The Godunov scheme and what it means for first order traffic flow models [1].

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ABSTRACT: In the present paper, we shall show that most recent discretizations of macroscopic first order traffic flow models are equivalent to Godunov's scheme, by analyzing the Riemann problem in the case of equilibrium flow-density relationship that are discontinuous relatively to the position. Further, it will be shown that the resulting formulas lead to the introduction of the local traffic supply and demand concepts. These concepts provide a unifying framework for the modelling of boundary conditions in the LWR model and correlatively the modelling of intersections. A few examples of resulting intersection and network models are discussed.

1 Introduction

This paper is about first order traffic flow models and the Godunov scheme. Let us first recall the basic equations of the first order traffic flow model, called hereafter the LWR model [LW 55], [RI 56]. The basic variables of this model are the following.

. $K(x,t)$: the density at point x and at time t.

. $Q(x,t)$: the flow at point x and at time t.

. $V(x,t)$: the space-time average speed at point x and at time t, defined by the relationship $Q = KV$.

The basic equations of the model are:

(1)
$$\frac{\partial K}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

(conservation of vehicles),

(2)
$$Q = KV$$

(definition equation of V),

(3)
$$V = V_e(K,x)$$

(equilibrium speed-density relationship). Of course these equations can be rewriten as

(4)
$$\frac{\partial K}{\partial t} + \frac{\partial}{\partial x} Q_e(K,x) = 0$$

with

$$Q_e(K,x) \stackrel{def}{=} KV_e(K,x)$$

the equilibrium flow-density relationship. Typically, the equilibrium flow-density has the following aspect.

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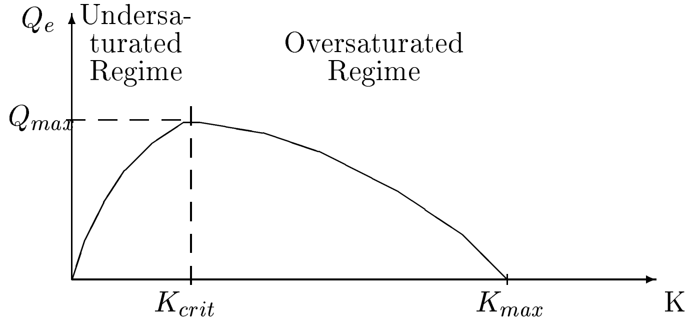


The explicit dependency on position x of both equilibrium relationships takes into account the variability of the physical and environmental parameters, for instance number of lanes, maximal allowed speed, etc .... In principle, equation (4) can only be written while assuming that both time and space scales are sufficiently large, in order that *local average* flow and densities may indeed be considered and described by reasonably regular functions. The use of model (4) for the building of traffic flow models on networks is quite old and various methods have been tried. Some earlier efforts implied an explicit solution of the LWR model (4), using the well-known shock-wave and rarefaction fan techniques, complete as in [MMS 81], or semi-discretized as in SIMAUT [INR 88] and Hilliges' model [HI 95] for instance. Other efforts implied a space-time discretization of the LWR model (4), such as SSMT [LE 84], [MBL 84], [CP 92], the model of Bui et al. [BNN 92], Daganzo's CELL model [DA 94], the urban part of METACOR [EHP 94] (whose corridor part is METANET [MP 90]), the flow model of INTEGRATION [VA 94], STRADA [BLL 95]. The INTEGRATION flow model is quite different from all the other models mentionned, since its discretization is a *particle discretization*.

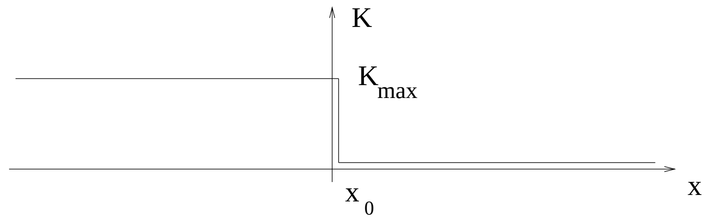
All these models compute the so-called *entropy solutions* of the LWR model, since this model does not admit unique solutions for given initial and boundary conditions. This choice is fairly natural but not completely self-evident, and we shall discuss it in section 2, refering to [AN 90] and [BNN 92]. The present paper concerns itself essentially with the Godunov scheme. This is the best first order scheme for the computation of the *entropy solution* of the LWR model.

This scheme has been introduced in 1959 [GO 59] and many first order schemes are derived from it. We shall discuss it in the sections 3 and 4 of this paper. Notably we shall show that the Godunov scheme can be extended to the case where the equilibrium flow-density relationship is discontinuous relatively to the space variable x. This point is definitely not an academic one. Indeed, if one needs to model a network, it is absolutely essential to be able to consider such discontinuities, if only at intersections. As a byproduct, we shall show that the various discretization schemes of (4) that have been introduced in the past are nothing more than the Godunov scheme under one guise or other. But only two of them ([LE 93] and [DA 93], [DA 94]) are compatible with discontinuous equilibrium flow-density relationships.

We shall then focuse our attention on the expression given by Daganzo in [DA 93], [DA 94] for the flow in Godunov's scheme. Indeed, this expression has a nice interpretation, since it defines locally the flow as the smallest of two quantities which, following [LE 95] and [BLL 95], we shall call *local traffic demand and supply*. First we shall precise the proper way to specify boundary conditions for simple links with the help of the above mentionned supply and demand functions (section 5). Though fairly obvious, this is a necessary step for the sequel. We shall show further how using the supply and demand functions make it is possible to construct various extensions of the basic LWR model, notably for the modelling of intersections (section 6) and the modelling of the dynamics of partial densities (section 7). Of course, these extensions contain many behavioural elements, and the benefit of the introduction of the local demand and supply concepts lies with the framework they provide for modelling, as well as a clarification of issues. These points and others, such as second order discretizations of the LWR model and assignment problems modelling will be evoed in the conclusion of the paper.

2 Background: entropy solutions of the LWR model

In this section, we shall discuss briefly entropy solutions of the LWR model. The fact that the basic requirement of unicity of solutions does not hold for (4) is well known. Any method devised to solve the LWR model must therefore specify which solution is computed. Let us recall first a basic example. Let us take as initial conditions at time $t = t_0$ the following:



modelling traffic allowed to start after having been stopped till $t = t_0$ for instance by a red light. With this initial condition, (4) admits an infinity of solutions. Let us describe three of them.

1. First solution: K is constant at all times, i.e. $K(x,t) = K(x,t_0)$ for all x and all $t \geq t_0$. At the discontinuity point x_0, the shock-wave speed $u = 0$ satisfies of course the well-known Rankine-Hugoniot condition

$$u = \frac{[Q]}{[K]}$$

   where the symbol [.] means as usual the difference of the limit values of the argument on both sides of the discontinuity. According to this solution the vehicles do not start moving at all!

2. The second solution is illustrated by the following chart of characteristics in the (x,t) plane.

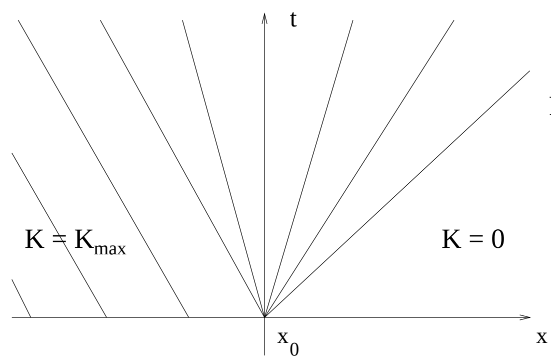


Figure 1

   It is the acceleration fan (or rarefaction fan). This is the customary solution.

3. Another solution yet could be built by imposing the trajectory of the leading vehicle as a (moving) boundary condition for the platoon. If $V(t)$ is the speed of this leading vehicle, the corresponding density is given as the solution of the following equation:

$$V(t) = V_e(K(t)) \quad .$$

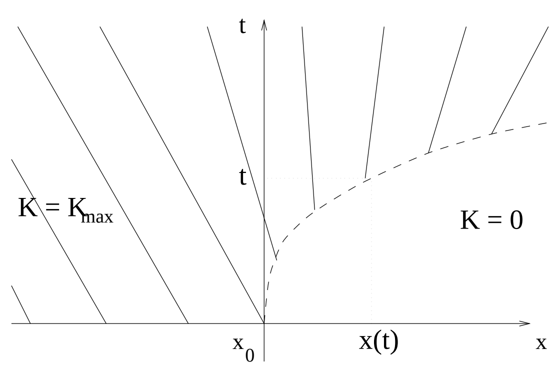


Figure 2

The solution obtained thus is consistent, in the sense that it satisfies the Rankine-Hugoniot condition, since if $x(t)$ is the position of the leading vehicle at time t, the shock speed u at point $(x(t), t)$ is:

$$u = \frac{[Q]}{[K]} = \frac{K(t)V(t) - 0}{K(t) - 0}$$

which is precisely equal to $V(t)$. Therefore it is also a solution of (4) in the weak sense.

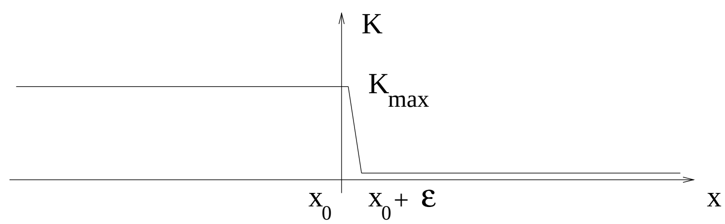
Among those three solutions, the first one is clearly inadequate, whereas the second can only be considered as a gross approximation of reality, since it implies irrealistic accelerations. Indeed, with the Greenshields equilibrium relationship,

$$V_e(K) = V_{max}(1 - \frac{K}{K_{max}})$$

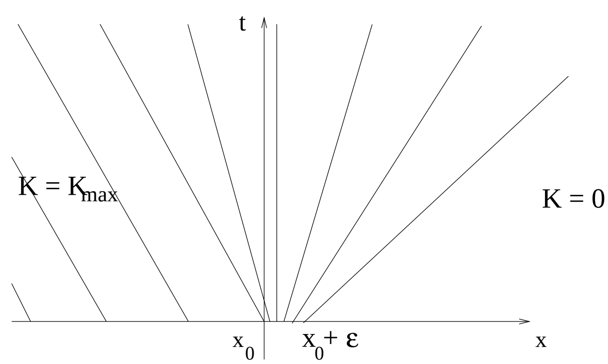
the initial acceleration of a vehicle starting from point x would be

$$\ddot{x} = -\frac{V_{max}^2}{2(x - x_0)} \quad .$$

From a physical point of view, the third solution is far more realistic, since the trajectory (and acceleration) of the first vehicle can be specified completely exogeneously and thus reflect exactly the average behaviour of drivers. Nevertheless, this solution is not stable in the sense that a slightly perturbed initial condition such as:



would result in a solution described by the following caracteristics chart:



which as a matter of fact converges to the rarefaction fan solution of figure 1 as the scale ϵ of the perturbation becomes vanishingly small.

Several reasons may be invoked to justify the choice that is nearly always made in favour of entropy solutions of (4). The main argument is precisely the consistency of entropy solutions. Indeed, they depend continuously on initial data, as shown for instance in chapter II of [GR 91], or chapter 16 of [SM 83]. This continuous dependency on initial data implies of course the unicity of the entropy solution.

As pointed out by Ansorge [AN 90] and later by Bui et al. [BNN 92], the entropy solutions of (4) that are of interest to traffic engineers, i.e. that are at least piecewise continuous, can only admit discontinuities such that the upstream density is smaller than the downstream density. Hence in the example of the starting of the (infinite) platoon, the first and third solution are precluded and only the second one is admissible. What we wish to emphasize here is that the choice of the entropy solution is essentially *a mathematically sound choice*. A solution allowing only finite accelerations (and consequently time-continuous speeds) as the third solution in the preceeding example might well be far more interesting, notably in situations in which the finiteness of accelerations plays an important role. The regulation of corridors, especially in congested conditions, where much of the delay suffered by the drivers results from alternating decelerations and accelerations, would be just such a situation.

Still, some sort of physical justification may be given for the choice of entropy solutions, by making reference to the derivation of Payne's model [PAY 71]. Indeed, starting with a microscopic follow-the-leader model of the following kind:

$$\dot{x}_n(t+T) = \Lambda(x_{n-1}(t) - x_n(t)) \quad ,$$

(with n the index of vehicles on a lane, $x_n(t)$ the position of vehicle n at time t, and T the reaction time), defining the following variables:

$x \overset{def}{=} x_n(t)$,

$V(x,t) \overset{def}{=} \dot{x}_n(t)$,

$y \overset{def}{=} (x_{n-1}(t) + x_n(t))/2$,

and making the following approximations:

$y \approx x + 1/2K(y,t) \approx x + 1/2K(x,t)$,

$x_{n-1}(t) + x_n(t) \approx 1/K(y,t) \approx \frac{1}{K(x,t)} - \frac{1}{2K(x,t)^3}\frac{\partial K}{\partial x}(x,t)$,

$\dot{x}_n(t+T) \approx V(x,t) + T\frac{dV}{dt}(x,t)$,

the Payne model for acceleration and speed dynamics results (admitting the Greenshields relationship):

$$\frac{dV}{dt} = \frac{1}{T}(V_e(K) - V - \frac{\nu}{K}\frac{\partial K}{\partial x}) \quad . \tag{5}$$

(with $V_e(K) \overset{def}{=} \Lambda(1/K)$). This equation, supplemented with (1) and (2) forms the Payne second order traffic flow model [PAY 71]. This model can therefore be related somewhat better to microscopic driver behavioral models than the LWR model.

Schochet [SC 88] has shown that, as T tends toward 0, Payne's system admits the limit

$$\left| \begin{array}{l} \frac{\partial K}{\partial t} + \frac{\partial Q}{\partial x} = 0 \\ Q = KV \\ V = V_e(K) - \frac{\nu}{K}\frac{\partial K}{\partial x} \end{array} \right. \tag{6}$$

or more concisely

$$\frac{\partial K}{\partial t} + \frac{\partial}{\partial x}Q_e(K) = \nu\frac{\partial^2 K}{\partial x^2} \quad . \tag{7}$$

Schochet has studied and demonstrated the existence and unicity of the solutions of Payne's system and of (7). The solutions of (7) can be considered as *viscosity solutions* for (4). As the anticipation factor ν tends towards 0, the solutions of (7) tend towards the *entropy solutions* of (4). This is a special case studied by Schochet of a general principle, i.e. that entropy solutions of hyperbolic conservation equations are the limit for vanishing viscosity of viscosity solutions. Now of course from a physical point of view, having T and ν tend towards 0 is really the same as to have the time and length units become arbitrarily large by a change of units in the equations of the traffic flow (it is straightforward to check that the length unit is equal to the square root of the product of the anticipation factor unit times the time unit). So, from a practical point of view, Schochet's result means that as the time- and space-scales taken into account become ever larger, the LWR becomes a better and better approximation of Payne's model (5) and of the viscosity model (7). So there exists a certain continuity and coherence between the follow-the-leader models, and the scale of macroscopic models (5), (7) and (4) (the solutions considered for the last one beeing the entropy solutions).

