Numerical simulation of scattering problems with Fourier-Integral operators.

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Abstract

In this paper, we present a general method for approximating oscillatory integrals arising in some scattering problems. In particular, a microlocal basis is used to approximate wave functions, by discretizing their wave-front. Then, we exhibit a result concerning the approximation of the scattering operator, which allows to retain only interactions between coefficients for which the product of the supports is near the wave-front of the kernel of the operator. Finally, we give an estimate of the algorithmic complexity of the method presented in the paper.

Approximation numérique de problèmes de diffraction incluant des opérateurs Fourier-Intégraux.

Résumé

Dans ce papier, nous présentons une méthode générale d'approximation numérique d'intégrales oscillantes intervenant dans divers problèmes de diffraction. En particulier, nous montrons une base microlocale de fonctions qui permet d'approcher les ondes que nous cherchons, en s'adaptant à leur front d'onde. Puis, nous exhibons un résultat d'approximation de l'opérateur, qui permet de ne retenir que les interactions entre des coefficients dont le produit des supports est proche du front d'onde du noyau de l'opérateur. Enfin nous donnons une estimation de la complexité algorithmique de la méthode exhibée.

1. Introduction. High frequency scattering problems are generally hard to simulate for many reasons. In this paper, we will focus on finite element methods in frequency domain. One of these reasons is that the linear systems which are involved are very large. Indeed, the mesh step has to be a portion of the wave length λ of the simulated phenomenon. This means $\mathcal{O}(\lambda^{-n})$ degrees of freedom where n is the dimension of the ambient space. Furthermore, in the case of pseudo-differential or Fourier-integral operators, the interaction is not local and each basis function interacts with each other. Thus, the number of coefficients to be computed is $\mathcal{O}(\lambda^{-2n})$. It is also the case when one uses integral equations for acoustics of electromagnetism.

First, we are going to present a few situations where pseudo-differential or Fourier-integral operators occur. The first case is the wave equation one, when the sound speed depends both on the geometric point (the material is not homogeneous) and on the direction of propagation of the wave (the material is not isotropic). For instance, this occurs in composite material. The equation then writes :

$$\partial_t^2 u + \int e^{ix.\xi} a(x, \frac{\xi}{|\xi|})|\xi|^2 \hat{u} d\xi = f. \tag{1}$$

In frequency domain it becomes :

$$-\int e^{ix.\xi}(\omega^2 - a(x, \frac{\xi}{|\xi|})|\xi|^2)\hat{u} d\xi = f \tag{2}$$

where we have kept the same notations for the functions and their Fourier transforms with respect to time, and ω is the pulsation at which we observe the scattering. The second case comes from the same idea, but we suppose that the medium is dispersive. Hence the equation becomes :

$$-\int e^{i(x.\xi-\phi(\xi))}(\omega^2 - a(x, \frac{\xi}{|\xi|})|\xi|^2)\hat{u} d\xi = f \tag{3}$$

These two situations are the ones we are going to investigate in this paper. The first case corresponds to pseudo-differential operators and the second to Fourier integral operators. We could also add the case of integral equations coming from harmonic acoustics or electromagnetism, since they also involve oscillating term or the Lipmann-Schwinger equation, especially when the term modelling the material oscillates with a characteristic step which is similar to the wave length.

At least in the case of integral equations, many works have already been done. V. Rokhlin presented in [9] a method based on the decomposition into Hankel and Bessel functions and on the use of addition formulae for the 2-D case. In [10] he presented the extension of the former method to the 3-D case, but in fact it happens that this paper has an error which makes the method as costly as the classical finite element discretization and thus of no use. In [3] F.X. Canning presented for the 2-D case, a method based on the discrete Fourier transform of packets of basis functions, and thus of packets of matrix coefficients.

It happened in this case that lots of transformed coefficients where small. For the 3-D case, it seems not so easy to implement the same method since, it was based on FFT which exist for segments but not for general 2D patches. But at least this method saves memory if not CPU time. In [6], the first author of this paper presented a method which is based on a discrete microlocalization of the functions which was leading to sparse matrices. We also mention the paper of B. Bradie, R. Coifman and A. Grossmann (see [2]) which deals with the oscillatory terms of acoustic integral equations. In this paper, they use local cosine transforms following basically the same idea as F.X. Canning. Our aim is here to extend the method proposed in [6] and [12] to the case of more general oscillatory integrals such as the ones presented above.
In section 2 we precisely set the problems we want to solve. In section 3, we present a discretization of the space of solutions we are looking for, thanks to an eikonal equation. In section 4, we evaluate the interaction coefficients between the basis functions, and show that most of them can be negected. In the last section, we evaluate the algorithmic and storage complexity.

2. Setting of the problems. Following what has been presented in the introduction, we will address two problems. The first one, which involves pseudo-differential operators is :
Let X be a regular bounded open set of $\mathbb{R}^n$. Given an elliptic pseudo-differential operator P of order one, we want to compute

$$Pu(x) = \int p(x,\xi) e^{ix.\xi} \hat{u}(\xi) d\xi \tag{4}$$

when

$$i\partial_t u - Pu = f \tag{5}$$

and f is oscillating at a given pulsation ω. Here, we consider first order problems instead of second order ones in the introduction for the sake of clearness, but it is well-known that one can always transform a higher order equation into a system of order one (see for instance [11]). The ellipticity of the operator which is the positivity of the principal symbol precisely means that the speed of propagation of the signal is always strictly positive.
The second problem involves Fourier-integral operators. Let X and Y be regular bounded open sets of $\mathbb{R}^n$. Given such an operator P of order one, we want to compute

$$Pu(x) = \int p(x,y,\xi) e^{i\phi(x,y,\xi)} u(y) d\xi dy \tag{6}$$

when

$$i\partial_t \int e^{i\phi(x,y,\xi)} u(y) dy d\xi - Pu = f \tag{7}$$

and f is oscillating at a given pulsation ω. Here, p is the amplitude of the operator and ϕ its phase. We impose ϕ and p to be 1-homogeneous in ξ. This corresponds to a

material which has a dispersion varying linearly with the frequency. It is often a correct assumption when a signal has its spectrum localized in frequency. We also impose the ellipticity of the operator which has the same meaning as in the previous case. Of course, the phase ϕ satisfies to usual property of non-degeneracy (see [7]) which is :
When $\partial_\xi \phi = 0$*, the differential forms* $d\partial\xi_i\phi$ *are lineary independent, where the* ξ_j *are the coordinates of* ξ*.*

3. Discretization. In this section, we will see that, using an eikonal equation coming from the fact that we observe a signal oscillation at pulsation ω, we are able to present a discretization of the functions we are looking for. The first point is to use a variational principle as in all finite element methods. Then, thanks to the eikonal equation and the ellipticity, we will obtain a finite manifold to discretize. We begin with the case of pseudo-differential operators which is the simpler.

