Multifractal Formalism and Anisotropic Selfsimilar Functions

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Abstract In this paper we prove that the conjectures of Frisch and Parisi in [9] and Arneodo et al in [1] (called the multifractal formalism for functions) may fail for some non homogenous selfsimilar functions in m dimension, $m \geq 2$. In these cases, we compute the correct spectrum of singularities and we show how the multifractal formalism must be modified.
Key words: Hölder exponent, Hausdorff dimension, spectrum of singularities, multifractal formalism, selfsimilar functions, wavelets.

1 Introduction

The multifractal formalism for functions is an heuristic principle which says that for a function $F : \mathrm{R}^m \mapsto \mathrm{R}$, the Hausdorff dimension $d(\alpha)$ of the set of points x where

$$|F(x+h) - F(x)| \sim |h|^\alpha$$

is equal to the Legendre transform of $\zeta(p) - m$

$$d(\alpha) = \inf_p (\alpha p - \zeta(p) + m)$$

where $\zeta(p)$ is the L^p-mean Hölder index

$$\int_{\mathbf{R}^m} \mid F(x+h) - F(x) \mid^p \, dx \sim |h|^{\zeta(p)} .$$

Recently many papers proved the validity of this conjecture for a large class of selfsimilar functions (see [2], [6], [11] and [12]). The self-similarity here means that locally the graph of the function F is a contraction of the global graph modulo an error which is more regular than the function F itself. This means that F satisfies:

$$F(x) = \begin{cases} \lambda_j F(S_j^{-1}(x)) \ + \ g(x) & \text{if } x \in S_j(\Omega) \\ g(x) & \text{if } x \notin \bigcup_{j=1}^d S_j(\Omega) \end{cases}$$

which can be written as

$$F(x) = \sum_{j=1}^{d} \lambda_j F(S_j^{-1}(x)) + g(x) \tag{1}$$

where

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- $|\lambda_1| < 1, \ldots, |\lambda_d| < 1$; $S_1, \ldots, S_d$ are contractive similitudes in a bounded open set Ω of R^m such that

$$S_i(\Omega) \subset \Omega \tag{2}$$

$$S_i(\Omega) \cap S_j(\Omega) = \emptyset \quad \text{if} \quad i \neq j\,. \tag{3}$$

- g is a C^k function with all derivatives of order less than k having fast decay.

- There exists $x_0 \in \Omega$ such that F is not $C^k(x_0)$.

The multifractal formalism was also proved in one dimension when the S_i are no more linear and two dimension when the S_i are analytic mappings of $z = x + iy$ (see [3]), and the fundamental idea is that in a certain sense (see Lemma 1 in [3]) locally these contractions are close to linear contractions in one dimension and "contract with the same rate" in each direction in two dimension. However, we will prove in this paper that for contractions that contract with different rates in each direction, the multifractal formalism for functions fails. Then we show how it must be modified in order to be adapted to a large class of non homogenous selfsimilar functions.

Let us first explain the terminology that will be used throughout this paper.

Definition 1 *A function $F : \mathrm{R}^m \to \mathrm{R}$ belongs to $C^\alpha(x_0)$ for $\alpha > 0$ if there exists a polynomial P of degree smaller than α such that*

$$|F(x) - P(x - x_0)| \leq C|x - x_0|^\alpha. \tag{4}$$

The Hölder exponent of F at x is defined by

$$\alpha(x) = \sup\{\beta : \quad F \in C^\beta(x)\}\,.$$

F belongs to $C^\alpha(\mathrm{R}^m)$ if (4) holds for any x in R^m with uniform constant C.

Definition 2 *The spectrum of singularities of F is the function $d(\alpha)$ which associates to each α the Hausdorff dimension of the set of points x where $\alpha(x) = \alpha$ (conventionally the dimension of the empty set is $-\infty$).*

Now, let $F \in L^p$ and define

$$\zeta(q) = \liminf_{|h| \mapsto 0} \frac{\log \int |F(x+h) - F(x)|^q \, dx}{\log |h|}\,; \tag{5}$$

and let us show that $\zeta(p)$ is linked to Sobolev's "smoothness" index. Let $s \geq 0$; if s is not an integer, $s = [s] + \sigma$ with $[s]$ integer and $0 < \sigma < 1$; let $p \geq 1$; F belongs to the space of Nikol'skij $H^{s,p}(\mathrm{R}^m)$ if $F \in L^p$ and for any multi-index γ such that $|\gamma| = [s]$ and $|h|$ small enough

$$\int |\partial^\gamma F(x+h) - \partial^\gamma F(x)|^p \, dx \leq C|h|^{\sigma p}. \tag{6}$$

Consider

$$\xi(p) = \sup\{s : F \in H^{s/p,p}\}.$$

Thus if $p \geq 1$ and $\zeta(p) < p$ then $\zeta(p) = \xi(p)$.

If $\zeta(p) \geq p$ then formula (5) must be modified as follows in order to be consistent with (6): if it is equal to p, one should use the same formula but with the gradient of F, and so on until $\zeta(p)$ falls between two integers multiplied by p.

$\xi(p)$ is also related to Besov's "smoothness" index. Let us recall that if ψ is a $C^k(\mathrm{R}^m)$ radial function with all moments of order less than k vanishing and all derivatives of order less than k are well localized and k large enough depending on the properties of F we want to analyze; then the wavelet transform of F at the position $b \in \mathrm{R}^m$ and the scale $a > 0$ is

$$C_{a,b}(F) = \frac{1}{a^m} \int_{\mathbf{R}^m} F(t)\psi\left(\frac{t-b}{a}\right) dt \ ; \tag{7}$$

Now, a function F belongs to the Besov space $B_p^{s,\infty}(\mathrm{R}^m)$ if (see [15]) its wavelet transform satisfies for a small enough

$$\int |C_{a,b}(F)|^p db \leq C a^{sp}. \tag{8}$$

And thanks to the imbeddings $H^{s+\epsilon,p}(\mathrm{R}^m) \hookrightarrow B_p^{s,\infty}(\mathrm{R}^m) \hookrightarrow H^{s-\epsilon,p}(\mathrm{R}^m)$, $\forall \ \epsilon > 0$, $p \geq 1$ and $s > 0$, we deduce that for $p \geq 1$

$$\xi(p) = \sup\{s \ : \ F \in B_p^{s/p,\infty}(\mathrm{R}^m)\} := \eta(p). \tag{9}$$

It is also well known (see [13]) that the Hölder regularity can be characterized in terms of estimates on the size of the wavelet transform. In fact we have:

- $F \in C^\alpha(\mathrm{R}^m)$ if and only if
$$|C_{a,b}(F)| \leq C a^\alpha.$$
- If $F \in C^\alpha(x_0)$, then
$$|C_{a,b}(F)| \leq C a^\alpha \left(1 + \frac{|b - x_0|}{a}\right)^\alpha. \tag{10}$$
- If (10) holds and if $F \in C^\varepsilon(\mathrm{R}^m)$ for an $\varepsilon > 0$, there exists a polynomial P such that if $| \ x - x_0 \ | \leq 1/2$,
$$|F(x) - P(x - x_0)| \leq C|x - x_0|^\alpha \log\left(\frac{1}{|x - x_0|}\right) \tag{11}$$
and so $F \in C^{\alpha - \varepsilon'}(x_0)$, $\quad \forall \varepsilon' > 0$.

The following formulas (the so-called multifractal formalism for functions) have been proposed for the computation of the spectrum of singularities $d(\alpha)$ (see [1] and [9])

$$d(\alpha) = \inf(\alpha p - \zeta(p) + m) \quad \text{or} \quad d(\alpha) = \inf(\alpha p - \xi(p) + m) \tag{12}$$

or

$$d(\alpha) = \inf(\alpha p - \eta(p) + m). \tag{13}$$

In the next section, we will prove the existence and uniqueness of the solution of equation (1) for non homogenous contractions and we compute its uniform regularity.

In the third section, we show that the previous relationships between the estimates on the size of the wavelet transform and the Hölder regularity are not compatible with non homogenous series: we obtain different lower and upper bound for the Hölder regularity for any non homogenous selfsimilar function. So we restrict to our couterexamples for the determination of the exact value of the Hölder regularity by estimating the increments of the function.

In the fourth section, we compute the spectrum of singularities for our couterexamples and we show that unlike the case of homogenous selfsimilar functions, the spectrum of singularities depends on the geometrical arrangement of the $S_j(\Omega)$.

In the fifth section, we compute $\zeta(p)$ and we prove that for our couterexamples, the multifractal formalism fails .

In the sixth section, we replace the Euclidean norm used in the definition of the Hölder regularity by another "norm" which will be compatible with the anisotropy, we make similar modifications for the notions that appear in the multifractal formalism and we give the characterizations of the modified Hölder regularity in termes of conditions on the size of an adapted wavelet transform.

Finally, in the seventh section, we prove the validity of the new multifractal formalism for a large class of non homogenous selfsimilar functions.

2 Anisotropic Selfsimilar Functions: existence, uniqueness and global Hölder regularity

For the convenience of the notations, we consider only the case $m = 2$ although the statements and proofs extend to the general case without any difficulties. Let s and t be two integers with $s < t$. We construct a kind of irregular Sierpinski carpet K as follows: we divide the unit square $\Re = [0,1]^2$ into a uniform grid of rectangles of height $1/t$ and width $1/s$, we choose a finite subset A of $\{0,1,\dots,s-1\} \times \{0,1,\dots,t-1\}$ and for each pair $\omega = (i,j) \in A$, we consider the affine map $S_\omega : \Re \to \Re$, given by $S_\omega(x_1,x_2) = \left(\frac{x_1}{s} + \frac{i}{s}, \frac{x_2}{t} + \frac{j}{t}\right)$ and mapping the unit square $\Re$ into the rectangle $\Re_\omega = [i/s,(i+1)/s] \times [j/t,(j+1)/t]$.