To conclude this section, we may consider that the entropy solution of the LWR model is:

- simpler from a methematical point of view (existence, unicity, continuous dependency on initial conditions),

- consistent with other macroscopic and microscopic models (at least at large time- and space-scales),

- nevertheless not the best and most realistic from a behavioral point of view.

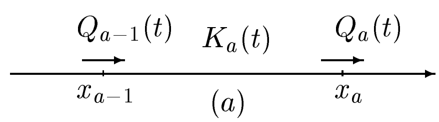
If we return one last time to the example of the starting vehicles of the beginning of this section, we note that indeed the third solution (figure 2) tends towards the second (figure 1) at infinity, or that equivalently if the time unit is such that the acceleration time of the leading vehicle can be neglected and the space unit suitably expanded, then both solutions are nearly identical (provided of course that the limit speed of the leading vehicle is the maximum speed).

3 The Godunov discretization scheme

We shall now recall some elements concerning the Godunov scheme, as applicable to the LWR model (4) on a homogeneous (infinite) lane. This scheme was introduced by Godunov [GO 59] for the resolution of nonlinear hyperbolic conservation equations of the following type:

$$(8) \qquad \left| \begin{array}{l} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \\ u(x,0) = u_0(x) \qquad \forall x \in \mathbb{R} \end{array} \right.$$

with u an unknown function of a real variable x. Usually the *flow function* f is assumed convex by numerical analysts, as for instance in the case of the celebrated inviscid Burger model, a simplified model for fluid dynamics, in which $f(u) = u^2/2$. Nevertheless most results carry over to concave flow functions, such as the equilibrium flow-density relationship of traffic flow models.



The description of the scheme is the following.

- The line is discretized into cells $(a) = [x_{a-1}, x_a]$ of lengths $l_a = x_a - x_{a-1}$.

- The time is discretized into time intervals $[t\Delta t, (t+1)\Delta t]$.

- At time $t\Delta t$, the solution u of (8) is approximated by a piecewise constant function $\tilde{u}$ defined as:

$$\tilde{u}(x, t\Delta t) \overset{def}{=} u_a^t \qquad (\forall x \in (a)) \quad .$$

- The computation of the approximation at time $(t+1)\Delta t$, $\tilde{u}(., (t+1)\Delta t)$ starting from the approximation $\tilde{u}(., t\Delta t)$ at time $t\Delta t$, requires the following two theoretical steps.

  1. Compute the *exact solution*, called Ξ, of (8) given the inital condition $\tilde{u}(., t\Delta t)$ at time $t\Delta t$:

$$(9) \qquad \left| \begin{array}{l} \frac{\partial \Xi}{\partial t} + \frac{\partial}{\partial x} f(\Xi) = 0 \\ \\ \Xi(x, t\Delta t) = \left\{ \begin{array}{ll} \tilde{u}(x, t\Delta t) & (\forall x \mathbb{R}) \\ u_a^t & (\forall (a),\ \forall x \in (a)) \end{array} \right. \end{array} \right.$$

  2. take the average of $\Xi(., (t+1)\Delta t)$ over every cell(a):

$$(10) \qquad u_a^{t+1} = \frac{1}{l_a} \int_{(a)} \Xi(y, (t+1)\Delta t)\, dy \quad .$$

- The two preceding steps can be simplified in the following:

(11) $$u_a^{t+1} = u_a^t + \frac{\Delta t}{l_a}(\Phi_{a-1}^t - \Phi_a^t) \quad .$$

with

(12) $$\Phi_a^t \stackrel{def}{=} \frac{1}{\Delta t}\int_{t\Delta t}^{(t+1)\Delta t} f[\Xi(x_a, s)]\, ds$$

the average flow crossing x_a from cell (a) to cell $(a+1)$ during time step $[t\Delta t, (t+1)\Delta t]$.

- Finally, if f is concave, the expression (12) can be replaced by the more tractable following half-closed expression due to Osher [OS 84]:

(13) $$\Phi_a^t = \begin{cases} Min_{u_a^t \le u \le u_{a+1}^t} f(u) & \text{if } u_a^t \le u_{a+1}^t \\ Max_{u_{a+1}^t \le u \le u_a^t} f(u) & \text{if } u_a^t \ge u_{a+1}^t \end{cases} \quad .$$

Bui et al. [BNN 92] applied this scheme directly to the modelization of traffic flow on a homogeneous lane. The resulting equations are:

$$\left| \begin{array}{lcl} K_a^{t+1} & = & K_a^t + \frac{\Delta t}{l_a}(Q_{a-1}^t - Q_a^t) \\ Q_a^t & = & \begin{cases} Min_{K_a^t \le \kappa \le K_{a+1}^t} Q_e(\kappa) & \text{if } K_a^t \le K_{a+1}^t \\ Max_{K_{a+1}^t \le \kappa \le K_a^t} Q_e(\kappa) & \text{if } K_a^t \ge K_{a+1}^t \end{cases} \end{array} \right. ,$$

which is exactly the translation of (11) and (13). Let us note that Q_a^t can equivalently be given by the following table:

(14)

$K_a^t \backslash K_{a+1}^t$	uc	oc
uc	Q_a^t	$Min[Q_a^t, Q_{a+1}^t]$
oc	Q_{max}	Q_{a+1}^t

in which the abbreviations *uc* and *oc* mean *undercritical* and *overcritical* respectively. The equivalence of this table with Osher's formula is straightforward.

- If $K_a^t \le K_{crit}$, $K_{a+1}^t \le K_{crit}$, $Q_e(.)$ is increasing on the interval $[K_{a+1}^t, K_a^t]$ or $[K_a^t, K_{a+1}^t]$ and consequently:

$$\begin{array}{lcl} Min_{K_a^t \le \kappa \le K_{a+1}^t} Q_e(\kappa) & = & Q_e(K_a^t) \\ Max_{K_{a+1}^t \le \kappa \le K_a^t} Q_e(\kappa) & = & Q_e(K_a^t) \end{array} \quad .$$

- If $K_{a+1}^t \le K_{crit}$, $K_a^t \le K_{crit}$, then

$$Max_{K_{a+1}^t \le \kappa \le K_a^t} Q_e(\kappa) = Q_e(K_{crit}) = Q_{max} \quad .$$

- If $K_a^t \le K_{crit} \le K_{a+1}^t$, Q_e being concave atteins its minimum on $[K_a^t, K_{a+1}^t]$ at one of the boundary points K_a^t, K_{a+1}^t. Hence:

$$Min_{K_a^t \le \kappa \le K_{a+1}^t} Q_e(\kappa) = Min[Q_e(K_a^t), Q_e(K_a^t)] \quad .$$

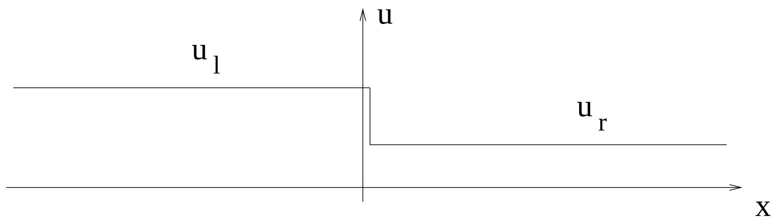
- If $K_a^t \ge K_{crit}$, $K_{a+1}^t \ge K_{crit}$, $Q_e(.)$ is decreasing on the interval $[K_{a+1}^t, K_a^t]$ or $[K_a^t, K_{a+1}^t]$ and consequently:

$$\begin{array}{lcl} Min_{K_a^t \le \kappa \le K_{a+1}^t} Q_e(\kappa) & = & Q_e(K_{a+1}^t) \\ Max_{K_{a+1}^t \le \kappa \le K_a^t} Q_e(\kappa) & = & Q_e(K_{a+1}^t) \end{array} \quad .$$

The key for establishing the Osher formula (13), which provides also the clue for the treatment of the space discontinuities of Q_e, is the *Riemann Problem.* This problem can be defined as: *find* u *such that*

$$(15)\qquad \left| \begin{array}{l} \frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0 \\ \\ u(x,0) = \left\{ \begin{array}{ll} u_l & \text{if } x < 0 \\ u_r & \text{if } x > 0 \end{array} \right. . \end{array} \right.$$

The initial condition of the Riemann Problem is illustrated hereafter



The Riemann Problem constitutes an abstraction of the problem (9) in the neighbourhood of each discretisation point x_a. Indeed, to solve problem (9), one has to solve locally, at each point x_a, a Riemann Problem, taking the point x_a as the space origin, the instant $t\Delta t$ as the time origin, and replacing u_l and u_r by u_a^t and u_{a+1}^t respectively. Of course this is only possible as long as the solution computed thus is not modified by the solution computed at the neighbouring points during the time-step. This implies that the cell length l_a must be greater than the product of the time-step by the greatest possible speed of wave-propagation. Hence:

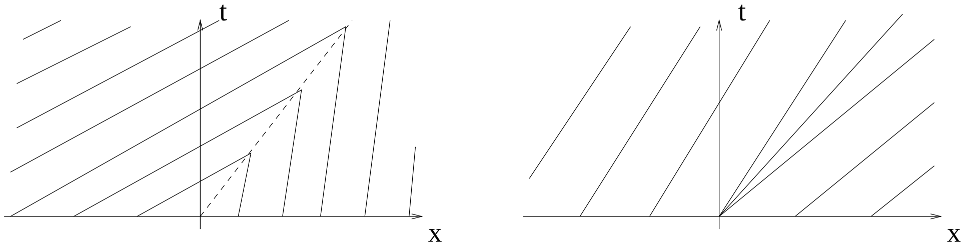
$$(16)\qquad l_a \geq \Delta t Max_u |f'(u)| \qquad (\forall(a))$$

This is precisely the condition under which the Osher formula is established. The resolution of the Riemann problem in the present context is trivial and the resulting solutions are described by the following characteristics charts. We use the notation

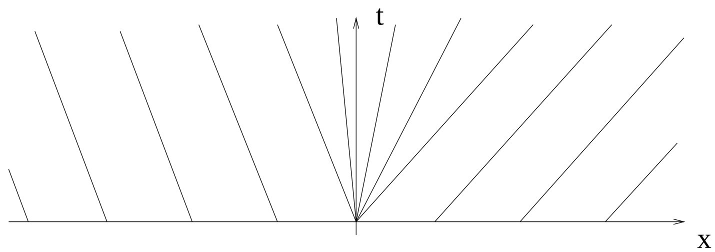
$$Q_l = Q_e(K_l)$$
$$Q_r = Q_e(K_r)$$

for the equilibrium flows upstream and downstream of the initial density discontinuity at $x = 0$. The flow $Q(0,t)$ through this point is given in every case and is identical to the flow given hereafter by the Osher formula.

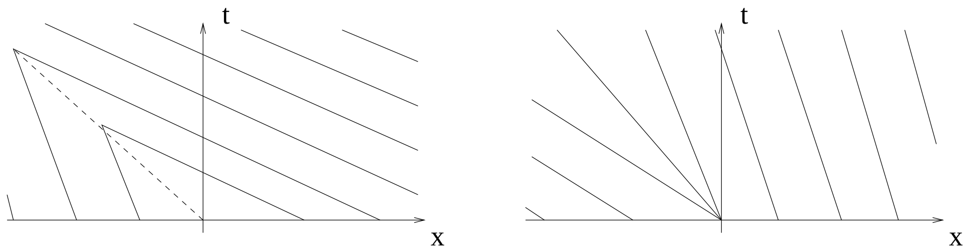
Case K_l and K_r undercritical: $Q(0,t) = Q_l$:



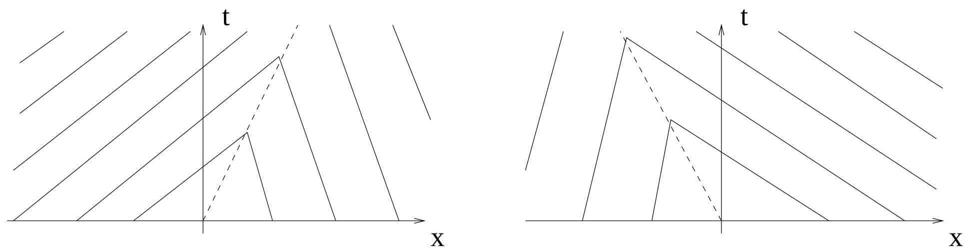
Case K_l overcritical and K_r undercritical: $Q(0,t) = Q_{max}$:



Case K_l and K_r overcritical: $Q(0,t) = Q_r$:



Case K_l undercritical and K_r overcritical: $Q(0,t) = Min[Q_l, Q_r]$:



The Osher formula is trivially equivalent to the solution of the Riemann problem.

To conclude, let us say a few words about some earlier models. In [CP 92], Roe's scheme was examined, and applied among others to the LWR model. This scheme is a variant of Godunov's method using an approximate Riemann solver instead of the exact solution of Riemann's problem. In [MBY 84], a shock-fitting Lax-Wendroff scheme was considered. But the resulting algorithm is quite intricate. In [HI 95] (section 4.1), the following simple formula is proposed:

$$Q_a^t = K_a^t \, V_e(K_{a+1}^t, a+1)$$

(in a sem-discretized context in fact, but easily discretized). This last formula has in principle the defect that there is no control over maximum flow or density values in the formulation; nevertheless interesting results are reported in [KLHK 95]. Finally, let us examine the SSMT formula [LE 84], which is described by the following sequential algorithm:

- Compute

$$u \stackrel{def}{=} \frac{Q_e(K_a^t) - Q_e(K_{a+1}^t)}{K_{a+1}^t - K_a^t}$$

  (u can be considered the shock-wave speed associated to the initial conditions of a Riemann problem with $u_l = K_a^t$, $u_r = K_{a+1}^t$.

- If $u \geq 0$, the flow, being determined by upstream conditions, is defined as:

$$Q_a^t = Q_e(K_a^t) \quad .$$

- If $u \leq 0$, the flow, being determined by downstream conditions, is defined as:

$$Q_a^t = Q_e(K_{a+1}^t) \quad .$$

- Finally, and whatever the value of u, if $K_l \geq K_{crit} \geq K_r$,

$$Q_a^t = Q_{max} \quad ,$$

  expressing the acceleration of vehicles in the rarefaction fan at maximum flow.

This algorithm is equivalent to the Osher formula and can be viewed as an implementation of the solution to the Riemann problem. Indeed, if K_a^t and K_{a+1}^t are undercritical, $u \geq 0$, hence $Q_a^t = Q_e(K_a^t)$, and if K_a^t and K_{a+1}^t are overcritical, $u \leq 0$, hence $Q_a^t = Q_e(K_{a+1}^t)$. If K_a^t undercritical and K_{a+1}^t overcritical, the sign of u is that of the difference $Q_e(K_{a+1}^t) - Q_e(K_a^t)$, and $Q_a^t = Q_e(K_a^t)$ if $u \geq 0$, $Q_a^t = Q_e(K_{a+1}^t)$ if $u \leq 0$, hence $Q_a^t = Min[Q_e(K_a^t), Q_e(K_{a+1}^t)]$. The case K_a^t overcritical and K_{a+1}^t overcritical needs no checking.

