Global existence of solutions to a singular parabolic/Hamilton-Jacobi coupled system with Dirichlet conditions

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Abstract

We study the existence of (distribution/viscosity) solutions of a singular parabolic/Hamilton-Jacobi coupled system. Our motivation stems from the study of the dynamics of dislocation densities in a crystal of finite size. The method of the proof consists in considering a parabolic regularization of the system, and then passing to the limit after obtaining some uniform bounds using in particular an entropy estimate for the densities. ***To cite this article: A. Names, C. R. Acad. Sci. Paris, Ser. I ••• (••••).***

Résumé

Existence globale de solutions pour un système couplé parabolique/Hamilton-Jacobi singulier avec condition de Dirichlet. Nous étudions l'existence de solutions mixtes (distribution/viscosité) pour un système couplé parabolique/Hamilton-Jacobi posé sur un interval. Notre motivation vient de l'étude de la dynamique de densités de dislocations dans un cristal de taille finie. L'idée de la preuve consiste à considérer une régularisation parabolique appropriée, et ensuite à passer à la limite en utilisant en particulier une estimation entropique pour les densités. ***Pour citer cet article : A. Names, C. R. Acad. Sci. Paris, Ser. I ••• (••••).***

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1. Version française abrégée

Pour tout temps $T > 0$, et l'interval spatial $I = (-1, 1)$, nous étudions le système parabolique/Hamilton-Jacobi suivant :

$$\begin{cases} \kappa_t \kappa_x = \rho_t \rho_x & \text{sur} \quad I_T := I \times (0, T), \\ \rho_t = \rho_{xx} - \tau \kappa_x & \text{sur} \quad I_T, \end{cases}$$

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qui est une version intégrée du modèle de Groma, Csikor et Zaiser [4] décrivant la dynamique de densités de dislocations dans un cristal. Les solutions physiquement acceptables, correspondant à des densités positives de dislocations, sont celles vérifiant

$$\kappa_x \geq |\rho_x| \quad \text{dans} \quad \mathcal{D}'(I_T).$$

Notre résultat principal est :

Théorème 1.1 (Existence globale). *Soit (ρ^0, κ^0) une donnée initiale sur I satisfaisant (4), (5). Alors il existe une fonction (ρ, κ) telle que pour tout $T > 0$, $(\rho, \kappa) \in (C(\overline{I_T}))^2$ avec $\rho \in C(I_T)$, solution de (1), (6), avec les conditions initiales (2), et les conditions de Dirichlet au bord (3).*

L'idée de la preuve du Théorème 1.1 consiste à considérer une régularization parabolique (8) du système (1), en ajoutant une petite viscosité $\varepsilon > 0$. Nous prouvons alors l'existence globale (Théorème 2.3) d'une solution au niveau ε, puis passons à la limite quand ε tend vers zéro.

2. Introduction and main results

Motivated by the study of the elastoviscoplastic properties of crystals, Groma, Csikor and Zaiser [4] have proposed a model describing the dynamics of dislocation densities. Dislocations are defects in a crystal structure that move when submitted to an exterior applied stress. We consider a one dimensional framework where the crystal is modelized by the interval $I := (-1, 1)$, and we consider two types of dislocation defects : the positive and negative ones (according to their Burgers vector, see [5]). Let θ^+ and θ^- represent the density of the positive and negative dislocations respectively. Indeed, we will work with the primitives (up to a constant) :

$$\rho^{\pm}_x = \theta^{\pm}, \quad \rho = \rho^+ - \rho^- \quad \text{and} \quad \kappa = \rho^+ + \rho^-.$$

For a given time $T > 0$, and $\tau \in \mathbb{R}$, a constant applied stress, we consider an integrated form of the model described in [4], namely :

$$\begin{cases} \kappa_t \kappa_x = \rho_t \rho_x & \text{on} \quad I_T := I \times (0, T), \\ \rho_t = \rho_{xx} - \tau \kappa_x & \text{on} \quad I_T, \end{cases} \tag{1}$$

with initial and boundary conditions

$$\rho(x, 0) = \rho^0(x), \quad \kappa(x, 0) = \kappa^0(x), \quad \forall x \in I, \tag{2}$$

$$\rho(\pm 1, t) = 0 \quad \text{and} \quad \kappa(\pm 1, t) = \pm 1, \quad \forall t \in (0, T). \tag{3}$$