3.1. Pseudo-differential operators. In the case of problem (4-5), we first recall that u satisfies

$$u(x) = \int a(x,\xi)e^{i\omega\psi(x,\xi)}d\xi \tag{8}$$

for given amplitude a and phase ψ. Hence, it is easy to check (see for instance [11]) by expanding (5) in decreasing powers of the pulsation ω, that, on the support of $a(x,\xi)$, for $(x,\xi) \in T^\star X$ satisfying $\nabla_\xi \psi = 0$,

$$1 = p_1(x, \nabla_x \psi(x,\xi)) \tag{9}$$

where p_1 is the principal symbol of the pseudo-differential operator P. This last equation is the eikonal equation. We denote by Λ_ψ the lagrangian manifold associated with ψ :

$$\Lambda_\psi = \{(x, \nabla_x\psi(x,\xi)) \in T^\star X, \nabla_\xi\psi(x,\xi) = 0\} \tag{10}$$

and we define

$$T^\star X_\alpha = \{(x,\xi) \in T^\star X, \alpha = p_1(x,\xi)\}\,. \tag{11}$$

To represent the function u, we want to discretize the part of its wavefront which is in the neighborhood of $T^\star X_1$, and combine local oscillatory terms like $e^{ix.\xi}$. In order to do that, we first prove the
THEOREM 3.1. *the manifold* $T^\star X_1$ *is a bounded manifold if* X *is.*
Proof :
Indeed, since the pseudo-differential operator P is elliptic, its principal symbol p_1 is coercive which means that there exists a strictly positive constant α such that

$$|p_1(x,\xi)| \geq \alpha|\xi|. \tag{12}$$

Thus, if (x,ξ) is in $T^\star X_1$, we have

$$1 \geq \alpha|\xi| \tag{13}$$

and so

$$T^\star X_1 \subset X \times B(O, \frac{1}{\alpha}) \tag{14}$$

where $B(O,r)$ stands for the ball centered at origin with radius r. This ends the proof. $\square$

The consequence of this theorem is that we are able to discretize $T^\star X_1$ with a finite number of degrees of freedom. More precisely, we will discretize a tubular neighborhood $\mathcal{N}$ of it in $T^\star X$ with a thickness of $\mathcal{O}(\omega^{-1/2})$. Now, we precise the way to discretize this manifold.

- We discretize X with a mesh step $h_X = \mathcal{O}(\omega^{-1/2})$, creating $\mathcal{O}(\omega^{\frac{n}{2}})$ points x_i.
- We discretize each fiber with a mesh step $h_F = \mathcal{O}(\omega^{-1/2})$, creating $\mathcal{O}(\omega^{\frac{n-1}{2}})$ directions ξ_j.
- We mesh $T^\star X_1$ with the relevant couples (x_i, ξ_j), thus having $\mathcal{O}(\omega^{n-1/2})$ degrees of freedom.

The next point is to present the basis of functions we will use to approximate the functions satisfying (8). First, we use the P1-Lagrange (see for instance [4]) basis functions on the mesh presented above. We denote them by $p_{ij}(x,\xi)$ where $p_{ij}(x,\xi) = 1$ when $(x,\xi) = (x_i,\xi_j)$ and $p_{ij}(x,\xi) = 0$ on the other points of the mesh. Then our basis functions will be :

$$q_{ij}(x) = \int p_{ij}(x,\xi) e^{i\omega(x-x_i).\xi} d\xi \tag{15}$$

Intuitively, integrating a Fourier integral distribution which satisfies the eikonal equation against q_{ij} means microlocalizing the distribution around (x_i, ξ_j).
Now, we are going to prove that the basis exhibited above is sufficient to represent the asymptotic behavior of the functions we are looking for. We want to find coefficients α_{ij} which minimize

$$E = \int \left| u(x) - \sum_{i,j} \alpha_{ij} q_{ij}(x) \right|^2 dx \tag{16}$$

The first point is the following. We make a change of variables such that the manifold

$$\Sigma_\psi = \{(x,\xi), \nabla_\xi \psi(x,\xi) = 0\} \tag{17}$$

is mapped to the manifold Λ_ψ which is the wavefront of u. We call x, η the new variables, and for (x,ξ) in Σ_ψ,

$$\eta = \nabla_x \psi(x,\xi). \tag{18}$$

Thus, if we denote, $\tilde{\psi}(x,\eta) = \psi(x,\xi)$, we have the implication

$$\left[\nabla_\eta \tilde{\psi}(x,\eta) = 0\right] \Rightarrow \left[\eta = \nabla_x \tilde{\psi}(x,\eta)\right] \tag{19}$$

In the high frequency limit, due to the eikonal equation, we just have to take into account the (x, η) which are in the neighborhood of the wavefront of u which is included into the manifold $\mathcal{N}$. Since the function

$$\chi(x, \eta) = \sum_{ij} p_{ij}(x, \eta) \tag{20}$$

is a cut-off function around $\mathcal{N}$, in the high frequency limit, we have that

$$E \sim \int \left| \int \sum_{ij} p_{ij}(x, \eta)\tilde{a}(x, \eta)e^{ik\tilde{\psi}(x,\eta)}d\eta - \sum_{ij} \alpha_{ij} p_{ij}(x, \eta)e^{ik(x-x_i).\eta}d\eta \right|^2 dx. \tag{21}$$

The symbol $\tilde{a}$ comes from the symbol a after the change of variables, it includes the jacobian of the transformation. Then, we have

PROPOSITION 3.2.

$$E \leq \int \left| \sum_{ij} p_{ij}(x, \eta) \left(\tilde{a}(x, \eta)e^{ik\tilde{\psi}(x,\eta)} - \alpha_{ij} e^{ik(x-x_i).\eta} \right) \right|^2 d\eta dx \tag{22}$$

since, thanks to eikonal equation η is in a bounded domain.

