On the projective evolution of 2D curves

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Abstract

In this paper, we investigate the evolution of curves of the projective plane according to a family of projective invariant intrinsic equations. This is motivated by previous work for the Euclidean [10, 11, 13] and the affine cases [20, 21, 3, 2] as well as by applications in the perception of two-dimensional shapes. We establish the evolution laws for the projective arclength and curvature. Among this family of equations, we define a "projective heat equation" [7] and establish the link with the projective evolution of curves in $\mathbf{R}^2$.

Résumé

Dans le présent article, nous étudions les familles de courbes du plan projectif évoluant suivant des équations intrinsèques projectivement invariantes. Cette démarche est motivée par les travaux précédents concernant les cas euclidien [10, 11, 13] et affine [20, 21, 3, 2]. ainsi que par les applications possibles en perception des formes 2D. Nous établissons les lois d'évolutions de l'abscisse curviligne et de la courbure projectives. Parmi ces équations, nous définissons une équation de la chaleur projective [7] et établissons le lien avec l'évolution projectivement invariante des courbes du plan réel $\mathbf{R}^2$.

1 Introduction

The use of partial differential equations and curve or surface evolution theory in image analysis became a major research topic in the past years (see [17]) leading to applications in image de-noising and de-blurring [18], in selective smoothing and edge detection [1, 16], in contrast enhancement [19], in shape segmentation [5]. Recently, applications were found in problems usually addressed by the computer vision community: intrinsic flows [13, 20] hold very good geometric smoothing properties and allow the computation of local differential invariants [8]. Motivated by the importance of projective geometry in computer vision, we found it natural to extend the Euclidean [13] and affine [20] cases to the projective one.

2 Geometric flows

Let $\mathcal{L}$ be a Lie group operating on some objects. A quantity q depending on these objects is called an *invariant* of $\mathcal{L}$ if, whenever a transformation $L \in \mathcal{L}$ changes q into q', we have $q' = \alpha(L)q$, where α is a function of L alone, i.e. does not depend on the object which is transformed. If $\alpha \equiv 1$, then q is called an *absolute invariant.*

Differential invariants are special invariants based on local transformations (see [12]).

Let $\mathcal{C} : \mathbf{R} \to \mathbf{R}^2$ be a plane curve of parameter p. The first and the second differential invariants for the Euclidean group $\{m \mapsto Rm + T \mid R \text{ rotation}, \ T \text{ translation}\}$ are the well known Euclidean arclength v and curvature κ defined by:

$$\left\{ \begin{array}{lcl} \frac{\partial v}{\partial p} & = & \|\frac{\partial \mathcal{C}}{\partial p}\| \\ \kappa & = & \|\frac{\partial^2 \mathcal{C}}{\partial v^2}\| \end{array} \right. \tag{1}$$

which are preserved by rotations and translations.

The corresponding invariants for the group of proper affine motions $\{m \mapsto Am + B \mid [A] > 0, B \in \mathbf{R}^2\}$ are the affine arclength s and curvature μ defined by:

$$\left\{ \begin{array}{lcl} \frac{\partial s}{\partial p} & = & [\frac{\partial \mathcal{C}}{\partial p}, \frac{\partial^2 \mathcal{C}}{\partial p^2}]^{1/3} \\ \mu & = & [\frac{\partial^2 \mathcal{C}}{\partial s^2}, \frac{\partial^3 \mathcal{C}}{\partial s^3}] \end{array} \right. \tag{2}$$

which are invariants for affine proper motions, and absolute invariants for special affine motions ($\{m \mapsto Am + B \mid [A] = 1, B \in \mathbf{R}^2\}$) .

Circles (and straight lines) are the only curves with constant Euclidean curvature. In the affine case, constant affine curvature is obtained for the conics ($\mu = 0$ for a parabola, $\mu > 0$ for an ellipse and $\mu < 0$ for an hyperbola).

