High frequency approximation of integral equations modelizing scattering phenomena.

Armel de La Bourdonnaye
CERMICS-INRIA
BP93
06902 Sophia-Antipolis Cedex
France

Abstract

In this paper, we present a new way of discretizing integral equations coming from high frequency wave propagation. Indeed, using the eikonal equation, we will write that the solution is locally the product of an amplitude by an oscillating function whose phase gradient modulus is the wave number. Discretizing in order to keep this relation, we will show that, is the limit of high frequencies, the matrices we obtain are sparse (as sparse as volumic finite-element methods, in fact), which is not the case with the classical way of discretizing for example with *P1-Lagrange* or H_{div} (see [11] or [13]) finite elements. More precisely, if N is the number of degrees of freedom, we lower the complexity from $\mathcal{O}(N^2)$ to $\mathcal{O}(N)$.

Approximation haute fréquence des équations intégrales venant des problèmes de diffraction.

Résumé

Dans ce rapport, nous présentons une nouvelle manière de discrétiser les équations intégrales qui viennent des phénomènes de propagation en régime harmonique à haute fréquence. En effet, utilisant l'équation eikonale nous allons écrire que localement la solution que l'on cherche s'écrit comme une amplitude multipliée par une fonction oscillante dont le gradient de la phase est de module proportionnel au nombre d'onde. En discrétisant de manière à conserver cette information, nous verrons que, dans la limite des hautes fréquences, les matrices que nous obtenons sont très creuses, ce qui n'est pas le cas lorque l'on discrétise de manière classique avec des éléments finis *P1* ou H_{div} (cf. [11] or [13]). Plus précisement, si N est le nombre de degrés de liberté, nous passons d'une complexité en $\mathcal{O}(N^2)$ à une complexité en $\mathcal{O}(N)$.

Introduction.

For solving scattering problems in frequency domain, there exists a large number of methods. Among them, the one of integral equations is known to be the most accurate. It is often chosen to validate other approaches (see [15] for instance). Nevertheless, its main drawback is that it cannot be used at high frequencies. Indeed, the resolution of integral equations with finite elements leads to full matrices with $\mathcal{O}(\frac{1}{\lambda^2})$ degrees of freedom, where λ is the wave-length.
Lots of studies have been done to reduce either the number of degrees of freedom or the number of significant coefficients in the matrix. In two space dimensions, V. Rokhlin [14] proposed a method using a decomposition in Hankel functions and addition formulae for these functions. This technique cannot be used in three dimensions because the corresponding addition formulae (Gegenbauer's formulae, [12], [17]) are much more complicated and cannot be treated in the same way. In [2], F.X. Canning post-treats the matrix with Fourier transforms, which allows him to neglect lots of coefficients, just keeking a few significant ones. Nevertheless, in this case too, the extension to three dimensions is not so easy, because Fast Fourier Transforms can be used only on regularly meshed parallelograms. Attempts have been performed using wavelets, following the idea of G. Beylkin, R. Coifman and V. Rokhlin [1], but the Green kernel of the Helmholtz equation is not enough decreasing to use this technique.
In [5] we presented a method using the coupling between integral equations (on an axisymmetric shape wrapping the scatterer) and volumic finite elements (between this shape and the scatterer). The technique we are going to describe here is more efficient but less general since it does not allow to treat heterogeneous media. It is based on the same idea as F.X. Canning's one [2]. Furthermore, it gives a mathematical explanation for it. Instead of localizing the basis functions only in space with a step proportional to the wave-length, we will localize both in space and in the direction of propagation (i.e. in the cotangent fiber bundle) with a larger spatial step in order to keep the number of degrees of freedom constant (or of the same order). We will show that this idea will lead to a really small number of significant interaction coefficients. This method will allow the treatment of scattering problems with much higher frequencies than before. Futhermore, it can be easily coupled with volumic or surfacic finite elements.

1 Position of the problem.

Here we will set some notations and write the problem we want to solve. Let Ω_i be a bounded open set of $\mathbb{R}^3$, Γ its boundary and Ω_e the interior of its complementary. (see figure 1).

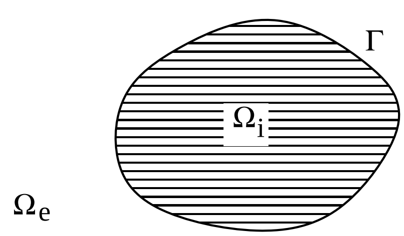


Figure 1: Position of the problem.

We suppose Γ is regular. We look for u in $H^1(\Omega_i) \cap H^1_{loc}(\Omega_e)$ solution of the Helmholtz equation with Neumann or Dirichlet boundary conditions. The function u satifies :

$$
\begin{aligned}
& \Delta u + k^2 u = 0 \text{ in } \Omega_i \cup \Omega_e && (1)\\
& \frac{\partial u}{\partial r} + \imath k u = o(\frac{1}{r}) \text{ for } r \to \infty && (2)\\
\text{either} \quad & \frac{\partial u}{\partial n} = g \text{ on } \Gamma && (3)\\
\text{or} \quad & u = f \text{ on } \Gamma && (4)
\end{aligned}
$$

with the Dirichlet right-hand side f in $H^{1/2}(\Gamma)$ and the Neumann right-hand side g in $H^{-1/2}(\Gamma)$. As usually, the wave number k satisfies $k = \frac{2\pi}{\lambda}$ where λ is the wave-length. We will use an integral equation. In the case of the Dirichlet problem the unknown is the jump of $\frac{\partial u}{\partial n}$ through Γ, which is in $H^{-1/2}(\Gamma)$: $p = \left[\frac{\partial u}{\partial n}\right]$ and the integral equation is (see for instance [4]) :

$$\forall q \ \in \ H^{-1/2}(\Gamma),$$

$$\int_{\Gamma\times\Gamma} \frac{e^{\imath k|x-y|}}{4\pi|x-y|} p(x)q(y)dxdy = \int_{\Gamma} q(x)f(x)dx \tag{5}$$

