac{\partial u}{\partial x}$ and the normal at $(0,0)$ is $(0,1)$ in the $(x,y)$ coordinates. We set $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$, $u_{xt} = \dfrac{\partial^2 u}{\partial x\partial t}$, and $u_{xxt} = \dfrac{\partial^3 u}{\partial x^2\partial t}$. In these coordinates, we have

$$\theta(t,x,u) = \arctan u_x, \quad \kappa(t,x,u) = \frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}} \quad \text{and} \quad n(t,x,u) = \frac{1}{\sqrt{1+u_x^2}}(-u_x,1)\in\mathbb{R}^2.$$

We recall that locally in a neighborhood of $(0,0)$, the function $u$ satisfies

$$u_t = c(t,x,u,\arctan u_x)\ \sqrt{1+u_x^2}\ . \tag{3.22}$$

For $t$ in a neighborhood of zero, let us define the curve $\gamma$ contained in $\Gamma$ by:

$$\gamma(t) = (t,0,u(t,0))$$

for which we have

$$\frac{d\gamma}{dt}(0) = (1,0,u_t(0,0)) = (1,0,c(0,0,0))$$

which is exactly the vector field $D_t$ evaluated at the origin, because we assume that $u_x$ vanishes at the origin. Therefore

$$D_t\tilde{\theta}(0,0,0) = \frac{d}{dt}(\tilde{\theta}\circ\gamma)(0) = \frac{d}{dt}\left(\arctan u_x(t,0)\right)_{|t=0} = u_{xt}(0,0)\ .$$

Similarly

$$D_t\tilde{\kappa}(0,0,0) = \frac{d}{dt}(\tilde{\kappa}\circ\gamma)(0) = \frac{d}{dt}\left(\frac{u_{xx}(t,0)}{(1+u_x^2(t,0))^{\frac{3}{2}}}\right)_{|t=0} = u_{xxt}(0,0)\ .$$

Derivating (3.22) with respect to $x$, we get

$$u_{xt} = (1,u_x)\cdot\partial_y c\ \sqrt{1+u_x^2} + \partial_\theta c\ \frac{u_{xx}}{\sqrt{1+u_x^2}} + c\frac{u_{xx}u_x}{\sqrt{1+u_x^2}} \tag{3.23}$$

which implies that at the origin we have

$$D_t\tilde{\theta}(0,0,0) = u_{xt}(0,0) = \tau\cdot\partial_y c + \kappa\partial_\theta c = D_\tau c\ .$$

Derivating now (3.23) with respect to $x$, we get at the origin

$$\begin{aligned} u_{xxt}(0,0) &= (0,u_{xx})\cdot\partial_y c + (1,0)\otimes(1,0):\partial^2_{yy}c + 2(1,0)\cdot\partial^2_{y\theta}c\ u_{xx} + \partial^2_{\theta\theta}c\ u_{xx}^2 + \partial_\theta c\ u_{xxx} + cu_{xx}^2 \\ &= \kappa n\cdot\partial_y c + \tau\otimes\tau:\partial^2_{yy}c + 2\kappa\tau\cdot\partial^2_{y\theta}c + \kappa^2\partial^2_{\theta\theta}c + c\kappa^2 + \partial_\theta c\ u_{xxx} \\ &= a_\kappa + \partial_\theta c\ \frac{\partial}{\partial x}\left(\frac{u_{xx}}{(1+u_x^2)^{\frac{3}{2}}}\right) \end{aligned}$$

with $a_\kappa$ given in (2.8). This shows that

$$D_t\tilde{\kappa}(0,0,0) = u_{xxt}(0,0) = a_\kappa + (\partial_\theta c)\ (\tau\cdot\partial_y\tilde{\kappa})\ .$$

This ends the proof of the Lemma.

**Lemma 3.4** *With the notation of Lemma 3.2, we have*

$$-\int_{\Gamma_t}(\partial_\theta c)\ (\tau\cdot\partial_y\tilde{\kappa})\ \partial_\kappa\varphi = \int_{\Gamma_t} D_\tau\left(\varphi\ \partial_\theta c\right) \quad \textit{with} \quad D_\tau = \tau\cdot\partial_y + \kappa\partial_\theta$$

*where the quantities in the integrals are evaluated at* $(t,y)$, $\theta(t,y)$, $\kappa(t,y)$.

**Proof of Lemma 3.4**
We consider a parametrization $\gamma$ by the curvilinear abscissa $s$ of the connected curve $\Gamma_t$, and set $\gamma(s)$ the point associated to $s \in \mathbb{R}/(L\mathbb{Z})$, with $L$ the length of $\Gamma_t$ and $\frac{d\gamma}{ds} = \tau$. Then we have $\tau \cdot \partial_y \tilde{\kappa} = \frac{d\kappa}{ds}(s)$. With an obvious abuse of notation, we denote by $\kappa(s)$, $\theta(s)$, respectively the curvature and the angle associated to $s$, i.e. $\kappa(s) = \kappa(t, \gamma(s))$, $\theta(s) = \theta(t, \gamma(s))$. For a general $C^1$ function $G(y, \theta, \kappa)$, we have

$$\frac{d}{ds}\left(G(\gamma(s), \theta(s), \kappa(s))\right) = \left\{\tau \cdot \partial_y + \kappa\partial_\theta + \frac{d\kappa}{ds}\partial_\kappa\right\} G \tag{3.24}$$

where we have in particular used the fact that $\frac{d\theta}{ds} = \kappa$ (i.e. $\frac{d\tau}{ds} = \kappa n$). We deduce that

$$\begin{aligned}
\int_{\Gamma_t} (\partial_\theta c)\ \frac{d\kappa}{ds}\ \partial_\kappa\varphi &= \int_{\mathbb{R}/(L\mathbb{Z})} ds\ (\partial_\theta c)\ \left\{\frac{d\kappa}{ds}\ \partial_\kappa\varphi + \kappa\partial_\theta\varphi + \tau\cdot\partial_y\varphi - D_\tau\varphi\right\}\\
&= \int_{\mathbb{R}/(L\mathbb{Z})} ds\ (\partial_\theta c)\ \frac{d\varphi}{ds} - \int_{\mathbb{R}/(L\mathbb{Z})} ds\ (\partial_\theta c)\ D_\tau\varphi\\
&= -\int_{\mathbb{R}/(L\mathbb{Z})} ds\ \left\{\left(\frac{d}{ds}(\partial_\theta c)\right)\ \varphi + (\partial_\theta c)\ D_\tau\varphi\right\}\\
&= -\int_{\mathbb{R}/(L\mathbb{Z})} ds\ \left\{(D_\tau(\partial_\theta c))\ \varphi + (\partial_\theta c)\ D_\tau\varphi\right\}\\
&= -\int_{\Gamma_t} D_\tau\left(\varphi\partial_\theta c\right)
\end{aligned}$$

where we have used (3.24) with $G = \varphi$ at the second line, we have made an integration by parts at the third line, and used (3.24) with $G = \partial_\theta c$ at the fourth line. This ends the proof of the Lemma.

**Lemma 3.5** *For a general $g$ solution of (2.6), we have*

$$(D_\tau g)_t + \partial_y \cdot (a_y D_\tau g) + \partial_\theta(a_\theta D_\tau g) + \partial_\kappa(D_\tau(a_\kappa g)) = 0\ . \tag{3.25}$$

**Proof of Lemma 3.5**
For two vector fields $V_1$, $V_2$ and $g$ a given distribution, we recall the definition of brackets $[V_1, V_2]$:

$$[V_1, V_2]g = V_1(V_2 g) - V_2(V_1 g)\ .$$

We first compute the following brackets of vector fields for general $g$:

$$\left\{\begin{array}{l}
[D_\tau, \partial_y\cdot(a_y\cdot)]\, g = (D_\tau(\partial_y\cdot a_y))g + (D_\tau a_y)\cdot\partial_y g + a_y\cdot[D_\tau, \partial_y]\, g\\[2ex]
[D_\tau, \partial_\theta\,(a_\theta\cdot)]\, g = (D_\tau(\partial_\theta a_\theta))g + (D_\tau a_\theta)\partial_\theta g + a_\theta\ [D_\tau, \partial_\theta]\, g
\end{array}\right.$$

and

$$\begin{cases} [D_\tau, \partial_y] = 0 \\ \\ [D_\tau, \partial_\theta] = -n \cdot \partial_y \\ \\ [D_\tau, \partial_\kappa] = -\partial_\theta \end{cases}$$

Applying the vector field $D_\tau$ to the equation

$$-g_t = a_0 g + \partial_y \cdot (a_y g) + \partial_\theta(a_\theta g) + \partial_\kappa(a_\kappa g)$$

we get

$$\begin{aligned} -(D_\tau g)_t &= D_\tau(a_0 g) + \partial_y \cdot (a_y D_\tau g) + \partial_\theta(a_\theta D_\tau g) + \partial_\kappa(D_\tau(a_\kappa g)) \\ &\quad + [D_\tau, \partial_y \cdot (a_y \cdot)]\, g + [D_\tau, \partial_\theta(a_\theta \cdot)]\, g + [D_\tau, \partial_\kappa]\,(a_\kappa g) \end{aligned}$$

i.e.

