Adiabatic switching for degenerate ground states

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Outline of the talk

- Plan of the presentation
  - Motivation: Green's functions in many-body perturbation theory
  - Some background material
  - The Gell-Mann and Low formula in a simple case
  - The degenerate case
  - Back to Green's functions
- References:
  - C. BROUDER, G. STOLTZ AND G. PANATI, Adiabatic approximation, Gell-Mann and Low theorem and degeneracies: A pedagogical example, *Phys. Rev. A* **72** (2008) 042102
  - C. BROUDER, G. PANATI AND G. STOLTZ, Many-body Green function of degenerate systems, *Phys. Rev. Lett.* **103** (2009) 230401
  - C. BROUDER, G. PANATI AND G. STOLTZ, Gell-Mann and Low formula for degenerate unperturbed states, *Ann. I. H. Poincare-Phy* **10**(7) (2010) 1285-1309

Physical motivation

- Gell-Mann and Low formula (*Phys. Rev.*, 1951) $\Psi_0 = \dfrac{U_\varepsilon(0,-\infty)\Phi_0}{\langle \Phi_0, U_\varepsilon(0,-\infty)\Phi_0 \rangle}$
- The two point Green's function is defined as

$$G(t,r;t',r') = -\mathrm{i}\left\langle \Psi_0, T[\psi_H(t,r)\psi_H^\dagger(t'x')]\Psi_0 \right\rangle$$

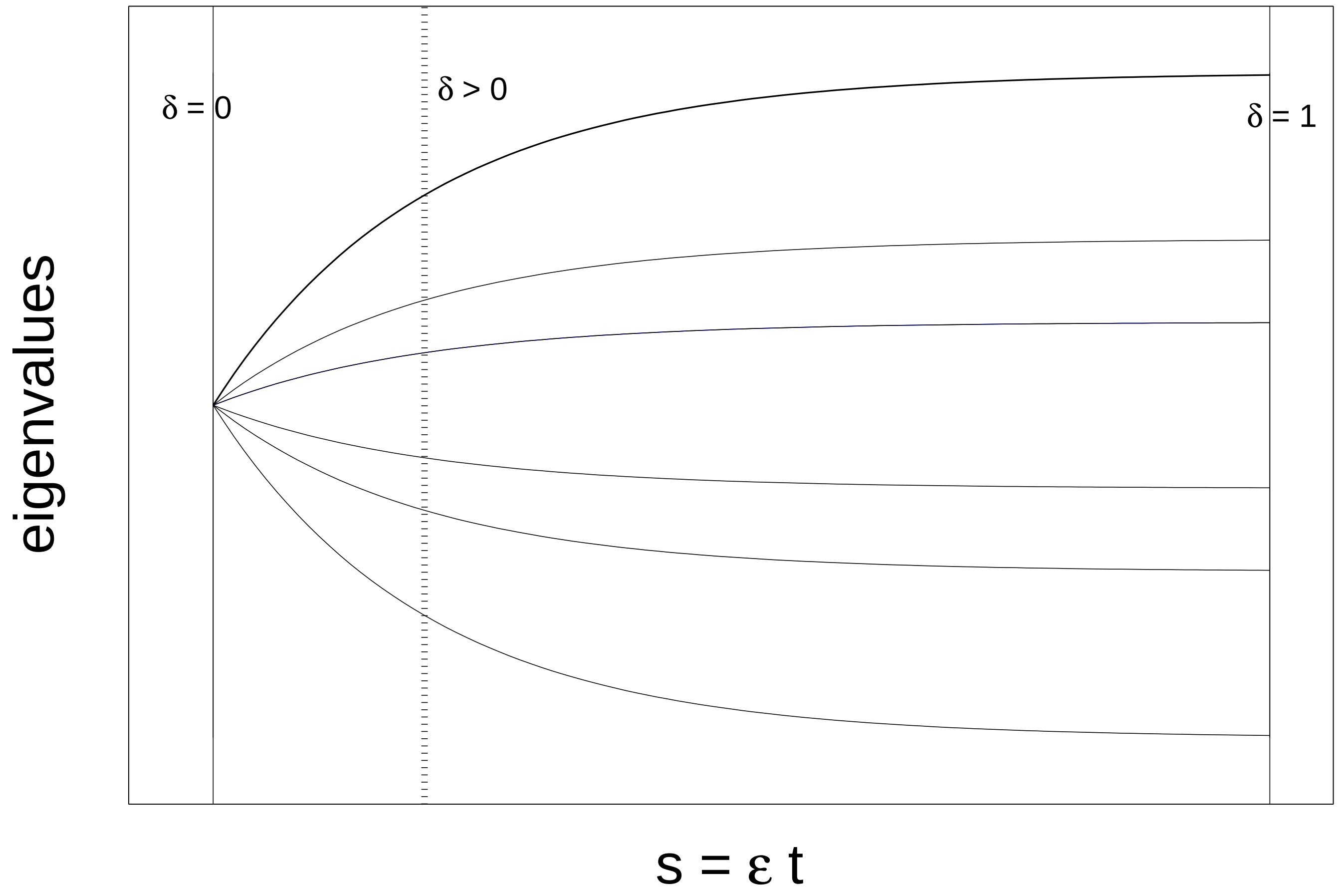
where Ψ_0 is the (unknown) ground state of some Hamiltonian $H_0 + V$