Given an initial plane curve $\mathcal{C}_0(p) : \mathbf{R} \to \mathbf{R}^2$, the associated geometric flow (see [15]) is the family of curves $\mathcal{C}(p,t) : \mathbf{R} \times [0, \tau) \to \mathbf{R}^2$ evolving according to the following law:

$$\left\{ \begin{array}{lcl} \frac{\partial \mathcal{C}(p,t)}{\partial t} & = & \frac{\partial^2 \mathcal{C}(p,t)}{\partial r^2} \\ \mathcal{C}(p,0) & = & \mathcal{C}_0(p) \end{array} \right. \tag{3}$$

where r is the group arclength (v for the Euclidean geometric flow, s for the affine one). Contrary to the classical heat flow $\mathcal{C}_t = \mathcal{C}_{pp}$, these flows are intrinsic (i.e. don't depend on the parameterization p of the initial curve). They are invariant for the considered Lie group. Their "smoothing" properties may be summarized as follow ([13, 20]): closed curves evolve toward a convex one and then disappear shrinking toward a circle point (Euclidean case) or an ellipse point (affine case).

For a given group, a plane curve is defined up to a group transformation by its group arclength and curvature. Hence, it is natural to study these flows through the evolution of the arclength and curvature. With $g_e = \frac{dv}{dp}$ and $g_a = \frac{ds}{dp}$, we have:

$$\left\{ \begin{array}{lcl} \frac{\partial g_e}{\partial t} & = & -g_e \kappa^2 \\ \frac{\partial \kappa}{\partial t} & = & -\kappa^3 - \frac{\partial^2 \kappa}{\partial v^2} \end{array} \right. \text{ and } \left\{ \begin{array}{lcl} \frac{\partial g_a}{\partial t} & = & -2 g_a \mu / 3 \\ \frac{\partial \mu}{\partial t} & = & \frac{4}{3}\mu^2 + \frac{1}{3}\frac{\partial^2 \mu}{\partial s^2} \end{array} \right. \tag{4}$$

3 Projective geometry

Like in equations (1) and (2), it is possible to define the projective arclength and curvature of a plane curve in $\mathbf{R}^2$. However, this leads to too complex expressions. The idea is to embed such a curve in the real projective plane $\mathcal{P}^2$. One can see $\mathcal{P}^2$ as the set of the lines of $\mathbf{R}^3$ going through the origin. An element of $\mathcal{P}^2$ is represented by its homogeneous coordinates (x, y, z) where (x, y, z) and $(\lambda x, \lambda y, \lambda z), (\lambda \neq 0)$ are different coordinate vectors of the same projective point.

Let $\mathbf{B}(p) : \mathbf{R} \to \mathcal{P}^2$ be a smooth curve of the projective plane. Using standard results of projective differential geometry [4], we change $\mathbf{B}(p)$ by a scale factor $\lambda(p)$ and characterize its projective arclength σ and curvature k introducing the Cartan point $\mathbf{A} = \lambda \mathbf{B}$, and the Cartan frame $(\mathbf{A}, \mathbf{A}^{(1)}, \mathbf{A}^{(2)})$ which satisfy the projective Frenet equations:

$$\begin{array}{rcl} \frac{d\mathbf{A}}{d\sigma} & = & \mathbf{A}^{(1)} \\ \frac{d\mathbf{A}^{(1)}}{d\sigma} & = & -k\mathbf{A} + \mathbf{A}^{(2)} \\ \frac{d\mathbf{A}^{(2)}}{d\sigma} & = & -\mathbf{A} - k\mathbf{A}^{(1)} \end{array} \tag{5}$$

and the condition:

$$|\mathbf{A}\mathbf{A}^{(1)}\mathbf{A}^{(2)}| = 1 \tag{6}$$

Note that $\mathbf{B}$ and $\mathbf{A}$ are different coordinate vectors of the same projective point. The point $\mathbf{A}^{(1)}$ is on the tangent to the curve in $\mathbf{A}$ and the line $\langle \mathbf{A}, \mathbf{A}^{(2)} \rangle$ is the projective normal. Functions k and σ are invariant under the action of the projective group and characterize the curve up to a projective transformation.

