A PRECONDITIONER FOR INTEGRAL EQUATIONS MODELING HELMHOLTZ EQUATION[1]

Armel de La Bourdonnaye
CERMICS
INRIA
06902 Sophia-Antipolis Cedex
France

Abstract

In this paper, we analyze a preconditioning operator for the scattering problem in acoustics. We first recall the integral equations we use to modelize the problem. Then, we study the case of a plane scatterer as a preliminary. We show that the preconditioning operator has a limit when the frequency grows to infinity. We repeat this study with a more general though compact geometry of the scatterer. We obtain the same result as in the plane case. Then, in order to understand the effects of the preconditioning operator, we apply this result to the sphere, as we can perform more analytical computation in this case. We can observe that it makes the higher part of the spectrum of the scattering operator much more empty. This fact explains the reduction of the number of iterations to reach convergence. Finally, we show some numerical experiments that reveal the actual efficiency of the preconditioner.

UN PRÉCONDITIONNEUR POUR LES ÉQUATIONS INTÉGRALES MODÉLISANT L'ÉQUATION DE HELMHOLTZ

Résumé

Dans ce papier, nous analysons un préconditionneur adapté aux problèmes de diffraction acoustique. Nous commençons par rappeler la formulation en équations intégrales que nous utilisons, puis nous étudions comme préliminaire le cas d'une surface plane comme objet diffractant. Nous voyons qu'alors le préconditionneur a une limite lorsque le nombre d'onde, et donc la fréquence, croîssent. Nous refaisons la même étude dans le cas d'une surface diffractante plus générale mais compacte. Nous obtenons exactement la même limite pour le préconditionneur. Nous appliquons ensuite le résultat à la sphère afin de pouvoir comprendre l'action du préconditionneur. Il apparait que celui-ci a pour effet de rendre le haut du spectre de l'opérateur plus creux. Ceci explique l'amélioration de la vitesse de convergence. Nous montrons dans la dernière partie des résultats numériques qui montrent la bonne qualité du préconditionneur obtenu.

[1]This work was supported by ONERA

Introduction

Numerical treatment of integral equations is known to be a hard problem because it leads to full non hermitian matrices which are often ill-conditioned. Many authors already contributed to improve this treament. Among them we can mention G. Markham [8] who studied conjugate gradient methods for complex and non hermitian systems or K. Chen [3] who presented a preconditioner for boundary integral equations which has proven to be efficient in the vicinity of the singularities of the boundary. In our paper, we present and analyze a new and efficient preconditioner for the solution integral equations arising from Helmholtz equations. We will first recall the mathematical problem under consideration and briefly present the numerical method. In a second part we will study the theoretical properties of the preconditioning matrix. We will study the case of the sphere with a special attention. Finally we will present numerical experiments showing the efficiency of the preconditioner.

1 The mathematical problem

We want to solve the classical problem of scattering. Given Ω a regular and bounded open set of $I\!R^3$ and Ω' the interior of its complementary, we are looking for u in $H^1(\Omega) \cap H^1_{loc}(\Omega')$ solution of the Helmholtz equation with Neumann boundary conditions. Let Γ be the boundary of Ω, n the outgoing unitary normal vector, and $g \in H^{-1/2}(\Gamma)$ the Neumann condition, then

$$\Delta u + k^2 u = 0 \text{ in } \Omega \cup \Omega' \tag{1}$$

$$\frac{\partial u}{\partial r} + iku = o(\frac{1}{r})\text{for } r \to \infty \tag{2}$$

$$\frac{\partial u}{\partial n} = g \text{ on } \Gamma \tag{3}$$

We will use integral representation of the solution. More precisely, we take as the unknown the jump of u through Γ :$\phi = [u]_{|\Gamma}$. Then the integral equation for ϕ writes, as shown by Hamdi [5], :

$$\forall \psi \ \in \ H^{1/2}(\Gamma)$$

$$\int_{\Gamma\times\Gamma} \frac{e^{ik|x-y|}}{4\pi|x-y|}(rot_\Gamma \psi(x).rot_\Gamma \phi(y) - k^2 n_x.n_y \psi(x)\phi(y))dxdy \quad = \tag{4}$$

$$\int_\Gamma \psi(x)g(x)dx \tag{5}$$

We discretize this variational formulation with a finite element method. This leads to a full complex and non hermitian matrix A. For solving the discretized equation, we use the preconditioned iterative method "Orthomin" (cf. [6]). The preconditioning matrix B is a submatrix of matrix A. More precisely, we keep only coefficients that represent interactions between degrees of freedom whose distance is less than a constant times the wavelength. On a mathematical point of view, it means that B is the discretization of variational formulation (5) in which the kernel $G(r) = \dfrac{e^{ikr}}{r}$ is replaced by $G_p(r) = G(r).\dfrac{\chi(kr)}{k}$. Here χ is a "cut-off" around 0 in $I\!R$, regular and with compact support.