- When the ground state Φ_0 of H_0 is known, G can be expressed in terms of expectations with respect to Φ_0 using the Gell-Mann and Low formula:

$$G(t,r;t',r') = -\mathrm{i}\lim_{\varepsilon\to 0}\frac{\left\langle \Phi_0, T[\psi_H(t,r)\psi_H^\dagger(t'x')U_\varepsilon(+\infty,-\infty)]\Phi_0 \right\rangle}{\langle \Phi_0, U_\varepsilon(+\infty,-\infty)]\Phi_0 \rangle}$$

- Formal expansions in terms of free-field Green's functions using Wick's theorem (Feynman diagrams)

The Gell-Mann and Low switching procedure requires some care when the ground state is degenerate... and this happens in many situations!



A pedagogical example

- Simplest possible system: Hamiltonian $H(t) = H_0 + \mathrm{e}^{-\varepsilon|t|} H_1$ with

$$H_0 = \begin{pmatrix} \mu - \delta & 0 \\ 0 & \mu + \delta \end{pmatrix}, \qquad H_1 = \begin{pmatrix} 0 & \alpha \\ \alpha & 0 \end{pmatrix}.$$

- Analytical computations can be performed
- The switching procedure is well defined when $\delta \neq 0$
- The switching procedure fails for almost all initial states when $\delta = 0$, and can be defined for two specific states only!

First, some background material...

- Fixed nuclei of charges z_m located at $R_m \in \mathbb{R}^3$ (Born-Oppenheimer approximation)
- Wavefunction $\psi((x_1,\sigma_1),\ldots,(x_N,\sigma_N)) \in \bigwedge_{i=1}^N \mathrm{L}^2(\mathbb{R}^3 \times \{-1,1\})$ with

$$\|\psi\|_{\mathrm{L}^2} = 1$$

- The spin variable will be omitted in the sequel
- Hamiltonian operator (in atomic units)

$$H = \sum_{i=1}^N \Big(-\frac{1}{2}\Delta_{x_i} + V_{\mathrm{nuc}}(x_i)\Big) + \sum_{1\le i<j\le N} \frac{1}{|x_i - x_j|}$$

with domain $D(H) = \bigwedge_{i=1}^N \mathrm{H}^2(\mathbb{R}^3) \subset \mathcal{H} = \bigwedge_{i=1}^N \mathrm{L}^2(\mathbb{R}^3)$ and where

$$V_{\mathrm{nuc}}(x) = -\sum_{m=1}^M \frac{z_m}{|x - R_m|}$$

Spectrum of a linear operator (1)