The plane curves with a constant projective curvature are (see [9]):

- If $k = k_0 = -3/32^{1/3}$: the exponential ($y = e^x$)
- If $k < k_0$: the general parabola ($y = x^m, m \notin \{2, \frac{1}{2}, -1\}$)

- If $k > k_0$: the logarithmic spiral ($\rho = e^{m\theta}, m \neq 0$)

4 Projective invariant intrinsic flows

The law $\mathbf{A}_t = \mathbf{A}_{\sigma\sigma}$ investigated in [7] could be thought of as a natural extension of the Euclidean and affine cases. Yet, this law raises some contradictions. For instance, according to the expression of k_t in [7], curves with a constant initial curvature should evolve keeping a constant curvature. Actually, it is not the case (see [9]).

The reason why it is so is that the Cartan point $\mathbf{A}(p,t)$ is some particular representant for the projective point $\mathbf{B}(p,t)$ and depends on the curve and its spatial derivatives at (p,t). As a result, one can't expect an arbitrary differential equation $\{\mathbf{A}(p,0) = \mathbf{A}_0(p); \mathbf{A}_t = f(p,t)\}$ to be such that $\mathbf{A}(p,t)$ will still be the Cartan point of the curve at time $t > 0$.

This leads us to consider the evolution law

$$\left\{ \begin{array}{lcl} \mathbf{A}(p,0) & = & \mathbf{A}_0(p) \quad (\mathbf{A}_0 \text{ Cartan point of the initial curve}) \\ \mathbf{A}_t(p,t) & = & \alpha\mathbf{A} + \beta\mathbf{A}^{(1)} + \gamma\mathbf{A}^{(2)} \end{array} \right. \tag{7}$$

where $f(p,t)$ has been decomposed on the Cartan frame, and to find out which conditions on (α,β,γ) will assure that $\mathbf{A}(p,t)$ remains the Cartan point.

We get the following result:

Proposition 1 *The differential equation (7) has a meaning (i.e.* $\mathbf{A}(p,t)$ *is the Cartan point of the curve at time t) if and only if:*

$$\begin{aligned} \alpha = \frac{1}{3+k_\sigma} \quad \Big[& -\frac{1}{3}k_{\sigma^3} - \frac{3}{2}k_{\sigma^2}\gamma_\sigma - k_\sigma(\frac{7}{3}k\gamma + \frac{17}{6}\gamma_{\sigma^2} + \beta_\sigma) - \frac{8}{3}k^2\gamma_\sigma \\ & + k(\gamma - \frac{5}{3}\gamma_{\sigma^3}) + \gamma_{\sigma^2}/2 - \gamma_{\sigma^5}/6 \ \Big] \end{aligned} \tag{8}$$

In this case, the projective arclength and curvature evolve according to:

$$\frac{g_t}{g} = \alpha + \beta_\sigma - \frac{1}{3}(k\gamma - \gamma_{\sigma^2}) \tag{9}$$

$$\begin{aligned} k_t = & -\alpha_{\sigma^2} + \frac{3}{2}\gamma_\sigma + \frac{\gamma_{\sigma^4}}{6} + k(\frac{2}{3}\gamma_{\sigma^2} - 2\alpha) \\ & + k_\sigma(\beta + \frac{7}{6}\gamma_\sigma) + \frac{\gamma}{3}(k_{\sigma^2} + 2k^2) \end{aligned} \tag{10}$$

where $g = \frac{d\sigma}{dp}$.