$$\begin{aligned} & -(D_\tau g)_t - \partial_y \cdot (a_y D_\tau g) - \partial_\theta(a_\theta D_\tau g) - \partial_\kappa(D_\tau(a_\kappa g)) \\ = \; & D_\tau(a_0 g) + [D_\tau, \partial_y \cdot (a_y \cdot)]\, g + [D_\tau, \partial_\theta(a_\theta \cdot)]\, g + [D_\tau, \partial_\kappa]\,(a_\kappa g) \\ = \; & (D_\tau a_0) g + a_0 D_\tau g \\ & +(D_\tau(\partial_y \cdot a_y)) g + (D_\tau a_y) \cdot \partial_y g \\ & +(D_\tau(\partial_\theta a_\theta)) g + (D_\tau a_\theta) \partial_\theta g - a_\theta \; n \cdot \partial_y g \\ & -\partial_\theta(a_\kappa g) \\ = \; & (D_\tau a_0 + D_\tau(\partial_y \cdot a_y) + D_\tau(\partial_\theta a_\theta) - \partial_\theta a_\kappa) g \\ & +(a_0 \tau + D_\tau a_y - a_\theta \; n) \cdot \partial_y g \\ & +(a_0 \kappa + D_\tau a_\theta - a_\kappa) \; \partial_\theta g \\ = \; & 0 \end{aligned}$$

because, on the other hand, we compute

$$\begin{cases} D_\tau a_\theta = a_\kappa - a_0 \kappa \\ D_\tau a_y = a_\theta \; n - a_0 \tau \\ D_\tau a_0 + D_\tau(\partial_y \cdot a_y) + D_\tau(\partial_\theta a_\theta) - \partial_\theta a_\kappa = 0 \end{cases}$$

where the last line is a consequence of the following computations

$$\begin{cases} \partial_y \cdot a_y + \partial_\theta a_\theta = 2n \cdot \partial_y c \\ D_\tau \left(\partial_y \cdot a_y + \partial_\theta a_\theta\right) = -2 \left(\kappa\tau \cdot \partial_y c - \kappa n \cdot \partial^2_{y\theta} c - \partial^2_{yy} c \cdot (\tau, n)\right) \\ D_\tau a_0 - \partial_\theta a_\kappa = \tau \cdot \partial_y a_0 - \partial_\theta(D_\tau a_\theta) = 2 \left(\kappa\tau \cdot \partial_y c - \kappa n \cdot \partial^2_{y\theta} c - \partial^2_{yy} c \cdot (\tau, n)\right) \; . \end{cases}$$

This ends the proof of the Lemma.

**Proof of Theorem 2.1**

Putting together the results of Lemmata 3.2, 3.3 and 3.4, we get for $g = g_\Gamma$

$$\begin{aligned} < g_t, \varphi > \; &= \; < g, -c\kappa\varphi + +cn \cdot \partial_y \varphi + (D_\tau c) \; \partial_\theta \varphi + a_\kappa \; \partial_\kappa \varphi + (\partial_\theta c) \; (\tau \cdot \partial_y \tilde{\kappa}) \; \partial_\kappa \varphi > \\ &= \; < g, -c\kappa\varphi + +cn \cdot \partial_y \varphi + (D_\tau c) \; \partial_\theta \varphi + a_\kappa \; \partial_\kappa \varphi - \partial_\theta c \; D_\tau \varphi - \varphi D_\tau(\partial_\theta c) > \\ &= \; < g, -a_0 \varphi + a_y \partial_y \varphi + a_\theta \partial_\theta \varphi + a_\kappa \partial_\kappa \varphi > \end{aligned}$$

which leads to equation (2.6).
Integrating equation (3.25) with respect to $\kappa$, leads to (2.9).
Finally, let us compute for $g_\Gamma$ defined in (2.4) and $\overline{\psi}(t,y,\theta,\kappa)=\psi(t,y,\theta)$

$$\begin{aligned}
<\hat{g_\Gamma},\psi> &= <D_\tau g,\overline{\psi}> \\
&= -<g_\Gamma, D_\tau\overline{\psi}> \\
&= -<g_\Gamma, \tau\cdot\partial_y\overline{\psi}+\kappa\partial_\theta\overline{\psi}+\frac{d\kappa}{ds}\partial_\kappa\overline{\psi}> \\
&= -\int_0^T dt\int_{\Gamma_t}\tau\cdot\partial_y\tilde{\psi}(t,y) \\
&= 0
\end{aligned}$$

where at the third line, we have used the fact that $\overline{\psi}$ is independent on $\kappa$, and we have used (3.24) with $G=\overline{\psi}$ at the fourth line. Finally we have set $\tilde{\psi}(t,y)=\overline{\psi}(t,y,\theta(t,y),\kappa(t,y))$. This proves that $\hat{g_\Gamma}=0$. This ends the proof of the Theorem.

**Proof of Propositon 2.2**
We start to multiply by $\tau$ the equation (2.6) satisfied by $g$. We get

$$0=(\tau g)_t+a_0\tau g+\partial_y\cdot(a_y\otimes\tau g)+\partial_\theta\left(a_\theta\tau g\right)-a_\theta ng+\partial_\kappa\left(a_\kappa\tau g\right)\ .$$

We compute

$$\begin{aligned}
& a_0\tau g+\partial_y\cdot(a_y\otimes\tau g)-a_\theta ng+\partial_\theta\left(c\kappa ng\right) \\
=\ & \partial_y\cdot(a_y\otimes\tau g)-a_\theta ng+n\partial_\theta(c\kappa g)+\kappa(\partial^2_{\theta\theta}c)\tau g+(\tau\cdot\partial^2_{y\theta}c)\tau g \\
=\ & (n\cdot\partial_y)(c\tau g)-(\tau\cdot\partial_y)((\partial_\theta c)\tau g)-(\tau\cdot\partial_y c)ng \\
& +nc\kappa\partial_\theta g+n\kappa(\partial_\theta c)g+\{\partial_\theta\left(\kappa(\partial_\theta c)\tau g\right)-\kappa(\partial_\theta c)ng-\kappa(\partial_\theta c)\tau\partial_\theta g\}+(\tau\cdot\partial^2_{y\theta}c)\tau g \\
=\ & ncD_\tau g-nc\tau\cdot\partial_y g-(\tau\cdot\partial_y c)ng+(n\cdot\partial_y)(c\tau g)-(\tau\cdot\partial_y)((\partial_\theta c)\tau g) \\
& +\partial_\theta\left(\kappa(\partial_\theta c)\tau g\right)-(\partial_\theta c)\tau D_\tau g+(\partial_\theta c)\tau(\tau.\partial_y g)+(\tau\cdot\partial^2_{y\theta}c)\tau g \\
=\ & \partial_\theta\left(\kappa(\partial_\theta c)\tau g\right)+(D_\tau g)\left(cn-\tau\partial_\theta c\right)-n(\tau\cdot\partial_y)(cg)+\tau(n\cdot\partial_y)(cg) \\
=\ & \partial_\theta\left(\kappa(\partial_\theta c)\tau g\right)+(D_\tau g)\left(cn-\tau\partial_\theta c\right)-\partial_y^\perp(cg)
\end{aligned}$$

where we have used the explicit expression of $a_0$ in the second line, the explicit expressions of $a_y$ and $a_\theta$ in the third line, and we have introduced the expression of $D_\tau g=\tau\cdot\partial_y g+\kappa\partial_\theta g$ in the fourth line. Therefore we get

$$0\ =\ (\tau g)_t-\partial_y^\perp(cg)+(D_\tau g)a_y+\partial_\theta\left((a_\theta+\kappa\partial_\theta c)\tau g-c\kappa ng\right)+\partial_\kappa\left(a_\kappa\tau g\right)\ .$$

This ends the proof of the Proposition.