K will be the unique non-empty compact set (see [10]) satisfying

$$K = \bigcup_{\omega \in A} S_\omega(K). \tag{14}$$

We have

$$\begin{aligned} K &= \{x \in \Re \ : \ (S_{\omega_1} \circ \cdots \circ S_{\omega_n})^{-1}(x) \in \bigcup_{\omega \in A} \Re_\omega \quad \forall\, (\omega_1,\dots,\omega_n) \in A^n\} \\ &= \bigcap_{n=0}^{\infty} \Big(\bigcup_{|\omega|=n} \Re_\omega \Big) \end{aligned}$$

where

$$\Re_\omega = (S_{\omega_1} \circ \cdots \circ S_{\omega_n})(\Re) \quad \text{for} \quad \omega = (\omega_1,\dots,\omega_n) \ .$$

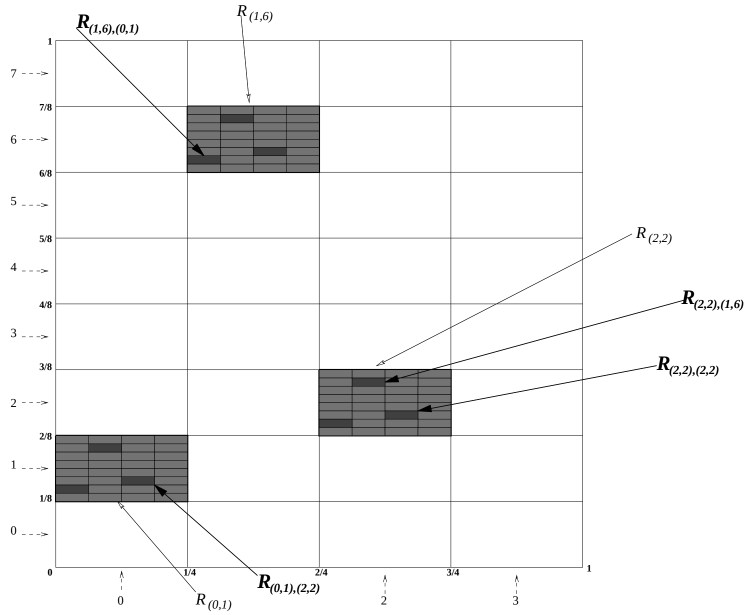

Figure 1: The two first steps of the construction of the Sierpinski Carpet associated to the subdivision $A = \{(0,1),(2,2),(1,6)\}$, $s = 4$ and $t = 8$

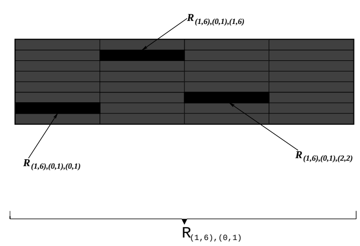

Figure 2: A zoom in for the third step of the construction of the Sierpinski Carpet associated to the subdivision $A = \{(0,1),(2,2),(1,6)\}$, $s = 4$ and $t = 8$

There is a natural onto application π from $A^{\mathbb{N}}$ to K given by

$$\begin{aligned} \pi(\omega_1, \ldots, \omega_n, \ldots) &= \lim_{n \mapsto \infty} S_{\omega_1} \circ \cdots \circ S_{\omega_n}(v) \quad \text{(for any } v \in \Re) \\ &= \bigcap_n \Re_{(\omega_1, \ldots, \omega_n)}. \end{aligned}$$

π will be a bijection in the case where the "separated open set condition"

$$\Re_\omega \cap \Re_{\omega'} = \emptyset \quad \text{if} \quad \omega \neq \omega' \tag{15}$$

holds.

Let g be a C^k function with all derivatives of order less than k well localized. We will call a "selfsimilar" function adapted to the subdivision A, a function F satisfying:

$$F(x) = \begin{cases} \lambda_\omega F(S_\omega^{-1}(x)) \;+\; g(x) & \text{if } x \in \Re_\omega \\ g(x) & \text{if } x \notin \bigcup_{\omega \in A} \Re_\omega \;. \end{cases}$$

With the conventions $F(T_\omega^{-1}(x)) = 0$ and $g(T_\omega^{-1}(x)) = 0$ for $x \notin \Re_\omega$, we can write

$$F(x) = \sum_{\omega \in A} \lambda_\omega F(S_\omega^{-1}(x)) \ + \ g(x). \tag{16}$$

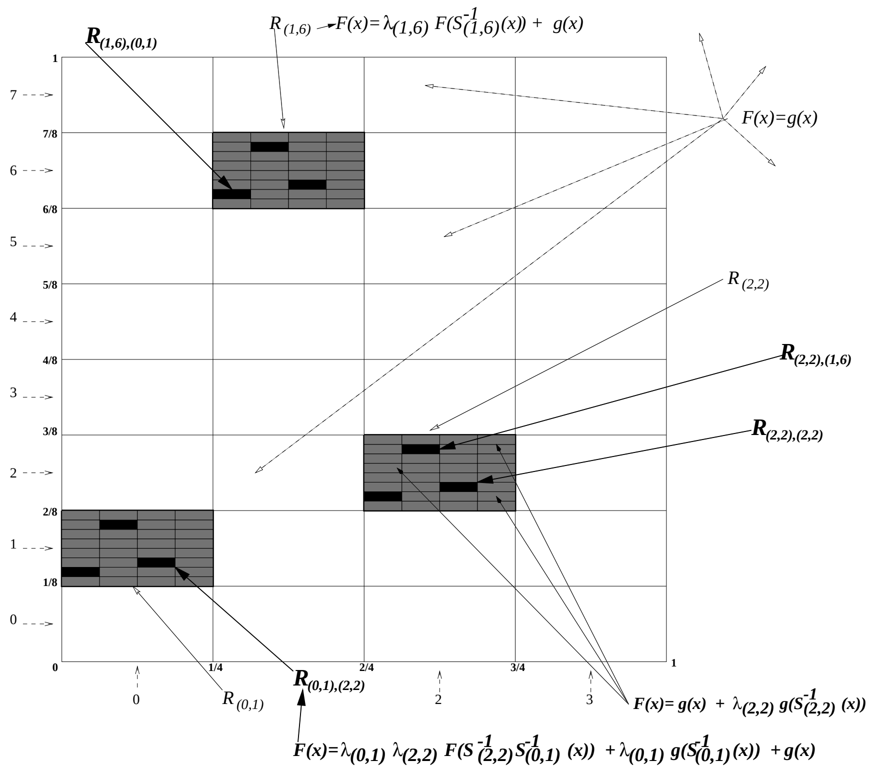

Figure 3: The two first steps of the construction of the "selfsimilar" function adapted to the subdivision $A = \{(0,1),(2,2),(1,6)\}$, $s = 4$ and $t = 8$

Remark: If $s = t$ then the maps S_ω are similitudes and this case was studied by Jaffard (see [12]) which proved the validity of the multifractal formalism for the associated selfsimilar functions.

Iterating (16), we obtain for any N:

$$\begin{aligned} F(x) \ &= \ \sum_{n=0}^{N-1} \sum_{(\omega_1,\ldots,\omega_n)} \lambda_{\omega_1} \cdots \lambda_{\omega_n} \ g\left(S_{\omega_n}^{-1} \cdots S_{\omega_1}^{-1}(x)\right) \\ &+ \quad \sum_{(\omega_1,\ldots,\omega_N)} \lambda_{\omega_1} \cdots \lambda_{\omega_N} \ F\left(S_{\omega_N}^{-1} \cdots S_{\omega_1}^{-1}(x)\right). \end{aligned} \tag{17}$$

We will now show the existence and the uniqueness in $L^1(\mathrm{R}^2)$ of the solution of (16) under some hypothesis on the λ_ω and then we will determine its global Hölder regularity.

Define

$$|\lambda|_{max} = max_{\omega \in A} |\lambda_\omega| \quad , \quad |\lambda|_{min} = min_{\omega \in A} |\lambda_\omega| \ ;$$

$$\alpha_{min} = -\frac{\log|\lambda|_{max}}{\log t} \quad \text{and} \quad \alpha_{max} = -\frac{\log|\lambda|_{min}}{\log t} \ .$$

Proposition 1 *Suppose that the "separated open set condition" (15) holds and that $\sum_{\omega\in A} \mid \lambda_\omega \mid < st$, then the functional equation (16) has a unique solution in $L^1(\mathrm{R}^2)$ given by the series*

$$F(x) = \sum_{n=0}^{\infty} \sum_{(\omega_1,\ldots,\omega_n)} \lambda_{\omega_1}\cdots\lambda_{\omega_n} \ g\left(S_{\omega_n}^{-1}\cdots S_{\omega_1}^{-1}(x)\right) . \tag{18}$$

If furthermore $t^{-k} < |\lambda|_{max} < 1$, then $F \in C^{\alpha_{min}}(\mathrm{R}^2)$.

Proof:
Distribution (18) verifies (16), its L^1 norm is bounded by

$$\begin{aligned}
& \sum_{n=0}^{\infty} \sum_{(\omega_1,\ldots,\omega_n)} \mid \lambda_{\omega_1}\cdots\lambda_{\omega_n} \mid \int \mid g(S^{-1}_{(\omega_1,\ldots,\omega_n)}(x)) \mid \ dx \\
= & \sum_{n=0}^{\infty} \sum_{(\omega_1,\ldots,\omega_n)} \mid \lambda_{\omega_1}\cdots\lambda_{\omega_n} \mid \int_{\Re_\omega} \mid g(S^{-1}_{(\omega_1,\ldots,\omega_n)}(x)) \mid \ dx \\
& \leq C \sum_{n=0}^{\infty} \sum_{(\omega_1,\ldots,\omega_n)} \mid \lambda_{\omega_1}\cdots\lambda_{\omega_n} \mid Area(\Re_{(\omega_1,\ldots,\omega_n)}) \\
& \leq C \sum_{n=0}^{\infty} s^{-n} \, t^{-n} \left(\sum_{\omega\in A} |\lambda_\omega|\right)^n \\
& \leq C'.
\end{aligned}$$

For the uniqueness of the solution of (16) in $L^1(\mathrm{R}^2)$, remark that if there was two solutions, it follows from the fact that (17) holds for any N that their difference is a distribution supported by K and is a solution of the homogeneous equation

$$F = \sum_{\omega\in A} \lambda_\omega F \circ S_\omega^{-1}. \tag{19}$$

But

$$\parallel F \circ S_\omega^{-1} \parallel_{L^1(\mathbf{R}^2)} = s^{-1} t^{-1} \parallel F \parallel_{L^1(\mathbf{R}^2)}$$

hence if $\sum_{\omega\in A} \mid \lambda_\omega \mid < st$, equation (19) has zero as a solution in $L^1(\mathrm{R}^2)$.