Next, we show the following lemma.

LEMMA 3.3. *Let $G(x, \eta)$ be a function, then,*

$$|\sum_{ij} p_{ij}G|^2 \leq (2n + 1) \sum_{ij} |p_{ij}G|^2 \tag{23}$$

where n is the dimension of the manifold X.

Proof :

Indeed, on each simplex of the mesh, we have exactly $2n + 1$ functions p_{ij} which are not null. Thus, a simple discrete version of Cauchy-Schwartz theorem leads to the result. □

Before using this lemma, we make the following remark. Since if

$$\Lambda_\psi \cap Supp(p_{ij}) = \emptyset, \tag{24}$$

the integral giving u on this set is fastly decreasing, then we can take $\alpha_{ij} = 0$ for such a 2-index.

On the other case, the phase in u is stationary and thus, in $Supp(p_{ij})$,

$$\tilde{\psi}(x, \eta) = Cte + \eta.(x - x_i) + \mathcal{O}(\omega^{-1}). \tag{25}$$

Indeed, since the phase is stationary, we can develop it around ξ, η. The η derivative is null, the x derivative is η and the higher order terms are $\mathcal{O}(\omega^{-1})$, since the mesh step is $\mathcal{O}(\omega^{-1/2})$. Thus, we can prove

PROPOSITION 3.4.

$$E \leq (2n+1) \sum_{(ij),\Lambda\cap Supp(p_{ij})\neq\emptyset} \int_{supp(p_{ij})} |b(x,\eta)-\alpha_{ij}|^2\, dx d\eta \tag{26}$$

The symbol b comes from $\tilde{a}$ by combination of the constant and $\mathcal{O}(\omega^{-1})$ terms in equation (25), since, multiplied by ω, these two terms cannot be considered as oscillatory ones.
Proof :
Indeed, since we have thrown out the supports which do not intersect the wave front of u, we are left, for the u part of the error E, with

$$b(x,\eta)e^{i\omega(x-x_i).\eta}. \tag{27}$$

Factorizing the oscillatory part, which is of module 1, and using lemma 3.3, we have the result since the functions p_{ij} are bounded by 1. $\square$

Now we can enounce the final result of this part.
THEOREM 3.5. *For a function u satisfying (8) and eikonal equation (9), if the mesh step h satisfies*

$$h = \frac{1}{C\omega^{1/2}}, \tag{28}$$

then, denoting by u_h the projection of u on the basis of the functions $q_{ij}(x)$, we have

$$||u-u_h||_{L_2} = \mathcal{O}(1/C)||u||_{L_2}. \tag{29}$$

The constant C represents the number of degrees of freedom by wave length in the physical space X.
Proof :
We are going to choose the coefficient α_{ij} such to minimize

$$\int_{supp(p_{ij})} |b(x,\eta)-\alpha_{ij}|^2\, dx d\eta \tag{30}$$

We proceed to the following change of variables :

$$y = x.\sqrt{\omega},\ \zeta = \eta.\sqrt{\omega}. \tag{31}$$

Thus, the new mesh step is $1/C$ and the jacobian is $1/\omega$. If we choose α_{ij} to be the mean value of b on the support of integration, we have that,

$$\int_{supp(p_{ij})} |b(x,\eta)-\alpha_{ij}|^2\, dx d\eta = \mathcal{O}(1/C^2)\int_{supp(p_{ij})} |b(x,\eta)|^2\, dx d\eta. \tag{32}$$

since $\tilde{b}(y,\zeta) = b(x,\eta)$ as a gradient proportional to $\tilde{b}$. $\square$

The meaning of this theorem is that, as classically done when approximating oscillatory functions, for a given requested accuracy of approximation, we only need a bounded number of degrees of freedom by wave length in the space manifold X. By this, we mean that the number of degrees of freedom is a constant times ω^n, when n is the dimension of X.

3.2. Fourier-integral operators. For Fourier-integral operators, as in (6-7), the situation is more complex than in previous section since these operators transform the wavefront of the functions to which they are applied. Thus we are in a "Petrov-Galerkin" situation where the searched functions are of one type and the test functions of another one. More precisely, as in previous section, we can take the function u of problem (6-7), as a locally finite sum of functions like

$$u(x) = \int a(x,\xi)e^{i\omega\psi(x,\xi)}d\xi. \tag{33}$$

Now, the problem is to compute Pu where P is the Fourier-integral operator. We will use a variational technique. It means that we want to compute $< Pu, v >$ where $< , >$ stands for the hermitian product, for all v in a class of functions which is similar to u. Namely, we want to compute $< Pu, v >$, taking into account the equation

$$i\partial_t < \int e^{i\phi}u, v > - < Pu, v >=< f, v > \tag{34}$$

at frequency ω for all u satisfying

$$u(x) = \int a(x,\xi)e^{i\omega\psi(x,\xi)}d\xi \tag{35}$$

and all v satisfying

$$v(x) = \int b(x,\eta)e^{i\omega\theta(x,\eta)}d\eta, \tag{36}$$

where I_ϕ is the Fourier integral operator of amplitude 1 and phase ϕ. The next point is to write something equivalent to the eikonal equation. In fact, the equation will involve both u and v together. It writes :

$$\omega = p_1(x, y, \nabla_x\theta(x,\eta), -\nabla_y\psi(y,\xi)) \tag{37}$$

for

$$\begin{cases} \nabla_\eta\theta(x,\eta) = 0 \\ \nabla_\xi\psi(y,\xi) = 0 \end{cases} \tag{38}$$

and

$$(x, y, \nabla_x\theta(x,\eta), -\nabla_y\psi(y,\xi)) \in \Lambda_\phi. \tag{39}$$

where p_1 is the principal symbol of the operator as defined in [8]. Let us recall that p_1 is defined on the neighborhood of the wavefront of the operator which is a lagrangian manifold of $T^\star(X \times Y)$ which we often replace by $T^\star X \times T^\star Y$.