All the formulas described in this section for the computation of the flows between cells or equivalently the flow at the origin in the Riemann problem are inadequate for the modelization of inhomogeneous conditions. For instance in the SSMT model, in which a facility for the modelling of intersections was provided, flow discontinuities could result from the use of the algorithm described above. These discontinuities went unnoticed for a long time, essentially because they occured infrequently (in saturated conditions), in intersections whose dynamics were complicated anyway, and because the input of the model was generally real data. The inclusion of a modified version of SSMT into METACOR as its urban part led to a correction which will be described in the next section, since it provides the link between Daganzo's flow formula and the Godunov scheme in the inhomogeneous case.

4 The Generalized Riemann Problem

4.1 introduction

We now adress the case where Q_e depends explicitly and discontinuously on x. Typically, we shall consider in this section that Q_e is piecewise constant relative to x, and as usual a concave function of K on the interval $[0, K_{max}(x)]$. The introduction of such a functional form is motivated by the necessity to be able to model:

- intersections,

- sections with variable number of lanes,

- incidents (implying local and temporary restrictions of capacity, speed, etc ...),

- any situation in which speed and/or capacity parameters are likely to vary.

Further, the discretization of traffic in situations such as refered to above implies the use of piecewise constant approximations of Q_e. Hence, the equilibrium flow-density relationship associated to cell (a) will hereafter been noted

(17) $$Q_e(K_a, a) \quad ,$$

to emphasize its dependency on both the average density K_a in cell (a) and on the cell itself. Typically, the physical parameters Q_{max}, V_{max}, K_{max}, K_{crit} would be functions of the cell (a), but the functional form itself might vary as well from one cell to the other. For instance the equilibrium relationship in a cell belonging to a urban link might have a different functional form from that of a cell belonging to say an access or exit ramp or a highway section. In this discontinuous context, the basic principles of the Godunov scheme remain the same as in the continuous case and the Godunov scheme requires the following steps for its basic iteration.

- The starting point is a piecewise constant approximation $\tilde{K}$ of the solution K of (4), at time $t\Delta t$, described by the approximate values K_a^t of $K(x, t\Delta t)$ for all x in cell a:

$$K(x, t\Delta t) \approx K_a^t \stackrel{def}{=} \tilde{K}(x, t\Delta t) \quad (\forall x \in a) \quad .$$

- One computes the exact solution (Ξ, Φ) of the LWR system (1), (2), (3) at time $(t+1)\Delta t$, with Ξ the density and Φ the flow, given the initial condition at time $t\Delta t$:

$$\Xi(x, t\Delta t) = K_i^t \quad (\forall x \in a) \quad .$$

The initial condition means $\Xi(., t\Delta t) = \tilde{K}(., t\Delta t)$

- One takes the average of $\Xi(.,(t+1)\Delta t)$ over cell (a), yielding K_a^{t+1}:

$$K_a^{t+1} = \int_{(a)} \Xi(x,(t+1)\Delta t)\, dx \quad .$$

Let us note the following:

> *It is essential that the set of discontinuity points of Q_e be included in the set of cell boundary points.*

As before, we get:

$$K_a^{t+1} = K_a^t + \frac{\Delta t}{l_a}(Q_{a-1}^t - Q_a^t) \quad ,$$

with Q_a^t the flow passing at exit point x_a of cell (a) during time interval $[t\Delta t,(t+1)\Delta t]$:

$$Q_a^t = \int_{t\Delta t}^{(t+1)\Delta t} \Phi(x_a,s)\, ds \quad .$$

The only remaining problem is that of estimating the Q_a^ts, which is achieved by solving a generalized Riemann problem.

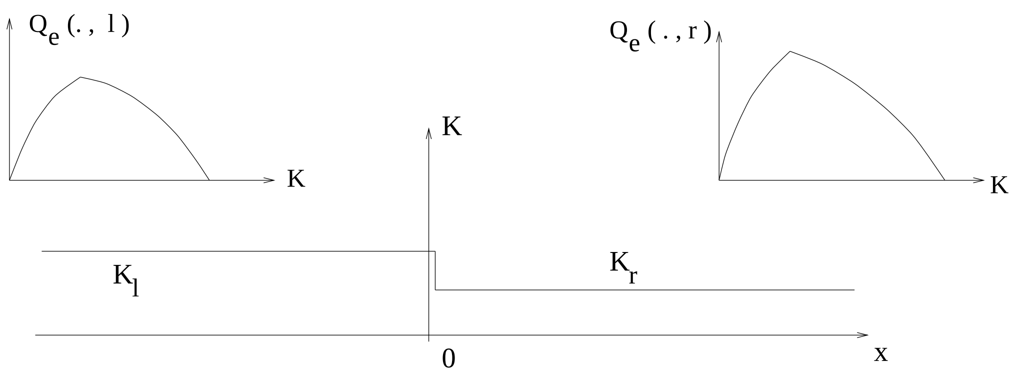
4.2 Principles of the solution

We consider now a Generalized Riemann Problem for the LWR system (1), (2), (3) with initial conditions:

$$K(x,0) = \begin{cases} K_l & \text{if } x < 0 \\ K_r & \text{if } x > 0 \end{cases}$$

and following equilibrium flow-density relationship:

$$Q_e(K,x) = \begin{cases} Q_e(K,l) & \text{if } x < 0 \\ Q_e(K,r) & \text{if } x > 0 \end{cases}$$



We must now describe the rules that will enable us to calculate the solution of this Riemann problem. Let us first consider the case where the dependency on x is smooth. The basic fact which enables us to compute the solutions of the LWR system (1), (2), (3) (i.e. (4)) is the existence of characteristic lines for the flow, i.e. lines along which the flow (but not the density) is constant. Indeed, if one looks for a line $(x(t),t)$, in the (x,t) plane, along which Q is conserved, one gets, starting from $\frac{dQ}{dt}=0$,

$$\frac{\partial}{\partial t}Q + \dot{x}\frac{\partial Q}{\partial x} = 0$$

or:

$$\frac{\partial Q_e}{\partial K}\frac{\partial K}{\partial t} + \dot{x}\frac{\partial Q}{\partial x} = 0$$

which, indentified with (4), yields:

$$\frac{dx}{dt} = \frac{\partial Q_e}{\partial K}(K(x,t),x) \quad . \tag{18}$$

This last equation of course must be combined with a rule of choice of the density since at any point x, for every value of the flow Q, there exist two values of the density such that

$$Q_e(K,x) = Q \quad .$$

Consequently we define:

$$\left| \begin{array}{l} Q_{eu}^{-1}(Q,x) = K \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} Q = Q_e(K,x) \\ K \leq K_{crit}(x) \\ (K \ \text{ undercritical}) \end{array} \right. \\ \\ Q_{eo}^{-1}(Q,x) = K \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} Q = Q_e(K,x) \\ K \geq K_{crit}(x) \\ (K \ \text{ overcritical}) \end{array} \right. \end{array} \right. \tag{19}$$

(the subscripts u and o meaning respectively under- and overcritical). Now, to compute for instance the characteristic line passing through point (x_0, t_0) of the (x,t) plane, we note the initial conditions:

$$K_0 \stackrel{def}{=} K(x_0, t_0)$$

and:

$$Q_0 \stackrel{def}{=} Q(x_0,t_0) = Q_e(K(x_0,t_0),x_0) \quad .$$

If K_0 is undersaturated, the characteristic line is described by the equation:

$$\left\{ \begin{array}{l} x(t_0) = x_0 \\ \frac{dx}{dt} = \frac{\partial Q_e}{\partial K}[Q_{eu}^{-1}(Q_0,x),x] \end{array} \right. \tag{20}$$

and by the equation:

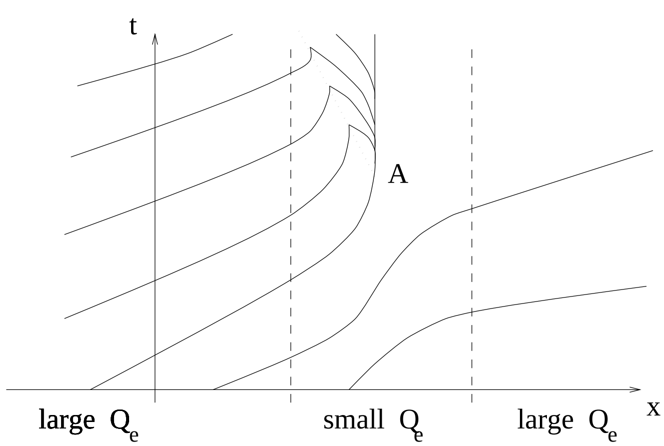
$$\left\{ \begin{array}{l} x(t_0) = x_0 \\ \frac{dx}{dt} = \frac{\partial Q_e}{\partial K}[Q_{eo}^{-1}(Q_0,x),x] \end{array} \right. \tag{21}$$

if K_0 is oversaturated.

What is to be highlighted here is the fact that as long as a characteristic line is not interrupted by a shock-wave, its associated density must vary continuously. As a consequence, since Q_e is smooth, what we shall call *the state of the flow* (i.e. whether it is under- or overcritical) will remain constant along a characteristic line. Let us emphasize that under- or overcriticality is understood here in the large sense, meaning that both concepts include criticality (i.e. $K = K_{max}$). The state of the flow along a characteristic line may only change at points x such that:

$$\frac{\partial Q_e}{\partial K}[Q_{eo}^{-1}(Q_0,x),x] = \frac{\partial Q_e}{\partial K}[Q_{eu}^{-1}(Q_0,x),x] = 0 \quad .$$

At such points which are inflection points, the characteristic need not be unique.



In the above figure an increasing flow is limited by a capacity restriction, characterized by a "small Q_e". There is no unicity of the characteristics issued from the inflection point A.

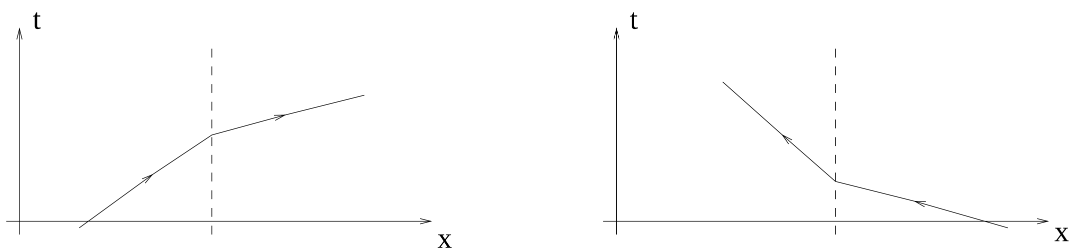
The case where Q_e depends discontinuously on x will be considered as a limit case of the smooth case, with $Q_e(K,.)$ varying very fast. The flow Q is continuous relative to x, except at shock-waves, as a consequence of the conservation equation (1). This holds true also at space discontinuity points of Q_e. At the crossing of such points the flow Q, but not the density K, is conserved. Another way to understand this is to consider a space-discontinuity point of Q_e as a very special, stationary, shock-wave, with shock-wave speed $u = 0$ and consequently the jump condition reduces to $[Q] = 0$.

Suppose now that a characteristic line crosses a space-discontinuity point of Q_e. The flow is the same on both sides of the discontinuity; in order to compute the relationship of densities on both sides of the discontinuity we impose the condition that the flow state be conserved while crossing the discontinuity. This *refraction condition* of characteristic lines can be justified two ways.

1. It is satisfied in the smooth case, hence it is consistent to suppose that it is satisfied in the discontinuous case as well, viewed as the limit of the continuous case.

2. If this condition were not satisfied, there could be no crossing of the discontinuity by the characteristic line. Indeed, the sign of the slope $\dot{x}(t)$ of the characteristic line, which is equal to the sign of
$$\frac{\partial Q_e}{\partial K}(K(x(t),t),x) \quad ,$$
must be the same on both sides of the discontinuity, which is to say again that the state of the flow must be conserved across the singularity.

The refraction of a shock-wave across a discontinuity has the following aspect:



Consequently, a characteristic line bearing a flow Q_0 and crossing a discontinuity at point x will be associated with density $Q_{e*}^{-1}(Q_0, x-)$ upstream and $Q_{e*}^{-1}(Q_0, x+)$ downstream of the discontinuity, where $*$ is equal to u or o depending on whether the characteristic line is associated with an undersaturated or oversaturated flow. Characteristic lines are therefore refracted (but not reflected: reflection conditions generate a shock-wave).

As a consequence of the preceeding analysis, we shall use the following rules for the computation of the solution of the Generalized Riemann Problem (these rules apply more generally for all problems with piecewise continuous dependency of Q_e on x):

- characteristic lines carry a constant flow and traffic state, and are described by equations (20) and (21),

- the density on a characteristic line is derived from the flow value by taking the reciprocal of the part of the equilibrium flow-density relationship corresponding to the traffic state of the characteristic line (equations (19)),

- across a spatial discontinuity of the equilibrium flow-density relationship, the flow value is conserved,

- along shock-wave lines, the usual Rankine-Hugoniot condition holds,

- shock-waves and rarefaction fans are introduced according to the usual rules for the computation of entropy solutions, wherever Q_e is continuous,

- in case of multiple possible solutions at the singularity, the solution maximizing the local flow is choosen.

This last rule is completely consistent with the conception of the solution of the LWR problem with a singularity as a limit case of the entropy solution of the LWR problem with a very rapidly varying Q_e. It may be conceived of as a generalization to the discontinuous case of the definition of the entropy solution of the LWR problem as the solution that, among all possible solutions, maximizes locally the flow. So we simply adapt the usual rules for the computation of entropy solutions and supplement them with rules for taking into account the spatial discontinuities of Q_e.

4.3 Effective solution of the Generalized Riemann Problem

Let us recall the definition of the Generalized Riemann Problem:

Find the solution $K(x,t)$, for $t \geq 0$ and all x, of the LWR system (1), (2), (3):

$$\left| \begin{array}{l} \frac{\partial K}{\partial t} + \frac{\partial Q}{\partial x} = 0 \\ Q = KV \\ Q = Q_e(K,x) \end{array} \right.$$

with initial conditions:

$$K(x,0) = \begin{cases} K_l & \text{if } x < 0 \\ K_r & \text{if } x > 0 \end{cases}$$

and Q_e given by:

$$Q_e(K,x) = \begin{cases} Q_e(K,l) & \text{if } x < 0 \\ Q_e(K,r) & \text{if } x > 0 \end{cases} \quad .$$

In the sequel, we shall also use frequently the following notation:

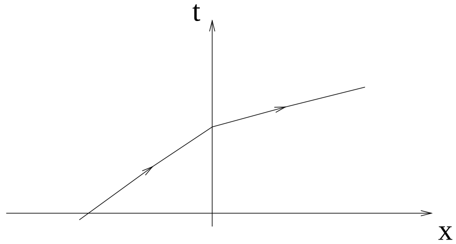
$$\left| \begin{array}{l} Q_l \stackrel{def}{=} Q_e(K_l,l) \\ Q_r \stackrel{def}{=} Q_e(K_r,r) \end{array} \right. ,$$

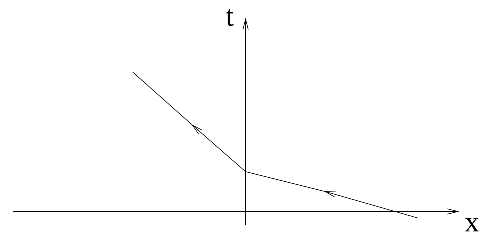
which are the flows upstream and downstream of the singularity. We shall now describe the various solutions of the Generalized Riemann Problem, corresponding to the possible relationships between the physical parameters upstream and downstream of the singularity (conventionnally placed at the origin). There are many cases to be considered, and the description of the solution for the various cases will be essentially graphical, with comments given only as needed. *The solutions will represented by their characteristics charts in the* (x,t) *plane*, as is custumary. The results do not depend on the specific functional forms of $Q_e(.,l)$ and $Q_e(.,r)$.