- Linear operator $A : D(A) \subset \mathcal{H} \to \mathcal{H}$ on a Hilbert space, with dense domain $D(A)$
- A is injective if $\mathrm{Ker}(A) = \{\phi \in D(A) \,|\, A\phi = 0\} = \{0\}$
- If A is injective, it is possible to define its inverse, which is an operator with domain

$$D(A^{-1}) = \mathrm{Ran}(A) = \left\{\psi \in \mathcal{H} \,\middle|\, \exists\, \phi \in D(A),\ \psi = A\phi\right\}$$

  such that $\phi = A^{-1}\psi \Leftrightarrow \psi = A\phi$
- A is invertible if it has a bounded inverse defined on $D(A^{-1}) = \mathcal{H}$
- If A is closed and one-to-one $D(A) \to \mathcal{H}$, the operator $A^{-1} : \mathcal{H} \to D(A)$ is automatically bounded by the closed graph theorem
- Resolvent set $\rho(A)$ = (open) set of $\lambda \in \mathbb{C}$ such that $\lambda - A$ is invertible
- The spectrum $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is closed

- The spectrum can be decomposed as $\sigma(A) = \sigma_{\rm p}(A) \cup \sigma_{\rm r}(A) \cup \sigma_{\rm c}(A)$, where ("by decreasing defaults of invertibility")
  - $\lambda \in \sigma_p(A)$ iff $\mathrm{Ker}(\lambda - A) \neq \{0\}$ [eigenvalues]
  - $\lambda \in \sigma_r(A)$ iff $\lambda - A$ is injective but $\overline{\mathrm{Ran}(\lambda - A)} \neq \mathcal{H}$ [the inverse is not uniquely defined]
  - $\lambda \in \sigma_c(A)$ iff $\lambda - A$ is injective, $\overline{\mathrm{Ran}(\lambda - A)} = \mathcal{H}$ but $\mathrm{Ran}(\lambda - A) \neq \mathcal{H}$ [the inverse is unbounded with dense domain; generalized eigenvalues]
- Other decomposition: $\sigma(A) = \sigma_{\rm d}(A) \cup \sigma_{\rm ess}(A)$, where the discrete spectrum $\sigma_{\rm d}(A) \subset \sigma_{\rm p}(A)$ = isolated eigenvalues of finite multiplicity
- Examples (necessarily infinite dimensional)
  - Residual spectrum: shift operator $\tau_{\rm d}$ on $l^2(\mathbb{N}, \mathbb{C})$ with $\tau_{\rm d}(z_0, z_1, z_2, \dots) = (0, z_0, z_1, \dots)$
  - Continuous spectrum: $A\psi(x) = x\psi(x)$ on $\mathrm{L}^2(\mathbb{R})$

- Adjoint of an unbounded operator = closed operator with domain

$$\begin{aligned} D(A^*) &= \left\{ \phi \in \mathcal{H} \,\middle|\, \forall \psi \in D(A),\ |\langle A\psi, \phi\rangle| \leq C_\phi \|\psi\| \right\} \\ &= \left\{ \phi \in \mathcal{H} \,\middle|\, \exists \varphi \in \mathcal{H},\ \forall \psi \in D(A),\ \langle A\psi, \phi\rangle = \langle \psi, \varphi\rangle \right\} \end{aligned}$$

  defined by $A^*\phi = \varphi$

- Symmetric operator: $\forall (\phi, \psi) \in D(A)^2,\ \langle A\phi, \psi\rangle = \langle \phi, A\psi\rangle$ (*i.e.* $A \subset A^*$)
- A symmetric operator is self-adjoint if $A = A^*$ (*i.e.* $D(A) = D(A^*)$)
- For self-adjoint operators, $\sigma(A) \subset \mathbb{R}$ and $\sigma_{\rm r}(A) = \emptyset$
- An operator V is H_0-bounded if $D(H_0) \subset D(V)$ and

$$\forall \phi \in D(H_0), \quad \|V\phi\| \leq a\|H_0\phi\| + b\|\phi\|$$

- Kato-Rellich criterion: If H_0 is self-adjoint and V is symmetric and H_0-bounded with relative bound $a < 1$, then $H = H_0 + V$ defined on $D(H) = D(H_0)$ is self-adjoint