For the Neumann problem, the unknown is the jump of u : $\phi = [u] \in H^{1/2}(\Gamma)$ and the integral equation is (following Hamdi, [9]):

$$\forall \psi \ \in \ H^{1/2}(\Gamma),$$

$$\int_{\Gamma\times\Gamma} \frac{e^{\imath k|x-y|}}{4\pi|x-y|}(rot_\Gamma\psi(x).rot_\Gamma\phi(y) - k^2 n_x.n_y\psi(x)\phi(y))dxdy = \int_\Gamma \psi(x)g(x)dx. \tag{6}$$

In our case, we will restrict the space of the right-hand side functions f ou g which correspond to a given incident wave. We will consider only traces of harmonic waves whose sources have a non zero distance from the scattering object. These functions are thus very regular (analytic if the surface Γ is analytic). It describes a space which is no longer dense in $H^{1/2}(\Gamma)$ or $H^{-1/2}(\Gamma)$.

2 Presentation and justification of the finite element basis.

With the restriction made above on the space of admissible incident waves, it is usual to consider (at least in the case where the scatterer is convex) that the solution of the integral equations problem can be written as :

$$u(x) = \sum_i a_i(x,k)e^{\imath k\phi_i(x)} \tag{7}$$

where the sum is locally finite and the $a_i(x,k)$ admit a development in $\frac{1}{k}$ (see [16]). In that case, the eikonal equation is

$$|\nabla\phi|^2 = 1 \tag{8}$$

We know that the formula (7) is not correct in what is called the *penumbra region* and in the neighborhood of the caustics. In the last case (see [8] or [10])

$$u(x) = e^{\imath k\phi(x)}(\frac{g(x)}{k^{1/3}}Ai(k^{2/3}\rho(x)) + \frac{h(x)}{k^{2/3}}Ai'(k^{2/3}\rho(x))) \tag{9}$$

where Ai is the Airy function. In the same way, we have an eikonal equation (see [16] for instance) :

$$\begin{aligned} |\nabla\phi|^2 + \rho|\nabla\rho|^2 = 1 \\ \nabla\phi.\nabla\rho = 0 \end{aligned} \tag{10}$$

We can see that, locally (i.e. on a region smaller than $\mathcal{O}(1/k^{1/3})$, see [7]), the solution is asymptotically equivalent to a finite sum of terms like

$$e^{\imath kx.\xi}a(x) \tag{11}$$

where ξ is a unit vector of R^3.

We will show that, locally, one can approach the solution by a function like the one of formula (11) where a will be compactly supported and with a limited regularity (for instance P_1) and ξ varies in a discrete set of vectors of the sphere S^2. What we are going to do is thus a microlocal discretization. Instead of having basis functions on the surface Γ whose support's characteristic length is $\mathcal{O}(\lambda)$, leading thus to $\mathcal{O}(k^2)$ degrees of freedom, we will discretize the cosphere bundle ΓS^2 with functions whose support size is $\mathcal{O}(\lambda^\alpha)$ on the surface multiplied by an oscillating term $e^{\imath x.\xi}$ where ξ describes a discrete set of values of S^2 whose cardinal is $\mathcal{O}(k^{2-2\alpha})$ and thus with a step of $\mathcal{O}(\lambda^{1-\alpha})$, where $0 < \alpha < 1$, in order to still have globally $\mathcal{O}(k^2)$ degrees of freedom. More precisely, we take $\Phi_{i,j}(x) = e^{ik\xi_j.(x-x_i)}\phi_i(x)$ as basis functions, where the points x_i describe the surface Γ and their number is $\mathcal{O}(k^{2\alpha})$, the vectors ξ_j describe the sphere S^2 and their number is $\mathcal{O}(k^{2-2\alpha})$, at last, the functions ϕ_i are piecewise C^∞ and globally C^{m-1}, their value is 1 in x_i and 0 in x_j, $j \neq i$, and their m first traces are null on the boundary of their supports. The diameter of the support of a basis function is thus $\mathcal{O}(\lambda^\alpha)$.