In the following we will discuss the behaviour of the preconditioner as k grows to infinity.

2 Preliminaries

In this section we will first study the case of a plane surface. Second we will recall some technical points of geometry

2.1 Case of a plane surface

We start with studying the case where Γ is a plane. The first interest in this simplification is that we can use Fourier transform and convolution so that we may hope optimal results. The second one is that all surfacic differential operators are much more simpler because the normal is the same everywhere. We will denote by $\mathcal{B}$ the operator made with the preconditioning kernel, so that $< \mathcal{B}\phi, \psi > = -k^2 < G_p * \phi, \psi > + < G_p * \nabla\phi, \nabla\psi >$ where $*$ denotes the convolution. In this section we will prove the following theorem.

Theorem 1 *Given positive s, s' verifying $1 \geq s + s' > 0$, there exists a positive constant α such that for all functions ϕ in $H^{s+1}(R^2)$ and ψ in $H^{s'+1}(R^2)$*

$$\left| < \mathcal{B}\phi, \psi > + C_\chi < (Id + \frac{\Delta_\Gamma}{k^2})\phi, \psi > \right| < \tag{6}$$

$$\alpha \left(\frac{||\phi||_{H^s} ||\psi||_{H^{s'}}}{k^{s+s'}} + \frac{||\phi||_{H^{s+1}} ||\psi||_{H^{s'+1}}}{k^{s+s'+2}} \right) \tag{7}$$

where C_χ is a constant which depends only on the cut-off.

This result means that, when k grows to infinity, $\mathcal{B}$ is equivalent to $Id + \dfrac{\Delta_\Gamma}{k^2}$. Before giving the proof of theorem 1, we will enonce a few lemmas. The detailed proofs are given in [4].

Lemma 1 *For all ξ in $\mathbb{R}^2$, $\lim_{k\to\infty} k^2 \hat{G}_p(\xi) = C_\chi$ where $\hat{G}_p$ is the Fourier transform of G_p and $C_\chi = \dfrac{1}{4\pi}\displaystyle\int_0^\infty e^{i\rho}\chi(\rho)d\rho$.*

This is a straightforward application of the Lebesgue dominated convergence theorem.

Lemma 2 *There exists a constant C' such that, for all ξ in $\mathbb{R}^2$,*
$|k^2\hat{G}_p(\xi) - C_\chi| < C' \min(1, \dfrac{\xi}{k})$

Proof :

As $k^2\hat{G}_p(\xi) - C_\chi = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\int_0^\infty \chi(\rho)\frac{e^{i\rho}}{4\pi}(e^{i\rho\frac{|\xi|}{k}\cos\theta} - 1)d\rho d\theta$, the result follows from the simple inequality :

$$|e^{i\rho\frac{|\xi|}{k}\cos\theta} - 1| < 2\min(1, \rho\frac{|\xi|}{k}).$$

The third lemma is just a technical one.

Lemma 3 *For all s, verifying $\frac{1}{2} \geq s \geq 0$ and for all $k > 0$,*

$$\max_\xi\left(\frac{\min(1, \frac{|\xi|}{k})}{(1+\xi^2)^s}\right) < \frac{1}{k^{2s}} \tag{8}$$

We won't prove this lemma, you have again to refer to [4] have the complete proof of it. We just want to stress the following point. When s is greater than $1/2$, we have the same kind of inequality with the exponent of k in the right-hand side of (8) will be kept fixed at 1.