Proof:

- **Step 1:** Let us first establish some preliminary properties. Using the fact that the independent variables p and t verify

$$\frac{\partial^2}{\partial t \partial p} = \frac{\partial^2}{\partial p \partial t}$$

it is quite immediate to show that the Lie bracket $[\frac{\partial}{\partial t}, \frac{\partial}{\partial \sigma}]$ equals:

$$[\frac{\partial}{\partial t}, \frac{\partial}{\partial \sigma}] = \frac{\partial^2}{\partial t \partial \sigma} - \frac{\partial^2}{\partial \sigma \partial t} = -\frac{g_t}{g}\frac{\partial}{\partial \sigma} \tag{11}$$

Applying this formula twice more, we obtain the expressions

$$\frac{\partial^3}{\partial t \partial \sigma^2} = -[\frac{g_t}{g}]_\sigma \frac{\partial}{\partial \sigma} - 2\frac{g_t}{g}\frac{\partial^2}{\partial \sigma^2} + \frac{\partial^3}{\partial \sigma^2 \partial t} \tag{12}$$

and

$$\frac{\partial^4}{\partial t \partial \sigma^3} = -[\frac{g_t}{g}]_{\sigma^2} \frac{\partial}{\partial \sigma} - 3[\frac{g_t}{g}]_\sigma \frac{\partial^2}{\partial \sigma^2} - 3\frac{g_t}{g}\frac{\partial^3}{\partial \sigma^3} + \frac{\partial^4}{\partial \sigma^3 \partial t} \tag{13}$$

that we will need later.

- **Step 2:** Let us now prove that equation (8) is necessary and that we have (9) and (10). Taking the derivative of equation (6) with respect to t at constant p, we get:

$$0 = |\mathbf{A}\mathbf{A}^{(1)}\mathbf{A}^{(2)}|_t = |\mathbf{A}_t\mathbf{A}^{(1)}\mathbf{A}^{(2)}| + |\mathbf{A}\mathbf{A}_t^{(1)}\mathbf{A}^{(2)}| + |\mathbf{A}\mathbf{A}^{(1)}\mathbf{A}_t^{(2)}| \tag{14}$$

From equations (7) and (5), the first determinant of the right hand side member equals α. Using also equation (11), we get

$$\mathbf{A}_t^{(1)} = \mathbf{A}_{t\sigma} = -\frac{g_t}{g}\mathbf{A}^{(1)} + \frac{\partial \mathbf{A}_t}{\partial \sigma} \tag{15}$$

from which we finally obtain the value of $|\mathbf{A}\mathbf{A}_t^{(1)}\mathbf{A}^{(2)}|$ in (14). In a similar way, we write

$$\mathbf{A}_t^{(2)} = \frac{\partial(k\mathbf{A} + \mathbf{A}_{\sigma\sigma})}{\partial t} = k_t\mathbf{A} + k\mathbf{A}_{\sigma\sigma} + \mathbf{A}_{t\sigma\sigma} \tag{16}$$

whose last term is obtained from equation (12) . Thus the value of the last determinant in (14) and finally equation (9) . From the Frenet formulas (5), we get the useful relation

$$\mathbf{A}_{\sigma^3} = -2k\mathbf{A}^{(1)} - (1 + k_\sigma)\mathbf{A} \tag{17}$$

Let us now compute $\frac{\partial}{\partial t}\mathbf{A}_{\sigma^3}$ in two different ways:

1. Using equation (17) , we have

$$\begin{aligned} \frac{\partial \mathbf{A}_{\sigma^3}}{\partial t} &= \frac{\partial}{\partial t}(-(1 + k_\sigma)\mathbf{A} - 2k\mathbf{A}^{(1)}) \\ &= -\frac{\partial k_\sigma}{\partial t}\mathbf{A} - (1 + k_\sigma)\mathbf{A}_t - 2k_t\mathbf{A}^{(1)} - 2k\mathbf{A}_t^{(1)} \end{aligned} \tag{18}$$

   where we know $\mathbf{A}_t$ and $\mathbf{A}_t^{(1)}$ from (7) and (15).