3 Asymptotics.

As in a classical finite-element method, we try to compute the interaction of two basis functions. Here, since we are interested in the high frequency limit, we will compute only the first term of the expansion in $\frac{1}{k}$. Thus, we are looking for an equivalent of

$$A(k) = \int_{\Gamma^2} \frac{e^{\imath k|x-y|}}{4\pi|x-y|}\phi_i(x)\phi_{i'}(y)e^{\imath k\xi_j.(x-x_i)}e^{\imath k\xi_{j'}.(y-x_{i'})}dxdy. \tag{12}$$

In order to take the homogeneity in λ into account, we rewrite the spatial function as $\phi_i(k^\alpha(x-x_i))$. We set $\tilde{x} = k^\alpha(x-x_i)$ and $\tilde{y} = k^\alpha(y-x_{i'})$. Then, we have :

$$A(k) = \frac{1}{k^{4\alpha}}\int \frac{e^{\imath|k(x_i-y_{i'})+k^{1-\alpha}(\tilde{x}-\tilde{y})|}}{4\pi|x-y|}\phi_i(\tilde{x})\phi_{i'}(\tilde{y})e^{\imath k^{1-\alpha}\xi_j.\tilde{x}}e^{\imath k^{1-\alpha}\xi_{j'}.\tilde{y}}d\tilde{x}d\tilde{y} \tag{13}$$

To evaluate the different terms, we will use stationary and non-stationary phase theorems ([3] or [6]). In (13), the amplitude is

$$\frac{\phi_i(\tilde{x})\phi_{i'}(\tilde{y})}{4\pi|x-y|}, \tag{14}$$

and the phase is

$$p = |k(x_i - y_{i'}) + k^{1-\alpha}(\tilde{x}-\tilde{y})| + k^{1-\alpha}(\xi_j.\tilde{x} + \xi_{j'}.\tilde{y}). \tag{15}$$

In order to perform stationary phase computations, we will restrain α to be inferior to 1/2.

3.1 Far field interactions.

We study first far field interactions which correspond to the cases where the supports of ϕ_i and $phi_{i'}$ are disjoint. Then, the amplitude is regular with C^{m-1} continuity. We will show a series of three propositions which cover the different kinds of interaction. We have first

Proposition 1 *Let's suppose that the phase is neither stationary in x nor in y. Then,*
(i) if it is not stationary on the lines of singularity of ϕ_i and $\phi_{i'}$,

$$A(k) = \frac{1}{k^{4\alpha}}\mathcal{O}(\frac{1}{k^{(2m+4)(1-\alpha)}}) \tag{16}$$

(ii) if it is not stationary on the lines of singularity of only one of the two functions ϕ_i and $\phi_{i'}$ and does not degenerate,

$$A(k) = \frac{1}{k^{4\alpha}}\mathcal{O}(\frac{1}{k^{(2m+3)(1-\alpha)}k^{1/2-\alpha}}) \tag{17}$$

(iii) If it is stationary on the lines of singularity of both ϕ_i and $\phi_{i'}$ and does not degenerate,

$$A(k) = \frac{1}{k^{4\alpha}}\mathcal{O}(\frac{1}{k^{(2m+2)(1-\alpha)}k^{1-2\alpha}}) \tag{18}$$

Proof :
Let us denote by $\vec{L}$ the vector $x_i - x_{i'}$. Then, we have the development $|k(x_i - x_{i'}) + k^{1-\alpha}(\tilde{x} - \tilde{y})| = kL + k^{1-\alpha}(\tilde{x} - \tilde{y}).\frac{\vec{L}}{L} + \mathcal{O}(k^{1-2\alpha})$. So, to the same order, the phase p is

$$P_1 = kL + k^{1-\alpha}\left((\tilde{x} - \tilde{y}).\frac{\vec{L}}{L} + \xi_j.\tilde{x} + \xi_{j'}.\tilde{y}\right). \tag{19}$$

We denote by $\mathcal{L}_x$ the operator

$$\mathcal{L}_x = k^{1-\alpha}\frac{\nabla_x p}{\imath|\nabla_x p|^2}.\nabla_x \tag{20}$$

which is well defined if the phase is regular in x. We introduce the same notations in y. This operator is bounded in k as one can see in formula (19). In case (i), we integrate m times by part in x and in y and we obtain :

$$A(k) = \frac{1}{k^{4\alpha}}\frac{1}{k^{2m(1-\alpha)}}\int e^{ip}\mathcal{L}_x^{\star m}\mathcal{L}_y^{\star m}\frac{\phi_i(\tilde{x})\phi_{i'}(\tilde{y})}{4\pi|x-y|}dxdy \tag{21}$$

where $\mathcal{L}^{\star m}$ denotes the adjoint of $\mathcal{L}$. As functions ϕ_i and $\phi_{i'}$ are both piecewise C^∞, and globally C^{m-1}, we can integrate by part once more in each variable. We have both an integral on the support of $\phi_i.\phi_{i'}$ and an integral on its boundary and its lines of singularity. Nevertheless on both terms we have again one order in $k^{1-\alpha}$. Now, for the integral on the support, we can again integrate it by part and we again obtain an order in $k^{1-\alpha}$. For the integral on the lines of singularity, if the phase is not stationary on them, we can integrate by part and we obtain another term in $k^{1-\alpha}$. If it is stationary we develop the phase up to the next order in k. Let x_0 be the point where the phase is stationary. In the neighborhood of this point,

$$\begin{aligned} p(\delta x) &= |k(x_0 - y) + k^{1-\alpha}\delta x| + \xi_i.(x_0 + \delta x), &(22)\\ &= k(|x_0 - y| + \xi.x_0) + k^{1-2\alpha}\frac{(\delta x \xi_i.n)^2}{2|x_0 - y|} + \mathcal{O}(\delta x^3 k^{1-3\alpha}). &(23)\end{aligned}$$

where n is the normal to Γ. Since the term of order $k^{1-\alpha}$ is null for the phase is stationary. In this case the stationary phase theorem says that the integral is $\mathcal{O}(k^{1/2-\alpha})$. This allows us to conclude the proof of the different cases of the proposition. $\Box$