As before, we define

$$(\Lambda_\phi)_\alpha = \{(x,y,\eta,\xi) \in \Lambda_\phi, p_1(x,y,\eta,\xi) = \alpha\}. \tag{40}$$

We have the theorem :

THEOREM 3.6. *the manifold $(\Lambda_\phi)_1$ is a bounded manifold if X and Y are.*

Proof :

Indeed, thanks to the assumed ellipticity of the operator, the principal symbol is coercive. Thus, as in previous section,

$$[(x,y,\eta,\xi)\in(\Lambda_\phi)_1] \Rightarrow \left[\sqrt{|\xi|^2+|\eta|^2}\leq\frac{1}{\alpha_c}\right] \tag{41}$$

where α_c is the constant of coercivity of the principal symbol of the operator. So,

$$(\Lambda_\phi)_1 \subset X\times B(0,\frac{1}{\alpha_c})\times Y\times B(0,\frac{1}{\alpha_c}). \tag{42}$$

This ends the proof of the theorem. □

As for pseudo-differential operators, this theorem implies that we can use a finite number of degrees of freedom to discretize our functions u and v.

Precisely, let $\mathcal{N}$ be a tubular neighborhood of the manifold $(\Lambda_\phi)_1$ in $T^\star X\times T^\star Y$ of thickness $\omega^{-1/2}$.

- We discretize the basis of $\mathcal{N}$ considered as a fiber bundle over $X\times Y$ with a mesh step $h_{XY}=\mathcal{O}(\omega^{-1/2})$, creating $\mathcal{O}(\omega^{\frac{n}{2}})$ couples $(x_\alpha,\ y_i)$.
- We discretize the fiber above each $(x_\alpha,\ y_i)$ with a mesh step $h_F=\mathcal{O}(\omega^{-1/2})$ by couples of directions (η_β,ξ_j), creating $\mathcal{O}(\omega^{\frac{n-1}{2}})$ directions η_β and $\mathcal{O}(\omega^{\frac{n-1}{2}})$ directions ξ_j.
- We mesh the subsets of $T^\star X$ and $T^\star Y$ with the relevant couples (x_α,η_β) and (y_i,ξ_j), thus having $\mathcal{O}(\omega^{n-1/2})$ degrees of freedom in $T^\star X$ and $\mathcal{O}(\omega^{n-1/2})$ degrees of freedom in $T^\star Y$.

Now, we define the functions $p_{\alpha\beta}(x,\eta)$ as in the previous section for $T^\star X$ and p_{ij} in the same manner. Finally we have,

$$\begin{cases} q_{\alpha\beta}(x)=\displaystyle\int p_{\alpha\beta}(x,\eta)e^{i\omega(x-x_\alpha).\eta}d\eta \\ q_{ij}(y)=\displaystyle\int p_{ij}(y,\xi)e^{i\omega(y-y_i).\xi}d\xi \end{cases} \tag{43}$$

The basis of functions $q_{\alpha\beta}$ will be used to approximate the functions v, and the basis of q_{ij} will be used to approximate the functions u. Since the functions u and v are not in the same class the result of approximation cannot be of the same kind as for pseudo-differential operators. For Fourier integral operators, we will try to estimate the quantity

$$E=|< I_\phi(u-u_h),v-v_h >| \tag{44}$$

where u_h and v_h are the projections of u and v on the bases q_{ij} and $q_{\alpha\beta}$ and I_ϕ is the Fourier integral operator of amplitude 1 and phase ϕ which is the phase of the operator P. We make two comments about this operator.

- First, in the case of pseudo-differential operators, the phase ϕ was $(x-y).\xi$. Thus the corresponding error E would have just been the modulus of the inner product $< u-u_h, v-v_h >$.
- Second, I_ϕ just transports the symbol of u from the wavefront of u to the wavefront of v, thus the error E is something like the error of approximation of the symbols.

Now, we will use the same machinery as for the pseudo-differential operators. We want to find coefficients δ_{ij} and $\delta_{\alpha\beta}$ which minimize

$$E = \left| \int e^{i\phi(x,y,\zeta)} \left[u(y) - \sum_{ij} \delta_{ij} q_{ij}(y) \right] . \left[v(x) - \sum_{\alpha\beta} \delta_{\alpha\beta} q_{\alpha\beta}(x) \right] dxdyd\zeta \right| \tag{45}$$

As previously, we make a change of variables which maps Σ_ϕ to Λ_ϕ, Σ_ψ to Λ_ψ and Σ_θ to Λ_θ. For the sake of simplicity, we keep the same notations for the new amplitudes, phases and variables. In the high frequency limit, we can limit ourselves to take into account only the variables x, y, η, ξ which are in the neighborhood of the wavefront of the operator. Since the function

$$\chi(x,y,\eta,\xi) = \sum p_{ij}(y,\xi) p_{\alpha\beta}(x,\eta) \tag{46}$$

is a cut-off around this wavefront, in the high frequency limit, we have that

$$\begin{aligned} E \sim \quad & \left| \int e^{i\phi(x,y,\zeta)} \left[\sum_{ij} p_{ij}(y,\xi) \left(a(y,\xi) e^{i\omega\psi(y,\xi)} - \delta_{ij} e^{i\omega(y-y_i).\xi} \right) \right] . \right. \\ & \left. \left[\sum_{\alpha\beta} p_{\alpha\beta}(x,\eta) \left(a(x,\eta) e^{i\omega\theta(x,\eta)} - \delta_{\alpha\beta} e^{i\omega(x-x_\alpha).\eta} \right) \right] \right| \end{aligned} \tag{47}$$

Then, we remark that if the set $S_{\alpha\beta ij} = Supp(p_{\alpha\beta} \otimes p_{ij})$ satisfies

$$S_{\alpha\beta ij} \cap (\Lambda_\theta \times \Lambda_\psi) = \emptyset, \tag{48}$$

then, the terms coming from u and v in this microlocal set are rapidly decreasing and thus, we can choose the corresponding coefficients δ_{ij} and $\delta_{\alpha\beta}$ to be null in this case.
Now, we have the
PROPOSITION 3.7.