- Consider $D(H_0^N) = \bigwedge_{i=1}^N \mathrm{H}^2(\mathbb{R}^3)$

$$H_0^N = \sum_{i=1}^N \Big(-\frac{1}{2}\Delta_{x_i} + V_{\rm nuc}(x_i) \Big), \qquad V^N = \sum_{1\leq i<j\leq N} \frac{1}{|x_i - x_j|}$$

- (Kato) Using the Hardy inequality

$$\forall \phi \in \mathrm{H}^1(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} \frac{|\phi(x)|^2}{|x|^2}\,dx \leq 4 \int_{\mathbb{R}^3} |\nabla \phi(x)|^2\,dx,$$

  it can be shown that $H^N = H_0^N + V^N$ is self-adjoint on $\mathcal{H}_N = \bigwedge_{i=1}^N \mathrm{L}^2(\mathbb{R}^3)$

- HVZ theorem: $\sigma_{\rm ess}(H^N) = [E^{N-1}, +\infty[$, where

$$E^{N-1} = \inf \sigma\Big(H^{N-1}\Big)$$

- If $N < Z+1$, then there are infinitely many eigenvalues of finite multiplicity below the essential spectrum

The Gell-Mann and Low formula in a simple case

- Consider, on a given Hilbert space $\mathcal{H}$,
  - a self-adjoint operator H_0, with dense domain $D(H_0) \subset \mathcal{H}$
  - a symmetric perturbation V, H_0-bounded with relative bound $a < 1$.
  - define $\widetilde{H}(\lambda) = H_0 + \lambda V$ with $\lambda \in [0, 1]$
- Switching function $f \in \mathrm{C}^2\Big((-\infty, 0], [0, 1]\Big)$
  - non-decreasing
  - $f, f'' \in L^1((-\infty, 0])$
  - $f(0) = 1$ and $\lim_{\tau \to -\infty} f(\tau) = 0$
  - for $\tau \in (-\infty, 0]$, define $H(\tau) = \widetilde{H}(f(\tau)) = H_0 + f(\tau)\, V$
- Denote by $U_\varepsilon(s, s_0)$ the unitary evolution generated by $H(\varepsilon s)$, *i.e.* the unique solution of the problem:

$$\mathrm{i}\frac{dU_\varepsilon(s, s_0)}{ds} = H(\varepsilon s)\, U_\varepsilon(s, s_0), \qquad U_\varepsilon(s_0, s_0) = \mathbb{I}$$

Switching procedure (2)

- Divergent phase as $\varepsilon \to 0$! Consider $V = 0$ and ϕ an eigenstate of H_0:

$$U_\varepsilon(s, s_0)\,\phi = \exp\left(-\frac{\mathrm{i}E_0(s - s_0)}{\varepsilon}\right)\,\phi$$

- Remove divergence by working in the interaction picture:

$$U_{\varepsilon,\mathrm{int}}(s, s_0) = \mathrm{e}^{\mathrm{i}sH_0}U_\varepsilon(s, s_0)\,\mathrm{e}^{-\mathrm{i}s_0H_0}.$$

- Macroscopic time $t = \varepsilon s$: unitary evolution

$$\mathrm{i}\,\varepsilon\,\frac{dU^\varepsilon(t, t_0)}{dt} = H(t)\,U^\varepsilon(t, t_0), \quad U^\varepsilon(t_0, t_0) = \mathbb{I},$$

so that, in the interaction picture, $U^\varepsilon_{\mathrm{int}}(t, t_0) = \mathrm{e}^{\mathrm{i}tH_0/\varepsilon}\,U^\varepsilon(t, t_0)\,\mathrm{e}^{-\mathrm{i}t_0H_0/\varepsilon}$

- Standard results show that, for $\psi \in D(H_0)$, the following limit exists:

$$U^\varepsilon_{\mathrm{int}}(t, -\infty)\,\psi = \lim_{t_0 \to -\infty} U^\varepsilon_{\mathrm{int}}(t, t_0)\,\psi$$