Then, we show

Proposition 2 *When the phase is stationary in one of the two variables only (we denote it by x) without degenerating, then :*
(i) If it is not stationary on the set of singularity of the shape function of the other variable ($\phi_{i'}$),

$$A(k) = \frac{1}{k^{4\alpha}}\mathcal{O}(\frac{1}{k^{(m+2)(1-\alpha)}k^{1-2\alpha}}) \tag{24}$$

(ii) If it is stationary on the set of singularity of $\phi_{i'}$,

$$A(k) = \frac{1}{k^{4\alpha}}\mathcal{O}(\frac{1}{k^{(m+1)(1-\alpha)}k^{3/2(1-2\alpha)}}) \tag{25}$$

Proof :
In y we do the same job as for the previous proposition. In x, we have to develop the phase to the next order in k. We denote by C_x the curvature operator of Γ in x. Its eigenvalues are $\dfrac{1}{R_1}$ and $\dfrac{1}{R_2}$ where R_1 and R_2 are the two radii of curvature of the surface. Hence, $x - x_i = \tau + \frac{1}{2}C_{x_i}(\tau,\tau).n_{x_i} + \mathcal{O}(|\tau|^3)$ where τ is tangent to the surface Γ in x_i. We use the same notations with primes (') for y. We perform the change of variables : $\tilde{\tau} = k^{\alpha}\tau$. Then, $|k(x-y)| = |k\vec{L} + k^{1-\alpha}(\tilde{\tau} - \tilde{\tau}') + k^{1-2\alpha}(C(\tilde{\tau},\tilde{\tau})n - C'(\tilde{\tau}',\tilde{\tau}')n')| + \mathcal{O}(|\tilde{\tau}|^3 + |\tilde{\tau}'|^3)$. The second order term of this expression is

$$t_2 = \frac{k^{1-2\alpha}}{2}\left((C(\tilde{\tau},\tilde{\tau})n - C'(\tilde{\tau}',\tilde{\tau}')n').\frac{\vec{L}}{L} + \left|\frac{\vec{L}}{L} \wedge (\tilde{\tau} - \tilde{\tau}')\right|^2\right) \tag{26}$$

The term of second order in the last part of the phase is

$$t_2{}' = \frac{k^{1-2\alpha}}{2}\left(\xi_j.nC(\tilde{\tau},\tilde{\tau}) + \xi_{j'}.n'C'(\tilde{\tau}',\tilde{\tau}')\right) \tag{27}$$

Thus, according to the stationary phase theorem, when the phase is not degenerated but is stationary in the variable τ', the integral in this variable leads to a term in $\dfrac{1}{k^{1-2\alpha}}$. This concludes the proof of the proposition. $\Box$

At last, we have

Proposition 3 *When the phase is stationary in the two variables, without degenerating,*

$$A(k) = \mathcal{O}(\frac{1}{k^2}) \tag{28}$$

Proof :
We proceed as in the proof of the previous proposition for the stationary variable. We then have $A(k) = \dfrac{1}{k^{4\alpha}}\mathcal{O}(\dfrac{1}{k^{(1-2\alpha).\frac{dim(\Gamma^2)}{2}}})$. This ends the proof of the proposition. $\Box$

We are now presenting the cases when the phase is stationary.

Proposition 4 *In $A(k)$, the phase is stationary when* $\dfrac{\vec{L}}{L} + \xi_j // n_{x_i}$ *and* $\dfrac{\vec{L}}{L} - \xi_{j'} // n_{x_{i'}}$ *with an accuracy of* $\mathcal{O}(k^{-\alpha})$.

Proof :
In formula (19), the term of first order is $\left((\tilde{x} - \tilde{y}).\frac{\vec{L}}{L} + \xi_j.\tilde{x} + \xi_{j'}.\tilde{y}\right)$ and the rest is $\mathcal{O}(k^{-\alpha})$. Then, projecting the gradient of this term on $T_{x_i}\Gamma \times T_{x_{i'}}\Gamma$, the product of the planes tangent to Γ at x_i and $x_{i'}$, we obtain that the tangential gradient is null if and only if $\dfrac{\vec{L}}{L} + \xi_j // n_{x_i}$ and $\dfrac{\vec{L}}{L} - \xi_{j'} // n_{x_{i'}}$. The relative accuracy comes from the terms in $k^{-\alpha}$ that we neglected. $\square$

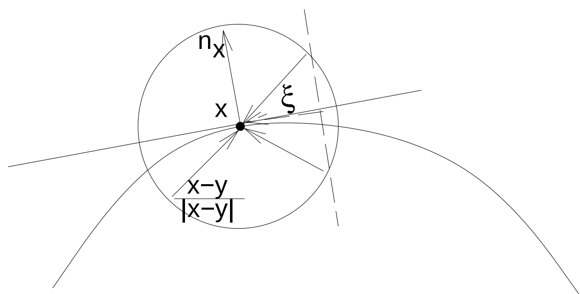


Figure 2:

Let us give a geometrical interpretation of these conditions. Saying that $\dfrac{\vec{L}}{L} + \xi_j // n_{x_i}$ means that ξ_j is either $-\dfrac{\vec{L}}{L}$, or its symmetric with respect to the plane tangent to Γ in x_i. The relative accuracy given in the proposition means that instead of considering x_i we can consider any x in the support of ϕ_i. Hence, ξ_i is the direction of a ray going from the support of ϕ_i to the support of $\phi_{i'}$ either directly or after a specular reflection on the surface Γ (see figure 2). The condition on $\xi_{j'}$ can be interpreted in the same way and the relative accuracy means that we can consider any y in the support of $\phi_{i'}$ instead of $x_{i'}$. So, the phase is stationary if we are in one of the four cases illustrated in the following figures (figures 3 to 6). When the phase is stationary only in one variable (x for instance), the interpretation is simpler. Indeed, we then have only two cases : transmission in x_i or reflection in x_i.