$$\begin{aligned} E \leq \ C_1 \sum_{\substack{ij,\alpha\beta \\ S_{\alpha\beta ij} \cap \Lambda_\theta \times \Lambda_\psi \neq \emptyset}} & \int |p_{ij}(y,\xi)(a(y,\xi) - \delta_{ij})| \, . \\ & |p_{\alpha\beta}(x,\eta)(b(x,\eta) - \delta_{\alpha\beta})| \, dxdyd\eta d\xi \end{aligned} \tag{49}$$

where C_1 is a constant which depends only on the dimension of the manifolds X and Y.
Proof :
Indeed, we use the same trick as in lemma 3.3, and the Cauchy-Schwartz theorem (since,

thanks to ellipticity and eikonal equation, variables ξ and η vary in a bounded domain) to obtain that

$$E \leq \begin{array}{c} C_1 \sum \int \left| p_{ij}(y,\xi)(a(y,\xi)e^{i\omega\psi(y,\xi)} - \delta_{ij}e^{i\omega(y-y_j).\xi}) \right| . \\ \left| p_{\alpha\beta}(x,\eta)(b(x,\eta)e^{-i\omega\theta(x,\eta)} - \delta_{\alpha\beta}e^{-i\omega(x-x_\alpha).\eta}) \right| dxdyd\eta d\xi. \end{array} \tag{50}$$

Now, we develop the phases around x_α and y_i. We have, in the microlocal supports,

$$\psi(y,\xi) = Cte + \xi.(y-y_i) + \mathcal{O}(\omega^{-1}), \tag{51}$$

$$\theta(x,\eta) = Cte + \eta.(x-x_\alpha) + \mathcal{O}(\omega^{-1}). \tag{52}$$

So the $\mathcal{O}(\omega^{-1})$ terms are not oscillatory in the spread of the microlocal supports and are joined to the symbols. Then we are left with oscillatory terms which we can factorize as

$$e^{i\phi}e^{-i\omega(x-x_\alpha).\eta}e^{i\omega(y-y_i).\xi} \tag{53}$$

and thus transformed to 1 by taking the modulus. □

At last, we can enounce the final theorem of this part.
THEOREM 3.8. *For u and v satisfying*

$$u(y) = \int a(y,\xi)e^{i\omega\psi(y,\xi)}d\xi \tag{54}$$

$$v(x) = \int b(x,\eta)e^{i\omega\theta(x,\eta)}d\eta \tag{55}$$

and the eikonal equation (37) if the mesh steps on $T^\star X$, $T^\star Y$ satisfy

$$h_X = \frac{1}{C_X\omega^{1/2}} \tag{56}$$

$$h_Y = \frac{1}{C_Y\omega^{1/2}} \tag{57}$$

Then, the error is bounded in the following way.

$$E \leq \mathcal{O}(\frac{1}{C_XC_Y})||u||_{L^2(Y)}||v||_{L^2(X)}. \tag{58}$$

Proof :
As for pseudo-differential operators, we have just to choose the coefficients δ as the mean values of v in the support of $p_{\alpha\beta}$ and of u in the support of p_{ij} to obtain the result. □

We can notice that this result is just the same as the result for pseudo-differential operators when we replace the Fourier integral operator by the former class.

4. Approximation. In the previous part we presented the way we discretize functions satisfying the pseudo-differential problem (4-5) or the Fourier integral one (6-7). Now, we are going to explain how we calculate Pu in the first case or $< Pu, v >$ in the second case. We begin with the classical variational technique for finite element which is :
Suppose $u_h = \sum \alpha_{i'j'} q_{i'j'}$, then for each q_{ij} we have to compute

$$\sum \alpha_{i'j'} < Pq_{i'j'}, q_{ij} > . \tag{59}$$

which leads to the linear algebra computation $\mathcal{P}\alpha$ where $\mathcal{P}$ is the matrix defined by

$$\mathcal{P}_{ij,i'j'} =< Pq_{i'j'}, q_{ij} > \tag{60}$$

and α is the vector formed by the coefficients $\alpha_{i'j'}$. For Fourier integral operators, we will compute

$$\sum < Pq_{ij}, q_{\alpha\beta} > \delta_{ij} \tag{61}$$

leading to the linear algebra computation $\mathcal{P}\delta$ where $\mathcal{P}$ is the matrix defined by

$$\mathcal{P}_{\alpha\beta,ij} =< Pq_{ij}, q_{\alpha\beta} > \tag{62}$$

and δ is the vector formed by the coefficients δ_{ij}. The complexity of the computation is at least $\mathcal{O}(\omega^{2n-1})$ for the computation of the matrix. We will present in the following sections a way to approximate these matrices which will decrease the complexity with a controlled error.

4.1. Pseudo-differential operators. We begin by studying the case of pseudo-differential operators. We will compute and evaluate the coefficients

$$a_{ij,i'j'} = \int p(x,\zeta) e^{i(x-y).\zeta} p_{ij}(x,\eta) p_{i'j'}(y,\xi) e^{-i\omega(x-x_i).\eta} e^{i\omega(y-x_{i'}).\xi} dx dy d\eta d\xi d\zeta \tag{63}$$

which corresponds to the interaction between basis functions and which form the matrix approximating the pseudo-differential operator in our space of approximation. We enounce the following theorem which we will prove in the sequel.
THEOREM 4.1. *In the evaluation of the coefficients $a_{ij,i'j'}$ of formula (63), we distinguish three cases :*
(i) If $i \neq i'$, then the coefficient $a_{ij,i'j'}$ is rapidly decreasing when ω goes to infinity.
(ii) Else, if $d(Supp(p_{ij}), Supp(p_{i'j'})) = \mathcal{O}(\omega^{-1/2})$, then

$$a_{ij,i'j'} = \mathcal{O}(\frac{1}{\omega^{3n/2-1}}). \tag{64}$$

(iii) Else,

$$a_{ij,i'j'} = \mathcal{O}(\frac{1}{\omega^{2n-1/2}}). \tag{65}$$

Proof :
We begin with the change of variables

$$(66)\qquad \begin{cases} \omega\tilde{\zeta} = \zeta, \\ \tilde{x} = \sqrt{\omega}(x - x_i), \\ \tilde{y} = \sqrt{\omega}(y - x_{i'}). \end{cases}$$