The main classification criterium is whether $Q_{max}(l)$ is greater than $Q_{max}(r)$ or not. The second criterium is the initial traffic state upstream and downstream of the singularity (under- or overcritical). The third criterium is the comparison between the smallest maximum flow $Min[Q_{max}(l), Q_{max}(r)]$ and the flows Q_l and Q_r. The reason for this is that in certain circumstances no characteristics at all may cross the singularity. This implies that in such a situation the characteristics either converge towards the singularity (the flow is then identical on both sides of the singularity, overcritical downstream and undercritical upstream) or diverge from it. In this last case (of which we shall see several examples later on), many solutions are possible, including the entropy solution maximizing the flow, which is then precisely equal to $Min[Q_{max}(l), Q_{max}(r)]$. With this value of the flow, the flow state varies continuously from overcritical upstream of the singularity through critical at the singularity, to undercritical downstream of the singularity. Hence 18 different subcases need to be considered.

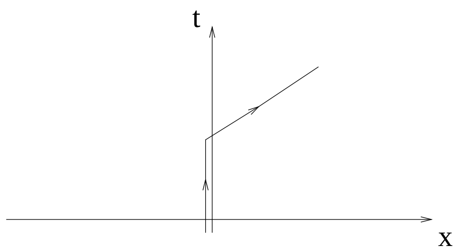
4.3.1 The general case $Q_{max}(l) \leq Q_{max}(r)$

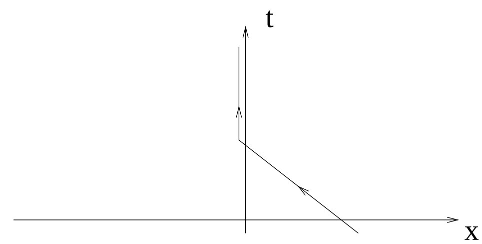
The refraction of characteristic lines will generally have the following aspect, which we shall use for the illustrations later on (the exact refraction angle depends on the specific functional forms of $Q_e(.,l)$ and $Q_e(.,r)$):



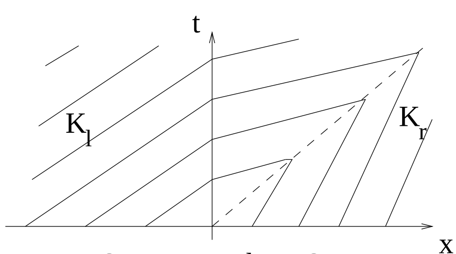


What is essential is that the refraction of a characteristic line is possible only if the flow Q it carries is smaller than $Q_{max}(l)$. The refraction in the limit case of a characteristic line carrying a flow value $Q = Q_{max}(l)$ is illustrated hereafter.

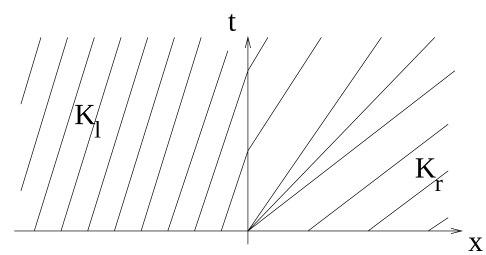




a. K_l**,** K_r **undercritical** The upstream characteristic lines can cross the discontinuity point unconditionnally. The solution depends on the relative values of Q_l and Q_r. The two resulting possibilities are illustrated hereafter.



Q_r greater than Q_l



Q_r smaller than Q_l

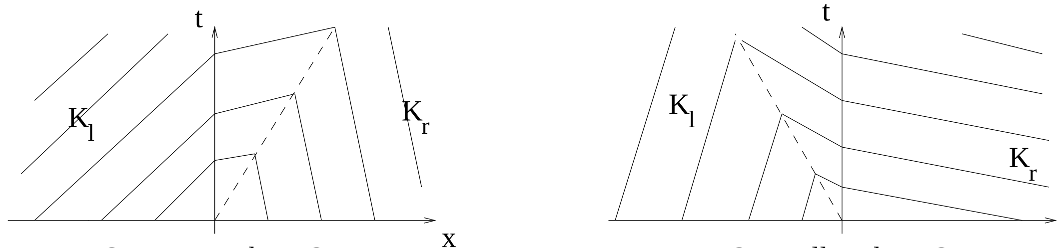
The characteristic lines carrying flow Q_l carry a density value

$$Q_{eu}^{-1}(Q_l, r)$$

on the right side of the discontinuity, and a density value K_l of course on the left side. The flow value at the singularity point is at all times:

$$Q(0,t) = Q_l \quad \forall t > 0 \quad .$$

b. K_l **undercritical,** K_r **overcritical** The crossover of characteristic lines is possible unconditionnally from left to right, and is possible from right to left only conditionnally to $Q_r \leq Q_{max}(l)$. Nevertheless, if $Q_r \geq Q_{max}(l)$, then $Q_r \geq Q_l$ as well, and in this case the crossover of characteristics occurs from left to right anyway. The following two possiblities result, depending on whether Q_l is greater or smaller than Q_r.



Q r greater than Q l   Q r smaller than Q l

The characteristic lines carrying a flow Q_l, when crossing the singularity, carry a density value

$$Q_{eu}^{-1}(Q_l, r)$$

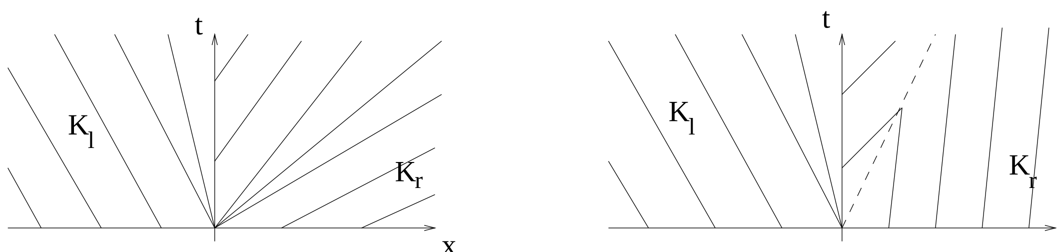
on the right-hand side of the discontinuity, and a density value K_l on the left side. The characteristic lines carrying a flow Q_r, when crossing the singularity, carry a density value

$$Q_{eo}^{-1}(Q_r, l)$$

on the left-side side of the discontinuity, and a density value K_r on the right side. The flow value at the singularity point is at all times:

$$Q(0,t) = \min[Q_l, Q_r] \quad \forall t > 0 \quad .$$

c. K_l overcritical, K_r undercritical Considering the slopes of the characteristics generated at $t = 0$, which are negative on the left-hand side of the singularity and positive on its right-hand side, it is clear that no characteristic line crosses the singularity. Hence, according to the last computational rule, the flow at the singularity is equal to the smallest maximal flow, i.e. $Q_{max}(l)$. Consequently there results an acceleration fan on the upstream side of the singularity, and a boundary condition $Q = Q_{max}(l)$ with undercritical traffic conditions on the downstream side of the singularity. The two resulting solutions are shown hereafter:



Q max (l) greater than Q r   Q max (l) smaller than Q r

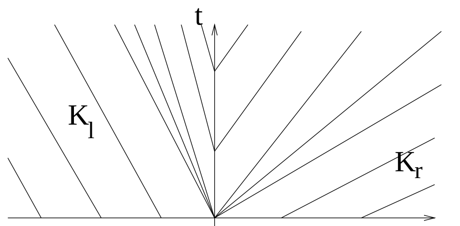
The density on the downstream side of the singularity, carried by the characteristics generated by the boundary condition $Q(0,t) = Q_{max}(l)$ is

$$Q_{eu}^{-1}(Q_{max}(l), r) \quad .$$

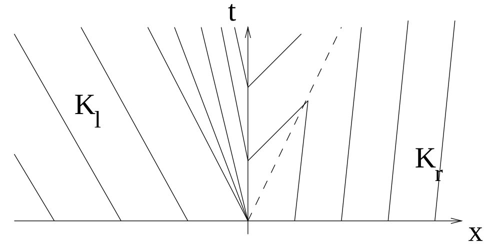
The flow at the singularity is of course

$$Q(0,t) = Q_{max}(l) \quad \forall t > 0 \quad .$$

To illustrate the non-unicity of solutions on this specific example of characteristics diverging initially from the singularity, corresponding solutions that do not maximize the flow locally (and do not assure continuity of the traffic state) are given hereafter.



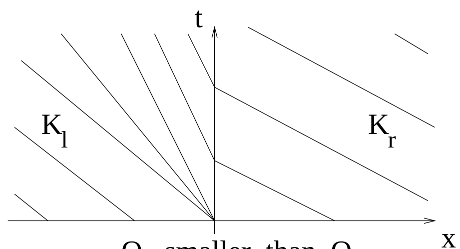
Q_0 greater than Q_r



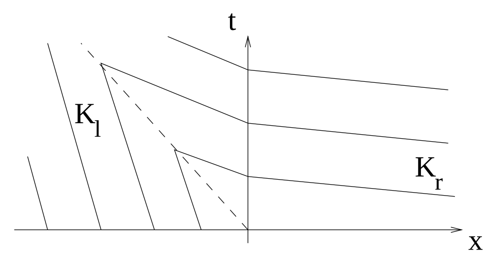
Q_0 smaller than Q_r

These solutions are characterized by a flow Q_0 at the singularity that satisfies to $Q_0 < Q_{max}(l)$.

c. K_l overcritical, K_r overcritical In this case we need only consider the crossing of the singularity by characteristics of negative slope, associated to an overcritical traffic state, generated on the downstream side of the singularity. The characteristics may only cross over if $Q_r \leq Q_{max}(l)$. Two subcases result from the satisfaction of this condition, depending on whether Q_l is smaller or greater than Q_r. The corresponding solutions are described by the following characteristics charts.



Q_l smaller than Q_r



Q_l greater than Q_r

In both cases,

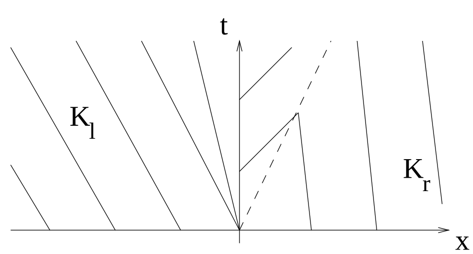
$$Q(0,t) = Q_r \quad \forall t > 0 \quad ,$$

and the density carried by the characteristics crossing the singularity is K_r on the right-hand side of the singularity and $Q_{eo}(Q_r, l)$ on its left-hand side.

Now, if $Q_r \geq Q_{max}(l)$, the situation is similar to the one analyzed above (case c: K_l overcritical, K_r undercritical, with $Q_r \geq Q_{max}(l)$). There is an acceleration fan upstream of the singularity, and characteristics carrying a flow $Q_{max}(l)$ and an undercritical density $Q_{eu}^{-1}(Q_{max}(l), r)$ are generated downstream of the singularity. The flow at the singularity is given by:

$$Q(0,t) = Q_{max}(l) \quad \forall t > 0 \quad .$$

The corresponding solution is illustrated hereafter:



Q_{max}(l) smaller than Q_r

4.3.2 Summary of the results for the case $Q_{max}(l) \leq Q_{max}(r)$

The values of the flow at the singularity resulting for the solution of the Generalized Riemann Problem are given in the following table:

(22)

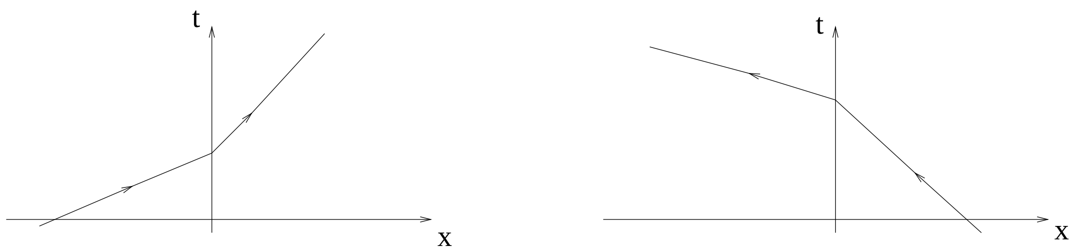
$K_l \backslash K_r$	uc	oc
uc	Q_l	$Min[Q_r, Q_l]$
oc	$Q_{max}(l)$	$Min[Q_r, Q_{max}(l)]$

In this table, the abbreviations *uc* and *oc* mean again *undercritical* and *overcritical* respectively. It is straightforward to check that if $Q_{max}(l) = Q_{max}(r)$, then this table is identical to table (14), summarizing the results of the Riemann Problem for the homogeneous LWR model. Indeed, if $Q_{max}(l) = Q_{max}(r)$, then $Q_r \leq Q_{max}(l) = Q_{max}(r)$, and it follows:

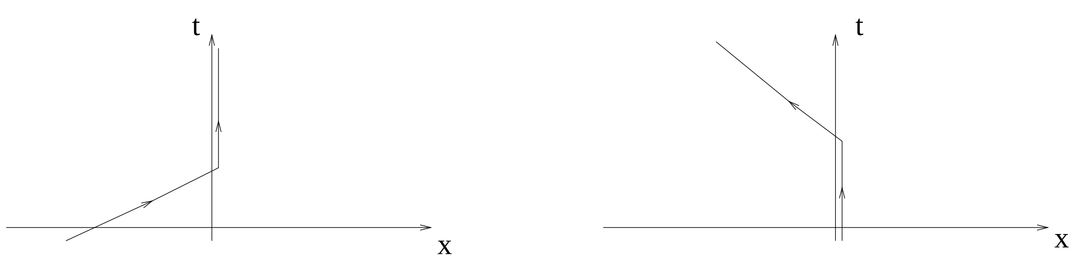
$$Min[Q_r, Q_{max}(l)] = Q_r \quad .$$

4.3.3 The general case $Q_{max}(l) \geq Q_{max}(r)$

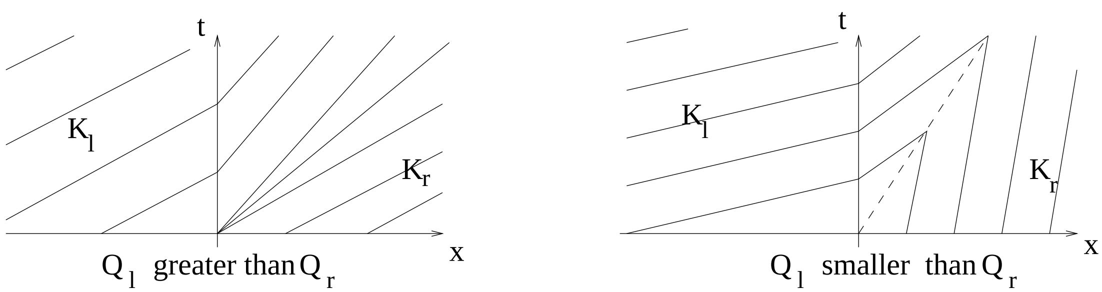
The refraction of characteristic lines will generally have the following aspect:



The refraction of a characteristic line is possible only if the flow Q it carries is smaller than $Q_{max}(r)$. The refraction in the limit case of a characteristic line carrying a flow value $Q = Q_{max}(r)$ is illustrated hereafter.



a. K_l**,** K_r **undercritical** In this case, we have to consider whether the characteristics generated on the left-hand side of the singularity can cross it or not. The crossover of characteristics is possible only if $Q_l \leq Q_{max}(r)$. If this condition is satisfied, the following solutions result:

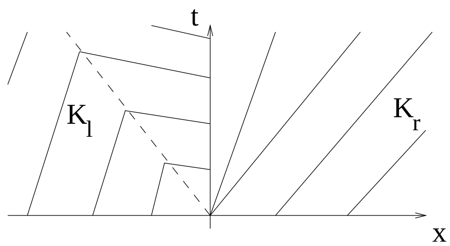


The flow at the singularity is:

$$Q(0,t) = Q_l \quad \forall t > 0 \quad ,$$

and the density carried by the characteristics crossing the singularity is K_l on the left-hand side of the singularity and $Q_{eu}^{-1}(Q_l, r)$ on its right-hand side.