Figure 3: Transmission in x, transmission in y

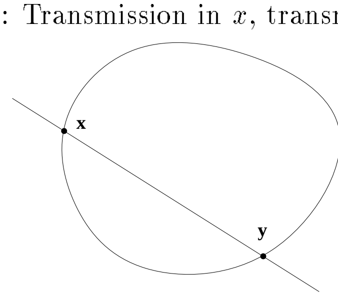


Figure 4: Transmission in x, reflection in y

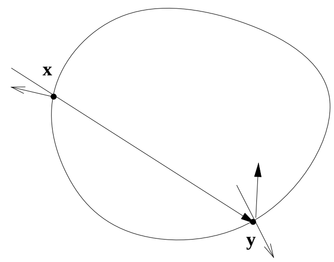


Figure 5: Reflection in x, transmission in y

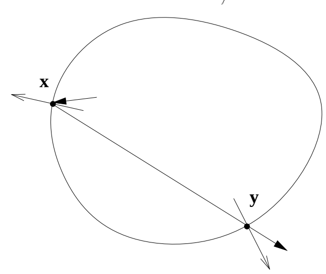
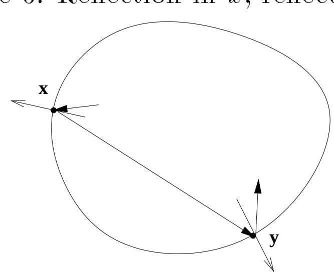


Figure 6: Reflection in x, reflection in y



Let us see now in which cases the phase is stationary on a line of singularity.

Proposition 5 *The phase is stationary in x along a line if and only if $\frac{x-x_{i'}}{|x-x_{i'}|}+\xi_j$ is orthogonal to the tangent to the line with an accuracy of $k^{-\alpha}$.*

Proof :
Indeed, we just have to write that the gradient of the phase (computed as before) is in the plane is orthogonal to the tangent to the line. □

This situation is illustrated on figure 7. Let us remark that, when the phase is degenerated, we may have a decay rate of $A(k)$ which is still slower. It depends on the value of α. Finally, we have shown that, for points which are not neighbors, the basis functions whose interactions are dominating are those corresponding to the four cases illustrated above.

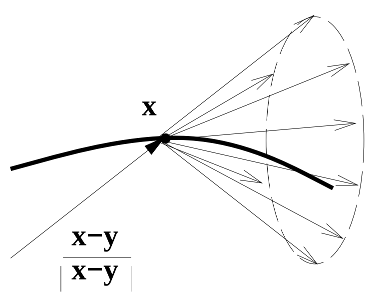


Figure 7: Stationary phase on a line

3.2 Near field interactions.

Now we are doing the same study in the case where the supports of the basis functions have non empty intersection. Thus, we can suppose that $x_i = x_{i'}$. So, we perform the change of variables : $\tilde{x} = k^\alpha x$ and $\tilde{y} = k^\alpha y$. Then $A(k)$ becomes

$$A(k) = \frac{1}{k^{3\alpha}} \int \frac{e^{\imath|k^{1-\alpha}(\tilde{x}-\tilde{y})|}}{4\pi|\tilde{x}-\tilde{y}|} \phi(\tilde{x})\phi'(\tilde{y}) e^{\imath k^{1-\alpha}\xi_j.\tilde{x}} e^{\imath k^{1-\alpha}\xi_{j'}.\tilde{y}} d\tilde{x}d\tilde{y} \tag{29}$$

As previously, we write : $x = \tau + \frac{1}{2}C(\tau,\tau)n$ and $y = \tau' + \frac{1}{2}C(\tau',\tau')n$ where τ is in the plane tangent to Γ at x_i, n is the unit vector normal to Γ at the same point and C is the curvature matrix. Writing $\tilde{\tau} = k^\alpha \tau$ and similarly for τ', we have

$$A(k) = \frac{1}{k^{3\alpha}} \int \frac{e^{\imath k^{1-\alpha}(|\delta|+\xi.\delta+\eta.\mu)+k^{1-2\alpha}(\xi.n\frac{(C-C')}{2}+\eta.n\frac{(C+C')}{2})+k^{1-3\alpha}\frac{(C-C')^2}{8\delta}}}{4\pi|\delta|} \Phi(\delta,\mu)d\delta d\mu \tag{30}$$

where we have set $\delta = \tilde{\tau}-\tilde{\tau}'$, $\mu = \tilde{\tau}+\tilde{\tau}'$, $C = C(\tilde{\tau},\tilde{\tau})$, $C' = C(\tilde{\tau}',\tilde{\tau}')$, $\xi = \frac{\xi_j+\xi_{j'}}{2}$, $\eta = \frac{\xi_j-\xi_{j'}}{2}$ and $\Phi(\delta,\mu)$ is a regular function. We show the following proposition :