Thus,

$$(67) a_{ij,i'j'} = \int p(x, \omega\tilde{\zeta}) e^{i\omega(x_i - x_{i'}).\tilde{\zeta} + i\sqrt{\omega}(\tilde{x}-\tilde{y})\tilde{\zeta} + i\sqrt{\omega}(-\tilde{x}.\eta + \tilde{y}.\xi)} \tilde{p}_{ij}(\tilde{x}, \eta) \tilde{p}_{i'j'}(\tilde{y}, \xi) d\tilde{x} d\tilde{y} d\eta d\xi d\tilde{\zeta}$$

For point (i), denoting by $\mathcal{L}$ the operator

$$(68)\qquad \mathcal{L} = \frac{-i(x_i - x_{i'}).\nabla_{\tilde{\zeta}}}{|x_i - x_i'|^2},$$

we can integrate by part with respect to $\mathcal{L}$ an infinite number of times since the operator $\mathcal{L}$ applied to the oscillatory part of the integrand is equivalent to ω times identity. Thus, $a_{ij,i'j'}$ is decreasing faster than any power of $\frac{1}{\omega}$ when $\omega \to \infty$. In the other cases, using the fact that the phase is stationary in the variables $\tilde{\zeta}$ and $\tilde{x}$ for $\tilde{x} = \tilde{y}$ and $\tilde{\zeta} = -\eta$, we have that

$$(69)\qquad a_{ij,i'j'} \sim \left(\frac{1}{\omega^{1/2}}\right)^n \int p(x, \omega\tilde{\zeta}) e^{i\sqrt{\omega}\tilde{y}.(\xi-\eta)} \tilde{p}_{ij}(\tilde{y}, \eta) \tilde{p}_{i'j'}(\tilde{y}, \xi) d\tilde{y} d\eta d\xi$$

In case (ii), the phase $\sqrt{\omega}\tilde{y}.(\xi - \eta)$ is equivalent to 1, and so the integrand is not oscillatory. Thus,

$$(70)\qquad a_{ij,i'j'} = \mathcal{O}\left(\frac{1}{\omega^{n/2}}.\omega.\frac{1}{\omega^n}\right)$$

since $p(x, \omega\eta) = \mathcal{O}(\omega)$ because the symbol of the operator is of order 1, and the volume of integration in $\tilde{y}$, ξ, η is proportional to $\left(\frac{1}{\omega^{1/2}}\right)^{2n}$.
In case (iii), the phase of the right hand side of expression (69) can be considered as oscillatory in $\tilde{y}$. Thus, defining $\mathcal{L}_y$ by

$$(71)\qquad \mathcal{L}_y = i\frac{(\eta - \xi).\nabla_{\tilde{y}}}{|\eta - \xi|^2},$$

we have that $\mathcal{L}_y$ applied to $e^{i\sqrt{\omega}\tilde{y}.(\xi-\eta)}$ is $\sqrt{\omega}$ times identity. So, each time we integrate by part with respect to $\mathcal{L}_y$ we gain a factor $\omega^{-1/2}$. For counting the number of times we can integrate by part, we make the following remark : since the functions p_{ij} are $P1$, we can integrate by part once without creating boundary terms. Then, for each new integration, we create one more boundary term of dimension one less than the former term. Thus, we can integrate $n+1$ times, leading to a factor of $\left(\frac{1}{\omega^{1/2}}\right)^{n+1}$. We are finally left with integrals the variables ξ, η other a volume of $\mathcal{O}\left(\frac{1}{\omega^{1/2}}\right)^{2n}$. It leads to

$$a_{ij,i'j'} = \mathcal{O}\left(\frac{1}{\omega^{n/2}}.\omega.\frac{1}{\omega^{(n+1)/2}}\frac{1}{\omega^{n}}\right). \tag{72}$$

This ends the proof of the theorem. $\square$

During the previous computation, we made the assumption that deriving the symbol in $\tilde{y}$, would not involve any perturbation in the evaluation of the magnitude of the coefficient. Let us check when this is true. For that let us recall that if the symbol p belongs to the class $S^1_{\rho,\delta}$, then

$$|\partial_x^\alpha p(x,\xi)| \leq C|\xi|^{1+\delta\alpha}. \tag{73}$$

Thus,

$$|\partial_{\tilde{y}}^\alpha p(x_i+\omega^{-1/2}\tilde{y},\omega\xi)| \leq C\frac{1}{\omega^{\alpha/2}}\omega^{1+\delta\alpha} = C\omega\omega^{(\delta-1/2)\alpha}. \tag{74}$$

Now, each time we integrate by part in $\tilde{y}$, we create a boundary term with no derivation of the symbol and a volumic term with derivating

$$p(x,\zeta)\tilde{p}_{ij}(\tilde{y},\eta)\tilde{p}_{i'j'}(\tilde{y},\xi). \tag{75}$$

In the second case, either we derivate basis functions, leading to a non-continuous integrand, or we derivate the symbol without modifying the regularity of the integrand. In that case, we gain a factor $\omega^{-1/2}$, with the integration by part, and a factor $\omega^{\delta-1/2}$ with the derivation of the symbol. Thus, in order not to alter the result of point (iii) of previous theorem, we just have to impose that the combination of the two factors leads to something not increasing with the frequency. This is reached for $\delta \leq 1$.
Let us comment the different cases of theorem 4.1. Case (i) just says that a pseudo-differential operator is local. Cases (ii) and (iii) mean that such an operator is microlocal, that is to say that the value of Pu in a point of the phase space depends only on the value of u in the neighborhood of this point.
Now, we are in position to give an efficient way to approximate the computation of the matrix $A = (a_{ij,i'j'})$ which discretizes the operator.
THEOREM 4.2. *If A is the matrix recalled above and $\tilde{A}$ is the submatrix of A built with the coefficients which correspond to case (ii) of theorem 4.1, setting the others to 0, we then have :*