# 4 Result in dimension $N \geq 2$

We now turn back to our main result in dimension $N \geq 2$. We use the notation of subsection 1.3. We consider a closed connected and oriented hypersurface $\Gamma_t$ for $t \in [0,T)$ for some fixed $T > 0$, with the normal pointing out the bounded set whose boundary is the hypersurface. At a point $y$ of $\Gamma_t$, we recall that we call $n(t,y)$ the unit normal and $K(t,y) \in \mathbb{R}^{N\times N}_{sym}$ the curvature matrix (negative for a ball).
We define for $(t,y,n,K) \in [0,T) \times \mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}^{N\times N}_{sym}$, the measure $g_\Gamma(t,y,n,K)$ by

$$< g_\Gamma, \varphi >= \int_0^T dt \int_{\Gamma_t} \varphi(t,y,n(t,y),K(t,y)) \tag{4.26}$$

for any test function $\varphi \in C^\infty_c\left([0,T) \times \mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}^{N\times N}_{sym}\right)$. Given any distribution $g$ (with compact support in the variable $K \in \mathbb{R}^{N\times N}_{sym}$), we also define formally the distribution $\hat{g}(t,y,n)$ by

$$\text{"}\ \hat{g} := \int_{\mathbb{R}^{N\times N}_{sym}} dK\ \ \{D_\tau g + n(I:K)g\}\ \text{"} \quad \text{with} \quad D_\tau = -\left(I - n\otimes n\right)\cdot \partial_y + K\cdot \partial_n$$

i.e. rigorously, for any $\psi \in C^\infty_c\left([0,T) \times \mathbb{R}^N \times \mathbb{S}^{N-1}\right)$

$$< \hat{g}, \psi >=< D_\tau g + n(I:K)g, \overline{\psi} > \quad \text{with} \quad \overline{\psi}(t,y,n,K) = \psi(t,y,n)\ . \tag{4.27}$$

Then we have

**Theorem 4.1 (Equivalence geometric motion/ linear transport, $N \geq 2$)** *Under the regularity assumption (1.2), if $(\Gamma_t)_t$ solves equation (1.1) on the time interval $[0,T)$, then the distribution $g_\Gamma(t,y,n,K)$ defined by (4.26) solves the following equation*

$$g_t + div\ (ag) + a_0 g = 0 \quad in \quad \mathcal{D}'\left((0,T) \times \mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}^{N\times N}_{sym}\right) \tag{4.28}$$

*with*

$$div\ (ag) = \partial_y \cdot (a_y g) + \partial_n \cdot (a_n g) + \partial_K : (a_K g) \qquad for \qquad a = (a_y, a_n, a_K) \tag{4.29}$$

*and*

$$\left\{\begin{array}{l} a_0 = c(I:K) + K : \partial^2_{nn} c - I : \partial^2_{ny} c, \quad a_y = cn + \partial_n c, \quad a_n = -(I - n\otimes n)\cdot \partial_y c,\\ \\ \begin{array}{rcl} a_K & = & cK^2 + K\cdot \partial^2_{nn} c \cdot K + (I - n\otimes n)\cdot \partial^2_{yy} c \cdot (I - n\otimes n)\\ & & +\,(n\cdot \partial_y c)\ \ K - K\cdot \partial^2_{ny} c\cdot (I - n\otimes n) - (I - n\otimes n)\cdot \partial^2_{yn} c \cdot K\\ & & +K\cdot \partial_y c \otimes n + n \otimes K \cdot \partial_y c\ . \end{array}\end{array}\right. \tag{4.30}$$

*Moreover we have for $\alpha := K\cdot n\ g$*

$$\alpha_t + div\ (a\otimes \alpha) + A_0 \cdot \alpha = 0 \quad in \quad \left(\mathcal{D}'\left((0,T) \times \mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}^{N\times N}_{sym}\right)\right)^N \tag{4.31}$$

*with*

$$-A_0 = cK + K\cdot \partial_{nn} c - (I - n\otimes n)\cdot \partial^2_{yn} c + n\otimes \partial_y c + \left(2\left(n\cdot \partial_y c\right) - a_0\right) I\ . \tag{4.32}$$

*Finally, for $g_\Gamma$ defined by (4.26) we have $K\cdot n\ g_\Gamma = 0$ and $\hat{g_\Gamma}$ defined in (4.27) satisfies $\hat{g_\Gamma} = 0$.*

Let now make a few comments on this Theorem.

**The invariant manifold.**
We first remark that in the expression (4.30) of $a_K$, the two last terms are new, in comparison to (2.8). Moreover these terms are the only terms not perpendicular to $n$. Their existence is due to our choice of writting the equations on $\mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}^{N\times N}_{sym}$, in order to preserve $K \cdot n = 0$ on the support of $g$ for all time, if it is true at the initial time (see the justification of equation (4.31)). This really means that we are interested in measures with support on the natural manifold

$$\mathcal{M} = \left\{(y,n,K) \in \mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}^{N\times N}_{sym}, \quad K\cdot n = 0\right\}$$

whose the dimension is the same as $\mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}^{(N-1)\times(N-1)}_{sym}$. This is obviously related to the fact (easy to check) that for any $t \in [0,T)$, the vector field

$$a^t(X) = a(t,X) \quad \text{with} \quad X = (y,n,K)$$

is tangent to the manifold $\mathcal{M}$. This means that it should be possible to represent (but probably less simple to write) the transport equation as some transport equation on the manifold $\mathcal{M}$, similar to equation (4.28). This also means that, while we keep our description on the space $\mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}^{N\times N}_{sym}$, there are several equivalent transport formulations, because for what we have in mind, in (4.30), $K$ can be replaced by $K\cdot(I - n\otimes n)$, or $(I-n\otimes n)\cdot K$, or even $(I-n\otimes n)\cdot K \cdot (I-n\otimes n)$.

**Explicit solution based on the characteristics**
It is known (see for instance Lions [22] or Leveque [21]) that that the solution $g$ of (4.28) is given (at least formally with the measure $g(0,\cdot)$ at the initial time) for $X=(y,n,K)$ by

$$g(t,Y) = g(0,X(0;t,Y))e^{-\int_0^t b_0(s;t,Y)\ ds} \quad \text{with} \quad b_0(s;t,Y) = (a_0 + \text{div } a)(s,X(s;t,Y))$$

and

$$\frac{d}{ds}X(s;t,Y) = a(s,X(s;t,Y)), \quad X(t;t,Y) = Y\ .$$

In particular, defining for $t \geq 0$

$$\hat{\Gamma}_t = \left\{ \begin{array}{l} (y,n(t,y),K(t,y)) \in \mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}^{N\times N}_{sym}, \\ \\ \text{for } y \in \Gamma_t \text{ with normal } n(t,y) \text{ and curvature } K(t,y) \end{array} \right\}$$

we have

$$\text{supp } g_\Gamma(t,\cdot) = \hat{\Gamma}_t\ .$$

Therefore, if $Y \in \hat{\Gamma}_0$, then $X(t;0,Y) \in \hat{\Gamma}_t$, which shows (at least formally) that the solutions $X(t;0,Y)$ are the characteristics of the evolution. In particular, we get, as a result, the value of the *evolution of the curvature*, which has to be put in relation with curvature estimates in Cannarsa, Frankowska [5], Cannarsa, Cardaliaguet [4] or Alvarez, Cardaliaguet, Monneau [1]. Moreover, at least the first two components $(cn + \partial_n c, -(I-n\otimes n)\cdot\partial_y c)$ of this vector field are similar to the characteristics of classical Hamilton-Jacobi equations with Hamiltonian $c(t,y,n)$, see Evans [9]).

**The mathematical difficulty for long time existence.**
The principal mathematical difficulty to solve equation (4.28), is the quadratic growth of the vector field $a$ as a function of the curvature $K$. This means that, even for initial data with compact support, the support of the solution can go to infinity in $K$ in finite time (which corresponds to the apparition of geometric singularities of the fronts, revealed by infinite curvature). On the contrary, in the particular case where the hypersurface is transported by a vector field $V(t, y)$, which means that the normal velocity is given by $c(t, y, n) = n \cdot V(t, y)$, then we can check that the vector field $a$ is at most linear in the curvature $K$, and $a_y = V$ is exactly the original vector field. This is natural, because its is well-known that linear transport equations do not create singularities in finite time. See also the Appendix (Subsection 6.3 and Remark 6.5) for indications to overcome these difficulties using a different transport formulation.

We do not know if there are conservative quantities (other than $K \cdot ng$, like maybe $\hat{g}$) that can be derived from this transport formulation of moving fronts. We do not know neither what is the best regularity that we can assume on $g$, in order to satisfy the natural constraints like the conserved quantities.

# 5 Proofs in dimension $N \geq 2$

We start with the following result which, for sake of completeness, is proved in the Appendix (Section 6).

**Lemma 5.1** *Let $\psi \in C^1_c\left((0, T) \times \mathbb{R}^N\right)$ and $(\Gamma_t)_t$ be a smooth evolution with normal velocity $c(t, y)$. Then we have*

$$(5.33) \qquad \frac{d}{dt}\left(\int_{\Gamma_t} \psi\right) = \int_{\Gamma_t} D_t\psi - c(I : K)\psi \qquad \textit{for} \qquad D_t\psi := \psi_t + cn \cdot \partial_y \psi$$

*where $\psi, n, K$ and $c$ are evaluated at the current point $(t, y)$, and $I : K$ denotes the trace of the matrix $K$.*

Then, we have the following result

**Lemma 5.2** *For any $\varphi \in C^\infty_c\left((0, T) \times \mathbb{R}^N \times \mathbb{S}^{N-1} \times \mathbb{R}^{N \times N}_{sym}\right)$, we have for $g = g_\Gamma$*

$$(5.34) \qquad < g_t, \varphi >=< g, -c(I : K)\varphi + cn \cdot \partial_y \varphi + (D_t\tilde{n}) \cdot \partial_n \varphi + (D_t\tilde{K}) : \partial_K \varphi >$$

*for any $C^1$ extension $\tilde{n}$ (resp. $\tilde{K}$), which restricted to $\Gamma$ is equal to $n$ (resp. $K$).*

**Proof of Lemma 5.2**
The proof is similar to the proof of Lemma 3.2, with (3.20) replaced by (5.33). This ends the proof of the Lemma.