Proposition 6 *(i) If the phase is neither stationary in μ nor in δ and is not stationary in μ on a line of singularity of ϕ or ϕ',*

$$A(k) = \mathcal{O}\left(\frac{1}{k^{3\alpha}k^{1-\alpha}k^{(m+2)(1-\alpha)}}\right) \tag{31}$$

(ii) If the phase is stationary neither in μ nor in δ and is stationary in μ on a line of singularity of ϕ and ϕ',

$$A(k) = \mathcal{O}(\frac{1}{k^{3\alpha}k^{1-\alpha}k^{(m+1)(1-\alpha)}k^{1/2-\alpha}}) \tag{32}$$

(iii) If the phase is stationary in μ but not in δ and is not degenerated,

$$A(k) = \mathcal{O}(\frac{1}{k^{3\alpha}k^{1-\alpha}k^{1-2\alpha}}) \tag{33}$$

(iv) If the phase is stationary both in μ and in δ,

$$A(k) = \mathcal{O}(\frac{1}{k^{\frac{3}{2}}}) \tag{34}$$

Proof :
For point *(i)*, we keep only the first-order term in k in the expansion of the phase. Then, in μ, we can use the same technique than for the far-field interactions and we obtain the term in $\frac{1}{k^{(m+2)(1-\alpha)}}$. In δ we use polar coordinates and we integrate by parts. The first term coming is in $\frac{1}{k^{1-\alpha}}$ taken in $\delta = 0$.
For point *(ii)*, still using the same technique, we have in μ a term $\frac{1}{k^{(m+1)(1-\alpha)}k^{1/2-\alpha}}$ due to the stationarity, nothing is changed in δ.
For point *(iii)* we still have the same order in δ. In μ, we have to evaluate, with the terms of second order in k in the phase,

$$\int e^{\imath k^{1-\alpha}(\eta.\mu)+k^{1-2\alpha}(\eta.nC(\mu,\mu))}\Phi(0,\mu)d\mu \tag{35}$$

Then, we obtain a term in $\frac{1}{k^{(1-2\alpha)}}$ for the integral in μ.
For point *(iv)*, we keep the previous result in μ. In δ, things become more complicated. Indeed, here, when the phase is stationary in δ, we have $\frac{\delta}{|\delta|} + \xi_T = 0$ where ξ_T is the orthogonal projection of ξ on the plane tangent to Γ in x_i. Then the phase is degenerated in direction ξ. Thus, we decompose δ in two directions (δ_1, δ_2), with δ_1 transverse to ξ. Then the phase is not degenerated in δ_1. Using the stationary phase theorem we gain a term in $\frac{1}{k^{1/2-\alpha}}$. For the variable δ_2, if $\alpha > 1/3$, the integral is of order 0 in k. Otherwise, by homogeneity, there is a factor of $\frac{1}{k^{1/6-\alpha/2}}$. $\square$

Lastly, let us see the geometrical meaning of the two cases of stationarity we have encountered for the near field interactions.

Proposition 7 *(i) The phase is stationary in μ if and only if $\xi_j - \xi_{j'}$ is parallel to the normal to Γ in x_i.*
(ii) The phase is stationary in μ and in δ if and only if $\xi_j = \xi_{j'}$ is tangent to the surface.

Proof :
Indeed the phase is stationary in μ if and only if η is normal to the surface and $\eta = \xi_i - \xi_j$. This ends the proof of point *(i)*.

The phase is stationary in δ, as we have already remarked, if and only if $\frac{\delta}{|\delta|} + \xi_T = 0$. This means that $|\xi_T| = 1$. So, ξ is in the tangent plane and its modulus is 1, since its projection is of modulus 1. Thus, necessarily $\xi_j = \xi'_j$ because both vectors have a modulus equal to 1. Then, $\xi = \xi_j$. We remark that the phase is then also stationary in μ. This ends the proof of the proposition. □

The situation of point *(i)* corresponds to the cases of transmission or reflection in x_i. The situation of point *(ii)* corresponds to the case where the wave is tangent to the surface.

4 Approximation of the operator.

Now, we are going to give a method to compute integral equations using the results of the previous section. We will approximate the exact matrix and give an evaluation of the error.
We denote by $\Phi_{i,j}$ the basis function introduced in section 2. We will retain in the matrix the interaction of $\Phi_{i,j}$ and $\Phi_{i',j'}$ if and only if :

$$\pi_{T_x} \frac{x_i - x_{i'}}{|x_i - x_{i'}|} + \xi_j \quad \leq \quad C k^{-\alpha} \tag{36}$$

$$\text{and } \pi_{T_{x'}} \frac{x_i - x_{i'}}{|x_i - x_{i'}|} + \xi_{j'} \quad \leq \quad C k^{-\alpha} \tag{37}$$

where π_{T_x} is the orthogonal projection onto the plane $T_{x_i}\Gamma$, and $\pi_{T_{x'}}$ is the corresponding in $x_{i'}$. In the opposite case, their interaction, which we have shown to be negligible, will be considered as null.

4.1 Accuracy of the approximation.

We now analyze the accuracy of this approximation, and how it behaves with respect to k. First we observe how the different terms computed in the previous section compare. We will suppose we use unstructured meshes on the surface. The shape functions on Γ will be $P_m - Lagrange$, then C^{m-1} and piecewise C^∞ with their m first traces null on the boundary of their support. We choose $\alpha = 1/2$. Doing this, we avoid considering the special cases where the phase is degenerating. Then, we have the following table which summarize the expansions of the previous section. It is easy to see that the rule exposed above (36 and 37) which determines which interactions are to be retained, says exactly that we have to keep only the terms which in the last two lines able of 1.
Now we choose an algebra norm on the matrices. We will choose the one coming from the euclidian norm.