$$||A|| = \mathcal{O}(\omega^{1-3n/2}), \tag{76}$$

$$||A-\tilde{A}|| = \mathcal{O}(\omega^{-3n/2}), \tag{77}$$

and thus, the relative error satisfies

$$\frac{||A-\tilde{A}||}{||A||}=(\omega^{-1}). \tag{78}$$

Proof :
Indeed, we take for instance,

$$||A||=\max_{ij}\sum_{i'j'}|a_{ij,i'j'}| \tag{79}$$

as an algebra norm. It is important to have an algebra norm, if we want to be able to deduce from the approximation of A something about the approximation of its inverse A^{-1}.
In each line of the matrix, there are $\mathcal{O}(1)$ coefficients corresponding to case (ii) and $\mathcal{O}(\omega^{(n-1)/2})$ coefficients corresponding to case (iii). Thus,

$$||A||=\mathcal{O}(1.\omega^{1-3n/2}) \tag{80}$$

and

$$||A-\tilde{A}||=\mathcal{O}(\omega^{(n-1)/2}.\omega^{1/2-2n}). \tag{81}$$

This completes the proof of the theorem. □

We finally have the numerically interesting
THEOREM 4.3. *The algorithmic and storage complexity of the computation of $\tilde{A}$ is $\mathcal{O}(\omega^{n-1/2})$.*
Proof :
Indeed, since we only retain $\mathcal{O}(1)$ coefficients by line in matrix $\tilde{A}$, we have the result for the storage complexity. For the computations, we have to recall that we keep the coefficient in the matrix only when the phase is stationary. Thus, we can numerically integrate with a finite number of Gauss integration points, this number not varying with the frequency. Hence the computation of a coefficient costs $\mathcal{O}(1)$ operations which ends the proof of the theorem. □

So, we see that the method proposed here to approximate pseudo-differential operators for propagation problems gives first a controlled error, which decreases when the frequency goes to infinity and second a gain in storage and computation complexity of at least $\mathcal{O}(\omega^{n+1/2})$.

4.2. Fourier-integral operators. Now, we go into the Fourier integral operator case. We want to compute the matrix $A=(a_{\alpha\beta,ij})$ where

$$a_{\alpha\beta,ij}=\int p(x,y,\zeta)e^{i\phi(x,y,\zeta)}p_{ij}(y,\xi)p_{\alpha\beta}e^{-i\omega(x-x_\alpha).\eta}e^{i\omega(y-y_i).\xi}dxdyd\xi d\eta d\zeta. \tag{82}$$

We will have the same kind of theorem as in the case of pseudo-differential operators.

THEOREM 4.4. *We denote by N the dimension of the space where ζ is, and by n the dimensions of X and Y.*

(i) If (x_α, y_j) is not the projection of a point of the lagrangian manifold of the operator Λ_ϕ, the coefficient $a_{\alpha\beta,ij}$ is rapidly decreasing.

(ii) Else, if $d(Supp(p_{\alpha\beta} \otimes p_{ij}),\ \Lambda_\phi)) = \mathcal{O}(\omega^{-1/2})$, then

$$a_{\alpha\beta,ij} = \mathcal{O}\left(\frac{1}{\omega^{N+n/2-1}}\right). \tag{83}$$

(iii) Else,

$$a_{\alpha\beta,ij} = \mathcal{O}\left(\frac{1}{\omega^{N+n-1/2}}\right). \tag{84}$$

We see that, applying this theorem to the pseudo-differential case would give the same kind of results as theorem 4.1 since is this case we would take $\Lambda_\phi = diag(T^\star X \times T^\star X)$.

Proof :

As previously, we proceed to the following change of variables :

$$\begin{cases} \omega\tilde{\zeta} = \zeta, \\ \tilde{x} = \sqrt{\omega}(x - x_i), \\ \tilde{y} = \sqrt{\omega}(y - x_{i'}). \end{cases} \tag{85}$$

This leads to a factor ω^{n-N} in front of the integral. Thus,

$$a_{\alpha\beta,ij} = \omega^{n-N} \int \tilde{p}(\tilde{x}, \tilde{y}, \omega\tilde{\zeta}) e^{i\omega\tilde{\phi}(\tilde{x},\tilde{y},\tilde{\zeta})} e^{i\omega^{1/2}(-\tilde{x}.\eta+\tilde{y}.\xi)} \tilde{p}_{\alpha\beta}(\tilde{x}, \eta)\tilde{p}_{ij}(\tilde{y}, \xi) d\tilde{x} d\tilde{y} d\xi d\eta d\tilde{\zeta}, \tag{86}$$

where the amplitude $\tilde{p}$ satisfies

$$\tilde{p}(\tilde{x}, \tilde{y}, \omega\tilde{\zeta}) = p(x_\alpha + \omega^{-1/2}\tilde{x}, y_i + \omega^{-1/2}\tilde{y}, \omega\zeta) \tag{87}$$

and the phase $\tilde{\phi}$ is given by,

$$\begin{aligned} \omega\phi(\tilde{x}, \tilde{y}, \tilde{\zeta}) &= \phi(x_\alpha + \omega^{-1/2}\tilde{x}, y_i + \omega^{-1/2}\tilde{y}, \omega\zeta) && (88) \\ &= \omega\phi(x_\alpha, y_i, \tilde{\zeta}) + \omega^{1/2}(\nabla_x\phi(x_\alpha, y_i, \tilde{\zeta})\tilde{x} + \nabla_y\phi(x_\alpha, y_i, \tilde{\zeta})\tilde{y}) + \mathcal{O}(1). && (89) \end{aligned}$$