The non-negativity of the densities θ^+ and θ^- at the initial time is interpreted in terms of the unknowns ρ^0 and κ^0 by :

$$\kappa^0_x \geq |\rho^0_x| \quad \text{on} \quad I. \tag{4}$$

Denote for $s, r \in \mathbb{N}$, $D^r_t D^s_x u = \frac{\partial^{s+r} u}{\partial t^r \partial x^s}$, and denote by ∂I the boundary of I, $\overline{I_T}$ the closure of I_T, and by $\mathcal{D}'(I_T)$ the space of distributions over I_T. We now introduce the notion of viscosity solution :

Definition 2.1 ***(Viscosity solution).*** *Assume $\rho \in C^1(I_T)$. A function $\kappa \in C(I_T)$ such that $x \mapsto \kappa(x, t)$ is non decreasing, is called a viscosity solution of the first equation of (1) if it satisfies $\forall \phi \in C^1(I_T)$:*
(i) for any local maximum $X_0 = (x_0, t_0) \in I_T$ of $\kappa - \phi$, we have : $\phi_t(X_0)\phi_x(X_0) \leq \rho_t(X_0)\rho_x(X_0)$,
(ii) for any local minimum $X_0 = (x_0, t_0) \in I_T$ of $\kappa - \phi$, we have : $\phi_t(X_0)\phi_x(X_0) \geq \rho_t(X_0)\rho_x(X_0)$.

The main result of this note is the following :

Theorem 2.2 ***(Global existence of a mixed solution).*** *Let $\rho^0, \kappa^0 \in C^\infty(\bar{I})$ satisfying (4) and*

$$D_x\rho^0, D_x\kappa^0 \in C_0^\infty(I). \tag{5}$$

Then there exists (ρ, κ) such that for every $T > 0$, $(\rho, \kappa) \in \left(C(\overline{I_T})\right)^2$ with $\rho \in C^1(I_T)$, is a solution of (1), (2) and (3). Moreover, this solution satisfies :

$$\kappa_x \geq |\rho_x| \quad on \quad \mathcal{D}'(I_T). \tag{6}$$

However, the solution has to be interpreted in the following sense :
(i) κ is a viscosity solution of $\kappa_t\kappa_x = \rho_t\rho_x$ in I_T,
(ii) ρ is a distributional solution of $\rho_t = \rho_{xx} - \tau\kappa_x$ in I_T,
(iii) the initial and the boundary conditions are satisfied pointwisely.

The principal difficulty we have to face is to deal with the first equation of (1), that we can formally rewrite as $\kappa_t = \rho_t\rho_x/\kappa_x$ which is singular as κ_x vanishes. The idea is to pass to the limit as $\varepsilon \to 0$ in the family of solutions $(\rho^\varepsilon, \kappa^\varepsilon)$, $\varepsilon > 0$, of the particular [1] parabolic regularization of (1), namely :

$$\begin{cases} \kappa_t^\varepsilon = \varepsilon\kappa_{xx}^\varepsilon + \dfrac{\rho_x^\varepsilon\rho_{xx}^\varepsilon}{\kappa_x^\varepsilon} - \tau\rho_x^\varepsilon & \text{on} \quad I_T \\ \rho_t^\varepsilon = (1+\varepsilon)\rho_{xx}^\varepsilon - \tau\kappa_x^\varepsilon & \text{on} \quad I_T, \end{cases} \tag{8}$$

with some initial data and the same boundary conditions

$$\rho^\varepsilon(x,0) = \rho^{\varepsilon,0}(x), \quad \kappa^\varepsilon(x,0) = \kappa^{\varepsilon,0}(x), \quad \forall x \in I, \tag{9}$$

$$\rho^\varepsilon(\pm 1, t) = 0 \quad \text{and} \quad \kappa^\varepsilon(\pm 1, t) = \pm 1, \quad \forall t \in (0,T). \tag{10}$$