Definition 1 *Let $A = (a_{i,j})$ be a squared matrix, we define its norm as :*

$$||A|| = \max_{||x||=1} ||Ax|| \tag{38}$$

where $||x||^2 = \sum_i x_i^2$.

Let us denote by A the full matrix of approximations and $\tilde{A}$ the approximated matrix. We can state the following proposition.

Far field interactions		Near field interactions	
prop. 1 case *(i)*	$\frac{1}{k^{(m+4)}}$		
prop. 1 case *(ii)*	$\frac{1}{k^{(m+7/2)}}$		
prop. 1 case *(iii)*	$\frac{1}{k^{(m+3)}}$		
prop. 2 case *(i)*	$\frac{1}{k^{(m/2+3)}}$	prop. 6 case *(i)*	$\frac{1}{k^{(m/2+3)}}$
prop. 2 case *(ii)*	$\frac{1}{k^{(m/2+5/2)}}$	prop. 6 case *(ii)*	$\frac{1}{k^{(m/2+5/2)}}$
prop. 3	$\frac{1}{k^2}$	prop. 6 case *(iii)*	$\frac{1}{k^2}$
		prop. 6 case *(iv)*	$\frac{1}{k^{3/2}}$

Table 1: Order of the different terms

Proposition 8

$$||A - \tilde{A}|| \leq \mathcal{O}(\frac{1}{k^{m/2}})||A|| \tag{39}$$

Proof :
We are going to evaluate the number of terms in the matrix for each case of the previous table. We have the table (2).

Far field interactions		Near field interactions	
prop. 1 case *(i)*	$\mathcal{O}(k^4)$		
prop. 1 case *(ii)*	$\mathcal{O}(k^{7/2})$		
prop. 1 case *(iii)*	$\mathcal{O}(k^3)$		
prop. 2 case *(i)*	$\mathcal{O}(k^3)$	prop. 6 case *(i)*	$\mathcal{O}(k^3)$
prop. 2 case *(ii)*	$\mathcal{O}(k^{5/2})$	prop. 6 case *(ii)*	$\mathcal{O}(k^{5/2})$
prop. 3	$\mathcal{O}(k^2)$	prop. 6 case *(iii)*	$\mathcal{O}(k^2)$
		prop. 6 case *(iv)*	$\mathcal{O}(k^{3/2})$

Table 2: Quantity of the different terms in the matrix

- Indeed, let us start with the top of the first column. The first case corresponds to the generic case. As we have $\mathcal{O}(k^2)$ degrees of freedom, we have the result.
- For the next case, it corresponds to a phase which is stationary on an edge only. Then, denoting by x, ξ the degree of freedom corresponding to the row of matrix we are considering, if we want the stationarity to occur in the x variable, we can choose $\mathcal{O}(k^{1/2})$ y on Γ and any direction of propagation, thus $\mathcal{O}(k)$ η. If at the opposite, there is no stationarity in x, then we can choose any y (thus $\mathcal{O}(k)$), but just $\mathcal{O}(k^{1/2})$ directions η.

- For the next case, the phase is stationary on an edge in x and on an edge in y. Hence for a given x, ξ, we can choose $\mathcal{O}(k^{1/2})$ y and then $\mathcal{O}(k^{1/2})$ η.

- Then, we have the cases where the phase is stationary on Γ. The first one is the generic situation. For a given x, ξ, if the stationarity occurs for the x variable, we can choose $\mathcal{O}(1)$ y and any η. Else, we choose any y and we have $\mathcal{O}(1)$ possible η.

- The next case corresponds to an additional stationarity on the edge. Thus, either the phase is stationary on Γ in x and we have $\mathcal{O}(1)$ y and $\mathcal{O}(k^{1/2})$ η, or we have $\mathcal{O}(k^{1/2})$ y and $\mathcal{O}(1)$ η.

- Then we have the case where the phase is stationary on $\Gamma \times \Gamma$. Here, for a given x, ξ, we have $\mathcal{O}(1)$ possible y and as many directions.

- For the near field interactions we use the same kind of arguments. For the first case, for a given x, ξ we have $\mathcal{O}(1)$ possible y and any η is convenient.

- For the next case, we still have the same number of possible y, but η is to be choosed among $\mathcal{O}(k^{1/2})$ values due to the stationarity on the lines of singularity.

- Finally, for the last two cases, nothing is changed in y, but we have $(\xi - \eta) \perp \Gamma$. Thus, we have $\mathcal{O}(1)$ possible η. This leads to $\mathcal{O}(k^2)$ coefficients in case *(iii)* and $\mathcal{O}(k^{3/2})$ in case *(iv)* since we have the additional constraint that ξ is tangent to Γ.