Let us now prove that $F \in C^{\alpha_{min}}(\mathrm{R}^2)$. For that we will use the Littlewood-Paley characterization. We split F as a sum

$$F(x) = \sum_{j\geq 0} F_j(x) \quad \text{where} \quad F_j(x) = \sum_{\omega\in A^j} \lambda_\omega g(S_\omega^{-1}(x)).$$

Let ψ be a function in the Schwartz class such that

$$\hat{\psi}(\xi) = 0 \quad \text{for} \quad \mid \xi \mid \leq 1 \quad \text{and} \quad \mid \xi \mid \geq 2s$$

$$\hat{\psi}(\xi) = 1 \quad \text{for} \quad 2 \leq \mid \xi \mid \leq s.$$

Set $\psi_l(x) = s^{2l}\psi(s^l x)$, $W_{l,j} = F_j * \psi_l$ and $h_{\omega,l} = (g \circ S_\omega^{-1}) * \psi_l$.
Recall that a function F belongs to $C^r(\mathrm{R}^2)$ if and only if

$$| F * \psi_l(x) | \leq C s^{-rl} \ \forall x \in \mathrm{R}^2.$$

We have

$$h_{\omega,l}(x) = s^{2l} \int g(S_\omega^{-1}(y)) \ \ \psi(s^l(x-y))\, dy.$$

Let $Pg_x(y)$ be the Taylor expansion of g at the order $k-1$ at the point x, i.e

$$Pg_x(y) = \sum_{|\gamma| \leq k-1} \frac{\partial^\gamma g(x)}{\gamma!} y^\gamma.$$

It follows from the cancellation of ψ and the fact that S_ω^{-1} is affine in each direction that for $\omega \in A^j$

$$h_{\omega,l}(x) = s^{2l} \int_{\mathbf{R}^2} \left(g(S_\omega^{-1}(y)) - P(g \circ S_\omega^{-1})_x(y-x)\right) \ \ \psi(s^l(x-y))\, dy$$

thus using the mean value theorem and the localization of g, we obtain

$$|h_{\omega,l}(x)| \quad \leq \quad \frac{C_N\, s^{2l} t^{kj}}{(1+|S_\omega^{-1}(x)|)^N} \int |x-y|^k \ |\psi(s^l(x-y))| \ (1 + s^j|x_1 - y_1| + t^j|x_2 - y_2|)^N\, dy$$

hence for $j \leq \sigma l$ with $\sigma = \log s / \log t$

$$|h_{\omega,l}(x)| \leq C_N \frac{s^{-kl} t^{kj}}{(1+|S_\omega^{-1}(x)|)^N} \ .$$

Thus for $j \leq \sigma l$

$$|W_{l,j}(x)| \leq C_N |\lambda|^j_{max} s^{-kl} t^{kj} \sum_{\omega \in A^j} \frac{1}{\left(1 + s^j|x_1 - (S_\omega(0))_1| + t^j|x_2 - (S_\omega(0))_2|\right)^N}$$

where $(S_\omega(0))_1$ and $(S_\omega(0))_2$ are the coordinates of $S_\omega(0)$.
We have the following lemma

Lemma 1 *For N large enough, there exists $C_N > 0$ such that for any $x \in \mathrm{R}^2$*

$$\sum_{\omega \in A^j} \frac{1}{\left(1 + s^j|x_1 - (S_\omega(0))_1| + t^j|x_2 - (S_\omega(0))_2|\right)^N} < C_N.$$

Lemma 1 is a consequence of the following one

Lemma 2 *Let $x \in K$ and D large enough and denote by $B_{j,D}(x)$ the set of $\omega \in A^j$ such that*

$$|x_1 - (S_\omega(0))_1| \leq D s^{-j}$$

and

$$|x_2 - (S_\omega(0))_2| \leq D t^{-j}.$$

The cardinality of $B_{j,D}(x)$ is bounded independantely of x and j by $4D^2$.

Proof:
The $\Re_\omega$ for $\omega \in B_{j,D}(x)$ are disjoints, thus they are all included in the rectangle

$$R_j = [x_1 - Ds^{-j}, x_1 + Ds^{-j}] \times [x_2 - Dt^{-j}, x_2 + Dt^{-j}]$$

hence

$$s^{-j}t^{-j} \; card \; B_{j,D}(x) \; \leq 4D^2 s^{-j} t^{-j}$$

whence Lemma 2.

Thanks to Lemma 1, we get for $0 \leq j \leq \sigma l$

$$|W_{l,j}(x)| \leq C_N s^{-kl} t^{kj} |\lambda|_{max}^j \,.$$

Hypothesis $t^{-k} < |\lambda|_{max}$ implies that

$$\begin{aligned} \sum_{0 \leq j \leq \sigma l} |W_{l,j}(x)| &\leq C s^{-kl} t^{k\sigma l} |\lambda|_{max}^{\sigma l} \\ &= C|\lambda|_{max}^{\sigma l} \\ &= C s^{-l\alpha_{min}}. \end{aligned}$$

On the other hand, for $j \geq \sigma l$

$$\begin{aligned} |W_{l,j}(x)| &\leq C \sup_x |F_j(x)| \\ &\leq C|\lambda|_{max}^j \end{aligned}$$

consequently

$$\begin{aligned} \sum_{j \geq \sigma l} |W_{l,j}(x)| &\leq C \sum_{j \geq \sigma l} |\lambda|_{max}^j \\ &\leq C|\lambda|_{max}^{\sigma l} \\ &= C s^{-l\alpha_{min}}. \end{aligned}$$

Hence

$$| \; F * \psi_l(x) \; | \leq C s^{-l\alpha_{min}} \;\; \forall x \in \mathrm{R}^2. \tag{20}$$

Whence Proposition 1.

3 Pointwise Hölder regularity

We want to estimate the Hölder regularity of F at every point.

Proposition 2 *If $x \notin K$ then F is C^k in a neighbourhood of x.*

Proof:
Let $x_0 \notin K$, if $x_0 \notin \Re$ then $F = g$ in a neighbourhood of x_0.
If $x_0 \in \Re$ then there exist N and $\omega = (\omega_1, \ldots, \omega_N) \in A^N$ such that $x_0 \in \Re_\omega \setminus \bigcup_{\omega' \in A} \Re_{\omega\omega'}$; in this neighbourhood of x_0, $F(x) = \sum_{n=0}^N \lambda_{\omega_1} \cdots \lambda_{\omega_n} \; g((S_{\omega_1} \circ \cdots \circ S_{\omega_n})^{-1}(x)) \in C^k(x)$.

Let us now compute the Hölder exponent $\alpha(x)$ of F at each point x of K; recall that

$$\alpha(x) = \sup\{\beta : \quad F \in C^{\beta}(x)\}. \tag{21}$$

For that, we will assume the "separated open set condition" (15) for the subdivision A. Define for $x \in K,\ \omega(= \omega(x)) \in A^{\mathbb{N}}$ by $\omega = \pi^{-1}(x)$.
If $\quad \omega = (\omega_1, \omega_2, \ldots, \omega_n, \ldots)$ with $\omega_l = (i_l, j_l) \in A \quad$ then $\quad x = \left(\sum_{l=1}^{\infty} \frac{i_l}{s^l}, \sum_{l=1}^{\infty} \frac{j_l}{t^l}\right)$.

For notational convenience, set

$$\omega(n,x) = (\omega_1, \ldots, \omega_n)\ ;\ \lambda_{\omega(n,x)} = \lambda_{\omega_1} \cdots \lambda_{\omega_n}$$

and

$$S_{\omega(n,x)} = S_{\omega_1} \circ \cdots \circ S_{\omega_n}.$$

Now let

$$a_n(x) = \frac{\log|\lambda_{\omega(n,x)}|}{\log t^{-n}}$$

and

$$a(x) = \liminf_{n \mapsto \infty} a_n(x).$$

Proposition 3 *Let F be a "selfsimilar" function adapted to a subdivision A satisfying the "separated open set condition". If $x \in K$ and $a(x)$ is not integer, then*

$$\alpha(x) \geq a(x)\ .$$

Proof:
Let $\epsilon > 0$, there exists n_0 so that $\ a_n(x) > a(x) - \epsilon \ $ for all $n \geq n_0$, implying

$$|\lambda_{\omega(n,x)}| \leq t^{-n(a(x)-\epsilon)}.$$

Let $h \in \mathbb{R}^2$, $|h| < t^{-n_0}$ and $n \in \mathbb{N}$ such that $t^{-n-2} \leq |h| < t^{-n-1}$, then thanks to the "separated open set condition"

$$\omega(n, x+h) = \omega(n,x).$$

Let $P_a g_x(h)$ be the Taylor expansion of g at the order $a = [a(x)]$ at the point x (where the notation [ ] denotes the integer part), i.e

$$P_a g_x(h) = \sum_{|\gamma| \leq a} \frac{\partial^{\gamma} g(x)}{\gamma!} h^{\gamma}.$$

Consider

$$PF_x(h) = \sum_{l=0}^{\infty} P_a(g \circ S^{-1}_{\omega(l,x)})_x(h)\ ;$$

$PF_x(h)$ is well defined because of the localization of the function g and all its derivatives of order less than k.