Concerning system (8), (9) and (10) we have the following global existence and uniqueness result :

Theorem 2.3 ***(Global existence of smooth solutions for the regularized system, [6]).*** *Let $\rho^{\varepsilon,0}, \kappa^{\varepsilon,0} \in C^\infty(\bar{I})$ satisfying the compatibility conditions :*

$$(1+\varepsilon)\rho_{xx}^{\varepsilon,0} = \tau\kappa_x^{\varepsilon,0} \quad and \quad (1+\varepsilon)\kappa_{xx}^{\varepsilon,0} = \tau\rho_x^{\varepsilon,0} \quad on \quad \partial I, \tag{11}$$

and

$$\kappa_x^{\varepsilon,0} > |\rho_x^{\varepsilon,0}| \quad on \quad \bar{I}. \tag{12}$$

Then there exists $(\rho^\varepsilon, \kappa^\varepsilon) \in (C^\infty(\bar{I} \times (0,\infty)))^2$ unique solution of (8), (9) and (10) for $T = \infty$, satisfying for $r, s \in \mathbb{N}$:

$$(D_t^r D_x^s \rho^\varepsilon, D_t^r D_x^s \kappa^\varepsilon) \in (C(\bar{I} \times [0,\infty)))^2, \quad 2r + s \leq 3 \tag{13}$$

and

$$\kappa_x^\varepsilon > |\rho_x^\varepsilon| \quad on \quad \bar{I} \times [0,\infty). \tag{14}$$

The boundary conditions (11) that we have imposed on the initial data of the regularized system are natural here. In fact, assume ρ^ε and κ^ε are sufficiently regular solutions of (8), (9) and (10). From (10), we know that ρ^ε and κ^ε are constants on $\partial I \times (0,T)$ and therefore $\rho_t^\varepsilon = \kappa_t^\varepsilon = 0$ on $\partial I \times (0,T)$ which, using (8) at time $t = 0$, immediately implies (11). The compatibility conditions (11), joint with the Hölder theory for parabolic equations imply the regularity (13).

[1] This comes from the natural parabolic regularization for the system satisfied by $\theta^\pm$, which is :

$$\theta_t^{\pm,\varepsilon} = \varepsilon\theta_{xx}^{\pm,\varepsilon} \pm \left(\left(\frac{\theta_x^{+,\varepsilon} - \theta_x^{-,\varepsilon}}{\theta^{+,\varepsilon} + \theta^{-,\varepsilon}} - \tau\right)\theta^{\pm,\varepsilon}\right)_x \quad \text{with} \quad \theta^{\pm,\varepsilon} = \frac{\kappa_x^\varepsilon \pm \rho_x^\varepsilon}{2}. \tag{7}$$

To do the proof of Theorem 2.2, we will apply Theorem 2.3 with initial conditions $\rho^{\varepsilon,0}$, $\kappa^{\varepsilon,0}$ constructed from ρ^0, κ^0. In fact, condition (5) is a sufficient technical condition to insure (11) and (12) for instance, with the special choice : $\rho^{\varepsilon,0}(x) = \frac{\rho^0(x)+\varepsilon\tau\psi(x)}{(1+\varepsilon)^2}$, $\kappa^{\varepsilon,0}(x) = \frac{\kappa^0(x)+\varepsilon x}{1+\varepsilon}$, with $\psi(x) = \frac{1}{4\tau^2}[1-\cos\tau(x^2-1)]$ when $\tau \neq 0$.