- In order for eigenstates to be stable during the switching procedure, some gap conditions are required
- The spectrum of $\widetilde{H}(\lambda) = H_0 + \lambda V$, $\lambda \in [0,1]$, consists of two disconnected pieces

$$\sigma\left(\widetilde{H}(\lambda)\right) = \sigma_N(\lambda) \cup \left(\sigma\left(\widetilde{H}(\lambda)\right) \backslash \sigma_N(\lambda)\right)$$

where $\sigma_N(\lambda) = \left\{\widetilde{E}_j(\lambda),\ j = 1, \ldots, N\right\} \subset \sigma_{\rm disc}\left(\widetilde{H}(\lambda)\right)$

- There is a uniform gap between the two parts of the spectrum, and between the elements of $\sigma_N(\lambda)$, in the sense that:

$$\Delta(\lambda) = \min_{j=1,\ldots,N}\left(\min\left\{\left|\widetilde{E}_j(\lambda) - E\right|,\ E \in \sigma(H(\lambda))\backslash\{\widetilde{E}_1(\lambda), \ldots, \widetilde{E}_N(\lambda)\}\right\}\right),$$

$$\delta(\lambda) = \min\left\{\left|\widetilde{E}_j(\lambda) - \widetilde{E}_i(\lambda)\right|,\ 1 \le i < j \le N\right\}$$

are bounded from below by a positive constant for all $\lambda \in [0,1]$

- For simplicity, eigenvalues $E_j(\tau) = \widetilde{E}_j(f(\tau))$ of multiplicity 1
- Then, for an eigenstate ψ_j of H_0 associated with $E_j(-\infty)$, if

$$\|P_j(-\infty) - P_j(0)\| < 1,$$

the limit

$$\Psi_j = \lim_{\varepsilon \to 0} \frac{U^{\varepsilon}_{\mathrm{int}}(0, -\infty)\psi_j}{\langle\, \psi_j \mid U^{\varepsilon}_{\mathrm{int}}(0, -\infty)\psi_j \,\rangle}$$

exists and is an eigenstate of $H_0 + V$ corresponding to the eigenvalue $E_j(0) = \widetilde{E}_j(1)$

- First proof due to Nenciu and Rasche (*Helvetica Physica Acta*, 1989)
- Extension to the case of eigenspaces of multiplicity higher than 1 provided some direction ϕ exists such that the denominator does not vanish...

First step of the proof: Geometric evolution

- Kato intertwining operator: $\dfrac{d\widetilde{A}(\lambda,\lambda_0)}{d\lambda} = \widetilde{K}(\lambda)\,\widetilde{A}(\lambda,\lambda_0)$ with $\widetilde{A}(\lambda_0,\lambda_0) = \mathbb{I}$
- Generator $\widetilde{K}(\lambda) = -\sum_{j=1}^{N+1} \widetilde{P}_j(\lambda)\,\dfrac{d\widetilde{P}_j}{d\lambda}(\lambda)$, with $\widetilde{P}_{N+1}(\lambda) = \mathbb{I} - \sum_{j=1}^{N} \widetilde{P}_j(\lambda)$
- Since $\widetilde{K}(\lambda)$ is uniformly bounded (gap, hence projectors smooth), the operator $\widetilde{A}(\lambda,\lambda_0)$ is well-defined and strongly continuous
- $\widetilde{A}(\lambda,\lambda_0)$ is unitary (since $K^* = -K$), and intertwines the spectral subspaces:

$$\widetilde{P}_j(\lambda) = \widetilde{A}(\lambda,\lambda_0)\widetilde{P}_j(\lambda_0)\widetilde{A}(\lambda,\lambda_0)^*$$

- Denoting by $A(s,s_0) = \widetilde{A}(f(s),f(s_0))$,

$$P_j(0)\,A(0,-\infty)\,\psi_j = A(0,-\infty)\,P_j(-\infty)\,\psi_j = A(0,-\infty)\,\psi_j,$$

so that $A(0,-\infty)\,\psi_j$ is an eigenstate of $H(0) = H_0 + V$

- Adiabatic evolution operator $U_{\mathrm{A}}(s, s_0)$ is defined as the unique solution of