We are now able to prove Theorem 1. Let s and s' satisfy $1 \geq s + s' > 0$. We are first going to show that there exists a positive constant α, such that for all ϕ in $H^{s+1}(R^2)$ and ψ in $H^{s'+1}(R^2)$

$$\Big| < k^2 G_p * \phi, \psi > - C_\chi < \phi, \psi > \Big| < \alpha \frac{||\phi||_{H^s} ||\psi||_{H^{s'}}}{k^{s+s'}}. \tag{9}$$

Indeed,

$$\begin{aligned} \Big| < k^2 G_p * \phi, \psi > - C_\chi < \phi, \psi > \Big| &= | < (k^2 \hat{G}_p - C_\chi)\hat{\phi}, \check{\hat{\psi}} > | & (10)\\ &\leq \alpha \frac{1}{k^{s+s'}} | < (1+\xi^2)^{s/2}\hat{\phi}, (1+\xi^2)^{s'/2}\check{\hat{\psi}} > | & (11)\\ &\leq \alpha \frac{1}{k^{s+s'}} ||\phi||_{H^s} ||\psi||_{H^{s'}}. & (12) \end{aligned}$$

Here, $\check{\hat{\psi}}(\xi) = \hat{\psi}(-\xi)$. The first inequality is obtained using the former lemmas. The second one directly comes from the definition of *Sobolev* norms by the mean of the Fourier transform.

The second point consists of showing a similar inequality with gradients :
Given positive s, s' satisfying $1 \geq s + s' > 0$, there exists a positive constant α, such that for all ϕ in $H^{s+1}(R^2)$ and ψ in $H^{s'+1}(R^2)$

$$\Big| < G_p * \nabla\phi, \nabla\psi > - \frac{C_\chi}{k^2} < \nabla\phi, \nabla\psi > \Big| < \alpha \frac{||\phi||_{H^{s+1}} ||\psi||_{H^{s'+1}}}{k^{s+s'+2}}. \tag{13}$$

To prove it you have just to apply the inequality (9) to $\nabla\phi, \nabla\psi$. We then prove the Theorem by adding inequalities (9) and (13).

2.2 Review of some fundamentals

In the next section we will consider the case where the boundary Γ is a curved surface. Before going into demonstrating a theorem similar to Theorem 1, we have to recall some well known technical points of geometry. We will restrict ourselves to closed compact and orientable manifolds. The main point we want to stress is the existence of a local parametrisation of special interest for our purpose.

Let Γ be the surface, x a point of Γ. Let's call t_1, t_2 an orthonormal basis of the tangent space $T_x\Gamma$ at x. We may suppose that t_1 and t_2 are eigenvectors of the curvature matrix. We complete this basis into a basis of $I\!R^3$ by adding the outer normal vector n.
Then we can locally define a parametrization of Γ by $T_x\Gamma$. Let U be an open neighborhood of x in Γ. We define θ to be the orthogonal projection of Γ onto $T_x\Gamma$. If we identify the tangent space with $I\!R^2$, then for each point y of U, there exists (τ_1, τ_2) such that $y = \theta^{-1}(\tau_1, \tau_2)$, and we have $y = x + \tau_1 t_1 + \tau_2 t_2 + (\frac{\tau_1^2}{2R_1} + \frac{\tau_2^2}{2R_2})n + \mathcal{O}(|\tau|^3)$, where R_1 and R_2 are the principal radii of curvature and τ is the vector $\tau_1 t_1 + \tau_2 t_2$. We then have a few properties.

Lemma 4

$$(i) \textit{if } y \in U, \ |x - y| = |\tau| + \mathcal{O}(|\tau|^3)$$

$$(ii) \ \frac{Dy}{D\tau} = 1 + \mathcal{O}(|\tau|^2)$$

where $\frac{Dy}{D\tau}$ *is the jacobian of the mapping* θ^{-1}.

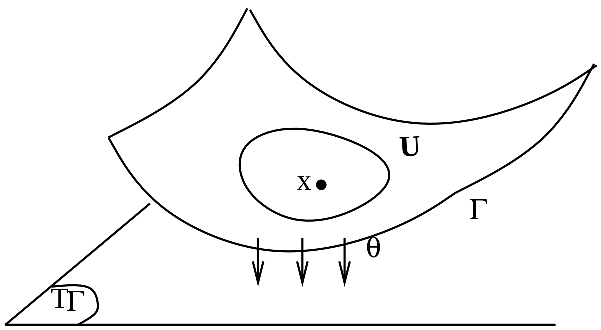

Figure 1:

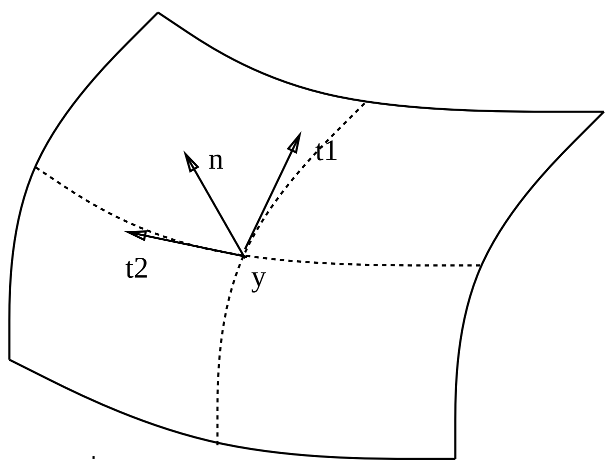

Figure 2: The local mapping of Γ onto $T_x\Gamma$.

The proofs are just exercises and may be found in [4].

3 Case of a compact curved surface

In this section, we will prove the following theorem, which is really similar to Theorem 1.

Theorem 2 *For $0 < s < 1$, there exists a positive constant C_1 such that for all functions ϕ in $H^{s+1}(\Gamma)$ and ψ in $H^1(\Gamma)$,*

$$| < \mathcal{B}\phi, \psi > + < C_\chi(Id + \frac{\Delta_\Gamma}{k^2})\phi, \psi > | \leq \tag{14}$$

$$C_1(\frac{1}{k^s}||\phi||_{H^s}||\psi||_{L^2} + \frac{1}{k^{s+2}}||\phi||_{H^{s+1}}||\psi||_{H^1}) \tag{15}$$

where C_χ is the same constant as before.

Before giving the proof of the theorem, we will again state some lemmas.

Lemma 5 *When k grows to infinity, then*

$$\int_\Gamma k^2 G_p(|x-y|)dx = C_\chi(1 + \mathcal{O}(\frac{1}{k^2})) \tag{16}$$

$$\int_\Gamma n_x.n_y k^2 G_p(|x-y|)dx = C_\chi(1 + \mathcal{O}(\frac{1}{k^2})) \tag{17}$$

where C_χ is the same constant as before. Furthermore, the convergences are uniform in y when k grows to infinity.

The skecht of the proof is the following. The first point to notice is that one can use the parametrization, because χ has a compact support, so that when k grows to infinity, the support of G_p will be small enough to be correctly parametrized. Then, we can write $x = y + \tau + L(\tau,\tau)n + \mathcal{O}(|\tau|^3)$. The last error term in this expansion is uniform in y, because of the compactness of the support of χ for k large enough and the compactness of the surface. It can therefore be replaced by a $\mathcal{O}(\frac{1}{k^3})$. The second point is to compute the integrals using the parametrisation and the evaluation of the Jacobian done in lemma 4. Thus, we have:

$$\int_\Gamma k^2 K_2(|x-y|)dx = \int_\Gamma k\chi(k|x-y|)\frac{e^{ik|x-y|}}{4\pi|x-y|}dx \tag{18}$$

$$= \int_{R^2} k\chi(k|\tau| + \mathcal{O}(\frac{1}{k^2}))\frac{e^{ik|\tau|}}{4\pi|\tau|}(1 + \mathcal{O}(\frac{1}{k^2}))d\tau \tag{19}$$

$$= \left(\int_{R^2} k\chi(k|\tau|)\frac{e^{ik|\tau|}}{4\pi|\tau|}d\tau\right)(1 + \mathcal{O}(\frac{1}{k^2})) \tag{20}$$

$$= C_\chi(1 + \mathcal{O}(\frac{1}{k^2})) \tag{21}$$

$$\tag{22}$$

The second part of the lemma is shown just in the same way, using the fact that $n_x.n_y = 1 + \mathcal{O}(|\tau|^2)$.

Using the same kind of demonstration, you can prove the following lemma.

Lemma 6 *For $0 < s < 1$, when k grows to infinity, $\int_\Gamma k^4 {G_p}^2(|x-y|)|x-y|^{2+2s}dx = \mathcal{O}(\frac{1}{k^{2s}})$, the convergence is still uniform in y.*

Before stating a few more lemmas we recall a definition of the $H^s(\Gamma)$ *Sobolev* norm.

Definition 1 *If $\phi \in H^s(\Gamma)$, for all s such that $0 < s < 1$, then $||\phi||^2_{H^s(\Gamma)} = ||\phi||^2_{L^2} + \int_{\Gamma\times\Gamma} \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{n+2s}}dxdy$ where n is the dimension of the manifold Γ.*

We refer to [7] for more information. We have to set a last lemma before proving theorem 2.

