A substructuring method for a harmonic wave propagation problem : Analysis of the conditioning number of the problem on the interfaces

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Abstract

In this paper, it is shown that the usual sub-domains methods which are efficient in the case of elliptic problems have some defects in the case of propagation problems. In a second part, a new method is presented and analysed. It is shown that this method is well conditioned and that the local problems are always regular. Furthermore, it can be applied to propagation in unbounded domains.

Une méthode de sous domaines pour un problème de propagation d'ondes harmoniques : Analyse du conditionnement du problème aux interfaces

Résumé

Dans ce papier, il est montré que les méthodes de sous-domaines usuelles qui sont efficaces dans le cas de problèmes elliptiques ont un certain nombre de défauts pour les problèmes de propagation d'ondes. Dans une deuxième partie, une nouvelle méthode est proposée et analysée. En particulier, cette méthode apparaît comme bien conditionnée et avec des problèmes locaux bien posés. De plus nous étendons la méthode aux domaines non bornés.

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1. Introduction. Wave propagation problems in domains like acoustics or electromagnetism for instance generally lead to huge linear systems, especially when one is interested in high frequencies, since one has to mesh the domain, with a mesh step proportional to the wavelength. When the problem is elliptic, people usually try to solve it by dividing it into subdomains. See for instance [17], [18], [19] or [12]. Hence, each of the sub-problems is practically solvable, and one has to make the sub-solutions compatible at the interfaces between the subdomains. One so has the "primal" or "dual" subdomain methods for instance. More precisely, for the Poisson problem, one may solve in each subdomain a Dirichlet problem, with the same Dirichlet data on the interface, and the compatibility relation has to insure that the normal derivatives coincide on the interface. On the other hand, one may solve local Neumann problems with the same Neumann data on the interface, and the compatibility relation has to insure that the traces coincide on the interface. For elasticity, one may solve local problems with imposed displacements, then forcing the normal constraints to be the same on both parts of the interface, or the opposite. For operators like the Helmholtz operator, we present a subdomain method which follows the same methodology as for elliptic operators, but which also avoids its drawbacks when it is applied to non-elliptic problems. This method uses the same basic idea as the one presented by B. Despres in [10] or [11], that is to use local problems with Robin boundary conditions. Nevertheless, in his papers B. Despres does not apply this method to unbounded problems with integral equations. Furthermore, he restricts himself to a unique numerical method. In our paper, we address general conjugate gradient like methods for bounded and unbounded scattering problems.
In section 2, we precisely set the problems we want to solve. Both acoustics and electromagnetism are involved, with various boundary conditions. In section 3 we analyze some classical subdomain methods when applied to the problems we consider, pointing out their main drawbacks which are namely that the subproblems may be singular and that the interface problem may be ill-conditioned. In section 4, we present a new way of setting the problem on the interfaces, which avoids the drawbacks of the former methods for the bounded domain problems. In section 5, we adapt this technique to unbounded domain scattering. In these cases, the domain is cut, and we represent the outside by an integral equation on the cutting surface. Thus, the surfacic-volumic coupled problem can be viewed as a subdomain one.

2. Position of the problem. In this paper we are mainly addressing two physical situations. The first is the acoustic one. Let Ω an bounded open set of $\mathbb{R}^3$ and Γ its boundary (see fig 1). We denote by $H^s(O)$ the Sobolev space of order s on the set O (see for instance [1] or [16]). We want to solve the following problem. Given a function g in $H^{-1/2}(\Gamma)$, find u in $H^1(\Omega)$, such that

$$(1)\qquad \begin{aligned} \Delta u + k^2 u = 0 \quad in \quad \Omega \\ \alpha u + \beta \frac{\partial u}{\partial n} = g \quad in \quad \Gamma \end{aligned}$$

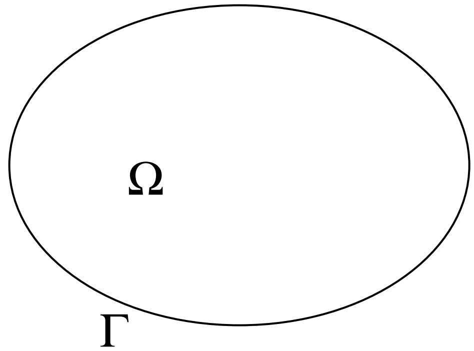

FIG. 1. *Geometrical situation*

Here n is the outward unitary normal vector of Γ. The numbers α and β are chosen to be real. Of course, they cannot be null in the same time. Note that, if β is null, then one has to choose g in $H^{1/2}$.
The second physical situation we consider is the propagation of harmonic electromagnetic waves. With the same geometry, we denote by $H_{curl}(\Omega)$ the space of vector fields E which belong to $L^2(\Omega)$ so that $curlE$ also belongs to $L^2(\Omega)$. We also denote by $H_{div}^{-1/2}(\Gamma)$ the tangential fields which are in $H^{-1/2}(\Gamma)$ and whose surfacic divergence is also in $H^{-1/2}(\Gamma)$. We fix a field E_0 in $H_{div}^{-1/2}(\Gamma)$, and we are looking for a field E in $H_{curl}(\Omega)$ satisfying :

$$(2) \qquad \begin{aligned} curlcurlE - k^2 E = 0 \quad in \quad \Omega \\ E \wedge n = -E_0 \quad in \quad \Gamma \end{aligned}$$