$$\mathrm{i}\,\frac{dU_{\mathrm{A}}(s,s_0)}{ds} = H_{\mathrm{A}}(s)\,U_{\mathrm{A}}(s,s_0), \quad U_{\mathrm{A}}(s_0,s_0) = \mathbb{I},$$

where the adiabatic Hamiltonian is $H_{\mathrm{A}}(s) = H(s) + \mathrm{i}K(s)$

- U_{A} is also an intertwiner
- A and U_{A} differ only by a phase, which commutes with the spectral projectors: Define

$$\Phi(s,s_0) = A(s,s_0)^* U_{\mathrm{A}}(s,s_0),$$

so that $U_{\mathrm{A}}(s,s_0) = A(s,s_0)\,\Phi(s,s_0)$. Then, $[\Phi(s,s_0), P_j(s_0)] = 0$

- The time-evolution of the phase matrix is then easily obtained and

$$U_{\mathrm{A}}(s,s_0)\,P_j(s_0) = \exp\left(-\mathrm{i}\int_{s_0}^{s} E_j(r)\,dr\right) A(s,s_0)\,P_j(s_0)$$

Second step: Adiabatic evolution (rescaling the dynamical phase)

- Important again to work in the interaction picture to remove the divergent (dynamical) phase: $U_{\mathrm{A,int}}(s, s_0) = \mathrm{e}^{\mathrm{i}sH_0}\, U_{\mathrm{A}}(s, s_0)\, \mathrm{e}^{-\mathrm{i}s_0 H_0}$
- It can be shown, through some limiting procedure, that
$$U_{\mathrm{A,int}}(0, -\infty)\, P_j(-\infty) = \exp\left(-\mathrm{i}\int_{-\infty}^{0} E_j(r) - E_0\, dr\right) A(0, -\infty)\, P_j(-\infty)$$
- Phase well-defined since $|E_j(r) - E_0| = \left|\widetilde{E}_j(f(r)) - \widetilde{E}_j(0)\right| \leq C f(r)$
- In the time-rescaled variable $t = \varepsilon s$,
$$U^{\varepsilon}_{\mathrm{A,int}}(0, -\infty) P_j(-\infty) = \exp\left(-\frac{\mathrm{i}}{\varepsilon}\int_{-\infty}^{0} E_j(\tau) - E_0\, d\tau\right) A(0, -\infty) P_j(-\infty).$$
- Eliminate the phase using
$$\frac{P_j(0)\, \psi_j}{\|P_j(0)\, \psi_j\|^2} = \frac{A(0, -\infty)\, \psi_j}{\langle\, \psi_j \mid A(0, -\infty)\, \psi_j\, \rangle} = \frac{U^{\varepsilon}_{\mathrm{A,int}}(0, -\infty)\, \psi_j}{\left\langle\, \psi_j \,\middle|\, U^{\varepsilon}_{\mathrm{A,int}}(0, -\infty)\, \psi_j\, \right\rangle},$$

- Compare the adiabatic and full evolutions in the rescaled time-variable:

$$\mathrm{i}\,\varepsilon\,\frac{dU_{\mathrm{A}}^{\varepsilon}(t,t_0)}{dt} = \Big(H(t)+\mathrm{i}\varepsilon K(t)\Big)U_{\mathrm{A}}^{\varepsilon}(t,t_0), \quad \mathrm{i}\,\varepsilon\,\frac{dU^{\varepsilon}(t,t_0)}{dt} = H(t)\,U^{\varepsilon}(t,t_0)$$

- Prove the uniform convergence $\lim_{\varepsilon\to 0}\|U^{\varepsilon}(0,-\infty)-U_{\mathrm{A}}^{\varepsilon}(0,-\infty)\| = 0$ (although $U^{\varepsilon}(0,-\infty), U_{\mathrm{A}}^{\varepsilon}(0,-\infty)$ do not have limits as $\varepsilon\to 0$)
- Strategy from (Teufel, *Adiabatic perturbation theory in quantum dynamics*, 2003):

$$U^{\varepsilon}(t,t_0) - U_{\mathrm{A}}^{\varepsilon}(t,t_0) = -U^{\varepsilon}(t,t_0)\int_{t_0}^{t} U^{\varepsilon}(t_0,t')K(t')U_{\mathrm{A}}^{\varepsilon}(t',t_0)\,dt'$$