Now, we just have to evaluate the importance of each line of Table 1 in the matrix. Then, we see that $||A|| = \mathcal{O}(1)$. On the other hand, for $A - \tilde{A}$, the leading term is the one of the fifth row of the former tables. Then, we see that $||A - \tilde{A}|| = \mathcal{O}(\frac{1}{k^{m/2}})$. This ends the proof of the proposition. $\Box$

4.2 Complexity.

We are now going to evaluate the complexity of the method presented here for the number of operations as for the memory requirement. First we show

Proposition 9 *The computation of an interaction coefficient of $\tilde{A}$ is done with $\mathcal{O}(1)$ operations.*

Proof :
For far-field interactions, the coefficients retained in $\tilde{A}$ correspond to cases where the phase is stationary. Then, with $\alpha = 1/2$, the following non zero term in the expansion of the phase is in k^0. We then have to integrate a function which is not oscillating. Thus we can integrate it numerically with $\mathcal{O}(1)$ points. For near field interactions, the leading-order term corresponds to $\delta = 0$, and as the phase is stationary in μ, with the same arguments as above, we can compute the coefficient with $\mathcal{O}(1)$ operations. $\Box$

Finally we can state the result :

Proposition 10 *With the method exposed here,*
(i) The complexity of the computation of the matrix $\tilde{A}$ is $\mathcal{O}(k^2)$.
(ii) The complexity in terms of memory requirements is $\mathcal{O}(k^2)$.

Proof :
Indeed, since for each degree of freedom, we keep in the corresponding line of the matrix only $\mathcal{O}(1)$ coefficients (see Table 2), the total number of coefficients is $\mathcal{O}(k^2)$. This proves point *(ii)*. Then, the previous proposition ends the proof of point *(i)*. $\square$

5 Conclusion.

We developped here a method to solve integral equations for scattering problems at high frequencies. This method keeps the interests of the integral equations since the accuracy is controlled and the matrix is computed once for all the incident waves. At the opposite, it has no longer the main drawback of the finite-element method in terms of CPU requirements and overall in terms of memory requirements. Indeed, for a classical discretization using finite elements the complexity is $\mathcal{O}(k^4)$. Thus, our method which is still not a *high frequency* one since the complexity increases with k, may be used in the same range of frequencies. In fact, we can qualify our method as an essentially *mid range frequency* one since it is not accurate for small k. Furthermore, it has the interest over the asymptotic methods like GTD or physical optics to have no special sensitivity to the geometry.
Some developments of this work seem worthwhile. Beyond the implementation and the necessary numerical tests, it is possible to study the pattern of the profile of matrix $\tilde{A}$ considering geometrical and mainly homological characteristics of the manifolds $(\Gamma S^2)^2$ which represent the retained interactions. We also intend to analyze the accuracy of the discretization. At last we want to study the possible hybridation of our method with others, and particularly those which, like the finite-element one, allow to take into account the heterogeneity of the medium.

References

[1] G. Beylkin, R. Coifman, and V. Rokhlin, *Fast wavelet transforms and numerical algorithms I*, Comm. Pure Appl. Math., XLIV (1991), pp. 141–183.

[2] F. X. Canning, *Sparse approximation for solving integral equations with oscillatory kernels*, Siam J. Sci. Stat. Comput., 13 (1992).

[3] J. Chazarain and A. Piriou, *Introduction à la théorie des équations aux dérivées partielles linéaires*, Gauthier-Villars, Paris, 1981.

[4] P. Colton and R. Kress, *Integral equation method in scattering theory*, Pure and Applied Mathematics, (1983).

[5] A. de La Bourdonnaye, *Accélération du traitement numérique de l'équation de Helmholtz par équations intégrales et parallélisation.*, PhD thesis, Ecole polytechnique, Palaiseau, France, 1991.

[6] J. J. Duistermaat, *Fourier integral operators*, Courant Institute of Mathematical Sciences, New York, 1973.

[7] V. Fock, *The distribution of currents induced by a plane wave on the surface of a conductor*, J. Phys., 10 (1946), pp. 130–136.

[8] V. Guillemin and D. Schaeffer, *Remarks on a paper of D. Ludwig*, Bull. of the A.M.S., 79 (1973).

[9] M. Hamdi, *Une formulation variationnelle par équations pour la résolution de l'équation de Helmholtz avec des conditions aux limites mixtes*, CRAS, Série II, t. 292 (1981), pp. 17–20.

[10] D. Ludwig, *Uniform asymptotic expansion of the field scattered by a convex object at high frequencies*, Comm. Pure Appl. Math., XX (1967), pp. 103–138.

[11] J. Nedelec, *Mixed finite elements in* $\mathbb{R}^3$, Numer. Mathematik, 35 (1980).

[12] A. F. Nikiforov and V. B. Uvarov, *Special fonctions of mathematical physics*, Birkhäuser, Basel Boston, 1988.

[13] S. Rao, D. Wilton, and A. Glisson, *Electromagnetic scattering by surface of arbitrary shape*, I.E.E.E. Trans. on antennas and propagation, AP-30 (1982), pp. 409–418.

[14] V. Rokhlin, *Rapid solution of integral equations of scattering theory in two dimensions*, Journal of Computational Physics, 86 (1990), pp. 414–439.

[15] B. Stupfel, R. L. Martret, P. Bonnemason, and B. Scheurer, *Combined boundary-element and finite-element method for the scattering problem by axisymmetrical penetrable objects*, in Mathematical and numerical aspects of wave propagation phenomena, G. Cohen, L. Halpern, and P. Joly, eds., SIAM, 1991, pp. 332–341.

[16] M. Taylor, *Pseudo differential operators*, vol. 34 of Princeton mathematical series, Princeton University Press, Princeton, 1981.

[17] G. N. Watson, *A treatise on Bessel functions*, Cambridge University Press, 1944.