In the same way, we define two unbounded problems. The first one is acoustics :

$$(3) \qquad \begin{array}{lll} \Delta u + k^2 u = 0 & in & \Omega \\ \alpha u + \beta \dfrac{\partial u}{\partial n} = g & in & \Gamma \\ \dfrac{\partial u}{\partial r} - iku = o(\dfrac{1}{r}) & when & r \longrightarrow +\infty \end{array}$$

with the same notations as before and r denoting the radius in the spherical coordinate system.
For electromagnetism, we set the problem :

$$(4) \qquad \begin{array}{lll} curlcurlE - k^2 E = 0 & in & \Omega \\ E \wedge n = -E_0 & in & \Gamma \\ curlE \wedge u_r - ikE_S = o(\dfrac{1}{r}) & when & r \longrightarrow +\infty \end{array}$$

where u_r is the unitary radius vector in the spherical coordinate system and E_S denotes the component of the electric field which is orthogonal to u_r.

3. Analysis of some usual methods. In this section, we first present the general framework of multidomain methods. One can find details for instance in [2], [9] or [12].

Then, we will study the usual ways people use these methods in the frame of elliptic operators. At last, we will briefly present how a volumic-surfacic coupled problem can be viewed as a multi-domain one, following [8].

3.1. General framework of multidomain methods. The first step of these methods is to divide Ω into a finite number of Ω_i satisfying :

$$Adh(\bigcup \Omega_i) = \Omega$$

$$\Omega_i \cap \Omega_j = 0 \text{ if } i \neq j$$

where $Adh(O)$ denotes the adherence of the set O. We will denote by Γ_{ij} the interface between Ω_i and Ω_j (see fig. 2). The second point, in the case of acoustics, is to notice

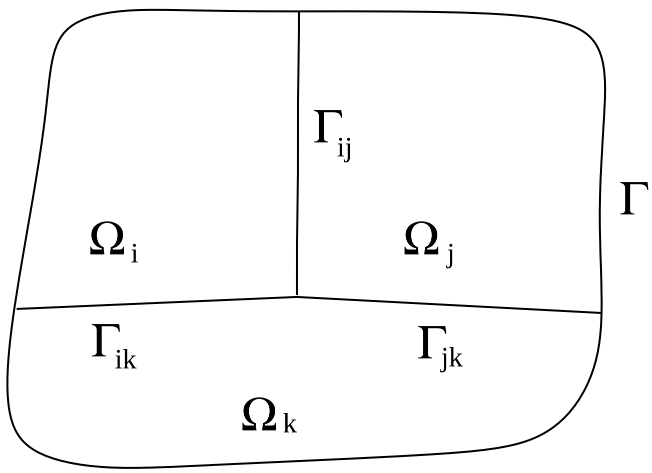

FIG. 2. *Geometrical situation*

that if u satisfies Helmholtz equation in each subdomain Ω_i, and the jumps of u and $\dfrac{\partial u}{\partial n}$ are null on the interfaces between subdomains, then u satisfies the Helmholtz equation in the whole set Ω. For the electromagnetism, one just has to replace the former continuity conditions by the continuity of the tangential trace of E and $curlE$.
Then, one can solve in each subdomain Ω_i, a Helmholtz equation with the global boundary condition on the part of $\partial\Omega_i$ which is common with Γ and for instance a Dirichlet condition p on the interfaces. The function p is the same for each subdomain, so that the jump of u is null through the interfaces. Then, in each subdomain, one has

$$\frac{\partial u}{\partial n} = S^i p + f_i \tag{5}$$

where S^i is the Steklov-Poincaré operator and f_i comes from the global boundary condition.