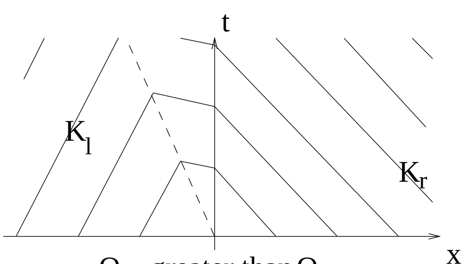
Now, if $Q_l \geq Q_{max}(r)$, the flow is necessarily critical at the singularity, since no characteristics may cross it. Hence, $Q(0,t) = Q_{max}(r)$ for all times $t > 0$, and the resulting solution can be described as:



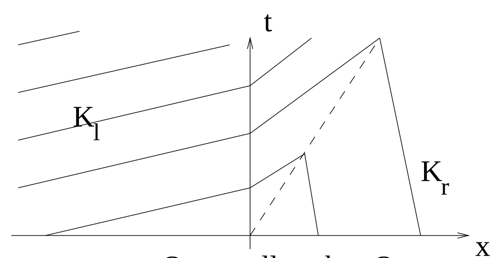
Q_{max} (r) smaller than Q_r

The characteristics generated upstream of the singularity by the boundary condition carry a flow $Q_{max}(r)$ and a density $Q_{eo}^{-1}(Q_{max}(r), l)$

b. K_l undercritical, K_r overcritical The characteristics generated on the right-hand side of the singularity can cross it unconditionnally, the characteristics generated on its left-hand side can cross it only if $Q_l \leq Q_{max}(r)$. Nevertheless, if $Q_l \geq Q_{max}(r)$, the crossover of characteristics occurs from the right to the left anyway, as a special case of the general case $Q_l \geq Q_r$. Consequently, the solution is described by the following chart:



Q_l greater than Q_r



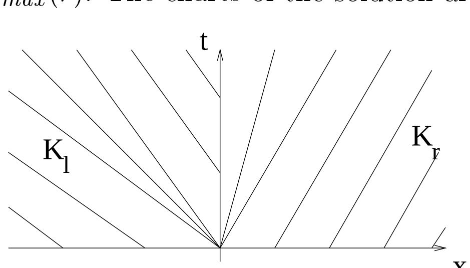
Q_l smaller than Q_r

The flow crossing the singularity is:

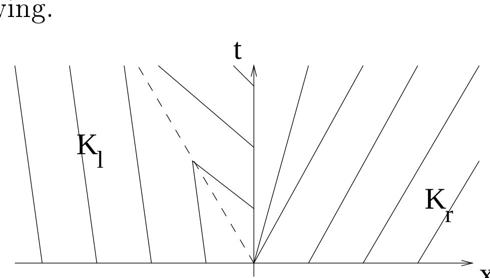
$$Q(0,t) = Min[Q_l, Q_r] \quad \forall t > 0 \quad ,$$

and the density carried by the characteristics after crossover is equal to $Q_{eu}^{-1}(Q_l, r)$ for the characteristics crossing the singularity from the left to the right, and to to $Q_{eo}^{-1}(Q_r, l)$ for the characteristics crossing the singularity from the right to the left.

c. K_l overcritical, K_r undercritical No characteristic can cross the singularity, which consequently defines a critical boundary condition $Q = Q_{max}(r)$. This boundary condition is associated to an acceleration fan on the downstream side of the singularity, and the flow upstream of the singularity is overcritical and equal to $Q_{max}(r)$. The exact solution depends on whether Q_l is smaller or greater than $Q_{max}(r)$. The charts of the solution are the following.



Q_l smaller than Q_{max} (r)



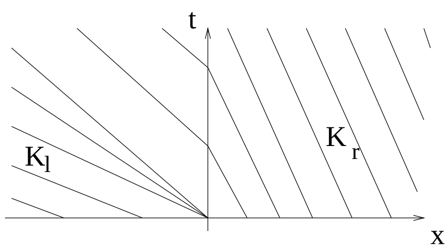
Q_l greater than Q_{max} (r)

The flow crossing the singularity is:

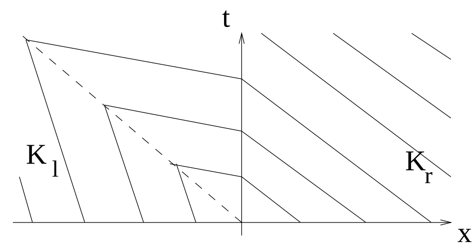
$$Q(0,t) = Q_{max}(r) \quad \forall t > 0 \quad ,$$

and the density carried by the characteristics generated by the boundary condition at the singularity is equal to $Q_{eu}^{-1}(Q_{max}(r), l)$.

d. K_l overcritical, K_r overcritical Only the characteristics generated on the downstream side of the singularity may cross it, and they can do so unconditionnally. Consequently, according to the respective values of Q_l and Q_r, we get the following solutions.



Q_l smaller than Q_r



Q_l greater than Q_r

The flow crossing the singularity is:

$$Q(0,t) = Q_r \quad \forall t > 0 \quad ,$$

and the density carried by the characteristics crossing the singularity is equal to $Q_{eo}^{-1}(Q_r, l)$ on the upstream side of the singularity.

4.3.4 Summary of the results for the case $Q_{max}(l) \geq Q_{max}(r)$

The values of the flow at the singularity that result for the solution of the Generalized Riemann Problem are given in the following table:

(23)

$K_l \backslash K_r$	uc	oc
uc	$Min[Q_l, Q_{max}(r)]$	$Min[Q_l, Q_r]$
oc	$Q_{max}(r)$	Q_r

As in the previous table (22), we use in this table the abbreviations *uc* and *oc* for *undercritical* and *overcritical* respectively. It is straightforward to check that if $Q_{max}(l) = Q_{max}(r)$, then this table is again identical to table (14), summarizing the results of the Riemann Problem for the homogeneous LWR model. Indeed, if $Q_{max}(l) = Q_{max}(r)$, then $Q_l \leq Q_{max}(r) = Q_{max}(l)$, and it follows:

$$Min[Q_l, Q_{max}(r)] = Q_l \quad .$$

Tables (22) and (23), which summarize the expression for the flow in Godunov's scheme resulting from the analysis of the Generalized Riemann Problem, were introduced in [LE 93] as a correction in the inhomogeneous case to the algorithm used previously in SSMT.

4.4 Relationship between various expressions of the flow in Godunov's scheme

A first point: if $Q_{max}(l) = Q_{max}(r)$, tables (22) and (23) are identical to table (14), i.e. to Osher's formula and consequently to all formulas equivalent to Osher's formula.

The second point is: they are also equivalent to Daganzo's formula introduced in [DA 94] and studied in [DA 93]. To check this point, let us define:

(24)

– the (upstream) traffic demand function

$$\Delta(K_l, l) \stackrel{def}{=} \begin{cases} Q_e(K_l, l)) & \text{if } \ K_l \leq K_{crit}(l) \ \ \text{(undercritical flow)} \\ Q_{max}(l) & \text{if } \ K_l \geq K_{crit}(l) \ \ \text{(overcritical flow)} \end{cases}$$

– the (downstream) traffic supply function

$$\Sigma(K_r, r) \stackrel{def}{=} \begin{cases} Q_{max}(r) & \text{if } \ K_r \leq K_{crit}(r) \ \ \text{(undercritical flow)} \\ Q_e(K_r, r) & \text{if } \ K_r \geq K_{crit}(r) \ \ \text{(overcritical flow)} \end{cases}$$

Following [BLL 95] and [LE 95] we shall prefer this terminology *demand* and *supply*, for reasons which will become completely self-evident in the next section 5.

Intuitively, the meaning of the traffic demand function is the following: it takes the value of the greatest possible outflow of the upstream half-line, if the downstream half-line were empty and of arbitrarily great capacity. Symetrically, the meaning of the traffic supply function is the following: it takes the value of the greatest possible inflow into the downstream half-line, if the upstream half-line were oversaturated and endowed with the same equilibrium flow-density relationship as the downstream half-line.

It is straightforward to check that with the definition (24), tables (22) and (23) can be replaced by the single following formula giving the flow $Q(0,t)$ through the singularity in the Generalized Riemann Problem:

(25) $$Q(0,t) = Min[\Delta(K_l, l), \Sigma(K_r, r)] \quad .$$

This formula expresses of course the fact that the Generalized Riemann Problem was solved by computing specifically those solutions that maximized the flow $Q(0,t)$. By definition, this flow must be smaller than, or equal to the traffic supply and demand. If $Q_{max}(l) \leq Q_{max}(r)$, the table (22) can be rewritten as:

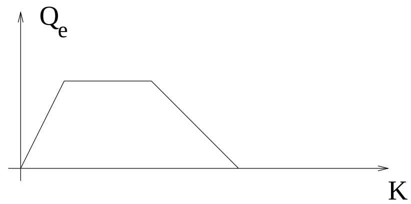
$\Delta(K_l, l) \backslash \Sigma(K_r, r)$	$Q_{max}(r)$	$Q_e(K_r(r)r)$
$Q_e(K_l(l)l)$	Q_l	$Min[Q_r, Q_l]$
$Q_{max}(l)$	$Q_{max}(l)$	$Min[Q_r, Q_{max}(l)]$

and every value of $Q(0,t)$ in this table appears indeed as the minimum of the corresponding values of the supply and demand given in the margins.

Similarly, if $Q_{max}(l) \geq Q_{max}(r)$, the table (23) can be rewritten as:

$\Delta(K_l, l) \backslash \Sigma(K_r, r)$	$Q_{max}(r)$	$Q_e(K_r(r)r)$
$Q_e(K_l(l)l)$	$Min[Q_l, Q_{max}(r)]$	$Min[Q_l, Q_r]$
$Q_{max}(l)$	$Q_{max}(r)$	Q_r

and every value of $Q(0,t)$ in this table appears also as the minimum of the corresponding values of the supply and demand given in the margins. To this author's knowledge, Daganzo was the first to introduce this specific expression (25) of the flow of Godunov's scheme, in [DA 94] where he used it in the context of a specific equilibrium flow-density relationship which was piecewise linear, as depicted hereafter:



Later on he generalized it in [DA 93] to general flow-density relationships, as mentionned already. Let us add that (25) is also linked to ideas expressed in [NE 93].

4.5 Discretization of traffic flow on an inhomogeneous lane

We return to the problem of the discretization with Godunov's scheme of the LWR system on an inhomogeneous lane, adressed to in subsection 4.1. In order to apply (25) to Godunov's scheme, the Generalized Riemann Problem is solved at all discretization points x_a in order to estimate the relevant integrals of Φ and Ξ. The resulting flows are indeed equal to the flows

$$Q_a^t = \int_{t\Delta t}^{(t+1)\Delta t} \Phi(x_a, s)\, ds$$

if the solution of the Generalized Riemann Problem at one extremity of a does not modify the solution at the other extremity during the time step $[t\Delta t, (t+1)\Delta t]$. To ensure that this condition to be satisfied, it suffices that the greatest possible propagation speed of shock-waves in cell a,

(26) $$U_{max}(a) \stackrel{def}{=} Max[\frac{\partial Q_e}{\partial K}(0,a), -\frac{\partial Q_e}{\partial K}(K_{max}(a),a)] \quad ,$$

be smaller than $l_a/\Delta t$:

(27) $$U_{max}(a) \leq \frac{l_a}{\Delta t} \quad .$$

This condition specifies that the spatial and temporal resolution of the model cannot be choosen arbitrarily. For a given time-step, there exists a lower limit to the admissible space-discretization step size.

It is convenient to associate to every cell a its traffic supply and demand functions, which following (24), will be defined as:

(28)

- the cell traffic demand function

$$\Delta_e(\kappa, a) \stackrel{def}{=} \begin{cases} Q_e(\kappa, a)) & \text{if } \kappa \leq K_{crit}(a) \quad \text{(undercritical flow)} \\ Q_{max}(a) & \text{if } \kappa \geq K_{crit}(a) \quad \text{(overcritical flow)} \end{cases}$$

- the cell traffic supply function

$$\Sigma_e(\kappa, a) \stackrel{def}{=} \begin{cases} Q_{max}(a) & \text{if } \kappa \leq K_{crit}(a) \quad \text{(undercritical flow)} \\ Q_e(\kappa, a) & \text{if } \kappa \geq K_{crit}(a) \quad \text{(overcritical flow)} \end{cases}$$

It results that, provided that (28) is satisfied,

(29) $$Q_a^t = Min[\Delta_e(K_a(t), a), \Sigma_e(K_{a+1}(t), a+1)] \quad .$$

The basic iteration of the Godunov scheme can be rewritten as:

- a first loop indexed on the cells, during which the cell traffic demands $\Delta(K_a(t), a)$ and supplies $\Sigma(K_a(t), a)$ are computed,

- a second loop indexed on the discretization points x_a, during which the flows are computed following above equation (28),

- a third loop indexed on the cells, during which the cell average densities are compute according to:

$$K_a^{t+1} = K_a^t + \frac{\Delta t}{l_a}(Q_{a-1}^t - Q_a^t)$$

which ensures the conservativity of the scheme.