- Define $\mathcal{K}(t) = -\mathrm{i}\varepsilon\,U^{\varepsilon}(t_0,t)F(t)U^{\varepsilon}(t,t_0)$ with $[H(t),F(t)] = K(t)$. Then

$$\mathcal{K}'(t) = U^{\varepsilon}(t_0,t)[H(t),F(t)]U^{\varepsilon}(t,t_0) - \mathrm{i}\varepsilon U^{\varepsilon}(t_0,t)F'(t)U^{\varepsilon}(t,t_0)$$

- Similar to $\int_0^t \mathrm{e}^{-i\tau/\varepsilon}d\tau = \mathrm{i}\varepsilon\left(\mathrm{e}^{-\mathrm{i}t/\varepsilon}-1\right)$ = highly oscillatory integral

- Expression of $F(t)$: useful to keep track of the dependence on the gap (required to understand the degenerate case)

$$F(t) = -\frac{1}{2}\left(\sum_{j=1}^{N+1} F_j(t) + G_j(t)\right), \quad F_j(t) = \frac{1}{2\mathrm{i}\pi}\oint_{\Gamma_j(t)} P_j^{\perp}(t)R(z,t)\dot{R}(z,t)\,dz$$

where $R(z,t) = (H(t)-z)^{-1}$ and $\Gamma_j(t)$ is a contour enclosing $E_j(t)$ and no other element of the spectrum

- Similar definitions for G_j, F_{N+1}, G_{N+1}
- Bounds $\|F(t)\| \leq C_F\,\dfrac{f'(t)}{f(t)}$ and

$$\int_{t_0}^{t}\|F'\| \leq C\left(\frac{1}{f(t_0)}\int_{t_0}^{t}\left(|f''|+(f')^2\right) + \frac{1}{f(t_0)^2}\int_{t_0}^{t}(f')^2\right)$$

The degenerate case

- Initial state is degenerate: $\widetilde{E}_j(0) = \widetilde{E}_k(0)$ for all $1 \leq j, k \leq N$
- Degeneracy splitting (for simplicity): $\mathcal{P}_0 V \mathcal{P}_0$ has non-degenerate eigenvalues and for any $\lambda^* > 0$, there exists α such that

$$\inf_{\lambda^* \leq \lambda \leq 1} \min_{k \neq l} \left| \widetilde{E}_k(\lambda) - \widetilde{E}_l(\lambda) \right| \geq \alpha > 0$$

- Let $(\psi_1, \ldots, \psi_N)$ be an basis of $\mathcal{E}_0$ which diagonalizes the bounded operator $\mathcal{P}_0 V \mathcal{P}_0 \big|_{\mathcal{E}_0}$. Then, if $\|P_j(-\infty) - P_j(0)\| < 1$, the limit

$$\Psi_j = \lim_{\varepsilon \to 0} \frac{U^{\varepsilon}_{\text{int}}(0, -\infty)\, \psi_j}{\langle\, \psi_j \mid U^{\varepsilon}_{\text{int}}(0, -\infty)\, \psi_j \,\rangle}$$

exists and is an eigenstate of $H_0 + V$ corresponding to $E_j(0) = \widetilde{E}_j(1)$

- Several extensions: decomposition to avoid the condition $\|P_j(-\infty) - P_j(0)\| < 1$; extension the case when $\mathcal{P}_0 V \mathcal{P}_0 \big|_{\mathcal{E}_0}$ has degenerate eigenvalues; existence of finitely many eigenvalue crossings

- Theorem II.6.1 in (KATO, *Perturbation Theory for Linear Operators*) shows that the eigenvalues $\widetilde{E}_j$ and projectors $\widetilde{P}_j$ are analytic functions of λ
- Initial states defined from $P_j^{\rm init} := \widetilde{A}(0,\lambda)\widetilde{P}_j(\lambda)\widetilde{A}(\lambda,0)$. Characterization?
- Eigenvectors satisfy $\widetilde{H}(\lambda)\,\phi_j(\lambda) = \widetilde{E}_j(\lambda)\,\phi_j(\lambda)$ with

$$\widetilde{E}_j(\lambda) = \sum_{n=0}^{+\infty} \lambda^n E_{j,n}, \qquad \phi_j(\lambda) = \sum_{n=0}^{+\infty} \lambda^n \varphi_{j,n}$$