The last point is to ensure that the jump of the normal derivative is null. For that, one has just to solve the equation on the interfaces :

$$(S^i + S^j)p = f_i - f_j \tag{6}$$

where i and j are indices such that there is an interface between Ω_i and Ω_j. The operators S^i and S^j are also referred to as Schur complements.
One could also do the opposite : solve a local Neumann problem, and then solve on the interfaces :

$$({S^i}^{-1} + {S^j}^{-1})p = \tilde{f}_i - \tilde{f}_j \tag{7}$$

Before presenting a few results about unbounded problems, we make a few remarks.
Remark 1. *Matrices corresponding to the S^i operators are full and one does not want to assemble them. So they are just defined in an implicit way. Namely, if in Ω_i we denote by x the unknowns which are not on the interface and by y those which are on it, then, the local problem with Neumann condition on the interface writes :*

$$\begin{bmatrix} A & B \\ B^T & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ p \end{bmatrix}. \tag{8}$$

So the matrix corresponding to the operator S^i is

$$\mathcal{S}^i = C - B^T A^{-1} B$$

and is called the Schur complement. It follows that usually one only factorize matrix A and the matrix corresponding to the operator S^i is only known implicitly. For instance one can just perform matrix-vector products with it.
Remark 2. *The consequence of the former remark is that the problem on the interface is solved iteratively. So the conditioning of the interface operator is of great importance.*
Remark 3. *The last remark is that it is a well-known fact that, even with ill-conditioned matrices the conjugate gradient like methods have a good behaviour is the upper part of the spectrum of the operator is sparse (see for instance [21] or [23]). Since operators S^i send $H^{1/2}$ to $H^{-1/2}$, they are unbounded. So the corresponding matrices are ill-conditioned and the higher part of their spectrum is dense and tends to infinity when the finite element mesh is refined. It follows that one would better use Neumann local problems if not using preconditioning matrices.*

3.2. Unbounded problems and subdomain methods. As one can not just discretize the Helmholtz equation on all the domain since it would lead to infinitely many degrees of freedom, one has, in a way or an other, to cut the domain at a certain distance from the scatterer and to impose boundary conditions on the cutting surface. The only condition which is exact comes from integral equations. Thus, the usual way of doing leads to a coupling between integral equations and volumic P.D.E.'s. The coupling

of finite element and boundary element methods has been studied for a while in various frameworks. We can find it described for example in Nedelec, [20] or Zienkiewicz, [25] or more recently in Costabel, [4] and Wendland, [24]. This method is of special interest for the study of complex bodies imbedded in a linear homogeneous medium. In the case of wave propagation, the study of a non-homogeneous body is presented in Levillain, [15] for an electromagnetic wave and in de La Bourdonnaye, [5] for an acoustic one.
This way of doing may be viewed as a domain decomposition technique. Indeed, there are two domains, one is bounded and contains the scatterer, the other is its complementary. This suggests to study the use of the Schur complement technique as presented above. In [8] we showed that the part of the complement which deals with the outside can be treated in a very simple manner, using well-known properties of the relevant integral operators.

3.3. Analysis of the previous methods applied to propagation problems. The first thing we have to point out is that with Helmholtz equation (and also with the harmonic electromagnetism equation : H.E.M.), the Dirichlet and Neumann boundary conditions may lead to singular problems when the frequency is a resonance. So when solving the local subproblems one can fall on such a frequency for one of the subdomains. The second point is mainly concerned with electromagnetism. Indeed, we have shown in [6] (see also in [7]) that, in the case of the coupling of an integral equation method on a surface wrapping an object and a volumic finite element method between the surface and the objet, the operator which sends the tangential trace of the electric field to its normal derivative is unbounded and of its reciprocal is unbounded. But this problem can be viewed as a subdomain method (as shown in [8]) where the wrapping surface is the interface. This implies that, the both methods presented above are ill conditioned and have the higher part of their spectrum dense and going to infinity when the mesh is refined.

4. A new method - Bounded domain scattering. In this section, we present another way of doing which avoids the two drawbacks presented in the former section. First, we change the local problem and present some of its properties. Namely, we use Robin boundary conditions like in [10] or [11] Second, we change the interface problem. Then, we will be able to comment about its properties related to the speed of convergence of the global iterative scheme.

4.1. The local problem. It is a well known that in the case of propagation, one obtains regular problems with mixing Neumann and Dirichlet boundary conditions with complex coefficients (see [3] or [13]). So, in our case, given an orientation of the normal vector n on the interface, we will solve in each subdomain Ω_i,

$$
(9) \qquad \begin{aligned}
\Delta u + k^2 u &= 0 \quad in \quad \Omega_i \\
\alpha u + \beta \frac{\partial u}{\partial n} &= g \quad \text{on} \quad \Gamma \\
\frac{\partial u}{\partial n} + iku &= p \quad \text{on} \quad \text{the interface I}
\end{aligned}
$$

for the acoustics and

$$
(10) \qquad \begin{aligned}
curlcurlE - k^2 E &= 0 \quad in \quad \Omega_i \\
E \wedge n &= -E_0 \quad in \quad \Gamma \\
curlE \wedge n + ikE_T &= p \quad \text{on} \quad \text{the interface I}
\end{aligned}
$$

for the electromagnetism, where E_T is the tangential part of E on a surface. It is easy to check that these two problems are injective. Indeed, suppose in the first case that g and p equal 0, then, multiplying the equation on Ω_i by $\bar{u}$, integrating on Ω_i and taking the imaginary part leads to :

$$\int_I ik|u|^2 = 0.$$

Then, u and $\frac{\partial u}{\partial n}$ are null on the interface and so u is null in Ω_i.
For the electromagnetism case, on the same way, one easily obtains that E_T and $curlE \wedge n$ are null. Using the fact that $divE$ is also null, leads to E and $\frac{\partial E}{\partial n}$ are null. So the field E is null in Ω_i.