2. On the other hand, we can use equation (13)

$$\frac{\partial \mathbf{A}_{\sigma^3}}{\partial t} = -[\frac{g_t}{g}]_{\sigma^2}\mathbf{A}_\sigma - 3[\frac{g_t}{g}]_\sigma \mathbf{A}_{\sigma^2} - 3\frac{g_t}{g}\mathbf{A}_{\sigma^3} + \frac{\partial^3 \mathbf{A}_t}{\partial \sigma^3} \quad (19)$$

where all the terms of the right hand side member are known from (9) and (5), except the last one which can be computed from equation (7) and the Frenet formulas.

Expressing (18) and (19) in the frame $(\mathbf{A}, \mathbf{A}^{(1)}, \mathbf{A}^{(2)})$ and equaling their coordinates, we get three relations: a tautology, equation (10) and condition (8).

- **Step 3:** We now have to prove that condition (8) is sufficient for the law (7) to be well-defined. More precisely, we have to show that $\mathbf{B}(p,t)$ defined by

$$\begin{cases} \mathbf{B}(p,0) &= \mathbf{A}_0(p) \ \ (\mathbf{A}_0 \text{ Cartan point of the initial curve}) \\ \mathbf{B}_t(p,t) &= \alpha\mathbf{A} + \beta\mathbf{A}^{(1)} + \gamma\mathbf{A}^{(2)} \\ & \quad (\mathbf{A}, \mathbf{A}^{(1)}, \mathbf{A}^{(2)}) \text{ Cartan frame of the curve at time } t \end{cases} \quad (20)$$

remains the Cartan point, i.e. $\mathbf{B}(p,t) = \mathbf{A}(p,t)$. Actually, the same method as in step 2 using $\mathbf{B} = \mathbf{A}/\lambda$ in (20) gives three evolution laws: g_t, k_t and λ_t. When α is given by (8), the evolution of λ becomes $\lambda_t = 0$, thus $\lambda(p,t) = \lambda(p,0) = 1$ and $\mathbf{B}(p,t) = \mathbf{A}(p,t)$.

□

Remark 1: Another way to see that a condition like (8) is necessary is to consider the surface $\mathcal{S} = \{\mathbf{A}(p,t)|(p,t)\}$ of $\mathbf{R}^3$. The reason why this is a well-defined surface is because there is no scale factor on $\mathbf{A}$ even though it represents a projective point of $\mathcal{P}^2$. Now, in order for (7) to be a well-defined PDE on $\mathcal{S}$, the vector $\mathbf{A}_t$ has to belong to the tangent plane $T_\mathcal{S}$ at $A(p,t)$. The right hand side contains the vector $\mathbf{A}^{(1)}$ which belongs to $T_\mathcal{S}$ but the vector $\alpha\mathbf{A} + \gamma\mathbf{A}^{(2)}$ does not in general belong to $T_\mathcal{S}$ unless α and γ are dependent. In fact the condition is even stronger since not only $\mathbf{A}_t$ must belong to $T_\mathcal{S}$ but also, as stated above, $\mathbf{A}$ must remain a Cartan point.

Remark 2: Note that $\mathbf{A}_t = \mathbf{A}_{\sigma\sigma}$ is the case $(\alpha, \beta, \gamma) = (-k, 0, 1)$, thus doesn't satisfy condition (8), hence the previously mentioned contradictions.