3. Sketch of the proof of Theorem 2.2

We need a framework where system (1) is stable under approximation. Roughly speaking, the C^1 regularity of ρ that appears in Theorem 2.2 is expected since it satisfies a parabolic equation (the second equation of (1)). In this case, considering the first equation of (1), we see that the right hand side $\rho_t\rho_x$ is continuous and hence, assuming $\kappa_x \geq 0$, we can interpret κ as a viscosity solution. This takes us in a natural way to the framework of viscosity solutions where the stability property is well satisfied (see [1, Lemma 2.3]). We want to show (as ε goes to zero) that $(\rho^\varepsilon,\kappa^\varepsilon) \to (\rho,\kappa)$ in $(L^\infty_{loc}(I_T))^2$, $\rho^\varepsilon_x \to \rho_x$ in $L^\infty_{loc}(I_T)$, and $\rho^\varepsilon_t \to \rho_t$ in $\mathcal{D}'(I_T)$ with $\rho_t \in C(I_T)$.

Step 1. (Convergence of κ^ε). Writing down the entropy associated to system (7)

$$S^\varepsilon(t) = \int_I \sum_\pm \theta^{\pm,\varepsilon}(x,t)\log\theta^{\pm,\varepsilon}(x,t)dx,$$

we show that $S^\varepsilon(t) \leq C(T)$ for $t \in [0,T]$, which implies that $\int_I \kappa^\varepsilon_x \log\kappa^\varepsilon_x \leq C_1(T)$, which gives the ε-uniform control of the modulus of continuity of κ^ε with respect to the variable x. On the other hand, remark that κ^ε satisfies $\kappa^\varepsilon_t - \varepsilon\kappa^\varepsilon_{xx} = f^\varepsilon$ where we will show that f^ε is ε-uniformly bounded. Then it is possible to deduce locally the ε-uniform control of the modulus of continuity of κ^ε with respect to the variable t. Indeed, we have $f^\varepsilon = \frac{\rho^\varepsilon_x}{\kappa^\varepsilon_x}A^\varepsilon_x$ with $A^\varepsilon = \rho^\varepsilon_x - \tau\kappa^\varepsilon$ satisfying $A^\varepsilon_t = (1+\varepsilon)A^\varepsilon_{xx} + \frac{\tau\rho^\varepsilon_x}{\kappa^\varepsilon_x}A^\varepsilon_x$. Hence, using interior estimates for parabolic equations (see [7, Proposition 7.1]), we obtain

$$A^\varepsilon \to A \quad \text{and} \quad A^\varepsilon_x \to A_x \quad \text{in} \quad L^\infty_{loc}, \tag{15}$$

and joint to (14), we conclude that f^ε is ε-uniformly bounded. Finally, using the fact that $\|\kappa^\varepsilon\|_{L^\infty(I_T)} \leq 1$, the convergence of κ^ε directly follows by Arzela-Ascoli Theorem.

Step 2. (Convergence of ρ^ε, ρ^ε_x and ρ^ε_t). Using similar arguments as in Step 1, particularly (14), and the fact that $\rho^\varepsilon_t - \varepsilon\rho^\varepsilon_{xx} = A^\varepsilon_x$, we deduce that $\rho^\varepsilon \to \rho$ in $L^\infty_{loc}(I_T)$. However, since ρ^ε satisfies a linear parabolic equation (the second equation of (8)), we can write $\rho^\varepsilon_x = \tau\kappa^\varepsilon + A^\varepsilon$, and $\rho^\varepsilon_t = \varepsilon\rho^\varepsilon_{xx} + A^\varepsilon_x$, therefore using (15), we deduce, with $\rho_t = A_x \in C(I_T)$, that $\rho^\varepsilon_x \to \rho_x$ in $L^\infty_{loc}(I_T)$, and $\rho^\varepsilon_t \to \rho_t = A_x$ in $\mathcal{D}'(I_T)$.