4.2. The problem on the interface. On the interface we will ensure that

$$\frac{\partial u}{\partial n} - iku$$

is continuous for the acoustics, and the corresponding condition for the electromagnetism. As in [13] or [10], we can show that the operator sending $\frac{\partial u}{\partial n} + iku$ on $\frac{\partial u}{\partial n} - iku$ is unitary. We show the following

Proposition 1. *Let B_i denotes the operator from $H^{-1/2}(\partial\Omega_i \backslash \Gamma)$ to itself, which maps*

$$\frac{\partial u}{\partial n} + iku \text{ to } \frac{\partial u}{\partial n} - iku$$

where u satisfies the Helmholtz equation in Ω_i with the global boundary conditions on Γ. Then B_i is unitary, and its spectrum has just one accumulation point which is 1.

Proof :

Indeed, let's compute $A = (B_i p, B_i q) - (p, q)$ where $(\ ,\ )$ stands for the $L^2(\partial\Omega_i \cap I)$

hermitian product. We denote by u (resp. v) the function which satisfies the Helmholtz equation in Ω_i and $\dfrac{\partial u}{\partial n} + iku = p$ (resp. q) on the interface. Then,

$$(11) \qquad (B_i p, B_i q) - (p,q) = 2ik \int_I \frac{\partial u}{\partial n}\bar{v} - u\frac{\partial \bar{v}}{\partial n}.$$

Taking into account the boundary conditions in 1, this quantity satisfies

$$(12) \qquad A = 2ik \int_{\partial\Omega_i} \frac{\partial u}{\partial n}\bar{v} - u\frac{\partial \bar{v}}{\partial n} = 2ik \int_{\Omega_i} \Delta u\bar{v} - u\Delta\bar{v}.$$

Using the fact that both u and v satisfy the Helmholtz equation in Ω_i we obtain that A is null which proves that B_i is a unitary operator.
Now let us denote by μ, ϕ an eigenvalue of B_i and its associated eigenfunction. We also denote by u the function which satisfies the Helmholtz equation in Ω_i and $\dfrac{\partial u}{\partial n} + iku = \phi$ on the interface I. Then, on I, we have

$$(13) \qquad \frac{\partial u}{\partial n} = ik\frac{1+\mu}{1-\mu}u.$$

As the operator which maps $u_{|I}$ to $\dfrac{\partial u}{\partial n}_{|I}$ has its inverse compact, the series of the $ik\dfrac{1+\mu}{1-\mu}$ has its only accumulation point at ∞. Thus the μ series has its only accumulation point in 1. $\square$

We now deduce from this property some qualities of the problem on the interface.
PROPOSITION 2. *The compatibility operator on the interface as defined in this section has the higher part of its spectrum sparse, so that it could behave well with a Gonjugate-Gradient like method.*
Indeed, this operator is the difference of the B_i operators on each side on the interface. So, it is of Fredholm type, and its spectrum has no accumulation point at infinity.
Now, we go into the analysis of the electromagnetic case (see also [11]). As for the acoustics we show the following
PROPOSITION 3. *Let B_i denotes the operator from $H^{-1/2}_{curl}(\partial\Omega_i\backslash\Gamma) \cap H^{-1/2}_{div}(\partial\Omega_i\backslash\Gamma)$ to itself, which maps*

$$curlE \wedge n + ikE_T \text{ to } \mathrm{curlE} \wedge \mathrm{n} - \mathrm{ikE_T}$$

where E satisfies system (2) in Ω_i with the global boundary conditions on Γ. Then B_i is unitary, and its spectrum has just two accumulation points which are 1 and -1.
Proof :
Indeed, let's compute $A = (B_i p, B_i q) - (p,q)$ where (,) stands for the $L^2(\partial\Omega_i \cap I)$ hermitian product. We denote by E (resp. F) the function which satisfies system 2 in Ω_i and $curlE \wedge n + ikE_T = p$ (resp. $curlF \wedge n + ikF_T = q$) on the interface. Then,

$$(14) \qquad (B_i p, B_i q) - (p,q) = 2ik \int_I (curlE \wedge n).\bar{F} - E.(curl\bar{F} \wedge n).$$