Finally, if β and γ are projective invariant intrinsic quantities, then α defined by equation (8) is a projective invariant intrinsic quantity too. Therefore, we get:

Corollary 1 *Let β and γ be some projective invariant intrinsic quantities, let α be defined by equation (8) . The differential equation (7) defines a projective invariant intrinsic flow. The evolution of the projective arclength and curvature of the curves is given by equations (9, 10).*

5 The projective "heat flow"

Among all the possible choices for (β, γ), it turns out that the simplest one $(0,1)$ is also the right one for a projective "heat flow" extending the Euclidean and affine cases. Some intuitive justification could be:

- $\beta \mathbf{A}^{(1)}$ is on the tangent in $\mathbf{A}$. Thus, the choice of β has no importance: changing β doesn't modify the family of curves obtained but only their parameterization p (see [20]).
- $(\beta, \gamma) = (0,1)$ are the components of $\mathbf{A}_{\sigma\sigma}$ on $(\mathbf{A}^{(1)}, \mathbf{A}^{(2)})$. The induced α could be considered as a corrected component of $\mathbf{A}_{\sigma\sigma}$ on $\mathbf{A}$.

However, the deep reason for this choice is that it gives the same flow as $\mathcal{C}_t = \mathcal{C}_{\sigma\sigma}$ in $\mathbf{R}^2$ (see next section). Consequently, we have directly from proposition 1 the following statement:

Proposition 2 *Let α be:*

$$\alpha = \frac{1}{9+3k_\sigma}(3k - 7kk_\sigma - k_{\sigma^3})$$

Let $\mathbf{B}_0(p)$ be a curve of $\mathcal{P}^2$ and $\mathbf{A}_0(p)$ its Cartan points. We define its projective heat flow as the solution of:

$$\left\{ \begin{array}{rcl} \mathbf{A}(p,0) & = & \mathbf{A}_0(p) \\ \mathbf{A}_t(p,t) & = & \alpha \mathbf{A} + \mathbf{A}^{(2)} \end{array} \right. \tag{21}$$

Let $g = \frac{d\sigma}{dp}$. The projective arclength and curvature evolve according to:

$$\frac{g_t}{g} = \frac{-1}{9+3k_\sigma}(8kk_\sigma + k_{\sigma^3}) \tag{22}$$

$$k_t = \frac{2}{3}k^2 + \frac{1}{3}k_{\sigma^2} - 2\alpha k - \alpha_{\sigma^2} \tag{23}$$

6 Going back to $\mathbf{R}^2$

Let us now justify our choice, explaining the link between:

- equation (21) on the Cartan point.
- the more natural equation $\mathbf{B}_t = \mathbf{B}_{\sigma\sigma}$ on any coordinate vector $\mathbf{B}$ in $\mathcal{P}^2$.
- and its previously mentioned [15] analog in the real plane $\mathbf{R}^2$: $\mathcal{C}_t = \mathcal{C}_{\sigma\sigma}$.

Proposition 3 *Given an initial curve in $\mathcal{P}^2$, let $\mathbf{B}_0(p)$ be any smooth enough coordinate vector of it.*

1. *The flow defined by*

$$\left\{ \begin{array}{rcl} \mathbf{B}(p,0) & = & \mathbf{B}_0(p) \\ \mathbf{B}_t(p,t) & = & \mathbf{B}_{\sigma\sigma} \end{array} \right. \tag{24}$$

is intrinsic and doesn't depend on the choice of $\mathbf{B}_0$ (i.e. $\mathbf{B}_0(p)$ and $\phi_0(p)\mathbf{B}_0(p)$ give the same family of curves for any choice of a smooth enough ϕ_0 strictly positive or negative).

2. *This flow is the projective heat flow defined by equation (21) up to a re-parameterization of the curves.*