This possibility of breaking down the scheme into three independent component parts is an essential feature, that allows for specific physical characteristics for each cell and constitutes the clue for the generalization to networks of the scheme.

4.6 Dissipativity of the Godunov scheme

Many proprieties of the Godunov scheme are discussed in [DA 94] from the traffic point of view, and there is no need here to expand on them. For a more mathematical point of view, the reader is refered to [LV 90] or to [GR 91], as mentionned before. It is nevertheless necessary to point out one of the less desirable features of the scheme, i.e. its dissipativity.

Let us consider for instance a single cell a, and let us suppose that the traffic flow is roughly stationnary, i.e. varying slowly from one time-step to the next. Then, if the flow is undercritical, we may consider the inflow Q_{a-1}^t as given (see section 5 below for a rigourous justification), whereas $Q_a^t = Q_e(K_a^t, a)$. It follows:

$$K_a^{t+1} = K_a^t + \frac{\Delta t}{l_a}(Q_{a-1}^t - Q_a^t) \quad ,$$

$$Q_a^{t+1} = Q_e(K_a^{t+1}, a) \approx Q_a^t + \frac{\partial Q_e}{\partial K}(K_a^t, a)\,[K_a^{t+1} - K_a^t] \quad ,$$

assuming $Q_{a-1}^t - Q_a^t$ small enough (here the near-stationnarity hypothesis intervenes). It follows:

$$Q_a^{t+1} \approx (1-\alpha)Q_a^t - \alpha Q_{a-1}^t$$

with

$$\alpha \stackrel{def}{=} \frac{\partial Q_e}{\partial K}(K_a^t, a)\,\frac{\Delta t}{l_a} = \frac{\frac{\partial Q_e}{\partial K}(K_a^t, a)}{V_{max}(a)}\,\frac{V_{max}(a)\Delta t}{l_a} \quad .$$

This is of course a smoothing process whose smoothing factor α can be described as the product of two terms

1. The term $V_{max}(a)\Delta t/l_a$ is of a geometric nature, and should be taken as near to 1 as possible in order to restrict the numerical dissipativity of the scheme. This term is limited by (27). In /Dag95.1/ it is taken to be 1 exactly; for complex networks this is usually not possible and the cell lengths and time-steps should be adjusted in order that α should be as near to 1 as possible.

2. The term

$$\frac{\partial Q_e}{\partial K}(K_a^t, a)/V_{max}(a)$$

depends on the traffic intensity, and is equal to one if $K_a^t = 0$, and small or nil (depending on the functional form of Q_e) for critical density $K_a^t \approx K_{max}(a)$.

A symetric result holds in the overcritical case, except that the roles of the entry and exit of the cell a are exchanged. We need to introduce the greatest backward propagation speed of perturbations, $W_{max}(a)$ defined as:

$$W_{max}(a) \stackrel{def}{=} -\frac{\partial Q_e}{\partial K}(K_{max}(a), a) \quad ,$$

(hence (26) can be restated as $U_{max}(a) = Max[V_{max}(a), W_{max}(a)]$). It follows that:

$$Q_{a-1}^{t+1} \approx (1-\beta)Q_{a-1}^t - \beta Q_a^t \quad ,$$

with

$$\beta \stackrel{def}{=} -\frac{\frac{\partial Q_e}{\partial K}(K_a^t, a)}{W_{max}(a)} \frac{W_{max}(a)\Delta t}{l_a} \quad .$$

Here the smoothing factor β is most favorable when the density is highest and the cell length smallest. It must be noted that in most cases, $W_{max}(a)$ will be smaller than $V_{max}(a)$, which means that there will always be a certain amount of residual smoothing for shock waves, of a factor $W_{max}(a)/V_{max}(a)$ in the most favorable case, i.e. for adjusted cell length $l_a \approx V_{max}(a)\Delta t$ and very high density.

This smoothing of shock-waves and undercritical flow might be thought of as a positive trait; indeed the formula obtained above is somewhat reminiscent of the TRANSYT [RO 69] smoothing formula. Nevertheless the problem here is that the modeller has little control over the geometric parts of smoothing factors, since cell lengths will usually result from other considerations such as desirable time-step, computational complexity, link lengths (the links must be subdivided into an integer number of cells!), etc ....

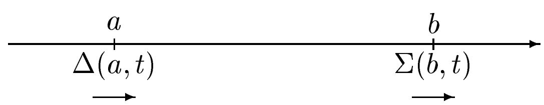
5 Proper boundary conditions for the LWR model of a link

We shall study in the next sections some extensions of the LWR model. We shall begin with the following remark: *the traffic demand and supply at the entrance and exit of a link provide the natural boundary conditions for the solution of equation (4) on the link.* More precisely, equation (4) on the link $[a, b]$ with initial conditions $K(x, t^0)$ for $x \in [a, b]$ supplemented with upstream demand $\Delta(a, t)$ and downstream supply $\Sigma(b, t)$ for $t \geq t^0$ will admit an unique (entropy) solution. For convenience's sake we shall consider *reduced demands* and *reduced supplies*:

$$\Delta(a, t) = \min(\Delta(a, t), Q_{max}) \quad ,$$

$$\Sigma(b, t) = \min(\Sigma(b, t), Q_{max}) \quad ,$$

where Q_{max} is the maximum flow in $[a, b]$.



The above-mentionned property is of course completely intuitive. A simple way to understand it is to consider the simple case where upstream demand $\Delta(a, t)$ and downstream supply $\Sigma(b, t)$ are constant, i.e. independant of t, and the initial condition $K(x, t^0)$ is homogeneous, i.e. independant of x. *This would be the analogue of the Riemann problem, this time for the description of the effect of boundary conditions.* Let us first point out that the equilibrium supply and demand functions are monotonous. Hence to every upstream demand it is possible to associate a unique undercritical density, and to every downstream supply it is possible to associate a unique overcritical density. Consequently we define the inverse equilibrium supply and demand functions (for a cell i):

(30) $$\Delta_e^{-1}(\delta, i) \stackrel{def}{=} \kappa \ \text{ iff } \ \delta = \Delta_e(\kappa, i) \ \text{ and } \ \kappa \in [0, K_{crit}(i)]$$

(31) $$\Sigma_e^{-1}(\delta, i) \stackrel{def}{=} \kappa \ \text{ iff } \ \delta = \Sigma_e(\kappa, i) \ \text{ and } \ \kappa \in [K_{crit}(i), K_{max}(i)] \quad .$$

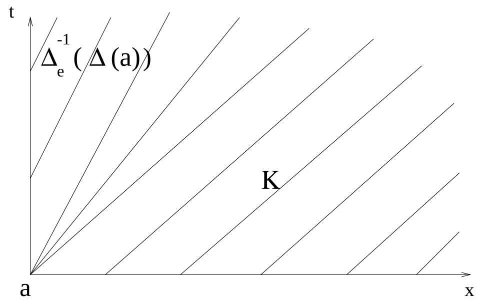
The computation rule for entry points is the following. The upstream reduced demand $\Delta(a)$ determines the boundary conditions at entry point a except when it is greater than the supply of the link $[a, b]$, $\Sigma_e(K(a), [a, b])$, which for simplicity's sake we shall abbreviate as $\Sigma_e(K)$. Hence:

- if $\Delta(a) \leq \Sigma_e(K)$, the density at entry point a is given by $\Delta_e^{-1}(\Delta(a))$ (here we skip the dependency of Δ_e^{-1} on $[a, b]$ as well) and the associated flow is $\Delta(a)$ (the associated characteristic lines are of positive slope),

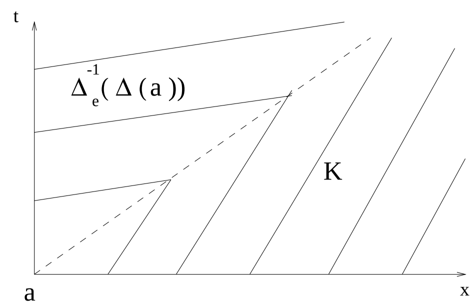
- if $\Delta(a) \geq \Sigma_e(K)$, the density at entry point a is given by K (which is necessarily overcritical) and the flow is $Q_e(K) = \Sigma_e(K)$ in this case. The associated caracteristic lines are of negative slope.

The resulting solutions are represented hereafter:

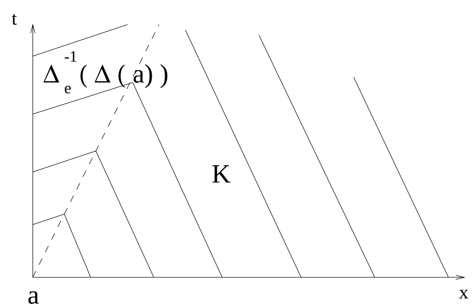
Case 1, $\Delta(a) \leq Q_e(K) \leq Q_{max} = \Sigma_e(K)$:



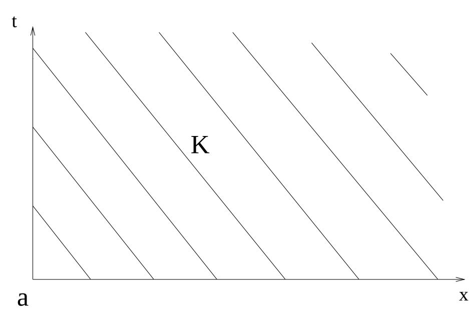
Case 2, $Q_e(K) \leq \Delta(a) \leq Q_{max} = \Sigma_e(K)$:



Case 3, $\Delta(a) \leq Q_e(K) = \Sigma_e(K) \leq Q_{max}$:



Case 4, $Q_e(K) = \Sigma_e(K) \leq \Delta(a) \leq Q_{max}$:

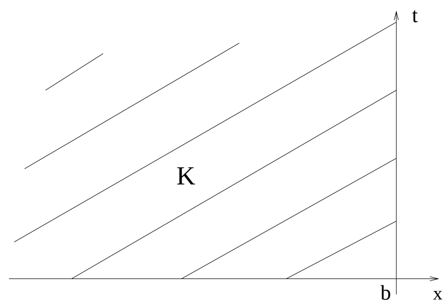


The situation at the other end b of the link is exactly symetric. The downstream reduced supply $\Sigma(b)$ determines the boundary conditions at exit point b except when it is greater than the demand of the link $[a, b]$, $\Delta_e(K(b), [a, b])$, abbreviated as $\Delta_e(K)$. Hence:

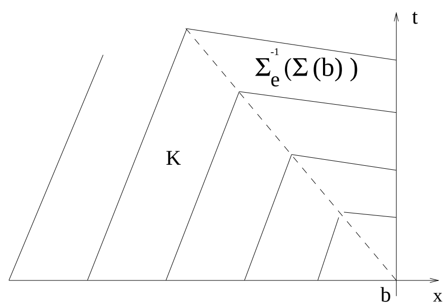
- if $\Sigma(b) \leq \Delta_e(K)$, the density at exit point b is given by $\Sigma_e^{-1}(\Sigma(b))$ (as previously we skip the dependency on $[a, b]$) and the associated flow is $\Sigma(b)$ (the associated caracteristic lines are of positive slope),

- if $\Sigma(b) \geq \Delta_e(K)$, the density at exit point b is given by K (which is necessarily undercritical) and the flow is $Q_e(K) = \Sigma_e(K)$ in this case. The associated caracteristic lines are of positive slope.

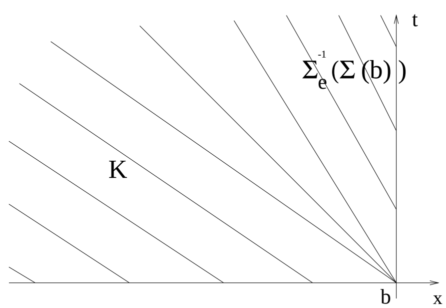
Case 1, $Q_e(K) = \Delta_e(K) \leq \Sigma(b) \leq Q_{max}$:



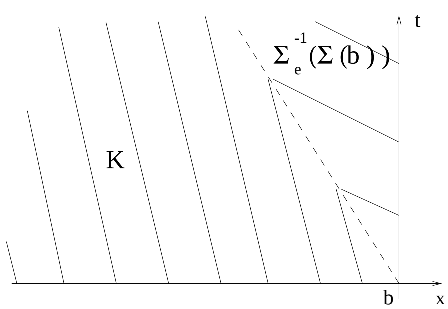
Case 2, $\Sigma(b) \leq Q_e(K) = \Delta_e(K) \leq Q_{max}$:



Case 3, $Q_e(K) \leq \Sigma(b) \leq Q_{max} = \Delta_e(K)$:



Case 4, $\Sigma(b) \leq Q_e(K) \leq Q_{max} = \Delta_e(K)$:



It is straightforward to check that by considering the solutions obtained in this section for entry and exit points as right-side and left-side solutions at a same point say $c \stackrel{def}{=} a = b$ and "pasting" them one retrives the various solutions of the Generalized Riemann Problem described in section 4.

6 Modelling intersections within the framework of the LWR model

6.1 Introduction

We shall now adress the problem of describing consistent first-order LWR models of networks. In the preceeding section, we have clarified the issue of boundary conditions for links within the framework of such a model. We have seen that the entry point, is characterized by the traffic supply of the link whereas its exit point is characterized by the link traffic demand. These quantities must be adjusted respectively to the upstream demand and the downstream supply to determine the dynamics of the link. If we wish to model an intersection, the problem becomes now: how to combine the different demands upstream of the intersection, and how to adjust them to the different supplies downstream of the intersection.

Before proceeding with the analysis, we must first precise the *scale* of the intersection model envisioned here. Essentially, we shall consider intersections modelled as geometrical points. As we shall see, these points can nevertheless be endowed with some physical characteristics and parameters if need be. Such a model is consistent with the spirit of the LWR model, in which the space scale should be reasonnably large. More sophisticated models are possible, we shall say a few words about them at the end of the section 7.3.