**Lemma 5.3** *We consider a local parametrization of the hypersurface $\Gamma$ as*

$$z = u(t, x) \quad \textit{for} \quad y = (x, z) \in U \subset \mathbb{R}^{N-1} \times \mathbb{R}$$

*where $U$ is an open set. Then we have with* $u_x = \dfrac{\partial u}{\partial x} \in \mathbb{R}^{N-1}$ *and* $u_{xx} = \dfrac{\partial^2 u}{\partial x^2} \in \mathbb{R}^{(N-1)\times(N-1)}_{sym}$

$$n(t,x,u) = \frac{1}{\sqrt{1+u_x^2}}(-u_x, 1) \in \mathbb{R}^{N-1}\times\mathbb{R} \quad \textit{and} \quad K(t,x,u) = F(u_{xx}, u_x) \in \mathbb{R}^{N\times N}_{sym}$$

*where for any* $M \in \mathbb{R}^{(N-1)\times(N-1)}_{sym}$, $p \in \mathbb{R}^{N-1}$, *we set*

$$F(M,p) = \frac{1}{\sqrt{1+p^2}} \left\{ \begin{array}{l} \mathcal{I}(M) + \dfrac{p\cdot M\cdot p}{(1+p^2)^2}\,(-p,1)\otimes(-p,1) \\ +\left(\dfrac{M\cdot p}{1+p^2}, 0\right)\otimes(-p,1) + (-p,1)\otimes\left(\dfrac{M\cdot p}{1+p^2}, 0\right) \end{array} \right\}$$

*with*

$$(\mathcal{I}(M))_{ij} = \begin{cases} M_{ij} & \textit{if} \quad i,j\in\{1,...,n-1\} \\ 0 & \textit{if} \quad i=n \quad \textit{or} \quad j=n\ . \end{cases}$$

**Proof of Lemma 5.3**

The only thing to prove is the expression of the curvature.
In the case where $n = (0,1) \in \mathbb{R}^{N-1}\times\mathbb{R}$, i.e. $u_x = 0$, it is clear that we have $F(M,0) = \mathcal{I}(M)$.
In the general case where $n \neq (0,1)$, i.e. $u_x \neq 0$, we need to make a rotation of the coordinates in the plane generated by $n$ and $(0,1)$, such that in the new coordinates the surface $\Gamma$ is represented by $Z = v(t,X)$ with $v_X = 0$. Then a full computation (based on the inverse function theorem) is possible.

We now make the details of the computation.
We drop the time coordinate which does not play any role in the present computation. Up to a translation we can assume that we work close to the origin where $u_x(0) \neq 0$. We choose an orthonormal basis $(e_1, ..., e_{N-1})$ such that

$$e_1 = \frac{u_x}{|u_x|}(0)$$

We set

$$x' = (x_2, ..., x_{N-1}) = X'$$

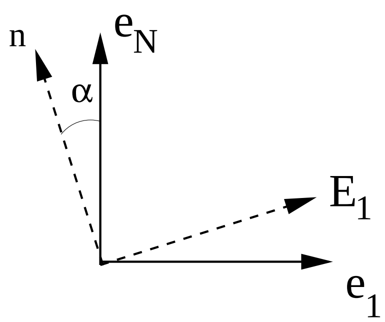


Figure 1: New coordinates by rotation

We will consider the unit vector $E_1$ such that orthonormal basis $(E_1, n(0))$ of the plane is obtained by a rotation of the basis $(e_1, e_N)$ (see Fig. 1), and define the new coordinates $X_1, X_N$ such that

$$X_1E_1 + X_N n(0) = x_1e_1 + x_Ne_N \quad \text{and} \quad X = (X_1, X', X_N)$$

i.e.

$$X_1 = x_1 \ \cos\alpha + x_N \ \sin\alpha, \quad X_N = -x_1 \ \sin\alpha + x_N \ \cos\alpha$$

for

$$\alpha = \text{angle } (e_N, n(0)) \in (0, \pi/2), \quad \text{i.e.} \quad \cos\alpha = \frac{1}{\sqrt{1+u_1^2}}, \quad \sin\alpha = \frac{u_1}{\sqrt{1+u_1^2}}$$

with $u_1 = \dfrac{\partial u}{\partial x_1}$. We see that

$$x_N = u(x_1, x') \Longleftrightarrow X_N = v(X_1, X')$$

for some new function $v$ which satisfies

$$v\left(x_1 \ \cos\alpha + u(x_1, x') \ \sin\alpha, x'\right) = -x_1 \ \sin\alpha + u(x_1, x') \ \cos\alpha \ .$$

We compute by derivation

$$\begin{cases} v_1(\cos\alpha + u_1 \sin\alpha) = -\sin\alpha + u_1\cos\alpha \\ \\ v_1 u_i \sin\alpha + v_i = u_i \cos\alpha, \quad i = 2, ..., N-1 \ . \end{cases}$$

In particular we have $v_X(0) = 0$. Derivating once again, we get at the origin

$$\begin{cases} v_{11} = \dfrac{u_{11}}{(1+u_1^2)^{\frac{3}{2}}} \\ v_{1i} = \dfrac{u_{1i}}{1+u_1^2}, \quad i = 2, ..., N-1 \\ v_{ij} = \dfrac{u_{ij}}{(1+u_1^2)^{\frac{1}{2}}}, \quad i,j = 2, ..., N-1 \ . \end{cases}$$

Then (with $E_1 = \cos\alpha \ e_1 + \sin\alpha \ e_N$, $E_1' = \cos\alpha E_1$)

$$\begin{aligned} K(1+u_1^2)^{\frac{1}{2}} &= v_{11} E_1 \otimes E_1 + \textstyle\sum_{i=2,\ldots,N-1} v_{1i}\left(E_1 \otimes e_i + e_i \otimes E_1\right) + \sum_{i,j=2,\ldots,N-1} v_{ij} e_i \otimes e_j \\ &= \textstyle\sum_{i,j=1,\ldots,N-1} u_{ij} e_i \otimes e_j + \sum_{i=1,\ldots,N-1} u_{1i}\left((E_1' - e_1) \otimes e_i + e_i \otimes (E_1' - e_1)\right) \\ &\quad + u_{11}\left(E_1' - e_1\right) \otimes \left(E_1' - e_1\right) \ . \end{aligned}$$

Using the fact that

$$E_1' - e_1 = \frac{u_1}{1+u_1^2}\left(-u_1 e_1 + e_N\right))$$

we see that we exactly get

$$K = F(u_{xx}, u_1 e_1)$$

which ends the proof of the Lemma.

**Lemma 5.4** *With the notation of Lemma 5.2, we have on* $\Gamma$

$$\begin{cases} D_t \tilde{n} &= -(I - n\otimes n)\cdot \partial_y c + K \cdot \partial_n c =: D_\tau c \\ \\ D_t \tilde{K} &= a_K - \partial_n c \cdot \partial_y \tilde{K} \end{cases}$$

*with* $a_K$ *given in (4.30), and on* $\Gamma$

$$(I - n\otimes n)\cdot \partial_y \tilde{n} = -K \ . \tag{5.35}$$

**Proof of Lemma 5.4**
We procceed exactly as in the proof of Lemma 3.3.
We consider $(t_0, y_0) \in \Gamma$. Up to a translation and rotations of the coordinates, we can assume that $t_0 = 0 = y_0$ and consider a local representation of $\Gamma$ as

$$z = u(t,x) \quad \text{for} \quad y = (x,z) \quad \text{with} \quad u_x(0,0) = 0$$

where $u_x$ denotes $\dfrac{\partial u}{\partial x}$ and the normal at $(0,0)$ is $(0,1)$ in the $(x,y)$ coordinates. We set $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$, $u_{xt} = \dfrac{\partial^2 u}{\partial x \partial t}$, and $u_{xxt} = \dfrac{\partial^3 u}{\partial^2 x \partial t}$. In these coordinates, the angle $n$ and the curvature $K$ are given by Lemma 5.3.