Lemma 7 *Let s satisfy $0 < s < 1$; there exists a positive constant C_1 such that,*

$$(i)\ ||\int_\Gamma k^2 K_2(|x-y|)(\phi(x)-\phi(y))dy||^2_{L^2} \leq C_1 \frac{||\phi||^2_{H^s}}{k^{2s}} \tag{23}$$

$$(ii)\ ||\int_\Gamma k^2 K_2(|x-y|)(\phi(x)n_x.n_y-\phi(y))dy||^2_{L^2} \leq C_1 \frac{||\phi||^2_{H^s}}{k^{2s}} \tag{24}$$

Proof :
For (i) we have

$$\begin{aligned}
& ||\int_\Gamma k^2 K_2(|x-y|)(\phi(x)-\phi(y))dy||^2_{L^2} && (25)\\
& \leq \int_\Gamma \left| \int_\Gamma k^2 K_2(|x-y|)(\phi(x)-\phi(y))dy \right|^2 dx && (26)\\
& \leq \int_\Gamma \left(\int_\Gamma k^4 K_2^2 |x-y|^{2+2s}dy \int_\Gamma \frac{|\phi(x)-\phi(y)|^2}{|x-y|^{2+2s}}dy \right) dx && (27)\\
& \leq \frac{C_1}{k^{2s}}||\phi||^2_{H^s} && (28)
\end{aligned}$$

The first inequality is set by using of *Cauchy-Schwarz* inequality, the second with the use of the previous definition and lemma 6. For (ii) we are going to evaluate $||\int_\Gamma k^2 K_2(|x-y|)(n_x.n_y-1)\phi(x)dx||_{L^2}$, then, combining it to (i), we will have our result.

$$\begin{aligned}
& ||\int_\Gamma k^2 K_2(|x-y|)(n_x.n_y-1)\phi(x)dx||^2_{L^2} \\
& \qquad = \int_\Gamma dy \left(\int_\Gamma k^2 K_2(|x-y|)(n_x.n_y-1)\phi(x)dx \right)^2 \\
& \qquad \leq \int_\Gamma dy ||\phi||^2_{L^2} \int_\Gamma k^2 \chi(k|x-y|)^2 \frac{(n_x.n_y-1)^2}{16\pi^2|x-y|^2}dx.
\end{aligned}$$

Yet, for x in a neighborhood of y, we have $n_x.n_y - 1 = \mathcal{O}(|x-y|)^2$ uniformly in y. Using again the change of variables $x \to \tau$, we have

$$\begin{aligned}
||\int_\Gamma k^2 K_2(|x-y|)(n_x.n_y-1)\phi(x)dx||^2_{L^2} & \leq C_2 ||\phi||^2_{L^2} \int_0^\infty k^2 \chi(kr)^2 \frac{r^4}{r^2} r dr && (29)\\
& \leq C_3 \frac{||\phi||^2_{L^2}}{k^2}. && (30)
\end{aligned}$$

Finally, $|| \int_\Gamma k^2 K_2(|x-y|)(n_x.n_y - 1)\phi(x)dx||^2_{L^2} \leq C_3 \frac{||\phi||^2_{L^2}}{k^2} \leq C_3 \frac{||\phi||^2_{H^s}}{k^{2s}}$. Adding this result to (i), we get (ii)•

We can now prove Theorem 2. First, we are going to consider $| < \int_\Gamma k^2 G_p \phi(x) n_x.n_y dx, \psi > -C < \phi, \psi > |$. We have

$$
\begin{aligned}
| < \int_\Gamma k^2 G_p \phi(x) n_x.n_y dx, \psi > -C < \phi, \psi > & \qquad (31) \\
\leq ||\psi||_{L^2} ||C\phi - \int_\Gamma k^2 G_p(|x-y|)\phi(x) n_x.n_y dx||_{L^2} & \qquad (32) \\
\leq ||\psi||_{L^2} (||(C - \int_\Gamma k^2 G_p n_x.n_y(|x-y|)dx)\phi||_{L^2} & \qquad (33) \\
+ || \int_\Gamma k^2 G_p(\phi(x) n_x.n_y - \phi(y))dx||_{L^2}). & \qquad (34)
\end{aligned}
$$

The part (33) is bounded from lemma 5, the part (34) with lemma 7.