A technical definition must be introduced at this point. In subsection 4.5, formula (28), the concepts of cell traffic supply and demand function were introduced. It is similarly possible to define *local traffic supplies and demands*, for piecewise continuous equilibrium flow-density relationships and piecewise continuous solutions of the LWR system (1), (2), (3). Let us first define the local equilibrium demand and supply:

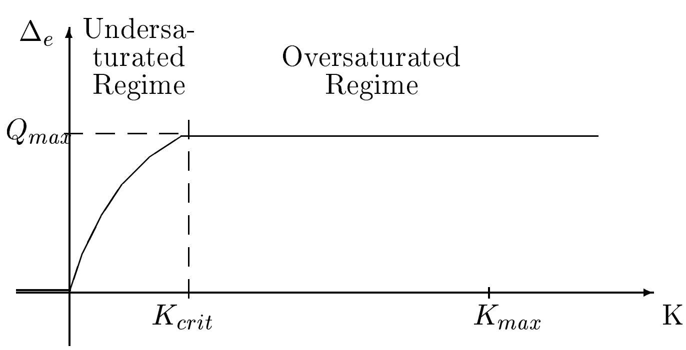
(32)
– the local equilibrium demand function

$$\Delta_e(\kappa, x) \stackrel{def}{=} \begin{cases} Q_e(\kappa, x-) & \text{if} \quad \kappa \le K_{crit}(x-) \quad \text{(undercritical flow)} \\ Q_{max}(x-) & \text{if} \quad \kappa \ge K_{crit}(x-) \quad \text{(overcritical flow)} \end{cases}$$

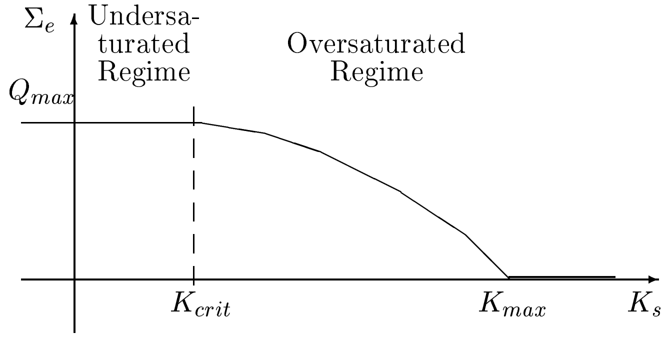
– the local equilibrium supply function

$$\Sigma_e(\kappa, x) \stackrel{def}{=} \begin{cases} Q_{max}(x+) & \text{if} \quad \kappa \le K_{crit}(x+) \quad \text{(undercritical flow)} \\ Q_e(\kappa, x+) & \text{if} \quad \kappa \ge K_{crit}(x+) \quad \text{(overcritical flow)} \end{cases} .$$

The traffic equilibrium demand function has the following aspect:



The equilibrium traffic supply function has the following aspect:



We can now define the local traffic demand and supply as

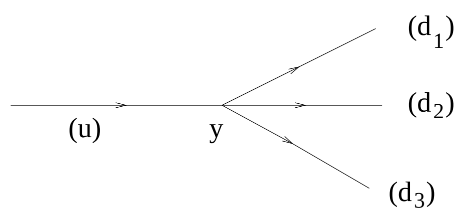
$$
\left|
\begin{array}{ll}
- & \text{the local demand} \\
 & \Delta(x,t) \stackrel{def}{=} \Delta_e(K(x-,t),x) \\
 & \\
- & \text{the local supply} \\
 & \Sigma(x,t) \stackrel{def}{=} \Sigma_e(K(x+,t),x) \quad .
\end{array}
\right.
\tag{33}
$$

Let us note that it follows from this definition that if (K, Q) is a piecewise continuous entropy solution of the LWR system, then at all points x and all times t:

$$Q(x,t) = Min[\Delta(x,t), \Sigma(x,t)] \quad .$$

6.2 Modelling diverges

Let us now consider first intersections of the diverge kind, composed of an upstream link (u) and several downstream links $(d1)$, ..., (di), ...



Given at a time t are the state of every link and the assignment coefficients p_i: p_i is the proportion of users in link (u) at intersection point y wishing to use link (d_i). The upstream demand is then given by:

$$\Delta_u \stackrel{def}{=} \Delta_e(K(y-,t;u),y;u) \quad ,$$

and the downstream supplies are given by:

$$\Sigma_i \stackrel{def}{=} \Sigma_e(K(y+,t;d_i),y;d_i) \quad .$$

Now we must specify what we understand exactly by *intersection model*. An intersection model consists of a set of rules enabling us to adjust upstream demands to downstream supplies. Such an adjustment requires two steps.

1. It is necessary to choose the order of the adjustment process, meaning to choose whether to compute first the total flow through the intersection, and then to divide it between the partial flows relative to the various entry and/or exit links, or to first compute these partial flows and then to agregate them.

2. It is necessary to use a rule for the splitting of flows, supplies and/or demands.

Both steps involve choices that are not intrinsical but of a phenomenological nature, directly linked to the kind of driver behaviour and physical conditions to be modelled. Let us illustrate this in the case of a diverge. If one chooses to compute the total outflow of (u) first, it is necessary to determine an equivalent downstream supply for (u), let Σ_{eq} be its name. Then the total outflow of (u), Q_u, will be given by:

$$Q_u = Min[\Delta_u, \Sigma_{eq}] \quad .$$

A simple rule for splitting the total flow according to downstream links (d_i) is the following:

$$QI_i \stackrel{def}{=} p_i Q_u \quad ,$$

(here we need to distinguish the *inflows* QI_i of links (d_i)). This rule means that the outflow is split exactly according to its composition, a rule that has a long history; it has been used already in [PA 90] for instance. We shall analyze it in the next section. Now there remains to check that $QI_i \leq \Sigma_i$ for all downstream links. From $Q_u \leq Min_i[\Sigma_i/p_i]$ it follows that the choice that maximizes the outflow Q is

$$Q_{eq} \stackrel{def}{=} Min_i\ [\Sigma_i/p_i] \quad .$$

This choice is in keeping with the flow-maximizing properties of the entropy solutions of the LWR model. The resulting intersection model has been described in [DA 94]. Let us note that if Σ_{eq} is nil, i.e. if one of the downstream supplies is nil, then the total (as well as the partial) outflows are nil. This illustrates precisely the kind of intersection that are modelled by the preceeding model: essentially intersections whose upstream link has a single lane, or behaves as if it did. There is no specific storage capability for the users whose destination is an oversaturated link. Let us note that this model works both for discretized and distributed models. *By distributed model we mean a model whose basic objects are undiscretized functions of continuous variables, in opposition to discretized models.* The discretized model is described by:

$$\begin{array}{rcl} Q_u^t & = & Min[\Delta_u^t, Min_i(\Sigma_i^t/p_i)] \quad , \\ QI_i^t & = & p_i Q_u^t \qquad \forall (di) \quad . \end{array}$$

A quite different model is the following, in which the demand Δ_u is split first according to:

$$\Delta_i \stackrel{def}{=} p_i \Delta_u \quad .$$

Thus partial demands relative to each downstream link are determined first, then the partial flows QI_i result from the adequation of demand to supply for each link:

$$QI_i = Min\ [\Delta_i, \Sigma_i] \quad .$$

Finally, the total outflow of the link (u) is given by:

$$Q_u = \sum_i QI_i \quad .$$

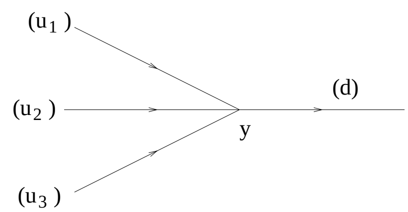
The resulting discretized model ([BLL 95]) is described by the following:

$$\begin{array}{rcl} QI_i^t & = & Min[p_i \Delta_u^t, \Sigma_i^t] \qquad \forall (di) \quad , \\ Q_u^t & = & \sum_i QI_i^t \quad . \end{array}$$

But this model is not easily adapted to distributed models. Indeed, since the assignment coefficients are only relative to the demand, they are not necessarily satisfied, i.e. $p_i \neq QI_i/QI_u$. There is a resulting change in the composition of the (u) link, which is easily accomodated if the model is discretized, but not if is distributed. In this last case, it is necessary to expand the basic LWR model in (u). This point will be discussed in next section. For the time being, let us note that the physical meaning of the model is that there exists a specific storage capacity in link (u) for the traffic flow relative to each exit link (d_i). This would typically be the case of a multilane link. If some supply Σ_i is nil, the corresponding partial flow Q_i becomes nil, but the other partial flows are not affected immediately.

6.3 Modelling merges

Let us consider the following merge, in which (u_i) are the upstream links and d is the downstream link;



the intersection is represented by the point y. Given are the traffic demands Δ_i of the upstream links and the traffic supply Σ_d of the downstream link:

$$\Delta_i \stackrel{def}{=} \Delta_e(K(y-,t;u_i),y;u_i) \quad ,$$

$$\Sigma_d \stackrel{def}{=} \Sigma_e(K(y+,t;d),y;d) \quad .$$

A possibility of modelling such an intersection is given by the following rule for splitting the supply Σ_d:

$$\Sigma_{d,i} = \alpha_i \Sigma_d$$

with α_i a split coefficient. The partial $(u_i) \longrightarrow (d)$ flow, denoted Q_i, is then equal to:

$$Q_i = Min[\Delta_i, \Sigma_{d,i}]$$

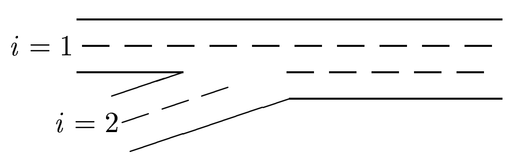
and the total flow is then equal to

$$Q = \sum_i Q_i \quad .$$

Various models can be proposed for the coefficients α_i. These coefficients reflect that proportion of the total traffic supply that the users coming from (u_i) perceive is accessible to them. So a possible simple model is:

$$\alpha_i = \frac{K_{max,i}}{K_{max}}$$

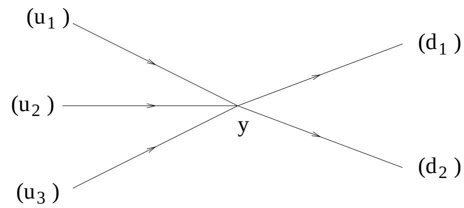
with K_{max} the maximum density of link (d) at point y and with $K_{max,i}$ the maximum density available to users coming from (u_i) at at the entrance of link (d). So the split coefficients in this model are proportional to the number of lanes the users coming from (u_i) can use when entering d. If usually, $\sum_i \alpha_i = 1$, it is possible that $\sum_i \alpha_i > 1$. In the following example



the natural values of the split coefficients are $\alpha_1 = \alpha_2 = 2/3$. The case $\sum_i \alpha_i > 1$ requires precautions whose description is outside the scope of this paper, the reader is refered to /BLL95/ for details. The preceeding merge model could be used indifferently for discretized or distributed models.

6.4 General intersections

The preceeding merge and diverge models can be combined into a general intersection model, which, owing to the limitations of the diverge model, is only applicable to discretized network models, or to extensions of the LWR model in a sense that will be described in the next section. Let us consider the following intersection:



The data are:

- the traffic demand Δ_i of link (u_i) at point y,
- the traffic supply Σ_j of link (d_j) at point y,
- the proportion p_{ij} of users of link (u_i) wishing to use link (d_j),
- the split coefficients α_{ij} of the supply Σ_j: $\alpha_{ij}\Sigma_j$ is the supply accessible to users from link (u_i) on link (d_j).

Of course, all these quantities must be understood to be time-dependent. The flow Q_{ij} of users going from link (u_i) to link (d_j) is given by:

$$Q_{ij} = Min[p_{ij}\Delta_i, \alpha_{ij}\Sigma_j] \quad ,$$

the total outflow QO_i of link (u_i) is:

$$QO_i = \sum_j Q_{ij} \quad ,$$

and the total inflow QI_j of link (d_j) is:

$$QI_j = \sum_i Q_{ij} \quad .$$

It is possible to impose bounds Φ_{ij} on the flows Q_{ij}, to represent the workings of traffic lights or to model conflicts within the intersection. The resulting model would be:

$$Q_{ij} = Min[p_{ij}\Delta_i, \Phi_{ij}, \alpha_{ij}\Sigma_j] \quad .$$

In the next section, we shall mention in a more general context an intersection model suitable for distributed LWR models as well.

7 Modelling partial flows for assignment problems within the framework of the LWR model

7.1 Links

Let us now consider the problem of splitting the flow between partial flows:

$$\begin{array}{rcl} K & = & \sum_d K^d \\ Q & = & \sum_d Q^d \end{array}$$

where the superscript d could, depending on the problem, represent local or global destinations, paths, or any other partition of the set of drivers. The simplest model , which has already been mentionned in the previous section, is:

(34) $$Q^d = Q\xi^d$$

with ξ^d the traffic composition:

$$\xi^d \stackrel{def}{=} K^d/K \quad . \tag{35}$$

The resulting flow model for the partial densities is:

$$\frac{\partial K^d}{\partial t} + \frac{\partial Q^d}{\partial x} = 0 \quad ,$$

(conservation of vehicles), which, supplemented with (34), (35), yields:

$$\frac{\partial K^d}{\partial t} + \frac{\partial}{\partial x}(K^d V) = 0 \quad . \tag{36}$$

In equation (36), the meaning of V is the following: V is the equilibrium speed

$$V \stackrel{def}{=} V_e(K, x)$$

associated to the solution (K, Q) of the LWR system for the global flow. Since V can be considered as given, (36) is of a very different nature than for instance the LWR equation (4). Indeed, let us derive the equations for the compositions. From (36), with $K^d = \xi^q K$ and (35), it follows:

$$\frac{\partial \xi^d K}{\partial t} + \frac{\partial \xi^d Q}{\partial x} = 0 \quad ,$$

hence

$$\frac{\partial \xi^d}{\partial t} + V\frac{\partial \xi^d}{\partial x} = 0 \quad . \tag{37}$$

This last equation is an advection equation with speed V; its significance is that *the compositions do not change along a trajectory*. In other words, the composition of the traffic entering the link at time t is the same as the composition of the traffic exiting the link at time $t + \tau(t)$, with $\tau(t)$ the effective travel time of users entering the link at time t. Another way to understand model (34), (35), is to notice that it is equivalent to the FIFO rule. Indeed, if one considers only two values of the superscript d, say $d = 1$ for the users entering the link before time t and $d = 2$ for the users entering the link after time t, then all users $d = 1$ will exit the link before time $t + \tau(t)$, and all users $d = 2$ will exit the link after time $t + \tau(t)$, as a consequence of the advection equation (37). Another way yet to view (36) or (37) is to note that these equation imply a *forward propagation* of the information relevant to partial densities, at the speed V of the traffic stream.

The partial densities equation (36) can be discretized several ways. Let us mention that in [DA 94], a scheme is proposed, in which the partial densities in a cell are split according to the time of entry in the cell of vehicles. The tracing of entry times is cumbersome, but permits a strict respect of the FIFO rule. In a simpler way, it is possible to simply discretize the continuity equation for partial densities and (34). The resulting scheme for partial densities is:

$$K_a^{d,t+1} = K_a^{d,t} + \frac{\Delta t}{l_a}(Q_{a-1}^{d,t} - Q_a^{d,t}) \quad ,$$

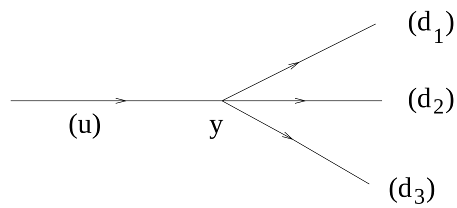
and

$$Q_a^{d,t} = Q_a^t K_a^{d,t} / K_a^t \quad .$$

$K_a^{d,t}$ is the mean partial density in cell a at time t of users having d as their destination, and $Q_a^{d,t}$ is the outflow of cell a of these users. This scheme is simple but implies, like Godunov's method, a strong amount of dissipativity, and therefore satisfies the FIFO rule only approximatively.