Using (17), we obtain

$$
\begin{aligned}
F(x+h) - PF_x(h) &= \sum_{l=0}^{n-1} \lambda_{\omega(l,x)} \; [g\left(S^{-1}_{\omega(l,x)}(x+h)\right) - P_a(g \circ S^{-1}_{\omega(l,x)})_x(h)] \\
&+ \lambda_{\omega(n,x)} \; F\left(S^{-1}_{\omega(n,x)}(x+h)\right) \\
&- \sum_{l \geq n} \lambda_{\omega(l,x)} \; P_a(g \circ S^{-1}_{\omega(l,x)})_x(h).
\end{aligned}
$$

It follows from the mean value theorem that the first term is in modulous bounded by

$$
\begin{aligned}
C \sum_{l=0}^{n-1} |\lambda_{\omega(l,x)}| \; t^{l(a+1)} \, |h|^{a+1} &\leq C|h|^{a+1} \sum_{l=0}^{n_0-1} |\lambda_{\omega(l,x)}| \; t^{l(a+1)} + C|h|^{a+1} \sum_{l=n_0}^{n-1} |\lambda_{\omega(l,x)}| \; t^{l(a+1)} \\
&\leq C'|h|^{a+1} + C|h|^{a+1} \sum_{l=n_0}^{n-1} t^{-l(a(x)-\epsilon)} \, t^{l(a+1)} \\
&\leq C'|h|^{a+1} + C|h|^{a+1} \, t^{n(a+1-a(x)+\epsilon)} \\
&\leq C''|h|^{a(x)-\epsilon}.
\end{aligned}
$$

Thanks to the boundedness of F, the second term will be bounded by $C|\lambda_{\omega(n,x)}|$, so by $Ct^{-n(a(x)-\epsilon)}$ i.e by $C|h|^{a(x)-\epsilon}$.

And the third term is bounded by

$$
\begin{aligned}
\sum_{l \geq n} |\lambda_{\omega(l,x)}| \sum_{|\gamma| \leq a} t^{l|\gamma|} \, |h|^{|\gamma|} &\leq \sum_{l \geq n} |\lambda_{\omega(l,x)}| \; t^{la} \, |h|^{a} \\
&\leq C|h|^{a} \sum_{l \geq n} t^{-l(a(x)-\epsilon)} \; t^{la}.
\end{aligned}
$$

which is bounded by $C'|h|^{a(x)-\epsilon}$ for $0 < \epsilon < a(x) - a$, $(a(x) > a$ because $a(x)$ is not integer).

Whence Proposition 3.

We shall now give an upper bound for the pointwise regularity $\alpha(x)$ of the "selfsimilar" function adapted to the subdivision A. Unfortunately, we can easily show that there are not "good" relationships between the regularity of such functions and the size of the wavelet transform, the reason is that unlike the wavelet transform which is homogenous in frequency, the contractions which appear in the non homogenous selfsimilar function contract with different rates in each direction. The two-microlocalization condition (10) gives up only $\alpha(x) \leq \sigma^{-1} a(x)$ which is much larger than the lower bound $a(x)$. Thus, the only method to determine the exact value of the pointwise Hölder regularity is to use Definition 1. Obviously, this argument is not easy, so we will restrict to our couterexamples for the multifractal formalism. We take $g(x) = \Lambda(x_1)\Lambda(x_2)$ with $\Lambda(u) = min(u, 1-u)$ if $u \in [0,1]$ and 0 else. Here g is C^1. We suppose that the λ_ω are positive and that

$$\forall \omega \in A\,, \quad \Re_\omega \subset [1/s, (s-1)/s] \times [1/t, 1/2] \tag{22}$$

or

$$\forall \omega \in A\,, \quad \Re_\omega \subset [1/s, (s-1)/s] \times [1/2, (t-1)/t]. \tag{23}$$

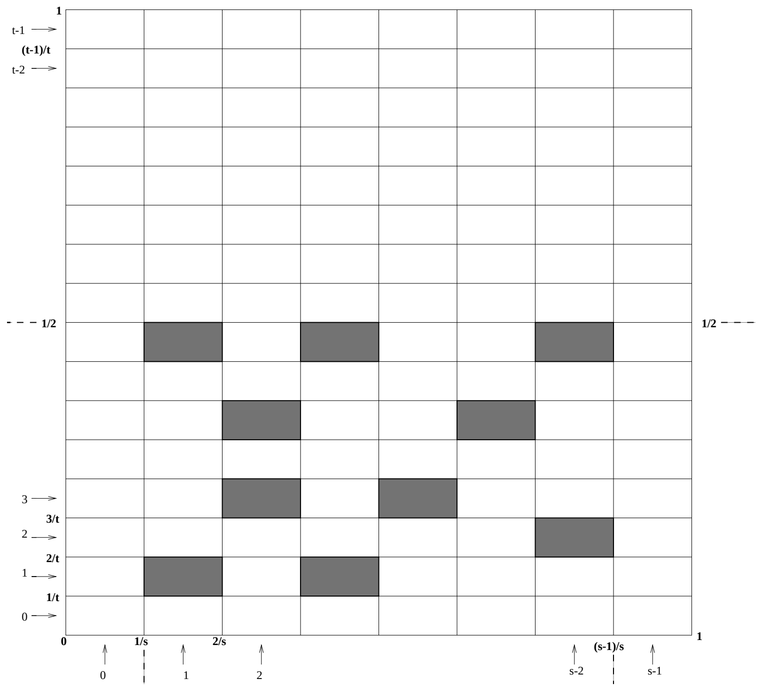

Figure 4: The condition (22) for the construction of the "selfsimilar" function

Remark: in the sixth section, we will prove that if we take anisotropic contractions in the wavelet transform, then we can find good relationships between estimates of the size of the wavelet transform and the "anisotropic Hölder regularity".

Proposition 4 *Let F be the L^1 solution of*

$$F(x) = \sum_{\omega \in A} \lambda_\omega F(S_\omega^{-1}(x)) \ + \ \Lambda(x_1)\Lambda(x_2)$$

with $\sum_{\omega \in A} |\lambda_\omega| < st$ and the asumptions (15) and (22) or (23). Then for $x \in K$ and $a(x) \leq 1$, we have

$$\alpha(x) \leq a(x).$$

Proof:
In the case of the asumption (22), we choose $h_n = (0, -t^{-(n+1)})$ with n large enough so that $\lambda_{\omega(n,x)} \geq t^{-n(a(x)+\delta)}$.
Since $\omega(l, x + h_n) = \omega(l, x)$ for any $l = 1, \ldots, n$ then

$$\begin{aligned} F(x+h_n) - F(x) &= \sum_{l=0}^{n-1} \lambda_{\omega(l,x)} \ [g\left(S^{-1}_{\omega(l,x)}(x+h_n)\right) - g\left(S^{-1}_{\omega(l,x)}(x)\right)] \\ &+ \ \lambda_{\omega(n,x)} \ [F\left(S^{-1}_{\omega(n,x)}(x+h_n)\right) - F\left(S^{-1}_{\omega(n,x)}(x)\right)]. \end{aligned}$$

Set $y_l = S^{-1}_{\omega(l,x)}(x) = (y_{l,1}, y_{l,2})$ for $l = 0, \ldots, n$. We have

$$y_l \in \Re_{\omega_{l+1}(x)} \subset [1/s, (s-1)/s] \times [1/2, (t-1)/t]$$

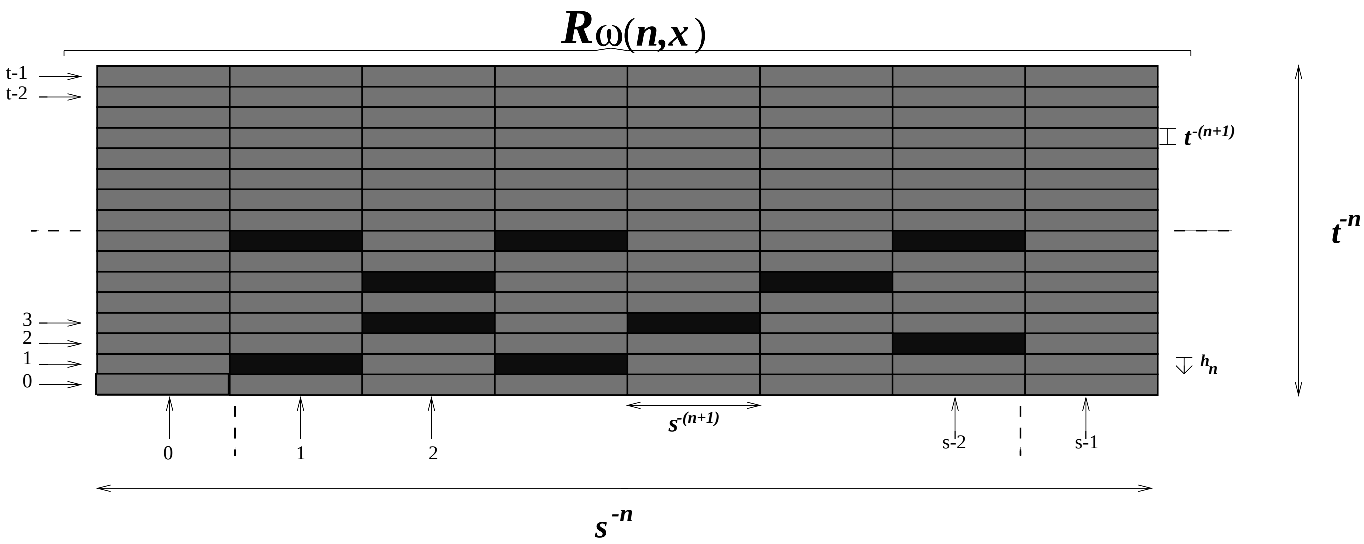

Figure 5: A geometric justification for the property $\omega(n, x + h_n) = \omega(n, x)$

and

$$S^{-1}_{\omega(l,x)}(x + h_n) = y_l + (0, -t^{l-(n+1)});$$

so for $0 \leq l \leq n-1$

$$\begin{aligned} g\left(S^{-1}_{\omega(l,x)}(x+h_n)\right) - g\left(S^{-1}_{\omega(l,x)}(x)\right) &= g(y_l + (0, -t^{l-(n+1)})) - g(y_l) \\ &= \Lambda(y_{l,1})\left(\Lambda(y_{l,2} - t^{l-(n+1)}) - \Lambda(y_{l,2})\right) \\ &= \Lambda(y_{l,1})\left(y_{l,2} - t^{l-(n+1)} - y_{l,2}\right) \quad \text{(because of (22))} \\ &= -t^{l-(n+1)}\Lambda(y_{l,1}). \end{aligned}$$