Taking into account the boundary conditions in 2, this quantity is equal to

$$(15) 2ik \int_{\partial\Omega_i} (curlE \wedge n).\bar{F} - E.(curl\bar{F} \wedge n) = 2ik \int_{\Omega_i} curl\ curl\ E.\bar{F} - E.curl\ curl\bar{F}.$$

Using the fact that both E and F satisfy system 2 in Ω_i we obtain that A is null which proves that B_i is a unitary operator.
Now let us denote by μ, ϕ an eigenvalue of B_i and its associated eigenfunction. We also denote by E the function which satisfies system 2 in Ω_i and $curlE \wedge n + ikE_T = \phi$ on the interface I. Then, on I, we have

$$curlE \wedge n = ikE_T \frac{1+\mu}{1-\mu}. \tag{16}$$

In the same manner as in [6], we can show that the operator which maps $E_{T|I}$ to $curlE \wedge n_{|I}$ is a direct sum of a compact operator and an operator with compact inverse modulo a Fredholm operator. Then, $ik\frac{1+\mu}{1-\mu}$ has its two only accumulation points at 0 and ∞. Thus the μ series has its two only accumulation points in 1 and -1. □

We now deduce from this property some qualities of the problem on the interface.
PROPOSITION 4. *The compatibility operator on the interface as defined in this section has the higher part of its spectrum sparse, so that it could behave well with a Gonjugate-Gradient like method.*
Indeed, this operator is the difference of the B_i operators on each side on the interface. So, it is of Fredholm type, and its spectrum has no accumulation point at infinity.

5. A new method - Unbounded domain scattering. In the case of the coupling between volumic and boundary formulations, we follow the general framework presented above and we focus on the differences with the bounded domain case.

5.1. local problems. First we wrap the scattering object Ω in a surface Γ_0 which plays in the present case the role of the interface. For acoustics, between Ω and Γ_0, we solve :

$$\begin{array}{lll} \Delta u + k^2 u = 0 & \textit{between} & \Omega \textit{ and } \Gamma_0 \\ \alpha u + \beta \dfrac{\partial u}{\partial n} = g & \text{on} & \Gamma \\ \dfrac{\partial u}{\partial n} + iku = p & \text{on} & \text{the interface } \Gamma_0 \end{array} \quad . \tag{17}$$

Outside Γ_0, we solve :

$$\begin{array}{lll} \Delta u + k^2 u = 0 & \textit{outside} & \Gamma_0 \\ \dfrac{\partial u}{\partial r} - iku = o(\dfrac{1}{r}) & \textit{when} & r \longrightarrow +\infty \\ \dfrac{\partial u}{\partial n} + iku = p & \text{on} & \text{the interface } \Gamma_0 \end{array} \quad . \tag{18}$$

For electromagnetism, between Ω and Γ_0, we solve :

$$(19)\qquad \begin{array}{rll} curlcurlE - k^2E = 0 & in & \Omega_i \\ E \wedge n = -E_0 & in & \Gamma \\ curlE \wedge n + ikE_T = p & \text{on} & \text{the interface } \Gamma_0 \end{array} .$$

Outside Γ_0, we solve

$$(20)\qquad \begin{array}{rll} curlcurlE - k^2E = 0 & in & \Omega_i \\ curlE \wedge u_r - ikE_S = o(\frac{1}{r}) & when & r \longrightarrow +\infty \\ curlE \wedge n + ikE_T = p & \text{on} & \text{the interface } \Gamma_0 \end{array} .$$

As the injectivity of problems (17) and (19) is just the same property as in the case of bounded problems, we will only deal with problems at the outside. We begin with acoustics. We first recall some properties that can be found in [14].

PROPOSITION 5. *If u is solution of problem (18), then :*

(i) : $u \sim A(\theta)\dfrac{e^{ikr}}{r}$ when r tends to infinity.

(ii) : the imaginary part of $\displaystyle\int_{S_R} \frac{\partial u}{\partial r}\bar{u}$ equals to $\int_S |A(\theta)|^2$

where θ denotes the angular coordinates in the spherical coordinate system, S_R is the sphere centered at origin with radius R, and $S = S_1$. Hence, we can enounce :

PROPOSITION 6. *The outside acoustics problem (18) is injective.*

Proof :

From Helmholtz equation, denoting by Ω_R the volume between S_R and Γ_0 we have :

$$(21)\qquad 0 = \int_{\Omega_R} \Delta u.\bar{u} + k^2|u|^2 = \int_{\Omega_R} -|\nabla u|^2 + k^2|u|^2 - \int_{\Gamma_0} \frac{\partial u}{\partial n}\bar{u} + \int_{S_R} \frac{\partial u}{\partial r}\bar{u}.$$

Taking the imaginary part, and then the limit $r \longrightarrow +\infty$ we obtain :

$$(22)\qquad 0 = \int_{\Gamma_0} |u|^2 + \int_S |A(\theta)|^2$$

Thus, u is null on Γ_0 and so is the normal derivative thanks to the boundary condition on Γ_0. Hence, it is a well known fact that $u = 0$ outside Γ_0. $\Box$

For the electromagnetism we first recall that

PROPOSITION 7. *If E is solution of problem (20), then :*

(i) : $E \sim A(\theta)\dfrac{e^{ikr}}{r}$ when r tends to infinity.