We recall that locally in a neighborhood of $(0,0)$, the function $u$ satisfies

$$u_t = c(t,x,u,n)\ \sqrt{1+u_x^2}\ . \tag{5.36}$$

For $t$ in a neighborhood of zero, let us define the curve $\gamma$ contained in $\Gamma$ by:

$$\gamma(t) = (t, 0, u(t,0))\ .$$

Then

$$\begin{aligned} D_t\tilde{n}(0,0,0) &= \frac{d}{dt}(\tilde{n}\circ\gamma)(0) = \frac{d}{dt}\left(n(t,0,u(t,0))\right)_{|t=0} \\ &= \frac{d}{dt}\left(\frac{1}{\sqrt{1+u_x^2(t,0)}}(-u_x(t,0),1)\right)_{|t=0} = (-u_{xt}(0,0),0)\ . \end{aligned}$$

Similarly

$$\begin{aligned} D_t\tilde{K}(0,0,0) &= \frac{d}{dt}(\tilde{K}\circ\gamma)(0) = \frac{d}{dt}\left(F(u_{xx}(t,0),u_x(t,0)\right)_{|t=0} \\ &= F'_M(\cdot, u_x(0,0))u_{xxt}(0,0) + F'_p(u_{xx}(0,0),u_x(0,0))u_{xt}(0,0) \\ &= \mathcal{I}(u_{xxt}) + (u_{xx}\cdot u_{xt},0)\otimes(0,1) + (0,1)\otimes(u_{xx}\cdot u_{xt},0) \end{aligned}$$

where all the quantities are evaluated at the origin $(0,0)$.
Derivating (3.22) with respect to $x$, we get

$$u_{xt} = (\partial_x c + u_x\partial_{y_N}c + \partial_n c\cdot n_x)\ \sqrt{1+u_x^2} + c\frac{u_{xx}\cdot u_x}{\sqrt{1+u_x^2}} \tag{5.37}$$

where

$$n_x = -\frac{1}{(1+u_x^2)^{\frac{3}{2}}}(-u_x,1)\otimes u_{xx}\cdot u_x + \frac{1}{\sqrt{1+u_x^2}}(-u_{xx},0)$$

which implies that at the origin we have

$$D_t\tilde{n}(0,0,0) = (-u_{xt}(0,0),0) = -(I-n\otimes n)\cdot\partial_y c + K\cdot\partial_n c = D_\tau c$$

and

$$(I-n\otimes n)\cdot\partial_y\tilde{n} = -K\ .$$

Derivating now (5.37) with respect to $x$, we get at the origin (with contraction in $n$ in expressions like $n_x \cdot \partial^2_{nx} c$ or $n_x \cdot \partial^2_{nn} c \cdot n_x$)

$$u_{xxt}(0,0) \;=\; \partial^2_{xx} c + \partial^2_{xn} c \cdot n_x + u_{xx} \partial_{y_N} c + n_x \cdot \partial^2_{nx} c + n_x \cdot \partial^2_{nn} c \cdot n_x + \partial_n c \cdot n_{xx} + c u_{xx} \cdot u_{xx}$$

and then at the origin

$$\mathcal{I}(u_{xxt}) \;=\; A + \mathcal{I}\left(\partial_n c \cdot n_{xx}\right)$$

with

$$A \;=\; (I - n \otimes n) \cdot \left\{\partial^2_{yy} c - \partial^2_{yn} c \cdot K + K(n \cdot \partial_y c) - K \cdot \partial^2_{ny} c + K \cdot \partial^2_{nn} c \cdot K + cK \cdot K\right\} \cdot (I - n \otimes n) \; .$$

Hence

$$D_t \tilde{K}(0,0,0) = B + C$$

with

$$B = A + K \cdot D_\tau c \otimes n + n \otimes K \cdot D_\tau c + K^2 \cdot \partial_n c \otimes n + n \otimes K^2 \cdot \partial_n c = a_K$$

where $a_K$ given in (4.30). and

$$C = \mathcal{I}\left(\partial_n c \cdot n_{xx}\right) - K^2 \cdot \partial_n c \otimes n - n \otimes K^2 \cdot \partial_n c \; .$$

On the one hand, let us compute at the origin

$$n_{xx} = -\left\{(0,1) \otimes u_{xx} \cdot u_{xx} + (u_{xxx}, 0)\right\}$$

and then at the origin (because $\partial_n c$ is orthogonal to $n$)

$$\partial_n c \cdot n_{xx} = -u_{xxx} \cdot \partial_n c \; .$$

On the other hand, let us compute at the origin

$$\begin{aligned} \left(F(u_{xx}, u_x)\right)_x \;&=\; F'_M(\cdot, u_x) u_{xxx} + F'_p(u_{xx}, u_x) u_{xx} \\ &=\; \mathcal{I}(u_{xxx}) + (u_{xx} \cdot u_{xx}, 0) \otimes (0,1) + (0,1) \otimes (u_{xx} \cdot u_{xx}, 0) \end{aligned}$$

and then

$$\begin{aligned} \partial_n c \cdot \left(F(u_{xx}, u_x)\right)_x \;&=\; \mathcal{I}(u_{xxx} \cdot \partial_n c) + K^2 \partial_n c \otimes n + n \otimes K^2 \cdot \partial_n c \\ &=\; \mathcal{I}(-\partial_n c \cdot n_{xx}) + K^2 \partial_n c \otimes n + n \otimes K^2 \cdot \partial_n c \\ &=\; -C \; . \end{aligned}$$

This shows that

$$D_t \tilde{K}(0,0,0) = a_K - \partial_n c \cdot \partial_y \tilde{K} \; .$$

This ends the proof of the Lemma.

**Lemma 5.5** *Let $\psi \in C^1(\mathbb{R}^N)$, $V \in (C^1(\mathbb{R}^N))^N$. Then we have*

$$-\int_{\Gamma_t} V \cdot (I - n \otimes n) \cdot \partial_y \psi = \int_{\Gamma_t} \psi \;\; \{(I - n \otimes n) \cdot \partial_y\} \cdot V + \int_{\Gamma_t} \psi \;(n \cdot V)\; (I : K) \; .$$

**Proof of Lemma 5.5**
We introduce for $\varepsilon > 0$

$$\Omega_\varepsilon = \Big\{x \in \mathbb{R}^N, \quad \exists y \in \Gamma_t, \quad x = y + rn(t,y) \quad \text{for some} \quad r \in (-\varepsilon, \varepsilon)\Big\} \ .$$

For $\varepsilon > 0$ small enough, and $x \in \Omega_\varepsilon$, there exists a unique $y = y(x)$ such that $y \in \Gamma_t$ and $x = y + rn(t,y)$. Then we can then extend the field $n$ on $\Omega_\varepsilon$ by

$$\tilde{n}(x) = n(t, y(x))$$

Then we have

$$\int_{\Gamma_t} V \cdot (I - n \otimes n) \cdot \partial_y \psi = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_{\Omega_\varepsilon} V \cdot (I - \tilde{n} \otimes \tilde{n}) \cdot \partial_y \psi$$

and

$$-\int_{\Omega_\varepsilon} V \cdot (I - \tilde{n} \otimes \tilde{n}) \cdot \partial_y \psi = \int_{\Omega_\varepsilon} \psi \ \{(I - \tilde{n} \otimes \tilde{n}) \cdot \partial_y\} \cdot V - \int_{\Omega_\varepsilon} \psi \ V \cdot \{(\partial_y \cdot \tilde{n}) \ \tilde{n} + (\tilde{n} \cdot \partial_y) \ \tilde{n}\} \ .$$

Hence

$$-\int_{\Gamma_t} V \cdot (I - n \otimes n) \cdot \partial_y \psi = \int_{\Gamma_t} \psi \ \{(I - n \otimes n) \cdot \partial_y\} \cdot V - \int_{\Gamma_t} \psi \ V \cdot \{(\partial_y \cdot \tilde{n}) \ \tilde{n} + (\tilde{n} \cdot \partial_y) \ \tilde{n}\} \ .$$

For our choice of the extension $\tilde{n}$, we have $(\tilde{n} \cdot \partial_y) \ \tilde{n} = 0$ and in the coordinates $(\tilde{n}, \tilde{n}^\perp)$, we see from (5.35) that $\partial_y \cdot \tilde{n} = -I : K$, which implies the result. This ends the proof of the Lemma.

**Lemma 5.6** *With the notation of Lemma 5.2, we have*

$$\int_{\Gamma_t} \partial_n c \cdot \partial_y \tilde{K} : \partial_K \varphi = \int_{\Gamma_t} D_\tau \cdot (\varphi \ \partial_n c) \quad \textit{with} \quad D_\tau = -(I - n \otimes n) \cdot \partial_y + K \cdot \partial_n$$

*where the quantities in the integrals are evaluated at* $(t,y)$, $n(t,y)$, $K(t,y)$.

**Proof of Lemma 5.6**
For a general smooth function $G$, we define $\tilde{G}(t,y) := G(t, y, \tilde{n}(t,y), \tilde{K}(t,y))$, and we have on $\Gamma$

$$(I - n \otimes n) \cdot \partial_y \tilde{G}(t,y) = \Big\{(I - n \otimes n) \cdot \partial_y - K \cdot \partial_n + (I - n \otimes n) \cdot \partial_y \tilde{K} : \partial_K\Big\} \, G \tag{5.38}$$

where we have in particular used (5.35). We deduce that

$$\begin{aligned}
\int_{\Gamma_t} \partial_n c \cdot \partial_y \tilde{K} : \partial_K \varphi &= \int_{\Gamma_t} \partial_n c \cdot \Big\{(I - n \otimes n) \cdot \partial_y \tilde{K} : \partial_K \varphi - K \cdot \partial_n \varphi + (I - n \otimes n) \cdot \partial_y \varphi + D_\tau \varphi\Big\} \\
&= \int_{\Gamma_t} \partial_n c \cdot (I - n \otimes n) \cdot \partial_y \tilde{\varphi} + \int_{\Gamma_t} \partial_n c \cdot D_\tau \varphi \\
&= \int_{\Gamma_t} \Big\{-\varphi \ \left((I - n \otimes n) \cdot \partial_y\right) \cdot \widetilde{\partial_n c} + \partial_n c \cdot D_\tau \varphi\Big\} \\
&= \int_{\Gamma_t} \{\varphi \ D_\tau \cdot \partial_n c + \partial_n c \cdot D_\tau \varphi\} \\
&= \int_{\Gamma_t} D_\tau \cdot (\varphi \partial_n c)
\end{aligned}$$

where we have used (5.38) with $G = \varphi$ in the second line, Lemma 5.5 at the third line, and (5.38) with $G = \partial_n c$ in the fourth line. This ends the proof of the Lemma.