$y_n + (0, -t^{-1}) \notin \bigcup_{\omega \in A} \Re_\omega$ thus $\quad F(y_n + (0, -t^{-1})) = g((y_n + (0, -t^{-1}))$ and from (16)

$$F(y_n) = \lambda_{\omega_{n+1}(x)} F(y_{n+1}) + g(y_n).$$

Thus

$$\begin{aligned} F\left(S^{-1}_{\omega(n,x)}(x+h_n)\right) - F\left(S^{-1}_{\omega(n,x)}(x)\right) &= F(y_n + (0, -t^{-1})) - F(y_n) \\ &= \Lambda(y_{n,1})\left(\Lambda(y_{n,2} - t^{-1}) - \Lambda(y_{n,2})\right) - \lambda_{\omega_{n+1}(x)} F(y_{n+1}) \\ &= -t^{-1}\,\Lambda(y_{n,1}) - \lambda_{\omega_{n+1}(x)} F(y_{n+1}). \end{aligned}$$

And since F is positive then

$$\begin{aligned} |F(x+h_n) - F(x)| &= \sum_{l=0}^{n-1} \lambda_{\omega(l,x)}\,\Lambda(y_{l,1})\,t^{l-(n+1)} \\ &+ \lambda_{\omega(n,x)}\,t^{-1}\,\left(\Lambda(y_{n,1}) + \lambda_{\omega_{n+1}(x)} F(y_{n+1})\right) \end{aligned}$$

but $\Lambda(y_{n,1}) \geq 1/s$, thus

$$\begin{aligned} |F(x+h_n) - F(x)| &\geq s^{-1}t^{-1}\lambda_{\omega(n,x)} \\ &\geq Ct^{-n(a(x)+\delta)} \\ &\geq C|h_n|^{a(x)+\delta}. \end{aligned}$$

In the case of the asumption (23), we choose $h_n = (0, t^{-(n+1)})$ and the proof is identical.

The lower and upper bounds for $\alpha(x)$ yield the following theorem

Theorem 1 *Let F be the L^1 solution of*

$$F(x) = \sum_{\omega \in A} \lambda_\omega F(S_\omega^{-1}(x)) \ + \ \Lambda(x_1)\Lambda(x_2)$$

with $\sum_{\omega \in A} |\lambda_\omega| < st$ and the asumptions (15) and (22) or (23). Then for $x \in K$ and $a(x) < 1$

$$\alpha(x) = a(x).$$

4 The spectrum of singularities

We want now to determine the Hausdorff dimension of the set of points x where $\alpha(x)$ is equal to a given $0 < \alpha < 1$.
For technical reasons, we shall assume another separation condition

$$\text{if} \quad \omega = (i,j) \in A \quad \text{then} \quad (i \pm 1, j) \notin A. \tag{24}$$

This condition requires that if column i of the grid contains points of K, the two adjacent columns do not.

On the sets of singularities E^α, we will concentrate a suitable family of probability measures with certain scaling properties and then use the Lemma below (see [7]) to estimate the dimension of these sets: each measure gives us an upper bound and one of them will gives the equality.

Lemma 3 *Let H^s be the Hausdorff measure of dimension s. Let μ be a probability measure on $\mathbb{R}^m$, $E \subset \mathbb{R}^m$ and C such that $0 < C < \infty$*

- *If* $\limsup_{r \to 0} \dfrac{\mu(B(x,r))}{r^s} < C \quad \forall x \in E$ *then* $H^s(E) \geq \frac{\mu(E)}{C}$.
- *If* $\limsup_{r \to 0} \dfrac{\mu(B(x,r))}{r^s} > C \quad \forall x \in E$ *then* $H^s(E) \leq \frac{2^s}{C}$.

For $q \in \mathbb{R}$, define $\tau(q)$ by $\sum_{\omega \in A} \lambda_\omega^q s^{\tau(q)} = 1$; i.e $\tau(q) = -\log(\sum_{\omega \in A} \lambda_\omega^q)/\log s$.
Set $P_\omega(q) = \lambda_\omega^q s^{\tau(q)}$ and let μ_q be a probability measure on K such that

$$\forall (\omega_1, \ldots, \omega_n) \in A^{\mathbb{N}}, \ \mu_q(\Re_{\omega_1,\ldots,\omega_n}) = P_{\omega_1}(q) \ldots P_{\omega_n}(q).$$

The construction of such measure by induction is straightforward (see [10], [14] or [16]).

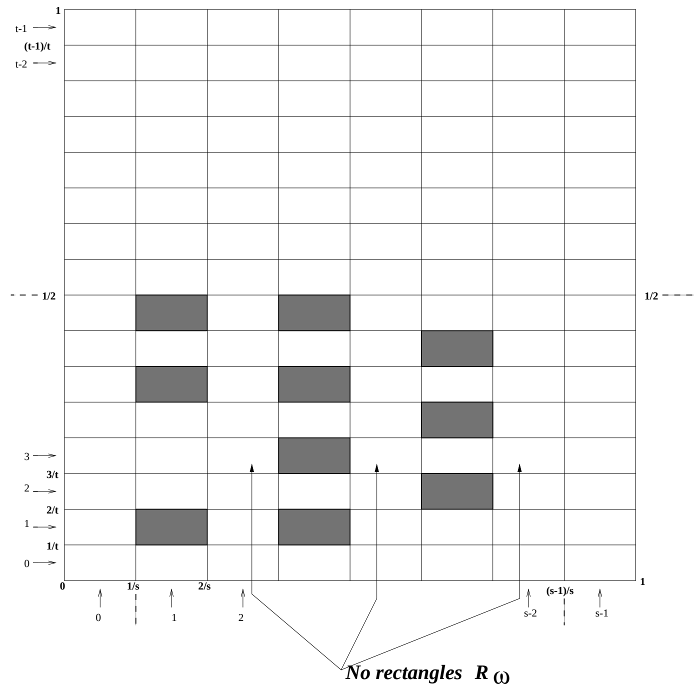

Figure 6: The condition (24) for the construction of the "selfsimilar" function

For $r > 0$ and $\omega = (\omega_1, \ldots, \omega_n, \ldots) \in A^{\mathbb{N}}$ with $\omega_n = (i_n, j_n) \in A$, define the approximate square $Q(\omega, r)$ with approximate side r by

$$\begin{aligned} Q(\omega, r) \quad &= \quad \left[\frac{i_1}{s} + \cdots + \frac{i_{k_1(r)}}{s^{k_1(r)}} \ , \ \frac{i_1}{s} + \cdots + \frac{i_{k_1(r)}}{s^{k_1(r)}} + \frac{1}{s^{k_1(r)}} \right] \\ & \qquad \times \left[\frac{j_1}{t} + \cdots + \frac{j_{k_2(r)}}{t^{k_2(r)}} \ , \ \frac{j_1}{t} + \cdots + \frac{j_{k_2(r)}}{t^{k_2(r)}} + \frac{1}{t^{k_2(r)}} \right] \end{aligned}$$

where $k_1(r)$ and $k_2(r)$ are the unique integers such that

$$s^{-(k_1(r)+1)} < r \leq s^{-k_1(r)}$$

and

$$t^{-(k_2(r)+1)} < r \leq t^{-k_2(r)}.$$

In [16], we have

$$\mu_q(Q(\omega, r)) = \prod_{l=1}^{k_1(r)} [\sum_{(i_l, j) \in A} P_{(i_l, j)}(q)] \prod_{l=1}^{k_2(r)} \frac{P_{\omega_l}(q)}{\sum_{(i_l, j) \in A} P_{(i_l, j)}(q)} \ . \tag{25}$$

By considering the two cases below, we will show that the spectrum of singularities depends on the geometrical arrangement of the $\Re_\omega$, $\omega \in A$.

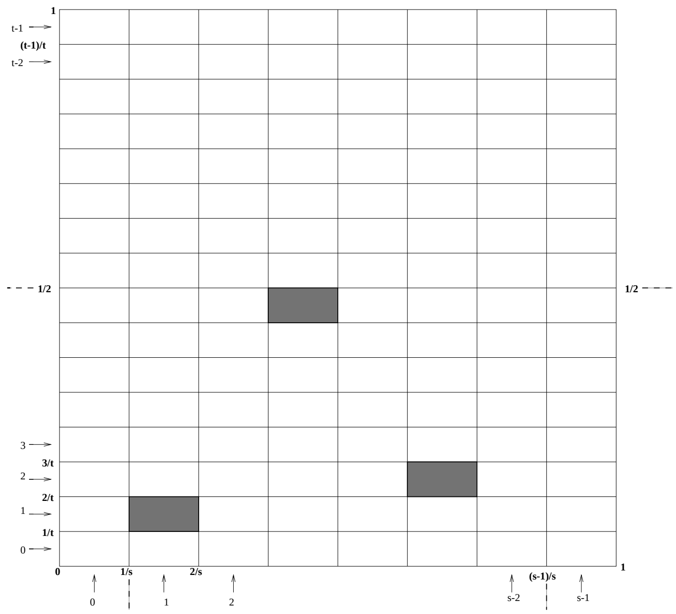

Figure 7: First case: each column of the grid contains at most one $\Re_\omega$

4.1 First case: each column of the grid contains at most one $\Re_\omega$

Proposition 5 *Assume that each column of the grid contains at most one* $\Re_\omega$ *.*
Let $\alpha < 1$ *and* $d(\alpha)$ *be the Hausdorff dimension of the set* E^α *of points* x *where* $\alpha(x) = \alpha$*.*
Then $d(\alpha)$ *is concave, equals to* $-\infty$ *outside* $[\alpha_{min}, \alpha_{max}]$ *and on this interval*

$$d(\alpha) = \inf_q (q\sigma^{-1}\alpha - \tau(q))$$

and it is analytic.