(ii) : the imaginary part of $\displaystyle\int_{S_R} curlE \wedge n.\bar{E}$ equals to $\int_S |A(\theta)|^2$.

Here $A(\theta)$ is a vector field which tangential to the sphere S. Hence, we can enounce :

PROPOSITION 8. *The outside electromagnetic problem (20) is injective.*

Proof :
As in the acoustic case, we have :

$$0 = \int_{\Omega_R} -|curlE|^2 + k^2|E|^2 - \int_{\Gamma_0} curlE \wedge n.\bar{E} + \int_{S_R} curlE \wedge n.\bar{E}. \tag{23}$$

Taking the imaginary part, and then the limit $r \longrightarrow +\infty$ we obtain :

$$0 = \int_{\Gamma_0} |E|^2 + \int_S |A(\theta)|^2 \tag{24}$$

Thus, E is null on Γ_0 and so is $curlE \wedge n$ thanks to the boundary condition on Γ_0. Hence, it is a well known fact that $E = 0$ outside Γ_0. □

5.2. The problem at the interface. On the interface Γ_0 we will constrain the same quantity as in bounded-domain problems to be continuous. Namely, if we denote by $[\,]$ the jump of a quantity at Γ_0, we impose

$$\left[\frac{\partial u}{\partial n} - iku\right] = 0 \tag{25}$$

in the acoustic case, and

$$[curlE \wedge n - ikE_T] = 0 \tag{26}$$

for the electromagnetism.
We define B operators as before. Since the "inside" problems are particular cases of the bounded problems, their B operators are still unitary. So we will focus in the following on the operators related to the outside domains.
We first recall a few facts about pseudo-differential operators (for more details, we refer to [22]). We denote by $S^{-\infty}$ the set of pseudo-differential operators whose symbols decrease faster than any polynomial at infinity. We recall that, if P is a pseudo-differential operator and Q is another one, Q is said to be a right parametrix of P (resp. left parametrix) if $P.Q - I$ (resp. $Q.P - I$) belongs to $S^{-\infty}$. Q is a parametrix of P if it is both a right and left parametrix.
Now we show a lemma using these facts.
LEMMA 1. *If $A(\theta)$ is the amplitude exhibited in former section, then both in the case of acoustics and electromagnetism, still denoting by A the operator which associates u or E to its amplitude, we have :*

$$A^\star A \in S^{-\infty}. \tag{27}$$

Proof :
Indeed, if u and v are solutions of problem (18), then

$$< Au, Av > \; = \int_{\Gamma_0\times\Gamma_0} G(x,y)\frac{\partial u}{\partial n}(x).\frac{\partial \bar{v}}{\partial n}(y) + \frac{\partial^2 G}{\partial n_x \partial n_y}(x,y)u(x).\bar{v}(y)dxdy \tag{28}$$

$$- \int_{\Gamma_0\times\Gamma_0} \frac{\partial G}{\partial n_x}(x,y)u(x).\frac{\partial \bar{v}}{\partial n}(y) + \frac{\partial G}{\partial n_y}(x,y)\frac{\partial u}{\partial n}(x).\bar{v}(y)dxdy \tag{29}$$

where the kernel $G(x,y)$ is the following :

$$G(x,y) = \frac{sink|x-y|}{k|x-y|}. \tag{30}$$

Since G is a even function of $|x-y|$, we can easily check that this kernel is analytic by considering its Taylor series in 0 for instance. Hence, its Fourier transform, which is the symbol of $A^\star A$ is exponentially decreasing, and thus $A^\star A$ is in $S^{-\infty}$. For the electromagnetic case, the same type of computations apply, and we are led to the same kernel G. Thus, the same results hold for electromagnetism. $\square$

The last point of this section is to show the same type of proposition as for the bounded-domain case for the operators B. Let us start with acoustics.