**Lemma 5.7** *For a general $g$ solution of (4.28), we have for $\alpha := K \cdot n\ g$*

$$\alpha_t + \partial_y \cdot (a_y \otimes \alpha) + \partial_n \cdot (a_n \otimes \alpha) + \partial_K : (a_K \otimes \alpha) + A_0 \cdot \alpha = 0 \tag{5.39}$$

*with $A_0$ given in (4.32).*

**Proof of Lemma 5.7**
We compute

$$\begin{aligned}
-\left(K \cdot n\ g\right)_t &= -K \cdot n\ g_t \\
&= K \cdot n\ \{a_0 g + \partial_y \cdot (a_y g) + \partial_n \cdot (a_n g) + \partial_K : (a_K g)\} \\
&= a_0\ K \cdot n\ g + \partial_y \cdot (a_y \otimes K \cdot n\ g) + \partial_n \cdot (a_n \otimes K \cdot n\ g) + \partial_K : (a_K \otimes K \cdot n\ g) \\
&\quad -(a_n g \cdot \partial_n)(K \cdot n) - (a_K g : \partial_K)(K \cdot n)
\end{aligned}$$

i.e. $\alpha := K \cdot n\ g$ satisfies

$$\begin{aligned}
&\alpha_t + a_0 \alpha + \partial_y \cdot (a_y \otimes \alpha) + \partial_n \cdot (a_n \otimes \alpha) + \partial_K : (a_K \otimes \alpha) \\
&= (a_n g \cdot \partial_n)(K \cdot n) + (a_K g : \partial_K)(K \cdot n) \\
&= K \cdot a_n\ g + a_K \cdot n\ g \\
&= -(A_0 - a_0 I) \cdot \alpha
\end{aligned}$$

with

$$-(A_0 - a_0 I) \quad = \quad cK + K \cdot \partial^2_{nn} c - (I - n \otimes n) \cdot \partial^2_{yn} c + n \otimes \partial_y c + 2\,(n \cdot \partial_y c)\, I$$

This ends the proof of the Lemma.

**Proof of Theorem 4.1**
The proof is completely similar to the one of Theorem 2.1.
Putting together the results of Lemmata 5.2, 5.4 and 5.6, we get for $g = g_\Gamma$

$$\begin{aligned}
< g_t, \varphi > &= < g, -c(I : K)\varphi + cn \cdot \partial_y \varphi + D_\tau c \cdot \partial_n \varphi + a_K : \partial_K \varphi - \partial_n c \cdot \partial_y \tilde{K} : \partial_K \varphi > \\
&= < g, -c(I : K)\varphi + cn \cdot \partial_y \varphi + D_\tau c \cdot \partial_n \varphi + a_K : \partial_K \varphi - \partial_n c \cdot D_\tau \varphi - \varphi D_\tau \cdot \partial_n c > \\
&= < g, -a_0 \varphi + a_y \partial_y \varphi + a_n \partial_n \varphi + a_K \partial_K \varphi >
\end{aligned}$$

which leads to equation (4.28).
Equation (5.39) is exactly (4.31).

Finally, let us compute still for $g_\Gamma$ defined in (4.26) and $\overline{\psi}(t,y,n,K)=\psi(t,y,n)$

$$
\begin{aligned}
<\hat{g_\Gamma},\psi> &= <D_\tau g + n(I:K)g,\overline{\psi}> \\
&= <g, n(I:K)\overline{\psi} - D_\tau\overline{\psi}> \\
&= <g, n(I:K)\overline{\psi} + (I-n\otimes n)\cdot\partial_y\overline{\psi} - K\cdot\partial_n\overline{\psi} + (I-n\otimes n)\cdot\partial_y\tilde{K}:\partial_K\overline{\psi}> \\
&= \int_0^T dt \int_{\Gamma_t}\Big\{n(I:K)\tilde{\psi}(t,y) + (I-n\otimes n)\cdot\partial_y\tilde{\psi}(t,y)\Big\} \\
&= 0
\end{aligned}
$$

where in the third line, we have used the fact that $\overline{\psi}$ is indenpendent on the curvature $K$, we have used in the fourth line equation (5.38) with $G=\overline{\psi}$ and $\tilde{\psi}(t,y)=\overline{\psi}(t,y,n(t,y),K(t,y))$. For the last line, we have used Lemma 5.5 with $V=e_i$, $i=1,...,N$, succesively. This proves that $\hat{g_\Gamma}=0$.
This ends the proof of the Theorem.

**Remark 5.8** *We remark that if $K\cdot ng=0$, then $(I-n\otimes n)\cdot K\cdot\partial_n g = K\cdot\partial_n g + (I:K)\,ng$. This last relation explains why the contribution of the term in $n(I:K)$ to the definition of $\hat{g_\Gamma}$ does not vanish in general, even if $\hat{g_\Gamma}=0$.*

# 6 Appendix

## 6.1 Proof of Lemma 5.1

**Proof of Lemma 5.1**
We consider a point $(t_0,y_0)\in\Gamma$, and a local parametrization $\gamma: B_{r_0}\longrightarrow\mathbb{R}^N$ of $\Gamma_{t_0}$ in a neighborhood of $y_0$, with $B_{r_0}=B_{r_0}(0)\subset\mathbb{R}^{N-1}$ and $r_0>0$. Then, in a neighborhood $U_{t_0,y_0}$, we can parametrize $\Gamma_{t+h}$ for $h$ small, by $\gamma^{t_0+h}$ defined by

$$\gamma^{t_0+h}(x)=\gamma(x)+r(h,x)n(x)$$

where $n(x)$ is the normal $n$ to $\gamma$ at the point $\gamma(x)$.
We will prove that formula (5.33) holds, assuming moreover that

$$\text{supp }\psi\subset U_{t_0,y_0}\ . \tag{6.40}$$

Finally, using a partition of unity, we recover the full formula (5.33) as in the Lemma.
We now prove (5.33) assuming (6.40).
We have $r(0,x)=0$ and $r_h(0,x)=c(t,\gamma(x))$ (where $r_h$ stands for $\dfrac{\partial r}{\partial h}$, $r_i$ stands for $\dfrac{\partial r}{\partial x_i}$, and $r_{ih}$ stands for $\dfrac{\partial^2 r}{\partial x_i\partial h}$). We compute the jacobian

$$J_h=|a_h|\quad\text{with}\quad <a_h,\cdot>=\det\,\left(\gamma_1^{t_0+h},...,\gamma_{N-1}^{t_0+h},\cdot\right)$$

where we denote

$$(6.41) \qquad \gamma_i^{t_0+h} = \frac{\partial \gamma}{\partial x_i}^{t_0+h} = \gamma_i + r_i n + r \frac{\partial n}{\partial x_i}$$

and similarly $\gamma_i = \dfrac{\partial \gamma}{\partial x_i}$ for $i = 1, ..., N-1$.

$$\begin{aligned} \int_{\Gamma_{t_0+h}} \psi(t_0+h,\cdot) &= \int_{B_{r_0}} dx\ \psi(t_0+h, \gamma^{t_0+h}(x)) J_h \\ &= \int_{B_{r_0}} dx\ \psi(t+h, \gamma(x) + r(h,x)n) J_h\ . \end{aligned}$$

Therefore

$$(6.42) \qquad \frac{d}{dh}\left(\int_{\Gamma_{t_0+h}} \psi(t+h,\cdot)\right)_{|h=0} = \int_{B_{r_0}} dx\ \left\{\left(\psi_t + (r_h(0,x))\ n\cdot\partial_y\psi\right) J_0 + \psi\ J_0'\right\}$$

with

$$J_0' = \left(\frac{dJ_h}{dh}\right)_{|h=0} = \frac{a_0\cdot a_0'}{|a_0|}$$

with $a_0' = \left(\dfrac{da_h}{dh}\right)_{|h=0}$. On the other hand, we have

$$|a_h|^2 = \det\ \left(\gamma_1^{t_0+h}, ..., \gamma_{N-1}^{t_0+h}, a_h\right)$$

which by derivation gives

$$2a_0\cdot a_0' = \det\ (\gamma_1, ..., \gamma_{N-1}, a_0') + \sum_{i=1}^{N-1} \det\ \left(\gamma_1, ..., \gamma_i^{t_0\prime}, ..., \gamma_{N-1}, a_0\right)$$

with (from (6.40))

$$\gamma_i^{t_0\prime} = \left(\frac{\partial \gamma_i}{\partial h}^{t_0+h}\right)_{|h=0} = r_{ih}(0,x)n + r_h(0,x)\frac{\partial n}{\partial x_i}\ .$$

Therefore

$$a_0\cdot a_0' = \sum_{i=1}^{N-1} \det\ \left(\gamma_1, ..., r_{ih}n + r_h\frac{\partial n}{\partial x_i}, ..., \gamma_{N-1}, a_0\right)\ .$$

Using in particular the fact that $a_0$ is parallel to $n$, we deduce that

$$a_0\cdot a_0' = r_h(0,x)|a_0|^2 k \quad \text{with} \quad k := \sum_{i=1}^{N-1} \frac{< \gamma_i, \frac{\partial n}{\partial x_i} >}{|\gamma_i|^2}\ .$$

A direct computation in local coordinates allows to see that $k = -I : K$, which gives

$$J_0' = -c(I:K)J_0\ .$$

Finally (6.42) implies

$$\frac{d}{dh}\left(\int_{\Gamma_{t_0+h}} \psi(t+h,\cdot)\right)_{|h=0} = \int_{B_{r_0}} dx\ J_0\left\{\psi_t + c\ n\cdot\partial_y\psi - c(I:K)\psi\right\}$$

$$= \int_{\Gamma_{t_0}} \left\{\psi_t + c\ n\cdot\partial_y\psi - c(I:K)\psi\right\}\ .$$

This ends the proof of the Lemma.