Proof:
In this case (25) implies that for $\omega = (\omega_1, \ldots, \omega_n, \ldots) \in A^{\mathbb{N}}$ and $r > 0$

$$\begin{aligned} \mu_q(Q(\omega, r)) &= \prod_{l=1}^{k_1(r)} P_{\omega_l}(q) \\ &= s^{k_1(r)\tau(q)} \prod_{l=1}^{k_1(r)} \lambda_{\omega_l}^q . \end{aligned} \tag{26}$$

Thus

$$\frac{\log \mu_q(Q(\omega, r))}{\log r} = \tau(q)\ \frac{k_1(r)}{\log r}\ \log s +\ q\ \frac{\log \prod_{l=1}^{k_1(r)} \lambda_{\omega_l}}{\log r} .$$

Since $\frac{k_1(r)}{\log r} \log s \mapsto -1$ as $r \searrow 0$, then

$$\liminf_{r \searrow 0} \frac{\log \mu_q(Q(\omega, r))}{\log r} = -\tau(q) + q\sigma^{-1} a(\pi(\omega)).$$

Denote by $B(x,r)$ the ball of center x and diameter $2r$. In [14] and [16], the following lemma was proved

Lemma 4 *If $\omega \in A^{\mathbb{N}}$ and $n \in \mathbb{N}$, then*

$$B(\pi(\omega), s^{-n}) \cap K \subset Q(\omega, s^{-n}) \subset B(\pi(\omega), (s+t)s^{-n}).$$

Thanks to Lemma 4

$$\liminf_{r \searrow 0} \frac{\log \mu_q(B(\pi(\omega), r))}{\log r} = \liminf_{r \searrow 0} \frac{\log \mu_q(Q(\omega, r))}{\log r} \tag{27}$$

whence

$$\limsup_{r \searrow 0} \frac{\mu_q(B(\pi(\omega), r))}{r^{-\tau(q)+q\sigma^{-1}a(\pi(\omega))+\epsilon}} = +\infty \quad \forall \epsilon > 0. \tag{28}$$

Let $E^\alpha = \{x : \alpha(x) = \alpha\}$; We can assume that $\alpha_{max} \leq 1$, so Theorem 1 implies that for $\alpha < 1$, $E^\alpha = \{\pi(\omega) : a(\pi(\omega)) = \alpha\}$. Equation (28) and the second part of Lemma 3 imply that

$$d(\alpha) \leq q\sigma^{-1}\alpha - \tau(q) \quad \forall q \in \mathrm{R}$$

so

$$d(\alpha) \leq \inf_q (q\sigma^{-1}\alpha - \tau(q)).$$

We will now prove that the previous infimum is reached. For that we will look for the good measure that will gives the equality.

We can easily show that $\tau(q)$ is strictly concave and analytic, so for $\beta \in]-\frac{\log \lambda_{max}}{\log s}, -\frac{\log \lambda_{min}}{\log s}[$, there exists a unique $q \in \mathrm{R}$ such that $\beta = \tau'(q)$. Hence for $\alpha \in]\alpha_{min}, \alpha_{max}[$ there exists a unique $q \in \mathrm{R}$ such that $\sigma^{-1}\alpha = \tau'(q)$.

With the probability $\tilde{\mu}_q = \mu_q \circ \pi$, the $X_j = \log P_{\omega_j}(q)$ are a sequence of i.i.d random variables; the strong law of large number implies that for $\tilde{\mu}_q - a.a\ \omega = (\omega_1, \ldots, \omega_n, \ldots) \in A^{\mathbb{N}}$

$$\begin{aligned} \frac{1}{n}\sum_{j=1}^{n} \log P_{\omega_j}(q) &\mapsto \sum_{\omega' \in A} P_{\omega'}(q)\ \log P_{\omega'}(q) \quad as\ n \mapsto \infty \\ &= \tau(q)\log s - q\tau'(q)\log s \end{aligned} \tag{29}$$

because

$$\begin{aligned} \sum_{\omega' \in A} P_{\omega'}(q)\ \log P_{\omega'}(q) &= \sum_{\omega' \in A} \lambda_{\omega'}^q s^{\tau(q)}\ (q \log \lambda_{\omega'} + \tau(q)\log s) \\ &= \tau(q)\log s + q\frac{\sum_{\omega' \in A} \lambda_{\omega'}^q \log \lambda_{\omega'}}{\sum_{\omega' \in A} \lambda_{\omega'}^q} \\ &= \tau(q)\log s - q\tau'(q)\log s. \end{aligned}$$

Thus for $\tilde{\mu}_q - a.a\ \omega \in A^{\mathbb{N}}$

$$\frac{\log \mu_q(Q(\omega, r))}{\log r} = \frac{k_1(r)}{\log r}\frac{1}{k_1(r)}\sum_{j=1}^{k_1(r)} \log P_{\omega_j}(q) \mapsto q\sigma^{-1}\alpha - \tau(q) \quad as\ r \searrow 0.$$

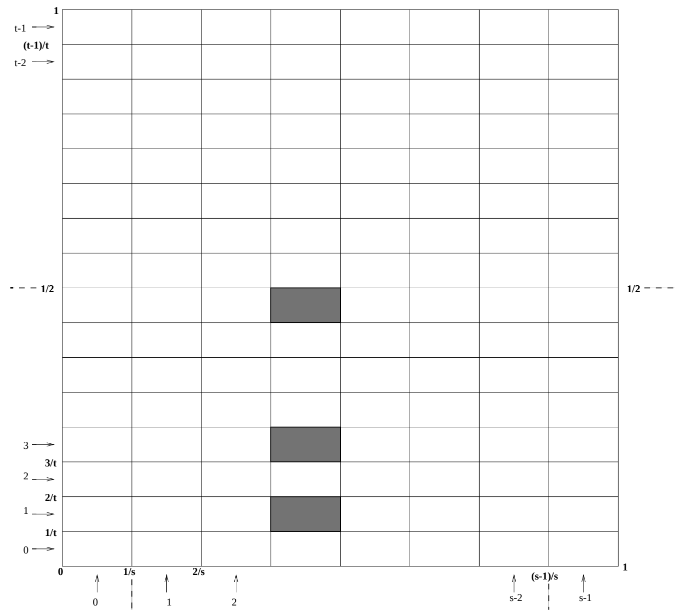

Figure 8: Second case: only one column containing all the $\Re_\omega$, $\omega \in A$

It follows from Lemma 4 that for $\tilde{\mu}_q - a.a$ $\omega \in A^{\mathbb{N}}$

$$\lim_{r \searrow 0} \frac{\log \mu_q(B(\pi(\omega), r))}{\log r} = q\sigma^{-1}\alpha - \tau(q). \tag{30}$$

Take $E = \{\pi(\omega) \ : \ \lim_{r\searrow 0} \frac{\log \mu_q(B(\pi(\omega),r))}{\log r} = q\sigma^{-1}\alpha - \tau(q)\}$; then (30) implies that $\mu_q(E) = 1$ and Lemma 3 yields that the Hausdorff dimension of E is equal to $q\sigma^{-1}\alpha - \tau(q)$. But for $\alpha < 1$, $\alpha \in]\alpha_{min}, \alpha_{max}[$, $E \subset E^\alpha$, whence $d(\alpha) \geq q\sigma^{-1}\alpha - \tau(q)$.

We conclude that $d(\alpha) = \inf_q (q\sigma^{-1}\alpha - \tau(q))$.

4.2 Second case: only one column containing all the $\Re_\omega$, $\omega \in A$

Proposition 6 *Assume that there is only one column containing all the $\Re_\omega$, $\omega \in A$. Then for $\alpha < 1$, $d(\alpha)$ is concave, equals to $-\infty$ outside $[\alpha_{min}, \alpha_{max}]$ and on this interval*

$$d(\alpha) = \inf_q (q\alpha - \sigma\tau(q))$$

and it is analytic.

Proof:

In this case, we have

$$\begin{aligned} \mu_q(Q(\omega, r)) &= \prod_{l=1}^{k_2(r)} P_{\omega_l}(q) \\ &= s^{k_2(r)\tau(q)} \prod_{l=1}^{k_2(r)} \lambda_{\omega_l}^q. \end{aligned} \tag{31}$$

Thus
$$\frac{\log \mu_q(Q(\omega,r))}{\log r} = \tau(q)\ \frac{k_2(r)}{\log r}\ \log s\ +\ q\,\frac{\log \prod_{l=1}^{k_2(r)} \lambda_{\omega_l}}{\log r}.$$
Since $\frac{k_2(r)}{\log r}\log t \mapsto -1$ as $r \searrow 0$, then
$$\liminf_{r\searrow 0} \frac{\log \mu_q(Q(\omega,r))}{\log r} = -\sigma\tau(q) + qa(\pi(\omega))$$
and thanks to Lemma 4
$$\liminf_{r\searrow 0} \frac{\log \mu_q(B(\pi(\omega),r))}{\log r} = \liminf_{r\searrow 0} \frac{\log \mu_q(Q(\omega,r))}{\log r} \tag{32}$$
whence
$$\limsup_{r\searrow 0} \frac{\mu_q(B(\pi(\omega),r))}{r^{-\sigma\tau(q)+qa(\pi(\omega))+\epsilon}} = +\infty \quad \forall \epsilon > 0. \tag{33}$$
It follows from Lemma 4 and (33) that
$$d(\alpha) \leq q\alpha - \sigma\tau(q) \quad \forall q \in \mathrm{R}$$
i.e
$$d(\alpha) \leq \inf_q (q\alpha - \sigma\tau(q)).$$

On the other hand, for $\alpha < 1$, $\alpha \in]\alpha_{min}, \alpha_{max}[$, there exists a unique $q \in \mathrm{R}$ such that $\alpha = \sigma\tau'(q)$, thus for $\tilde{\mu}_q - a.a\ \omega \in A^{\mathbb{N}}$
$$\frac{\log \mu_q(Q(\omega,r))}{\log r} = \frac{k_2(r)}{\log r}\frac{1}{k_2(r)}\sum_{j=1}^{k_2(r)} \log P_{\omega_j}(q) \mapsto q\alpha - \sigma\tau(q) \quad as\ r \searrow 0$$
whence by argument similar to the one of the first case, we deduce that
$$d(\alpha) = \inf_q (q\alpha - \sigma\tau(q)).$$

Remark: The spectrum of singularities of the second case is different from the one of the first case, whereas for an homogenous selfsimilar function (i.e if $s = t$) the spectrum of singularity and the L^p-mean Hölder index $\zeta(p)$ are the same in the two cases; this means that they don't depend on the choice of the $\Re_\omega$. We will now prove that unlike $d(\alpha)$, the L^p-mean Hölder index $\zeta(p)$ does not depend on the geometrical arrangement of the chosen $\Re_\omega$ and so the multifractal formalism will fail.