3. *Let λ be the Cartan scale factor ($\mathbf{A} = \lambda\mathbf{B}$). (σ, k, λ) define $\mathbf{B}$ up to a projective transformation. Their evolution is given by:*

$$\begin{aligned}
\frac{g_t}{g} &= \frac{-1}{9+3k_\sigma}(8kk_\sigma + k_{\sigma^3} + 18\Lambda_{\sigma^2}) \\
k_t &= \frac{2}{3}k^2 + \frac{1}{3}k_{\sigma^2} - 2Pk - P_{\sigma^2} - 2k_\sigma\Lambda_\sigma \\
\Lambda_t &= \frac{-1}{9+3k_\sigma}[k_{\sigma^3} + 3k_\sigma(\Lambda_\sigma^2 - 3\Lambda_{\sigma^2}) + 4k(k_\sigma - 3) + 9(\Lambda_\sigma^2 - \Lambda_{\sigma^2})] \\
&\text{where } g = \frac{d\sigma}{dp},\ \Lambda = \log|\lambda|,\ P = \Lambda_\sigma^2 - \Lambda_{\sigma^2} - k + \Lambda_t
\end{aligned} \tag{25}$$

Proof:

- **Step 1:** Let $\mathbf{B}(p,t)$ be the solution of (24) for a given coordinate vector $\mathbf{B}_0$ of the initial curve (chosen such that all the necessary derivatives are defined). Any other smooth coordinate vector of the initial curve can be written $\phi_0(p)\mathbf{B}_0(p)$ where ϕ_0 is also smooth enough and $\phi_0(p) \neq 0, \forall p$. Let $\phi(p,t)$ be the solution of:

$$\left\{\begin{aligned} \phi(p,0) &= \phi_0(p) \\ \phi_t(p,t) &= \phi_{\sigma\sigma} - 2(\phi_\sigma)^2/\phi \end{aligned}\right. \tag{26}$$

  Let us define $\mathbf{B}'(p,t) = \phi\mathbf{B}$. $\mathbf{B}(.,t)$ and $\mathbf{B}'(.,t)$ are different coordinate vectors for the same curves of $\mathcal{P}^2$ thus have the same projective arclength : $\sigma' = \sigma$. It is then straightforward to check that $\mathbf{B}'$ verifies

$$\mathbf{B}'_t = (-2\phi_{\sigma'}/\phi)\mathbf{B}'_{\sigma'} + \mathbf{B}'_{\sigma'\sigma'}$$

  which gives the same curves as $\mathbf{B}'_t = \mathbf{B}'_{\sigma'\sigma'}$ up to a re-parameterization in space. $\mathbf{B}'$ is the solution of (24) with initial condition $\phi_0\mathbf{B}_0$. Finally both initial conditions $\mathbf{B}_0$ and $\phi_0\mathbf{B}_0$ give the same flow.

- **Step 2:** Any choice of $\mathbf{B}_0$ giving the same solution, let us take $\mathbf{B}_0 = \mathbf{A}_0$. With $\mathbf{A} = \lambda\mathbf{B}$, equation $\mathbf{B}_t = \mathbf{B}_{\sigma\sigma}$ becomes $\mathbf{A}_t = \alpha_1\mathbf{A} + \beta_1\mathbf{A}^{(1)} + \mathbf{A}^{(2)}$ where $(\alpha_1, \beta_1, 1)$ verifies condition (8). The tangential component of the velocity being of no importance, this latest equation gives the same family of curves as $\mathbf{A}_t = \alpha_2\mathbf{A} + \mathbf{A}^{(2)}$ where α_2 is the correct value of α for $(\beta,\gamma) = (0,1)$: $\mathbf{B}_t = \mathbf{B}_{\sigma\sigma}$ gives the projective heat flow defined by (21).

- **Step 3:** The method used in step 2 of the proof of proposition 1 applied on law (24) with $\mathbf{A} = \lambda\mathbf{B}$ gives equations (25). λ and its derivatives now appear in the components (α,β,γ) of $\mathbf{A}_t$ in the frame $(\mathbf{A}, \mathbf{A}^{(1)}, \mathbf{A}^{(2)})$. In the previous case, we wanted λ to be the constant 1. Hence the condition

(8) on α. Here, λ is free and the same equation that gave condition (8) now leads to the evolution law of λ. Note that the quantity P used to get a somehow shorter writing of equations (25) is in fact the first component α of $\mathbf{A}_t$.