Hence, we have

$$
\begin{aligned}
| < \int_\Gamma k^2 G_p(|x-y|)\phi(x) n_x.n_y dx, \psi > -C < \phi, \psi > | & \qquad (35) \\
\leq C_1 ||\psi||_{L^2} ||(\frac{||\phi||_{L^2}}{k^2} + \frac{||\phi||_{H^s}}{k^s}) & \qquad (36) \\
\leq C_1 ||\psi||_{L^2} \frac{||\phi||_{H^s}}{k^s}. & \qquad (37)
\end{aligned}
$$

For the term concerning the gradients in (15), we proceed just in the same way.

$$
\begin{aligned}
& | < \int_\Gamma G_p(|x-y|) rot_\Gamma \phi(x) dx, rot_\Gamma \psi > - \frac{C}{k^2} < rot_\Gamma \phi, rot_\Gamma \psi > | \\
& \leq ||\psi||_{H^1} || \frac{C}{k^2} rot_\Gamma \phi - \int_\Gamma G_p(|x-y|) rot_\Gamma \phi(x) dx||_{L^2} \\
& \leq ||\psi||_{H^1} (||(\frac{C}{k^2} - \int_\Gamma G_p(|x-y|)dx) rot_\Gamma \phi||_{L^2} \\
& + || \int_\Gamma G_p(|x-y|)(rot_\Gamma \phi(x) - rot_\Gamma \phi(y))dx||_{L^2})
\end{aligned}
$$

As above, the first part is bounded from lemma 5, the second one from lemma 7. Thus, $| < \int_\Gamma G_p(|x-y|) rot_\Gamma \phi(x) dx, rot_\Gamma \psi > - \frac{C}{k^2} < \phi, \psi > |$

$$
\begin{aligned}
& \leq C_1 ||\psi||_{H^1} ||(\frac{||\phi||_{H^1}}{k^4} + \frac{||\phi||_{H^{s+1}}}{k^{s+2}}) \\
& \leq C_1 ||\psi||_{H^1} \frac{||\phi||_{H^{s+1}}}{k^{s+2}}.
\end{aligned}
$$

Adding the computations done for each of the two components of the initial expression, we end the proof of the theorem•

The two terms of the dominant part of the inequation of theorem 2 do not seem to be of the same order. Nevertheless when one uses finite elements, the typical size h of an element is chosen to be proportional to 1/k, so that, for such meshes, $||\nabla u||_{L^2}$ is equivalent to $k||u||_{L^2}$; then the two terms in the right hand side of (15) appear to be of the same order in practice.

4 The spherical case

In this section we are going to show the interest of the previous results when Γ is a sphere. We consider here the sphere because both the operator and the limit of the preconditioner for large k may be diagonalized in the same basis. So, we are going to study the spectrum of the two operators and analyse the effect of the preconditioning.
Let us first recall some fundamental points about special functions on the sphere. Referring to [2] or [9] for more details. Let θ, ϕ be the spherical coordinates, with θ the polar angle

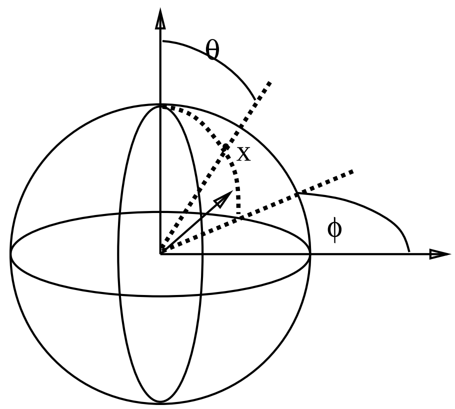

Figure 3:

(cf. fig. 3); then, there exists an orthonormal basis of $L^2(\Gamma)$ built with eigenvectors of the Laplace-Beltrami operator Δ_Γ on the sphere, called spherical harmonics and noted $Y_{n,l}(\theta, \phi)$ with $n = 0, ..., +\infty$ and $l = -n, ..., n$. It is a well-known fact that

$$\Delta_\Gamma Y_{n,l}(\theta, \phi) = -n(n+1)Y_{n,l}(\theta, \phi). \tag{38}$$

So,

$$(I + \frac{\Delta_\Gamma}{k^2})Y_{n,l} = (1 - \frac{n(n+1)}{k^2})Y_{n,l}. \tag{39}$$

Furthermore, we introduce spherical Hankel an Bessel functions noted h_n and j_n. It is also a well known fact that $h_n(kr)Y_{n,l}$ is a solution of the Helmholtz equation outside the sphere with the Sommerfeld radiation condition, and that $j_n(kr)Y_{n,l}$ is a solution of the Helmholtz equation inside the sphere. If we note A the integral equation operator we are interested in, it may be shown that

$$AY_{n,l} = h'_n(k)j'_n(k)\frac{k}{i}Y_{n,l} \tag{40}$$

(cf. [4] or [1]).