5 The failure of the Multifractal Formalism

We will now prove that the equivalent formulas (12) and (13) for $\alpha < 1$, that have been proposed for the computation of the spectrum of singularities $d(\alpha)$ fail for the two previous cases.

In order to compute $\zeta(q) = \liminf_{|h|\mapsto 0} \frac{\log \int |F(x+h)-F(x)|^q\,dx}{\log |h|}$, we need to find good upper and lower bounds for $S_p(h) = \int |F(x+h) - F(x)|^p\,dx$.

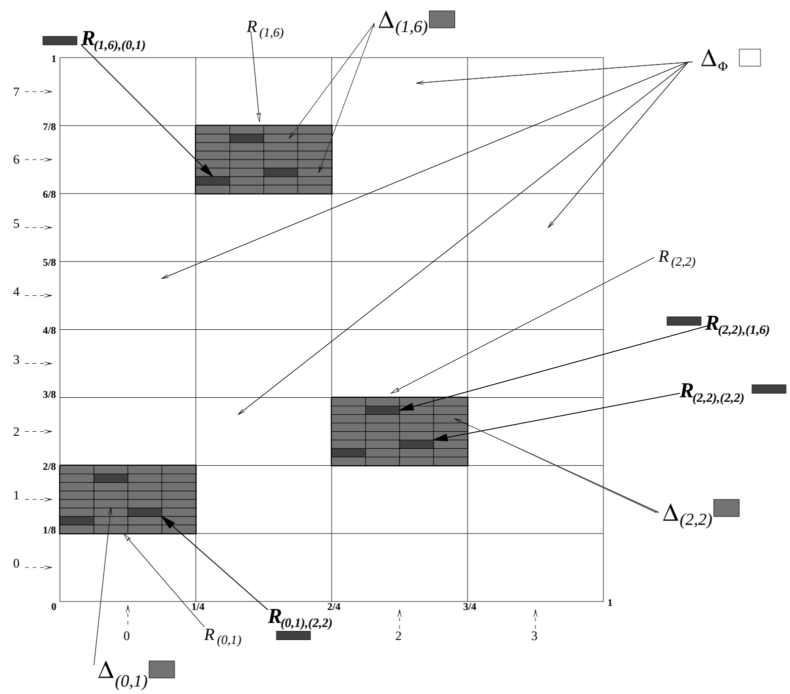

Figure 9: Δ_ω for $\omega \in A = \{(0,1),(2,2),(1,6)\}$, $s=4$ and $t=8$

Lemma 5 *Let $p > 0$ such that $\sigma\tau(p) + 1 + \sigma < p$. For any $\epsilon > 0$, there exist $C > 0$ and a sequence of $h_N \neq (0,0)$ with $\lim_{N\mapsto\infty} |h_N| = 0$ so that*

$$S_p(h_N) \geq C|h_N|^{\sigma\tau(p)+1+\sigma+\epsilon} .$$

Proof:
For $n \in \mathbf{N}^*$ and $\omega = (\omega_1, \ldots, \omega_n) \in A^n$, let $\Delta_\omega = \Re_\omega \setminus \bigcup_{\omega'\in A} \Re_{\omega\omega'}$ where the notation $\Re_{\omega\omega'}$ denotes the rectangle $\Re_{(\omega_1,\ldots,\omega_n,\omega')}$. Set $\Delta_\emptyset = \Re \setminus \bigcup_{\omega'\in A} \Re_{\omega'}$.

We have

$$S_p(h) \;=\; \int_{\Delta_\emptyset} |F(x+h) - F(x)|^p \, dx \;+\; \sum_{\omega\in A} \int_{\Re_\omega} |F(x+h) - F(x)|^p \, dx.$$

By iteration, we get for any integer N

$$S_p(h) = \sum_{n=0}^{N} \sum_{\omega\in A^n} \int_{\Delta_\omega} |F(x+h) - F(x)|^p \, dx \;+\; \sum_{\omega\in A^{N+1}} \int_{\Re_\omega} |F(x+h) - F(x)|^p \, dx.$$

Consider

$$\Delta'_{h,\omega} = \{x \in \Delta_\omega \;:\; x + h \notin \Delta_\omega\}.$$

Observe that asumptions (15) and (22) or (23) imply that for $\omega \in A^n$, if $|h| < t^{-(n+2)}$ and $x \in \Delta'_{h,\omega}$ then $x + h \in \bigcup_{\omega'\in A} \Delta_{\omega\omega'}$.

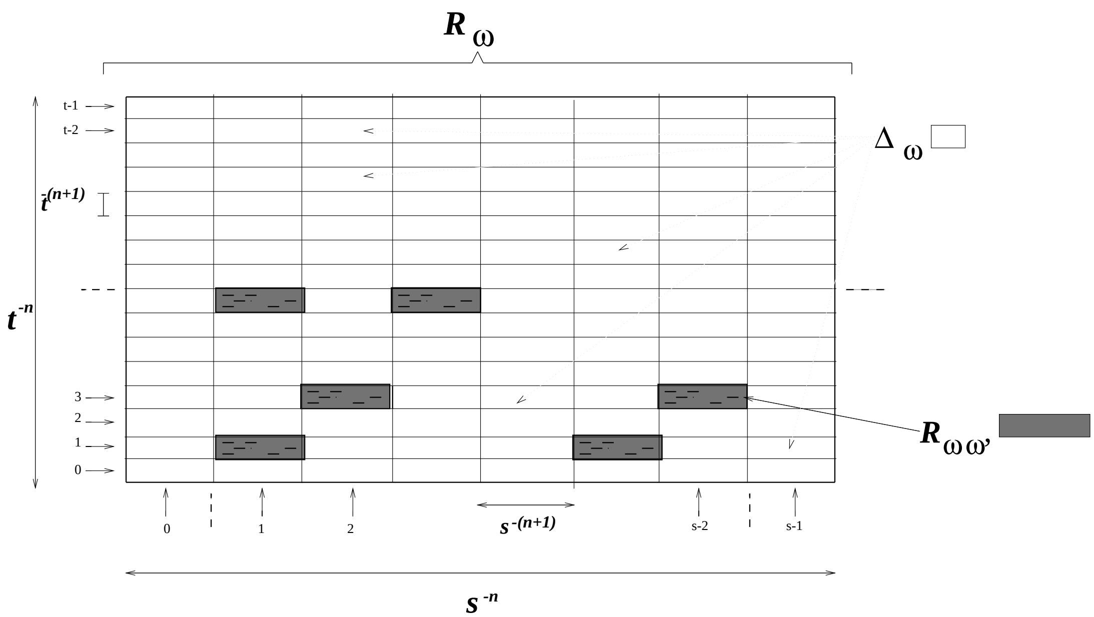

Figure 10: Δ_ω, $\Re_\omega$, $\Re_{\omega,\omega'}$

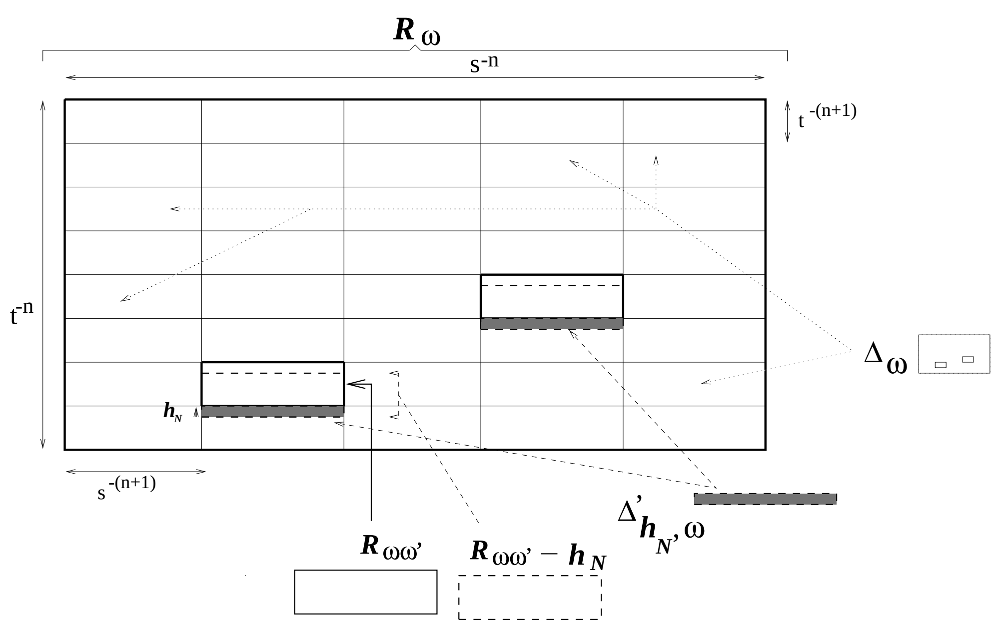

Figure 11: Δ_ω, $\Re_\omega$, $\Re_{\omega,\omega'}$, $\Delta'_{h_N,\omega}$ in the case of asumption (22)