7.2 The diverge intersection revisited

Let us now return to the problem of modelling a diverge like the following



The first diverge model which we examined in the subsection 6.2, in which the outflow of the link (u) is described precisely by a relation similar to (34) (with $d \equiv d_i$), is in complete keeping with the model (36). On the other hand, the second model in which the demand was split according to compositions, but not necessarily the outflow, is not compatible with (37). Indeed, the outflow would be determined independently both by the previous inflow, according to (37), and by the conditions downstream of the diverge, leading to a contradiction. The forward propagation of the traffic coposition explains the difficulty of modelling diverges. The model describing the partial densities (equations (34) to (37)) needs to be modified in order to take into account the following elements:

- locally, at any point x, the partial demand associated to trafic d, $\Delta^d(x)$, should be proportional to the total demand $\Delta(x)$ and the composition:

  (38) $$\Delta^d(x) = \xi^d \Delta(x)$$

  where the total demand $\Delta(x)$ is determined as usual according to (33).

- locally, a rule for the splitting of the total supply $\Sigma(x)$ into partial demands $\Sigma^d(x)$, with

  $$\Sigma(x) = \sum_d \Sigma^d(x)$$

  should be provided. (The total supply results from (33) as well).

A possible model along these lines would be the following:

(39) $$\Sigma^d(x) = M[\frac{K^d_{max}(x) - K^d(x)}{K_{max}(x) - K(x)}]\Sigma(x) \quad ,$$

with

. $K^d_{max}(x)$ the maximum density for vehicles of destination d (representing the storage capacity of preselection lanes associated to downstream link d),

. $K^d_{max}(x)$ the maximum density,

. M is defined as:
$$\begin{array}{rcll} M(\kappa) & = & \kappa & \text{if} \quad \kappa \geq 0 \quad , \\ & & 0 & \text{if} \quad \kappa \leq 0 \quad . \end{array}$$

The idea here is to consider that $\Sigma(x)^d$ should be proportional to the *residual storage capacity* $K^d_{max}(x) - K^d(x)$. Such a model limits the partial density $K^d(x)$ to its capacity $K^d_{max}(x)$. The dynamics of the resulting model are obtained by combining the conservation equation for partial densities with the adjustment of partial supplies and demands.

(40) $$\begin{array}{l} \frac{\partial K^d}{\partial t} + \frac{\partial Q^d}{\partial x} = 0 \quad , \\ Q^d(x,t) = Min\,[\Sigma(x,t)^d, \Delta(x,t)^d] \quad . \end{array}$$

The total flow and density result from

$$Q(x,t) = \sum_d Q^d(x,t) \qquad (41)$$
$$K(x,t) = \sum_d K^d(x,t) \quad .$$

The global supplies and demands result from

$$\Sigma(x,t) = \Sigma_e(K(x+,t),x) \qquad (42)$$
$$\Delta(x,t) = \Delta_e(K(x-,t),x) \quad .$$

The partial supplies and demands result from

$$\Sigma^d(x,t) = M[\frac{K^d_{max}(x) - K^d(x,t)}{K_{max}(x) - K(x,t)}]\Sigma(x,t) \qquad (43)$$
$$\Delta^d(x,t) = \frac{K^d(x,t)}{K(x,t)}\Delta(x,t) \quad .$$

The resulting system, (40), (41), (42) (43) is a system of conservation laws. The dynamics of the partial densities are fairly independant, nevertheless they are linked through the global density and its associated local supply and demand. For instance, a shock-wave affects all partial densities, and the Rankine-Hugoniot conditions hold for all of them with the same shock-wave speed. At low densities, the system simplifies to the simpler FIFO model (36). The interaction mechanism depicted here is in sharp contrast to other previous multilane models such as [MBY 84], since it aims essentially to allow differential storage behavior, depending on the lane. The structure of the model is well adapted to the description of large diverges with storage capacity.

The discretized version of system (40), (41), (42) (43) is straightforward, and consists of three loops:

loop 1

$$\begin{aligned}
K^t_a &= \sum_d K^{d,t}_a \\
\Delta^t_a &= \Delta_e(K^t_a, a) \\
\Sigma^t_a &= \Sigma_e(K^t_a, a) \\
\Delta^t_a &= \Delta^{d,t}_a \frac{K^{d,t}_a}{K^t_a} \\
\Sigma^{d,t}_a &= M[\frac{K^d_{max,a} - K^{d,t}_a}{K_{max,a} - K^t_a}]\Sigma^t_a \\
\Delta^{d,t}_a &= \frac{K^{d,t}_a}{K^t_a}\Delta^t_a
\end{aligned}$$

loop 2

$$\begin{aligned}
Q^{d,t}_a &= Min\,[\Delta^t_a, \Sigma^{d,t}_{a+1}] \\
Q^t_a &= \sum_d Q^{d,t}_a
\end{aligned}$$

loop 3

$$K^{d,t+1}_a = K^{d,t}_a + \frac{\Delta t}{l_a}[Q^{d,t}_{a-1} - Q^{d,t}_a]$$

7.3 General intersections

The model of a general intersection described in subsection 6.3 requires very little modification to include partial flows. With the same notations, it suffices to introduce also:

- the assignment coefficients ϖ^d_{ij} (proportion of users coming from link (u_i), with final destination d, who use link (d_j)),
- the composition ξ^d_i of the traffic flow leaving link (u_i).

Here superscripts d represent again ultimate destinations. We define the proportion p_{ij} of the trafic leaving link (u_i) for link (d_j), which is given by:

$$p_{ij} = \sum_d \xi^d_i \varpi^d_{ij} \quad .$$

As previously, the flows are given by

$$Q_{ij} = Min[p_{ij}\Delta_i, \alpha_{ij}\Sigma_j] \quad ,$$

and the partial flows Q_{ij}^d are determined using a simple proporionality rule as

$$Q_{ij}^d = Q_{ij}\frac{\xi_i^d \varpi_{ij}^d}{p_{ij}} \quad .$$

Indeed, the fraction of the $(u_i) \rightarrow (d_j)$ of ultimate destination d is

$$\frac{\xi_i^d \varpi_{ij}^d}{p_{ij}} \quad .$$

Such an intersection model is perfectly suitable in a discretized context, since the cells provide the ncessary storage capacity if some downstream links are oversaturated. Indeed, if one considers a network with intersections modelled as above and a discretization with just one cell per link, models of the exit-function kind ([MN 78], [FLTW 89]) result, with nevertheless some provision for traffic supply modelling at intersections. In a distributed context, the above model is only compatible with a non FIFO flow model of the kind of the system (40), (41), (42), (43). It can be noted also that the intersection model proposed in [HI 95], chapter 6, presents some similarities with the above model.

It is possible to design more complicated intersection models, endowed with various physical parameters, notably storage capacities, and with various dynamical variables describing the numbers of vehicles stored in the intersection, split according to final destination, and to the entry and exit points of the intersection. Such models can provide proper boundary conditions as defined in section 5 for all upstream and downstream links, even in a distributed context. Therefore they can be used with links modelled by the LWR model and the FIFO (36) to build discrete or continuous time models of networks. The *exchange zone* of [BLL 95] provides such an intersection model. It is closely linked to the intersection concept of METACOR [EHP 94]. These intersection models cannot garantie the satisfaction of the FIFO rule, neither globally, which would not have much physical meaning anyway, nor if one considers only the traffic joining one entrance to one exit of the intersection.

8 Conclusion.

In this paper we have shown that the Godunov scheme is well adapted to the discretization of the LWR system and the approximate solution of its entropy solution, especially in the presence of discontinuities of the equilibrium flow-density relationship. This raises two questions. The first is: would not a higher order approximation scheme be applicable and even preferable? For instance, Van Leer's method (see [GR 91], pp 201 and following, or [LV 90], section 16.3) follows the same principles as Godunov's method, but with piecewise linear (discontinuous) approximation of the density. But solving the analogue of the Riemann problem explicitly might proove a difficult task, because then this problem is no longer scale-invariant. Approximate methods would have to be used. Further, in order to benefit from the gain in precision, it would be necessary to diminish the cell size, since the slope of the linear approximation in every cell would be limited by the constraints $0 \leq K \leq K_{max}$. Apart from the fact that small cells are not in keeping with the spirit of the LWR model, the gain in precision might well be more than compensated by the loss in computational tractability. Another question is: should the entropy solutions of the LWR system be thus privileged. The answer to this question depends on the range of application of the model. For problems in which the finiteness of accelerations is important, such as corridor control, an adaptation of the Godunov scheme emulating bounded accelerations seems advisable. Such an adaptation might take the form of a bound on the flow between cells, following the analysis of sections 3 and 4.1. Of course, one of the nice feature of the Godunov scheme, its scale invariance (i.e. the fact that the discretization has the same form whatever the cell size), would be lost as a consequence of such an adaptation.

If we turn now to the concepts of traffic supply and demand as they result from Dagazo's flow formula, they provide effectively a tool and a rationale for modelling intersections and networks within the framework of the LWR model, or eventually related models. Nevertheless the analysis of traffic demand and supply is in itself insufficient and must be completed with experimentally supported models describing driver behaviour, as reflected in the various split coefficients. Further research in this direction should therefore put emphasis on the experimental testing of these intersection split coefficients.

References

[AN 90] ANSORGE, R. *What does the entropy condition mean in traffic flow theory.* Transportation Research 24B. 133-144. 1990

[BLL 95] BUISSON, C., LEBACQUE, J.P., LESORT, J.B. *Macroscopic modelling of traffic flow and assignment in mixed networks.* Proc. of the Berlin ICCCBE Conf. 1367-1374. 1995. (ed. PAHL, P.J., WERNER, H.)

[BNN 92] BUI, D.D., NELSON, P., NARASIMHAN, S.L. *Computational realizations of the entropy condition in modelling congested traffic flow.* Report 1232-7. Texas Transportation Institute. 1992.

[CP 92] CHIN JIAN LEO, PRETTY, R.L. *numerical simulation of macroscopic traffic models.* Transportation Research 26B. 3: 207-220, 1992.

[DA 93] DAGANZO, C.F. 1993. em A finite difference approximation of the kinematic wave model. Institute of Transportation Studies, Report UCB-ITS-PRR-93-11 (published in Transportation Research 29B, 4: 261-276. 1995).

[DA 94] DAGANZO, C.F. *The cell transmission model 1: a dynamic representation of highway traffic consistent with the hydrodynamic theory.* Transportation Research 8B. 4: 269-287. 1994. *The cell transmission model 2: network traffic simulation.* Transportation Research 29B. 2: 79-93. 1995.

[EHP 94] ELLOUMI, E., HADJ SALEM, H., PAPAGEORGIOU, M. *METACOR, a macroscopic modelling tool for urban corridors.* TRISTAN II Int. Conf., Capri, 1994.

[FLTW 89] FRIESZ, T.L., LUQUE, J., TOBIN, R.L., WIE, B.W. *Dynamic network traffic assignment considered as a continuous time optimal control problem.* Op. Res. 37, 6: 893-901. 1989.

[GR 91] GODLEWSKI, E., RAVIART, P.A.: *Hyperbolic systems of conservation laws.* SMAI. Ellipses. 1991.

[GO 59] GODUNOV, S. : *A difference scheme for numerical computation of discontinuous solutions of equations of fluid dynamics.* Mat. Sb. 47(89): 271-306. 1959.

[HI 95] HILLIGES, M. : *Ein phänomenologisches Modell des dynamischen Verkehrsflusses in schnellstraßennetzen.* PHD dissertation. Institut für theoretische Physik der Universität Stuttgart. Shaker Verlag. 1995.

[INR 88] *Les modèles INRETS de simulation.* Synthèse INRETS 12, 1988.

[KLHK 95] KÜNE, R.D., LANGBEIN-EUCHNER, K., HILLIGES, M., KOCH, N. *Model based optimization of traffic control systems.* Proc. of the Berlin ICCCBE Conf. 1431-1438. 1995. (ed. PAHL, P.J., WERNER, H.)

[LE 84] LEBACQUE, J.P. *Semimacroscopic simulation of urban traffic.* Int. 84 Minneapolis Summer Conference. AMSE. 1984.

[LE 93] LEBACQUE, J.P. *Les modèles macroscopiques de trafic.* Annales des Ponts. 3^{rd} trim., 67: 28-45. 1993.

[LE 95] LEBACQUE, J.P. *L'échelle des modèles de trafic: du microscopique au macroscopique.* Annales des Ponts. 1^{st} trim., 74: 48-68. 1995.

[LV 90] LEVEQUE R.L. *Numerical methods for conservation laws.* Birkhäuser Verlag. 1990.

[LW 55] LIGHTHILL, M.H., WHITHAM, G.B. On kinematic waves II: *A theory of traffic flow on long crowded roads.* Proc. Royal Soc. (Lond.) A 229: 317-345. 1955.

[MBL 84] MICHALOPOULOS, P.G., BESKOS, D.E., LIN J.K. *Analysis of interrupted traffic flow by finite difference methods.* Transportation Research 18B, 4/5: 409-421. 1984.

[MBY 84] MICHALOPOULOS, P.G., BESKOS, D.E., YAMAUCHI, Y. *Multilane traffic flow dynamics: some macroscopic considerations.* Transportation Research, 18B, 4/5: 377-395. 1984

[MN 78] MERCHANT, D.K., NEMHAUSER, G.L. *1.A model and an algorithm for the dynamic traffic assignment problem.*
2. Optimality conditions for a dynamic assignment model.
Transportation Science 12: 183-199 & 200-207. 1978 .

[MMS 81] MICHALOPOULOS, P.G., STEPHANOPOULOS, George, STEPHANOPOULOS, Gregory. *An application of shockwave theory to traffic signal control.* Transportation Research, 15B: 35-51. 1981.

[MP 90] MESSNER, A., PAPAGEORGIOU, M. *METANET: a macroscopic modelling simulation for motorway networks.* Technische Universität München. 1990.

[NE 93] NEWELL, G.F. *A simplified theory of kinematic waves in highway traffic, parts I, II, III.* Transportation research 27 B. 4: 281-287, 289-303, and 305-313. 1993.

[OS 84] OSHER, S. 1. *Riemann solvers, the entropy condition, and difference approximations.* SIAM J. Num. Anal. 21: 217-235. 1984.
2. *Convergence of generalized MUSCL schemes.* SIAM J. Num. Anal. 22: 947-961. 1985.

[PA 90] PAPAGEORGIOU, M. *Dynamic modelling, assignment and route guidance in traffic networks.* Transportation Research 24 B. 6: 471-495. 1990.

[PAY 71] PAYNE, H.J. 1971. *Models of freeway traffic and control.* Simulation Council Proceedings 1: ch 6 .

[RI 56] RICHARDS, P.I. *Shock-waves on the highway.* Opns. Res. 4: 42-51. 1956

[RO 69] ROBERTSON, D.I. *TRANSYT: A traffic network study tool.* Road Research Report LR 253. Crowthorne (G.B.). 1969 .

[SC 88] SCHOCHET, S. *The instant response limit in Whitham's nonlinear traffic model: uniform well-posedness and global existence.* Asymptotic Analysis 1: 263-282. 1988.

[SM 83] SMOLLER, J. *Shock-waves and reaction-diffusion equations.* Springer Verlag, N.Y., 1983.

[VA 94] VAN AERDE, M. *INTEGRATION: A model for simulating integrated traffic networks.* Transportation Systems Research Group. Queen's University, Kingston, Ontario. 1994.