To analyze these results, we are going to give an equivalent to $h'_n(k)j'_n(k)$ when n is going to infinity. We have $h'_n(k) \sim i\dfrac{(2n)!(n+1)}{2^n n! k^{n+1}}$ *and* $j'_n(k) \sim \dfrac{2^n n! n k^n}{(2n+1)}$. Thus $h'_n(k)j'_n(k)\dfrac{k}{i} \sim \dfrac{n(n+1)}{2n+1} \sim \dfrac{n}{2}$. Hence, the action of the preconditioner is to transform the highest eigenvalues of the operator into the smallest. Indeed, when $n >> k$, the eigenvalues of the preconditioner are equivalent to $\dfrac{n^2}{k^2}$ and they transform the eigenvalue of $Y_{n,l}$, equivalent to $\dfrac{n}{2}$, to a value equivalent to $\dfrac{k^2}{2n}$. This is not reducing the condition number because, instead of having eigenvalues going to infinity, one has them going to zero as $\frac{1}{n}$. Nevertheless, it improves the rate of convergence, because the highest eigenvalues of A have a high mutiplicity. $h'_n(k)j'_n(k)\frac{k}{i}$ is of multiplicity $2n+1$. So, the upper part of the spectrum of A is very dense and when one applies the preconditioning operator you make the upper part of the spectrum of AB^{-1} be much more empty. This point as been shown by F.X. Roux (cf. [10]) and others to be a really important one in the improvement of the rate of convergence.

5 Numerical experiments

First we show the repartition of the spectrum for both the preconditioned and the unpreconditioned operator when Γ is a sphere of radius 1 and k=14.7 in fig. 4 and 5. In these figures we

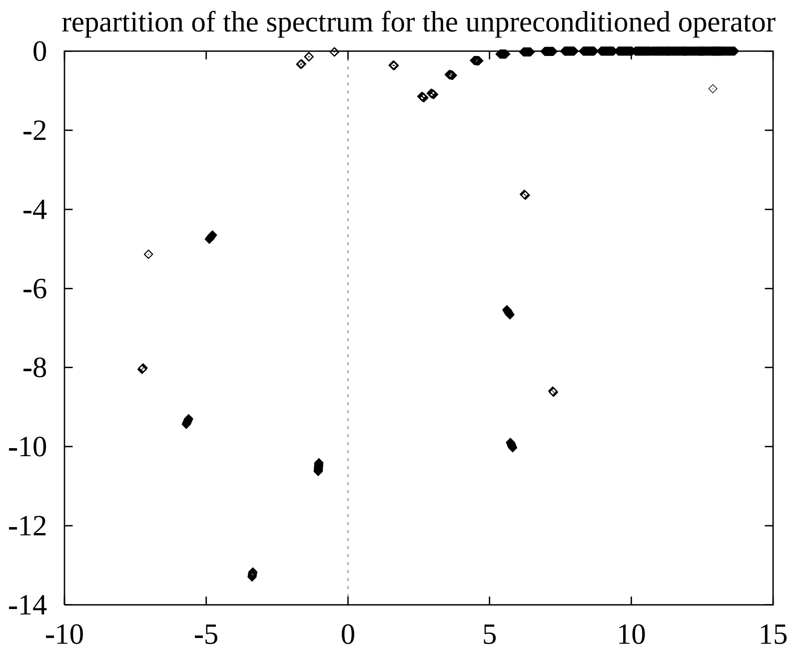

Figure 4:

clearly see that for the unpreconditioned operator, the spectrum gets denser and denser when the eigenvalues are growing and that, at the opposite, for the preconditioned one, the spectrum is more and more empty as the eigenvalues grows.

The second test aims to show the real efficiency of the preconditioning method in the improvement of the rate of convergence. We still consider a sphere of radius 1, and for various values of the frequency $\frac{kc}{2\pi}$, we show the number of iterations needed to reach convergence. In tables 5 and 5, ϵ is the value of the residual under which we say that we have reached convergence. The mesh has 1026 degrees of freedom. These numerical results really clearly show the strong efficiency of our method.