□

Let $\mathcal{C}_0(p) = (x_0, y_0)$ be a real plane curve, it is then easy to prove that:

Corollary 2 *The flow defined by* $\{\mathcal{C}(p,0) = \mathcal{C}_0\ ;\ \mathcal{C}_t = \mathcal{C}_{\sigma\sigma}\}$ *is a projective invariant flow. It gives the same family of curves through the map* $(\frac{x}{z}, \frac{y}{z})$ *as the projective heat flow (21) with initial curve* $(x_0, y_0, 1)$*. Let* $\mathcal{C}(p,t) = (x,y)$ *and* λ *be the Cartan scale of* $(x, y, 1)$*, the evolution of the projective arclength and curvature of* $\mathcal{C}$ *is given by equations (25).*

Proof: Let $\mathbf{B} = (x, y, 1)$. If $\mathcal{C}_t = \mathcal{C}_{\sigma\sigma}$ then $\mathbf{B}$ follows law (24) just because:

1. Its projective arclength is the same as the one of $\mathcal{C}$
2. Its first two components are the ones of $\mathcal{C}$ and its third component is constant.

As a result, the parameters (σ, k, λ) of $\mathcal{C}$, which are the ones of $\mathbf{B}$, evolve according to (25).
□

Note that this was already proved in [14], even though the argument in [15] about the relationship between different coordinate vectors is incorrect (see proposition 3 above: the scaling factor ϕ doesn't remain the initial one and the curves have to be re-parameterized)

7 Curves with constant projective curvature

A first property of the projective heat flow, common with the Euclidean and affine cases, is that initial curves with a constant projective curvature evolve remaining with a constant curvature. Moreover, we see that they keep their initial curvature: $k_t = 0$. Hence, they evolve remaining the initial one up to an homography. It is easy to solve equation (24) completely and to see that Logarithmic spirals rotate and shrink, general parabolas and the exponential go to a line in the limit (figure 1).

8 Conclusion

We have established a link between the invariant projective flow defined in $\mathbf{R}^2$ [15, 14] and the one defined in $\mathcal{P}^2$ [7]. We have defined the projective heat equation in three equivalent ways: $\mathbf{A}_t = \alpha\mathbf{A} + \mathbf{A}^{(2)}$ (α given by equation (8)) or $\mathbf{B}_t = \mathbf{B}_{\sigma\sigma}$ in $\mathcal{P}^2$, and $\mathcal{C}_t = \mathcal{C}_{\sigma\sigma}$ in $\mathbf{R}^2$. As expected, the connection is not

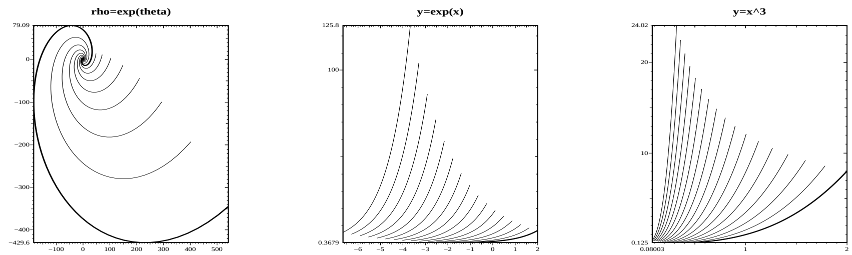

Figure 1: Evolution of the curves with a constant projective curvature. In bold, the initial curve

trivial but simple enough. The advantage of the definition in $\mathcal{P}^2$ [7] which we have modified here to make it entirely correct is that: a) it does not depend on the particular coordinates used to represent $\mathcal{P}^2$ and b) it has allowed us to establish the evolution of the projective arclength and curvature. There remains to see if it is possible to define a projective scale-space as in the Euclidean and affine cases. Of particular interest would be to compare our approach with the one developed by Dibos [6].

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