PROPOSITION 9. *Let B denote the operator from $H^{-1/2}(\Gamma_0)$ to itself which maps $\frac{\partial u}{\partial n}+iku$ to $\frac{\partial u}{\partial n}-iku$ where u satisfies the Helmholtz equation outside Γ_0 and the radiation condition at infinity. Then B is essentially unitary, which means that $B^\star$ is a left parametrix of B, and its spectrum as just one accumulation point which is 1.*

Proof :

As for the bounded domain case we set

$$L = < Bp, Bq > - < p,q > . \tag{31}$$

Then denoting by u, (resp. v) the solution of Helmholtz equation with radiation condition which satisfies $\frac{\partial u}{\partial n} + iku = p$ (resp. $= q$), we have :

$$\begin{aligned} L &= 2ik \int_{\Gamma_0} \frac{\partial u}{\partial n}\bar{v} - u\frac{\partial \bar{v}}{\partial n} && (32) \\ &= 2ik \int_{\Omega_R} \Delta u\bar{v} - u\Delta\bar{v} + 2ik \int_{S_R} \frac{\partial u}{\partial r}\bar{v} - u\frac{\partial \bar{v}}{\partial r} && (33) \end{aligned}$$

Due to the Helmholtz equation, the volumic integral is null, thus letting R grow to infinity, we have :

$$L = \int_S -4k^2|A(\theta)|^2 d\theta. \tag{34}$$

From the previous lemma, we deduce that

$$B^\star B - I = A^\star A \in S^{-\infty}. \tag{35}$$

This proves the first part of the proposition. Thus B is unitary modulo a regularizing operator. To analyze its spectrum the method is just the same as for bounded domains. The only difference is that 1 is never an eigenvalue, since it would correspond to a null eigenvalue for an outside Dirichlet problem, which is not possible. $\square$

Finally, for acoustics we can enounce

PROPOSITION 10. *The compatibility operator on the interface Γ_0 for coupled volumic-surfacic problems has the higher part of its spectrum sparse, so that it could behave well with a Conjugate-Gradient like method.*

For electromagnetism, we prove the same type of result.

PROPOSITION 11. *Let B denote the operator from $H^{-1/2}_{div}(\Gamma_0) \cup H^{-1/2}_{curl}(\Gamma_0)$ to itself which maps $curlE \wedge n + ikE_T$ to $curlE \wedge n - ikE_T$ where E satisfies the electromagnetic Helmholtz equation outside Γ_0 and the radiation condition at infinity. Then B is essentially unitary, which means that $B^\star$ is a left parametrix of B, and its spectrum has only two accumulation points which are 1 and -1.*

Proof :

As previously, we set

$$L = < Bp, Bq > - < p, q > . \tag{36}$$

Then denoting by E, (resp. F) the solution of electromagnetic Helmholtz equation with radiation condition which satisfies $curlE \wedge n + ikE_T = p$ (resp. $= q$), we have :

$$(37)\ L = 2ik \int_{\Gamma_0} curlE \wedge n.\bar{F} - E.curl\bar{F} \wedge n$$

$$(38)\quad = 2ik \int_{\Omega_R} curlcurlE.\bar{F} - E.curlcurl\bar{F} + 2ik \int_{S_R} curlE \wedge u_r.\bar{F} - E.curl\bar{F} \wedge u_r$$

Due to the Helmholtz equation, the volumic integral is null, thus letting R grow to infinity, we have :

$$L = \int_S -4k^2 |A(\theta)|^2 d\theta. \tag{39}$$

As for acoustics, from the previous lemma, we deduce that

$$B^\star B - I = A^\star A \in S^{-\infty}. \tag{40}$$

Thus B is unitary modulo a regularizing operator. Again, to analyze its spectrum we use the same method as for bounded domains. And here, again the only difference is that 1 and -1 are never eigenvalues, since it would correspond to a null eigenvalue for an outside metallic problem, which is not possible. □

Finally, we have

PROPOSITION 12. *The compatibility operator on the interface Γ_0 for coupled volumic-surfacic problems has the higher part of its spectrum sparse, so that it could behave well with a Conjugate-Gradient like method.*

6. conclusion. After having given the general framework of subdomain methods as well as the way they are usually used for elliptic problems, we presented their two main drawbacks when they are applied to propagation of harmonic waves. Indeed, the subproblems may be singular and the interface problem does not behave well as far as the speed of convergence is concerned, especially in the case of electromagnetic waves. Then we introduced a new formulation that keeps the general framework of subdomain methods but corrects the above mentionned defaults. In fact the local problems are made coercive by the mean of a dissipative boundary condition. Furthermore, we have developed in this paper a Schur complement which is unitary is the case of bounded domains and essentially unitary in the case of unbounded domains. This leads in every case to Fredholm operators which are known to behave well with respect to Gradient-Conjugate like algorithms. Of course, it is possible to use some bounded domains with an unbounded one and to mix our studies on both cases.