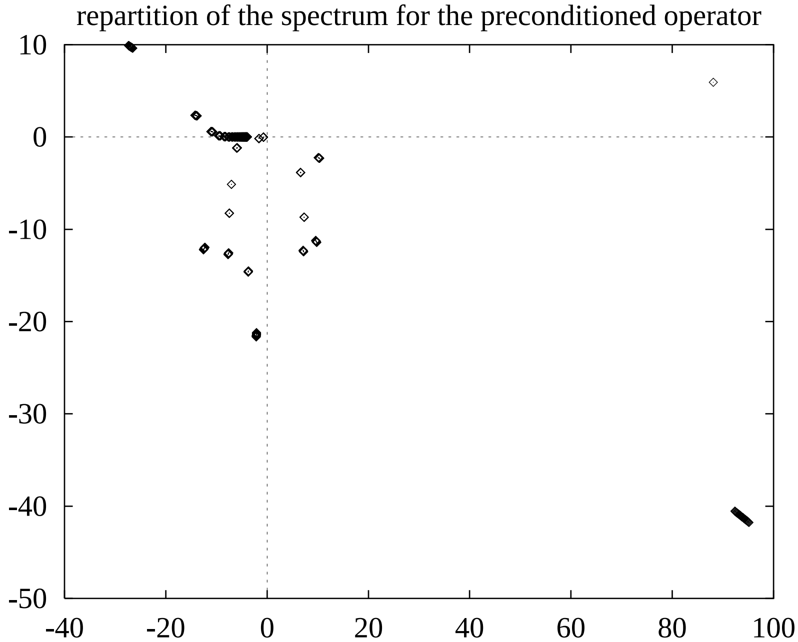

Figure 5:

ϵ^2	500Hz	600Hz	700Hz	750Hz	780Hz
10^{-4}	> 300	185	85	265	> 300
10^{-5}	> 300	> 300	> 300	> 300	> 300
10^{-6}	> 300	> 300	> 300	> 300	> 300

Table 1: number of iterations with the unpreconditioned algorithm.

ϵ^2	500Hz	600Hz	700Hz	750Hz	780Hz
10^{-4}	12	2	3	2	2
10^{-5}	25	4	4	4	5
10^{-6}	> 50	10	9	12	21

Table 2: number of iterations for the preconditioned algorithm.

Conclusion

In this paper, we have presented a preconditioning method for an integral equation problem which is really efficient from a numerical point of view. The explanation given for this, is that the preconditioning operator makes the upper part of the spectrum much more empty. It is an important fact that the mathematical result obtained in the plane case still remains true for a more general surface.

References

[1] T. Abboud, *Etude mathématique et numérique de quelques problèmes de diffraction d'ondes électromagnétiques*, PhD thesis, Ecole polytechnique, 1991.

[2] M. Abramowitz and I. A. Stegun, eds., *Handbook of mathematical functions*, National Bureau of Standards, U.S. Government, Washington D.C., 1964.

[3] K. Chen, *Conjugate gradient methods for the solution of boundary integral equations on a piecewise smooth boundary*, J. Comput. Phys., 97 (1991), pp. 127–143.

[4] A. de La Bourdonnaye, *Accélération du traitement numérique de l'équation de Helmholtz par équations intégrales et parallélisation.*, PhD thesis, Ecole polytechnique, Palaiseau, France, 1991.

[5] M. Hamdi, *Une formulation variationnelle par équations pour la résolution de l'équation de Helmholtz avec des conditions aux limites mixtes*, CRAS, Série II, t. 292 (1981), pp. 17–20.

[6] P. Joly, *Méthodes de gradient conjugué*, Tech. Rep., Université Pierre et Marie Curie, Laboratoire d'analyse numérique, 1984.

[7] J. L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, vol. 1, Dunod, Paris, 1968.

[8] G. Markham, *Conjugate gradient type methods for indefinite, asymmetric, and complex systems.*, IMA J. Numer. Anal., 10 (1990), pp. 155–170.

[9] A. F. Nikiforov and V. B. Uvarov, *Special fonctions of mathematical physics*, Birkhäuser, Basel Boston, 1988.

[10] F. Roux, *Spectral analysis of the interface operators associated with the preconditioned saddle-point principle domain decomposition method*, in Domain Decomposition Conference (Norfolk), 1991.