## 6.2 Case of a curve transported in dimension $\overline{N}$

In this Subsection, we consider a curve $\overline{\Gamma}_t$ transported in $\mathbb{R}^{\overline{N}}$ along a vector field $\overline{c}(t,\overline{y}) \in \mathbb{R}^{\overline{N}}$, in dimension $\overline{N} \geq 2$. Here we use the notation $\overline{N}$ for the dimension rather than $N$, because $\overline{N}$ can be different from $N$, as in the application given in Subsection 6.3. We also emphase the fact that we do not consider the case of "normal velocity", possibly depending on the unit tangent vector to the curve $\overline{\Gamma}_t$, that we call $\overline{\tau} \in \mathbb{S}^{\overline{N}-1}$.
We call $\overline{\kappa} \in \mathbb{R}^{\overline{N}}$ the curvature vector $\overline{\kappa} = \dfrac{d\overline{\tau}}{ds}$ of the curve and set

$$g_{\overline{\Gamma}}(t,\overline{y},\overline{\tau},\overline{\kappa}) = \delta_{\overline{\Gamma}_t}(\overline{y})\delta_0(\overline{\tau}-\overline{\tau}(t,\overline{y}))\delta_0(\overline{\kappa}-\overline{\kappa}(t,\overline{y}))\ . \tag{6.43}$$

Then we have:

**Proposition 6.1 (Linear transport equation in the phase space for geometric transport of a curve in $\mathbb{R}^{\overline{N}}$)**
*Assume that a smooth curve $\overline{\Gamma}_t$ is transported in $\mathbb{R}^{\overline{N}}$ along a smooth vector field $\overline{c}(t,\overline{y}) \in \mathbb{R}^{\overline{N}}$. Then the distribution $g = g_{\overline{\Gamma}}$ defined in (6.43) satisfies*

$$g_t + div\ (\overline{a}g) + \overline{a}_0 g = 0 \tag{6.44}$$

*with*

$$\left\{\begin{array}{l} \overline{a}_0 = \overline{c}\cdot\overline{\kappa}, \quad \overline{a}_{\overline{y}} = (I-\overline{\tau}\otimes\overline{\tau})\cdot\overline{c}, \quad \overline{a}_{\overline{\tau}} = (I-\overline{\tau}\otimes\overline{\tau})\cdot(\overline{\tau}\cdot\partial_{\overline{y}}\overline{c}) \\ \\ \overline{a}_{\overline{\kappa}} = (I-\overline{\tau}\otimes\overline{\tau})\cdot\left\{\overline{\kappa}\cdot\partial_{\overline{y}}\overline{c} + \overline{\tau}\otimes\overline{\tau} : \partial^2_{\overline{y}\overline{y}}\overline{c}\right\} - (\overline{\kappa}\otimes\overline{\tau}+\overline{\tau}\otimes\overline{\kappa})\cdot(\overline{\tau}\cdot\partial_{\overline{y}}\overline{c})\ . \end{array}\right.$$

The proof of Proposition 6.1 uses the following result (analogous to Lemma 5.1)

**Lemma 6.2** *For a smooth curve $\overline{\Gamma}_t$ transported in $\mathbb{R}^{\overline{N}}$ along a smooth vector field $\overline{c}(t,\overline{y},\overline{\tau}) \in \mathbb{R}^{\overline{N}}$ (which is not assumed perpendicular to $\overline{\tau}$), we have for any smooth function $\psi$*

$$\frac{d}{dt}\left(\int_{\overline{\Gamma}_t}\psi\right) = \int_{\overline{\Gamma}_t} D_t\psi - \overline{c}\cdot\overline{\kappa}\psi \quad \textit{with} \quad D_t\psi = \psi_t + \overline{c}\cdot(I-\overline{\tau}\otimes\overline{\tau})\cdot\partial_{\overline{y}}\psi\ .$$

**Sketch of the proof of Proposition 6.1**
To get equation (6.44), we simply follows the lines of our approach for hypersurfaces. For a general vector field $\overline{c}(t,\overline{y},\overline{\tau})$ (which here depends on $\overline{\tau}$), we get

$$< g_t,\varphi > = < g, -\overline{c}\cdot\overline{\kappa}\varphi + \overline{c}\cdot(I-\overline{\tau}\otimes\overline{\tau})\cdot\partial_{\overline{y}}\varphi + D_t\tilde{\overline{\tau}}\cdot\partial_{\overline{\tau}}\varphi + D_t\tilde{\overline{\kappa}}\cdot\partial_{\overline{\kappa}}\varphi >$$

with

$$
(6.45) \qquad \left\{ \begin{array}{l}
D_t \tilde{\overline{\tau}} = D_\tau \overline{c} \quad \text{with} \quad D_\tau = (I - \overline{\tau} \otimes \overline{\tau}) \cdot (\overline{\tau} \cdot \partial_{\overline{y}} + \overline{\kappa} \cdot \partial_{\overline{\tau}}) \\[2ex]
D_t \tilde{\overline{\kappa}} = a_{\overline{\kappa}} + (I - \overline{\tau} \otimes \overline{\tau}) \cdot \left( \dfrac{d\overline{\kappa}}{ds} \cdot \partial_{\overline{\tau}} \overline{c} \right) \\[2ex]
\begin{array}{lcl} a_{\overline{\kappa}} & = & (I - \overline{\tau} \otimes \overline{\tau}) \cdot \left\{ \overline{\kappa} \cdot \partial_{\overline{y}} \overline{c} + \overline{\tau} \otimes \overline{\tau} : \partial^2_{\overline{y}\overline{y}} \overline{c} + 2\overline{\tau} \cdot \partial^2_{\overline{y}\overline{\tau}} \overline{c} \cdot \overline{\kappa} + \overline{\kappa} \otimes \overline{\kappa} : \partial^2_{\overline{\tau}\overline{\tau}} \overline{c} \right\} \\ & & -(\overline{\kappa} \otimes \overline{\tau} + \overline{\tau} \otimes \overline{\kappa}) \cdot (\overline{\tau} \cdot \partial_{\overline{y}} \overline{c} + \overline{\kappa} \cdot \partial_{\overline{\tau}} \overline{c}) \ . \end{array}
\end{array} \right.
$$

Indeed, we see that in the expression of $< g_t, \varphi >$, there is a term

$$
(I - \overline{\tau} \otimes \overline{\tau}) \cdot \left( \frac{d\overline{\kappa}}{ds} \cdot \partial_{\overline{\tau}} \overline{c} \right) \cdot \partial_{\overline{\kappa}} \varphi
$$

where $\partial_{\overline{\tau}} \overline{c}$ is a matrix. Because of this new term, we can not use the trick applied in the case of hypersurfaces, where we rewrote this term, essentially as $\partial_{\overline{\tau}} \overline{c}$ times the term $\dfrac{d\overline{\kappa}}{ds} \cdot \partial_{\overline{\kappa}} \varphi$, and then conclude by an integration by parts.
Nevertheless, in the case where the vector field $\overline{c}$ is independent on $\overline{\tau}$, we easily get that $g$ solves equation (6.44).

## 6.3 Application to another transport formulation of a curve moving in the plane

Let us remark that the formalism of Subsection 6.2 can be applied to the particular case of a curve $\Gamma_t$ moving in the plane with a normal velocity $c(t, y, \theta)$, to which we associate the corresponding curve in the space/angle coordinates
(6.46)

$$
\overline{\Gamma}_t = \left\{ \begin{array}{l} \overline{y} = (y, \theta(t,y)) \in \mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z})) \\[2ex] \text{for} \quad y \in \Gamma_t \quad \text{with unit tangent vector} \quad \tau(t,y) = (\sin\theta(t,y), -\cos\theta(t,y)) \end{array} \right\} .
$$

The curve $\overline{\Gamma}_t$ moves with velocity

$$
(6.47) \qquad \overline{c}(t, \overline{y}) = a'(t, y, \theta) \quad \text{with} \quad \overline{y} = (y, \theta) \in \mathbb{R}^2 \times (\mathbb{R}/(2\pi\mathbb{Z}))
$$

where $a' = (a_y, a_\theta)$ is defined in Theorem 2.1 ($N = 2$). Here this velocity does not depend on $\overline{\tau}$, the unit tangent vector to $\overline{\Gamma}_t$, which is given by

$$
\overline{\tau}(t, \overline{y}) = \frac{1}{\sqrt{1 + \kappa^2}} (\tau, \kappa)
$$

where $\kappa(t, y)$ is the (scalar) curvature of the original curve $\Gamma_t$.