REFERENCES

[1] R. A. Adams, *Sobolev Spaces*, vol. 65 of Pure and applied mathematics, Academic Press, 1975.

[2] V. Agoshkov, *Poincaré-Steklov operators and domain decomposition methods in finite dimensional spaces*, in Domain decomposition methods for partial differential equations, R. Glowinski, G. H. Golub, G. A. Meurant, and J. Périaux, eds., SIAM, Philadelphia, 1988, pp. 73–112.

[3] A. Bamberger and T. Ha Duong, *Diffraction d'une onde acoustique par une paroi absorbante: nouvelles équations intégrales*, Math. Meth. in the Appl. Sci., 9 (1987), pp. 431–454.

[4] M. Costabel, *Symmetric methods for the coupling of finite elements and boundary elements*, in Boundary elements IX, Brebbia, Wendland, and Kuhn, eds., 1987, pp. 411–420.

[5] A. de La Bourdonnaye, *Accélération du traitement numérique de l'équation de Helmholtz par équations intégrales et parallélisation.*, PhD thesis, Ecole polytechnique, Palaiseau, France, 1991.

[6] ——, *Décomposition de $H_{div}^{-1/2}(\Gamma)$ et nature de l'opérateur de Steklov-Poincaré du problème extérieur de l'électromagnétisme*, Note aux C.R.A.S., t. 316, Série I (1993), pp. 369–372.

[7] ——, *Diverses formulations du couplage entre méthode volumique et méthode surfacique pour l'équation de Helmholtz et l'électromagnétisme*, Tech. Rep., CERMICS, 1993.

[8] ——, *Some formulations coupling volumic and integral equation methods for Helmholtz equation and electromagnetism*, Num. Math, (accepted).

[9] Y. H. de Roeck, *Résolution sur ordinateurs multiprocesseurs de problèmes d'élasticité par décomposition de domaines*, PhD thesis, Université Paris IX Dauphine, 1991.

[10] B. Despres, *Domain decomposition method and the Helmholtz problem*, in Mathematical and numerical aspects of wave propagation phenomena, Strasbourg, France, 1991, pp. 44–52.

[11] B. Despres, P. Joly, and J. E. Roberts, *A domain decomposition method for the harmonic Maxwell equations*, in Iterative methods in linear algebra, IMACS, 1992, pp. 475–484.

[12] M. Dryja and O. B. Widlund, *Towards a unified theory of domain decomposition algorithms for elliptic problems*, in Proceedings of the third international symposium of domain decompostion methods, T. Chan and R. Glowinski, eds., SIAM, 1989.

[13] J. Grégoire, J. Nedelec, and J. Planchard, *Problèmes relatifs à l'équation de Helmholtz*, Bulletin de la D.E.R., E.D.F., (1974), pp. 15–32.

[14] M. Hamdi, *Une formulation variationnelle par équations pour la résolution de l'équation de Helmholtz avec des conditions aux limites mixtes*, CRAS, Série II, t. 292 (1981), pp. 17–20.

[15] V. Levillain, *Couplage éléments finis-équations intégrales pour la résolution des équations de Maxwell en milieu hétérogene.*, PhD thesis, Ecole polytechnique, 1991.

[16] J. L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, vol. 1, Dunod, Paris, 1968.

[17] P. L. Lions, *On the Schwartz alternating method 1*, in International symposium on domain decomposition methods for partial differential equations, SIAM, 1988, pp. 1–42.

[18] ———, *On the Schwartz alternating method 2 : stochastic approach and order properties*, in International symposium on domain decomposition methods for partial differential equations, SIAM, 1989, pp. 47–70.

[19] ———, *On the Schwartz alternating method 3*, in International symposium on domain decomposition methods for partial differential equations, SIAM, 1990.

[20] J. Nedelec and C. Johnson, *On the coupling of boundary integral and finite element methods*, Mathematics of computation, 35 (1980), pp. 1063–1079.

[21] F. X. Roux, *Spectral analysis of the interface operators associated with the preconditioned saddle-point principle domain decomposition method*, in Domain Decomposition Conference (Norfolk), 1991.

[22] M. Taylor, *Pseudo differential operators*, vol. 34 of Princeton mathematical series, Princeton University Press, Princeton, 1981.

[23] A. Van der Sluis and H. Van der Vorst, *The rate of convergence of conjugate gradients*, Numerische Mathematik, 48 (1986), pp. 543–560.

[24] W. Wendland, *On the coupling of finite elements and boundary elements*, in Discretization methods in structural mechanics, Kuhn and Mang, eds., Springer, 1990, pp. 405–414.

[25] O. C. Zienkiewicz, D. W. Kelly, and P. Bettess, *The coupling of the finite element method and boundary solution procedures*, Int. J. Num. Meth. Eng., 11 (1977), pp. 355–375.