Step 3. (Passing to the limit and boundary conditions). We rewrite system (8) in terms of A^ε, we get :

$$\begin{cases} \kappa^\varepsilon_t\kappa^\varepsilon_x = \varepsilon\kappa^\varepsilon_x\kappa^\varepsilon_{xx} + \rho^\varepsilon_x A^\varepsilon_x & \text{on} \quad I_T \\ \rho^\varepsilon_t = \varepsilon\rho^\varepsilon_{xx} + A^\varepsilon_x & \text{on} \quad I_T. \end{cases}$$

Using Steps 1 and 2, we can pass to the limit in the above system, using in particular the stability property for viscosity solutions in order to pass to the limit in the first equation. Our result then directly follows. The only thing left is to recover the boundary conditions. This is made by the equicontinuity of ρ^ε and κ^ε with respect to x near $\partial I \times [0,T]$, and the equicontinuity of ρ^ε and κ^ε with respect to t near $I \times \{t = 0\}$.

4. Sketch of the proof of Theorem 2.3

Step 1. (A lower bound on κ_x^ε). We have the following comparison principle for system (8) (which gives a stronger version than inequality (6) for system (1)).

Proposition 4.1 ***(A comparison principle for system (8)).*** *Let $(\rho^\varepsilon, \kappa^\varepsilon)$ be the solution given by Theorem 2.3. Choose $\beta = \beta(\varepsilon, \tau) > 0$ large enough. Let $M(x,t) := \cosh(\beta x)\{\kappa_x^\varepsilon(x,t) - \sqrt{\gamma^2(t) + (\rho_x^\varepsilon(x,t))^2}\}$ for $(x,t) \in \overline{I_T}$, where $\gamma(0) = \frac{\gamma_0}{2}$ for some $\gamma_0 \in (0,1)$, with $\kappa_x^{\varepsilon,0} \geq \sqrt{\gamma_0^2 + (\rho_x^{\varepsilon,0})^2}$ on I, and*

$$\frac{\gamma'}{\gamma} \leq -\left(c_0 + \|\rho_{xxx}^\varepsilon(.,t)\|_{L^\infty(I)}\right), \quad c_0 = c_0(\varepsilon, \beta, \tau). \tag{16}$$

Then $m(t) := \min_I M(x,t)$ satisfies $m(t) \geq \gamma^2(t)$ for all $t \in [0,T]$. In particular, we have

$$\kappa_x^\varepsilon(x,t) \geq \sqrt{\gamma^2(t) + (\rho_x^\varepsilon(x,t))^2} \quad on \quad \overline{I_T}. \tag{17}$$

The idea of the proof of Proposition 4.1 is to write the partial differential inequality satisfied by $M(x,t)$ derived from system (8), and to deduce that $m(t)$ satisfies the following ordinary differential inequality in the viscosity sense :

$$m_t \geq b_0 m + b_1$$

for some coefficients b_0, b_1 depending in particular on γ, γ' and ρ_{xxx}. On the other hand, we can show that

$$(\gamma^2)_t \leq b_0 \gamma^2 + b_1,$$

and we conclude by comparison.

Comments on the strategy of the proof of Theorem 2.3. Recall that we work on I_T. The term E that will appear in the sequel may certainly vary from line to line but always has the form $E = E(T) = ce^{cT}$, $c > 0$ is a positive constant independent of time but depending on ε. By applying a fixed point argument, we can show the existence of a local smooth solution $(\rho^\varepsilon, \kappa^\varepsilon)$ of (8), (9) and (10) for $T > 0$ small enough. This solution satisfies inequality (17) of Proposition 4.1 which somehow linearizes the first equation of (8) and may leads to a set of *a priori* estimates on the solution. We remark form inequality (16) that we need to have a good control on $\|\rho_{xxx}^\varepsilon(.,t)\|_{L^\infty(I)}$ in order to prevent γ from vanishing at a finite time. Otherwise, we can not guarantee the long time existence of $(\rho^\varepsilon, \kappa^\varepsilon)$. In fact, using Hölder estimates for parabolic equations [9], we get :

$$\|\rho_{xxx}^\varepsilon(.,t)\|_{L^\infty(I)} \leq \frac{E}{\gamma(t)}, \tag{18}$$

which, if plugged in (16), does not prevent γ from vanishing, and here is the principal difficulty in treating system (8). More *a priori* estimates concerning system (8) can also be obtained, namely for $t \in (0,T)$:

$$\|\rho_{xxx}^\varepsilon\|_{BMO_p(I\times(0,t))} \leq E \quad \text{and} \quad \|\rho_{xxx}^\varepsilon\|_{W_2^{2,1}(I\times(0,t))} \leq \frac{E}{\gamma^4(t)}, \tag{19}$$

with $W_2^{2,1}(I_T) = \{u \in L^2(I_T), (u_t, u_x, u_{xx}) \in (L^2(I_T))^2\}$, and the parabolic bounded mean oscillation space BMO_p is now recalled.