Applying now Proposition 6.1 with $\overline{N} = 3$, we get the following result

**Corollary 6.3 (A transport equation associated to a curve, with at most linear growth in its coefficients)**
*For the curve $\overline{\Gamma}_t$ defined in (6.46), the distribution $g_{\overline{\Gamma}}$ satisfies equation (6.44) with coefficients $\overline{a}, \overline{a}_0$ given in (6.45) where $\overline{c}$ is defined in (6.47) with $\overline{N} = 3$. Moreover the vector field $\overline{a}$ has at most a linear growth in the curvature vector $\overline{\kappa}$ defined in Subsection 6.2.*

The advantage here to consider $\overline{\Gamma}_t$ in place of $\Gamma_t$, is that the curve $\overline{\Gamma}_t$ always stays regular for all time (this is the wavefront, see for instance Osher et al. [24] for the wavefront associated to curves and Leung, Qian, Osher [20] for the wavefront associated to surfaces), while $\Gamma_t$ can become singular in finite time. At the level of the transport equation (6.44) satisfied by $g_{\overline{\Gamma}}$ in Corollary 6.3, the nice property is that the vector field $\overline{a}$ has at most a linear growth in $\overline{\kappa}$. The consequence is the existence of solutions for all time for the solutions of (6.44) in that case.

**Remark 6.4** *From the previous point of view, it is natural to ask if there are some relations between first order evolution of curves in $\mathbb{R}^3$, and second order evolution of curves in the plane.*

**Remark 6.5** *Similarly, if could be interesting for hypersurfaces $\Gamma_t$ to consider*

$$\overline{\Gamma}_t = \left\{ \begin{array}{l} \overline{y} = (y, n(t,y)) \in \mathbb{R}^N \times \mathbb{S}^{N-1} \\ \\ \textit{for} \quad y \in \Gamma_t \quad \textit{with unit normal vector} \quad n(t,y) \end{array} \right\}$$

*and to try to write a transport equation for a measure*

$$g_{\overline{\Gamma}}(t, \overline{y}, \overline{n}, \overline{K}) = \delta_{\overline{\Gamma}_t}(\overline{y})\delta_0(\overline{n} - \overline{n}(t,\overline{y}))\delta_0(\overline{K} - \overline{K}(t,\overline{y}))$$

*where $\overline{n}(t,\overline{y})$ defines the tangent space (of dimension $N-1$) to $\overline{\Gamma}_t$ at $\overline{y}$, and $\overline{K}(t,\overline{y})$ defines its curvature.*
*It would be also interesting to see how to extend this method for the evolution of general submanifolds of arbitrary codimensions.*

**Aknowledgements**
I would like to thank Radan Sedláček for enlighting discussions which were motivating this work and T. Hochrainer for mentionning Remark 2.3. We also thank unknown referees for their suggestions improving the presentation of the manuscipt.

# References

[1] O. Alvarez, P. Cardaliaguet, R. Monneau, *Existence and uniqueness for dislocation dynamics with nonnegative velocity*, Interfaces Free Bound. **7** (4) (2005), 415-434.

[2] O. Alvarez, P. Hoch, Y. Le Bouar, R. Monneau, *Dislocation dynamics: short time existence and uniqueness of the solution*, Archive Rational Mech. Anal. **181** (3), (2006).

[3] R. Buckdahn, P. Cardaliaguet, M. Quincampoix, *A representation Formula for the Mean Curvature Motion*, SIAM J. Math. Anal. **33** (2001), 827-.

[4] P. Cannarsa, P. Cardaliaguet, *Perimeter estimates for the reachable set of control problems*, J. Convex. Anal. (to appear).

[5] P. Cannarsa, H. Fankowska, *Interior sphere property of attainable sets and the time optimal control problems*, ESAIM: Control, Optimisation and calculus of Variations **12** (2006), 350-370.

[6] A. El-Azab, *The boundary value problem of dislocation dynamics*, Modelling Simul. Mater. Sci. Eng. **8** (2000), 37-54.

[7] A. El-Azab, *Statistical mechanics treatment of the evolution of dislocation distributions in single crystals*, Phys. Rev. B, Vol. **61**, no. 18 (2000), 11956-11966.

[8] A. El Hajj, N. Forcadel, *A convergent scheme for a non-local coupled system modelling dislocations densities dynamics*, preprint.

[9] L.C. Evans, *Partial Differential equations*, Graduate studies in Mathematics, Vol. 19, American Mathematical Society, providence, Rhode Island, (1998).

[10] I. Groma, P. Balogh, *Link between the individual and continuum approaches of the description of the collective behaviour of dislocations*, Mat. Sci. Eng. A **234-236** (1997), 249-252.

[11] I. Groma, P. Balogh, *Investigation of dislocations pattern formation in a two-dimensional self-consistent field approximation*, Acta mater. **47** (1999), no. 13, 3647-3654.

[12] T. Hochrainer, *A continuum theory of dislocation motion*, In N.M. Ghoniem, editor, Proc. of 2nd Int. Conf. on Multiscale Materials Modeling, Oct. 11-15, 2004, pages 210-212, Los Angeles, US. Univ. of California, Los Angeles.

[13] T. Hochrainer, M. Zaiser, *Fundamental of a continuum theory of dislocations*, pp. 1-12, Proceedings of Science, (SMPRI 2005) International conference on Statistical Mechanics of Plasticity and related Instabilities, August 29-September 2, 2005, Indian Institute of Science, Bangalore, India; Editorial board: M. Zaiser, S. Dattagupta, A. El-Azab, S.B. Krupanidhi, S. Noronha, S.A. Shivashankar, to appear.

[14] T. Hochrainer, M. Zaiser, P. Gumbsch, *A three-dimensional continuum theory of dislocation systems: kinematics and mean-field formulation*, Philosophical Magazine **87** (8 & 9), (2007), 1261-1282.

[15] C. Imbert, R. Monneau, *Homogenization of first order equations with $(u/\varepsilon)$-periodic Hamiltonians. Part II: application to dislocations dynamics*, preprint.

[16] A.M. Kosevich, *Dynamical theory of dislocations*, Soviet physics Uspekhi **7** (1965), 837-854.

[17] E. Kröner, *Kontinuumstheorie der Versetzungen und Eigenspannungen*, Erg. Angew. Math. **5**, 1-179, Berlin: Springer, (1958).

[18] E. Kröner, *Benefit and shortcomings of the continuous theory of dislocations*, Int. J. Solids and Structures **38** (2001), 1115-1134.

[19] R.W. Lardner, *Mathematical theory of dislocations and fracture*, Mathematical Expositions **17**, University of Toronto Press, (1974).

[20] S. Leung, J. Qian, S. Osher, *A level set method for three-dimensional paraxial geometrical optics with multiple point sources*, Comm. Math. Sci. **2** (2004), no. 4, 643-672.

[21] R.J. LeVeque, *Numerical methods for conservation laws*, Lectures in Mathematics ETH Zurich, Birkhäuser Verlag, Basel, (1990).

[22] P.-L. Lions, *Generalized solutions of Hamilton-Jacobi equations*, Research Notes in Mathematics **69**. Pitman (Advanced Publishing Program), Boston, Mass.-London, (1982).

[23] T. Mura, *Continuous distribution of moving dislocations*, Philosophical Magazine **8** (1963), 843-857.

[24] S. Osher, L.-T. Cheng, M. Kang, H. Shim, Y.-H. Tsai, *Geometric Optics in a Phase-Space-Based Level Set and Eulerian Framework*, J. Comp. Physics **179** (2002), 622-648.

[25] V.I. Oseledets, *On a new way of writting the Navier-Stokes equation : the Hamiltonian formalism*, Comm. Moscow Math. Soc. (1988); translated in Russ. Math. Surveys **44** (1989), 210-211.

[26] B. Perthame, *Kinetic formulation of conservation laws*, Oxford Univ. Press, (2002).

[27] R. Sedláček, J. Kratochvil, E. Werner, *The importance of being curved: bowing dislocations in a continuum description*, Philosophical Magazine **83** (2003), no. 31-34, 3735-3752.

[28] H.M. Soner, N. Touzi, *Dynamics programming for stochastic target problems and geometric flows*, J. Eur. Math. Soc. **4** (2002), 201-236.

[29] H.M. Soner, N. Touzi, *Level set characterization of stochastic target problems*, Comm. Partial Differential Equations **27** (2002), 2031-2053.

[30] H.M. Soner, N. Touzi, *A stochastic representation for mean curvature type geometric flows*, The Annals of probability **31** (2003), no. 3 , 1145-1165.

[31] M. Zaiser, T. Hochrainer, *Some steps towards a continuum representation of 3d dislocation systems*, Scripta Mater. **54** (2005), 717-721.