Thus, as before, if $\nabla_{\tilde{\zeta}}\phi(x_\alpha, y_i, \tilde{\zeta})$ cannot be null, we can integrate by part an infinite number of times with respect to $\tilde{\zeta}$ and that concludes the proof of point (i) of the theorem. In the other cases, we perform the change of variables from the manifold Σ_ϕ to Λ_ϕ, denoting by ζ_x, ζ_y the new variable. The phase is stationary in ζ_x, $\tilde{x}$ for $\eta = \nabla_x\phi(x_\alpha, y_i, \tilde{\zeta}) = \zeta_x$ and $\nabla_{x\zeta}\phi\tilde{x} + \nabla_{y\zeta}\phi\tilde{y} = 0$. So, we have that

$$a_{\alpha\beta,ij} \sim \omega^{n-N}\frac{1}{\omega^{n/2}} \int \tilde{p}(\tilde{x}, \tilde{y}, \omega\tilde{\zeta}) e^{i\omega^{1/2}(\zeta_y+\xi)\tilde{y}} \tilde{p}_{\alpha\beta}(\tilde{x}, \eta)\tilde{p}_{ij}(\tilde{y}, \xi) d\tilde{y} d\xi d\eta \tag{90}$$

In case (ii), the phase of the integrand is $\mathcal{O}(1)$ and so we just to integrate a non oscillatory term in a volume of $\mathcal{O}(\omega^{-n})$. Thus, in this case,

$$a_{\alpha\beta,ij} = \mathcal{O}(\frac{1}{\omega^{N-n/2}}\omega\frac{1}{\omega^{n}}) = \mathcal{O}(\frac{1}{\omega^{N+n/2-1}}) \tag{91}$$

since we have choosen the amplitude of the operator to be homogeneous of degree 1.
In case (iii), the phase is oscillatory in $\tilde{y}$. So, we can perform integrations by parts with respect to this variable. As before, each integration gives a factor of $\omega^{-1/2}$ and we can integrate $n+1$ times, giving a global factor of $\dfrac{1}{\omega^{(n+1)/2}}$. Hence, in case (iii),

$$a_{\alpha\beta,ij} = \mathcal{O}(\frac{1}{\omega^{N-n/2}}\frac{1}{\omega^{(n+1)/2}}\omega\frac{1}{\omega^{n}}) = \mathcal{O}(\frac{1}{\omega^{N+n-1/2}}). \tag{92}$$

This concludes to proof of the theorem. □

Now, as for the pseudo-differential case, we can exhibit a way of approximating Fourier integral operators in the case of propagation problems at high frequency.
THEOREM 4.5. *If A is the complete matrix discretizing the operator in the basis choosed for it, and $\tilde{A}$ is the submatrix built with the coefficients of A which correspond to case (ii) of theorem 4.4, we have that*

$$||A|| = \mathcal{O}(\omega^{1-n/2-N}), \tag{93}$$

$$||A-\tilde{A}|| = \mathcal{O}(\omega^{-n/2-N}), \tag{94}$$

and thus, the relative error satisfies

$$\frac{||A-\tilde{A}||}{||A||} = (\omega^{-1}). \tag{95}$$

Proof :
Indeed, since there are only $\mathcal{O}(1)$ coefficients by line in matrix A corresponding to case (ii) of theorem 4.4 and $\mathcal{O}(\omega^{(n-1)/2})$ corresponding to case (iii) of the same theorem, we have that

$$||A|| = \mathcal{O}(1.\omega^{1-n/2-N}) \tag{96}$$

and

$$||A-\tilde{A}|| = \mathcal{O}(\omega^{(n-1)/2}\omega^{1/2-n-N}). \tag{97}$$

This allows to conclude the proof. □

Now, for the complexity, we have the

THEOREM 4.6. *The algorithmic and storage complexity of the computation of $\tilde{A}$ is $\mathcal{O}(\omega^{n-1/2})$.*

Proof :

Indeed, as in the case of pseudo-differential operators, since we only keep $\mathcal{O}(1)$ coefficients by line in $\tilde{A}$, we have the result for the storage complexity, and since the coefficients corresponding to case (ii) of theorem 4.4 have non oscillatory integrands, they only need $\mathcal{O}(1)$ operations each to be computed with a given accuracy. □

5. Conclusion. In this paper, we have presented a technique of discrete microlocalization which is well-suited to diffraction problems using pseudo-differential or Fourier integral operators. Let us summarize our results for Fourier integral operators since it appeared in all the paper that the pseudo-differential operators could be thought of as a subcase of them.

- First, we have exhibited a space of microlocal approximation built with functions whose wave-front was in the neighborhood of the wave-front of the operator and we projected these functions on the two factors $T^\star X$ and $T^\star Y$ creating the basis of the $p_{\alpha\beta}$ and the one of the p_{ij}.
- Second, we have shown that the coefficients of interaction where important only when $Supp(p_{\alpha\beta} \otimes p_{ij})$ was in the neighborhood of the wave-front of the operator.
- Third, we have got that the relative error made on the matrix when setting to 0 all the interactions which where not important was $\mathcal{O}(\frac{1}{\omega})$.
- Fourth, the complexity of the computation and storage of the approximated matrix is $\mathcal{O}(\omega^{n-1/2})$ which has to be compared with the $\mathcal{O}(\omega^{2n})$ for a classical method.

Finally, we make a few comments and give some perspectives. As we did in [5], it is easy to take caustics and grazing rays into account, because, using Fourier-Airy operators we still obtain an eikonal equation and the basis functions presented in this paper give a correct approximation for that kind of waves. In fact, the point is that the characteristic size of our discretization is $\sqrt{\lambda}$ and caustics or degenerated waves differ from a generic wave only on a length greater than $\lambda^{1/3}$ where λ is the wavelength. So our discretization does not descriminate between generic and degenerated waves.

This method can also take into account singular geometries. For instance in the case of edges, instead of asking the gradient of the phase to be null, we would ask for its projection on the tangent space of the edge to be null, and we would be able to carry on with our technique.

If one wants a greater quality in the approximation of the operator, one has to use basis functions which are more regular than just P1. Indeed, the magnitude of what is neglected is related to the number of times that we can integrate by part in the case (iii) of theorems 4.1 and 4.4. This method could be extended to the time domain propagation. In that case, we would have to fix a maximum pulsation ω_{max} related to the time step.

We would then have to mesh the manifold corresponding to the eikonal inequality :

$$|p_1 x, y, \eta, \xi| \leq \omega_{max} \tag{98}$$

and then to adapt the techniques presented in this paper. We may also use an extension of our method to compute solutions for non-linear wave equations, using paraproducts defined by Bony (see [1]).

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