- Hierarchy of equations. First order condition

$$(H_0 - E_0)\,\varphi_{j,1} = (E_{j,1} - V)\,\varphi_{j,0}$$

- A necessary condition for this equation to have a solution is that the right-hand side belongs to $\mathcal{E}_0^\perp$
- This requires $E_{j,1} = \langle\,\varphi_{j,0}, V\varphi_{j,0}\,\rangle$ and $\forall k \neq j,\ \langle\,\varphi_{k,0}, V\varphi_{j,0}\,\rangle = 0$ so that the basis diagonalizes $\mathcal{P}_0 V \mathcal{P}_0\big|_{\mathcal{E}_0}$

- Geometric and adiabatic evolutions: unchanged (the regularity of the projectors follows from the analytic continuation at $\lambda = 0$)
- Adiabatic limit: decomposition of the evolution into

$$U^\varepsilon(0,t_0) - U^\varepsilon_{\mathrm{A}}(0,t_0) = -U^\varepsilon(0,t_0)\int_{t_0}^{T} U^\varepsilon(t_0,t)K(t)U^\varepsilon_{\mathrm{A}}(t,t_0)\,dt - U^\varepsilon(0,t_0)\int_{T}^{0} U^\varepsilon(t_0,t)K(t)U^\varepsilon_{\mathrm{A}}(t,t_0)\,dt$$

  - an evolution on $[T,0]$, for Hamiltonians operators with (small) gaps of order $f(T)$; bound in $C\varepsilon(1+f(T)^{-2})$
  - an evolution on the time-frame $(-\infty,T]$, with T small enough so that the unitary evolutions are not very different; bound in $Cf(T)$
  - choose T such that $f(T) = \varepsilon^{1/3}$ to have a final bound in $\varepsilon^{1/3}$

Physical extensions

- Operator A expressed in the Heisenberg picture $A_{\mathrm{hsnbrg}}(t) = \mathrm{e}^{\mathrm{i}tH} A\, \mathrm{e}^{-\mathrm{i}tH}$ and, in the interaction picture, $A_{\mathrm{int}}(t) = \mathrm{e}^{\mathrm{i}tH_0} A\, \mathrm{e}^{-\mathrm{i}tH_0}$
- Correlation function $C_{A,B}(t,t') = \langle \psi \,| T\, [A_{\mathrm{hsnbrg}}(t) B_{\mathrm{hsnbrg}}(t')] |\, \psi \rangle$
- Technical lemma: For fixed t, t',

$$\mathrm{s-}\lim_{\varepsilon \to 0} U_{\varepsilon,\mathrm{int}}(t,0)^* A_{\mathrm{int}}(t)\, U_{\varepsilon,\mathrm{int}}(t,t')\, B_{\mathrm{int}}(t')\, U_{\varepsilon,\mathrm{int}}(t',0) = A_{\mathrm{hsnbrg}}(t) B_{\mathrm{hsnbrg}}(t')$$

- Using the Gell-Mann and Low formula, it can then be shown that

$$C_{A,B}(t,t') = \lim_{\varepsilon \to 0} \frac{\langle \psi_0 \,| T\, [A_{\mathrm{int}}(t)\, B_{\mathrm{int}}(t')\, U_{\varepsilon,\mathrm{int}}(+\infty, -\infty)] |\, \psi_0 \rangle}{\langle \psi_0 \,| U_{\varepsilon,\mathrm{int}}(+\infty, -\infty) |\, \psi_0 \rangle}.$$

- Formal extension to the case when A, B are field operators
- Basis for a perturbative treatment of the Green's function, where the operators $U_{\varepsilon,\mathrm{int}}(+\infty, -\infty)$ in the numerator and denominator are expanded using Feynman diagrams.