Definition 4.2 ***(Parabolic bounded mean oscillation space).*** *A function $u \in L^1_{loc}(I_T)$ is said to be of bounded mean oscillation, $u \in BMO_p(I_T)$, if the quantity : $\|u\|_{BMO_p(I_T)} = \sup_{Q \subset I_T} \left(\frac{1}{|Q|}\int_Q |u - m_Q(u)|\right)$ is finite. Here $Q = Q_r(x_0, t_0) = \{(x,t);\, |x - x_0| < r,\, t_0 - r^2 < t < t_0\}$ with $r > 0$, and $m_Q(u) = \frac{1}{|Q|}\int_Q u$. The BMO_p space is a Banach space whose elements are defined up to an additive constant.*

Step 2. (A parabolic Kozono-Taniuchi inequality). We seek to find an estimate on $\|\rho^\varepsilon_{xxx}(.,t)\|_{L^\infty(I)}$ better than (18). In fact, the space L^∞ lies in between the spaces BMO_p and $W^{2,1}_2$, and it seems natural to estimate the L^∞ norm by interpolation between these two spaces. Indeed, this is the goal of the next result.

Proposition 4.3 ***(A parabolic Kozono-Taniuchi inequality.)*** *Let $u \in W^{2,1}_2(I_T)$, then there exists a constant E such that, for all $t \in (0,T)$, the following estimate holds (with $\log^+ a = \max(0, \log a)$) :*

$$\|u\|_{L^\infty(I\times(0,t))} \leq E\|u\|_{BMO_p(I\times(0,t))}\left(1 + \log^+ \|u\|_{W^{2,1}_2(I\times(0,t))}\right). \tag{20}$$

The proof is an adaptation of the Kozono-Taniuchi inequality which is shown on the whole space $\mathbb{R}^n$ in the elliptic case [8, Theorem 1]. It is worth mentioning that the original type of the logarithmic Sobolev inequality was found in [2], and [3]. Using inequality (20) together with (19), we obtain :

$$\|\rho^\varepsilon_{xxx}(.,t)\|_{L^\infty(I)} \leq E\left(1 + \log^+ \frac{E}{\gamma^4(t)}\right). \tag{21}$$

Step 3. (Long time existence). Using the sharper estimate (21) on $\|\rho^\varepsilon_{xxx}(.,t)\|_{L^\infty(I)}$, we can choose γ solution of the following ODE :

$$\frac{\gamma'}{\gamma} = -E\left(1 + \log^+ \frac{1}{\gamma}\right),$$

with some new constant $E = E(T)$. From Proposition 4.1, we finally deduce that :

$$\kappa^\varepsilon_x(.,t) \geq \gamma(t) \geq e^{-e^{e^{c(t+1)}}} > 0 \quad \text{for} \quad t \in [0,T], \tag{22}$$

where $c > 0$ is a positive constant independent of time. Indeed, the time T being arbitrary, we see that (22) is true for all time $t \geq 0$. From (22), the following *a priori* estimates on ρ^ε and κ^ε can be obtained :

$$\|D^s_x\rho^\varepsilon(.,t)\|_{L^\infty(I)} \leq e^{e^{e^{c(t+1)}}}, \quad \|D^s_x\kappa^\varepsilon(.,t)\|_{L^\infty(I)} \leq e^{e^{e^{c(t+1)}}}, \quad \forall s \in \mathbb{N},\ s \leq 3, \quad \forall t \geq 0. \tag{23}$$

The above *a priori* estimates (22) and (23) permit to show the long time existence by time iteration.

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