Take $h_N = (0, t^{-(N+2)})$ in the case where asumption (22) holds (and $h_N = (0, -t^{-(N+2)})$ for asumption (23)), then

$$S_p(h_N) \geq \sum_{n=1}^{N} \sum_{\omega \in A^n} \int_{\Delta'_{h_N,\omega}} |F(x+h_N) - F(x)|^p \, dx.$$

For $x \in \Delta'_{h_N,\omega}$ with $\omega = (\omega_1, \dots, \omega_n) \in A^n$, we have $\omega(n, x+h_N) = \omega(n,x) = \omega$, hence

$$\begin{aligned} |F(x+h_N) - F(x)|^p &= |\sum_{j=0}^{n} \lambda_{(\omega_1,\dots,\omega_j)} \, g(S^{-1}_{(\omega_1,\dots,\omega_j)}(x+h_N)) \\ &\quad + \lambda_{\omega(n+1,x+h_N)} \, g(S^{-1}_{(\omega_1,\dots,\omega_n,\omega_{n+1}(x+h_N))}(x+h_N)) \\ &\quad - \sum_{j=0}^{n} \lambda_{(\omega_1,\dots,\omega_j)} \, g(S^{-1}_{(\omega_1,\dots,\omega_j)}(x))|^p. \end{aligned}$$

If $j \leq n-1$ then

$$\begin{aligned} & g(S^{-1}_{(\omega_1,\dots,\omega_j)}(x+h_N)) \; - \; g(S^{-1}_{(\omega_1,\dots,\omega_j)}(x)) \\ & = \Lambda\left((S^{-1}_{(\omega_1,\dots,\omega_j)}(x+h_N))_1 \right) [\Lambda\left((S^{-1}_{(\omega_1,\dots,\omega_j)}(x+h_N))_2 \right) \; - \; \Lambda\left((S^{-1}_{(\omega_1,\dots,\omega_j)}(x))_2 \right)] \\ & \geq s^{-1}[(S^{-1}_{(\omega_1,\dots,\omega_j)}(x+h_N))_2 - (S^{-1}_{(\omega_1,\dots,\omega_j)}(x))_2] \\ & \geq s^{-1} t^j t^{-(N+2)}. \end{aligned}$$

And for $j = n$, $S^{-1}_{(\omega_1,\dots,\omega_n)}(x) \in \Delta_\emptyset$ and $S^{-1}_{(\omega_1,\dots,\omega_n)}(x+h_N) \in \Delta_{\omega_{n+1}(x+h_N)}$, so since $h_N = (0, t^{-(N+2)})$ and $n \leq N$ then $S^{-1}_{(\omega_1,\dots,\omega_n)}(x) \in [1/s, (s-1)/s] \times [0, 1/2]$, hence

$$g(S^{-1}_{(\omega_1,\dots,\omega_n)}(x+h_N)) \; - \; g(S^{-1}_{(\omega_1,\dots,\omega_n)}(x)) \geq s^{-1} t^n t^{-(N+2)}.$$

And since the λ_ω and g are positive, then taking $r(n,p) = 1$ if $p \geq 1$ and $(n+1)^{p-1}$ if $0 < p < 1$, we get for $x \in \Delta'_{h_N,\omega}$ with $\omega \in A^n$

$$\begin{aligned} |F(x+h_N) - F(x)|^p &\geq \left(\sum_{j=0}^{n} \lambda_{(\omega_1,\dots,\omega_j)} \, [\, g(S^{-1}_{(\omega_1,\dots,\omega_j)}(x+h_N)) \; - \; g(S^{-1}_{(\omega_1,\dots,\omega_j)}(x)) \,] \right)^p \\ &\geq \sum_{j=0}^{n} \lambda^p_{(\omega_1,\dots,\omega_j)} \, s^{-p} t^{jp} t^{-(N+2)p} \, r(n,p). \end{aligned}$$

Hence

$$\begin{aligned} S_p(h_N) &\geq \sum_{n=1}^{N} \sum_{\omega \in A^n} (Area \Delta'_{h_N,\omega}) \sum_{j=0}^{n} \lambda^p_{(\omega_1,\dots,\omega_j)} \, s^{-p} t^{jp} |h_N|^p \, r(n,p) \\ &\geq \sum_{n=1}^{N} \sum_{\omega \in A^n} s^{-n} |h_N| \sum_{j=0}^{n} \lambda^p_{(\omega_1,\dots,\omega_j)} \, s^{-p} t^{jp} |h_N|^p \, r(n,p). \end{aligned}$$

Let a denotes the cardinality of A. It follows from the equality

$$\sum_{\omega\in A^n}\sum_{j=0}^{n} t^{jp}\ \lambda^p_{(\omega_1,\dots,\omega_j)} = \sum_{j=0}^{n} t^{jp}a^{n-j}(\sum_{\omega\in A}\lambda^p_\omega)^j$$

that

$$S_p(h_N)\geq C_p|h_N|^{p+1}\sum_{n=1}^{N} s^{-n}r(n,p)\sum_{j=0}^{n} t^{jp}a^{n-j}(\sum_{\omega\in A}\lambda^p_\omega)^j\ .$$

If

$$t^p\sum_{\omega\in A}\lambda^p_\omega > a \tag{34}$$

we obtain

$$S_p(h_N)\geq C_p|h_N|^{p+1}\sum_{n=1}^{N} s^{-n}r(n,p)t^{np}(\sum_{\omega\in A}\lambda^p_\omega)^n$$

and if

$$s^{-1}t^p\sum_{\omega\in A}\lambda^p_\omega > 1 \tag{35}$$

we get

$$\begin{aligned} S_p(h_N) &\geq C_p|h_N|^{p+1}s^{-N}r(N,p)t^{Np}(\sum_{\omega\in A}\lambda^p_\omega)^N \\ &\geq C_p|h_N|^{p+1}|h_N|^{\sigma}|h_N|^{-p}|h_N|^{\sigma\tau(p)+\epsilon} \\ &\geq C_p|h_N|^{\sigma\tau(p)+1+\sigma+\epsilon}. \end{aligned}$$

Remark that $\sigma\tau(p)+1+\sigma<p$ is equivalent to $t^{-1}s^{-1}t^p\sum_{\omega\in A}\lambda^p_\omega>1$ and thus yields (35), and since the cardinality of A is smaller than st, it yields (34).
Whence, for $p>0$ such that $\sigma\tau(p)+1+\sigma<p$, we obtain

$$\zeta(p)(=\liminf_{|h|\mapsto 0}\frac{\log S_p(h)}{\log|h|})\leq\sigma\tau(p)+1+\sigma\ . \tag{36}$$

Now, we shall give the exact value of $\zeta(p)$ and we will show that unlike $d(\alpha)$, $\zeta(p)$ (or $\xi(p)$) is the same for the two previous cases and so it doesn't depend on the geometrical arrangement of the chosen rectangles $\Re_\omega$. This fact gives also a reason for the failure of the multifractal formalism.
In order to give good upper bound for $S_p(h)$, we will assume that $|\lambda|_{min}>1/t$, (i.e the Hölder regularity of any point is smaller than 1).

Lemma 6 *Let $p>0$ such that $\sigma\tau(p)+1+\sigma<p$. Then for any $\epsilon>0$, there exists $C>0$ such that for $|h|$ small enough*

$$S_p(h)\leq C|h|^{\sigma\tau(p)+1+\sigma-\epsilon}\ .$$

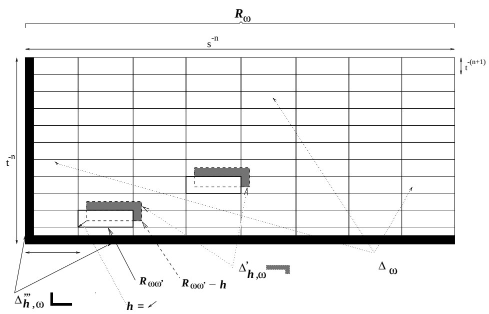

Figure 12: Δ_ω, $\Re_\omega$, $\Re_{\omega,\omega'}$, $\Delta'_{h,\omega}$, $\Delta''_{h,\omega} = \Delta_\omega \setminus \Delta'_{h,\omega}$

Proof:
Let $t^{-(N+2)} \leq |h| < t^{-(N+1)}$ and define $\Delta''_{h,\omega} = \Delta_\omega \backslash \Delta'_{h,\omega}$, i.e

$$\Delta''_{h,\omega} = \{x \in \Delta_\omega \ : \ x + h \in \Delta_\omega\} \ .$$

We have

$$\begin{aligned} S_p(h) &= \sum_{n=0}^{N} \sum_{\omega \in A^n} \int_{\Delta'_{h,\omega}} |F(x+h) - F(x)|^p \, dx \ + \ \sum_{n=0}^{N} \sum_{\omega \in A^n} \int_{\Delta''_{h,\omega}} |F(x+h) - F(x)|^p \, dx \\ &\quad + \sum_{\omega \in A^{N+1}} \int_{\Re_\omega} |F(x+h) - F(x)|^p \, dx. \end{aligned}$$

For $x \in \Delta''_{h,\omega}$ with $\omega \in A^n$

$$|F(x+h) - F(x)|^p = \Big| \sum_{j=0}^{n} \lambda_{(\omega_1,\ldots,\omega_j)} \left[g(S^{-1}_{(\omega_1,\ldots,\omega_j)}(x+h)) - g(S^{-1}_{(\omega_1,\ldots,\omega_j)}(x)) \right] \Big|^p \ .$$

Consider $r'(n,p) = 1$ if $0 < p < 1$ and $(n+1)^{p-1}$ if $p \geq 1$.
Thanks to the fact that g is C^1 (uniformly Lipschitz)

$$|F(x+h) - F(x)|^p \leq \sum_{j=0}^{n} r'(n,p) |\lambda_{(\omega_1,\ldots,\omega_j)}|^p t^{jp} |h|^p \ .$$

Thus

$$\begin{aligned}\sum_{n=0}^{N}\sum_{\omega\in A^n}\int_{\Delta''_{h,\omega}}|F(x+h)-F(x)|^p\,dx &\leq C\sum_{n=0}^{N}\sum_{\omega\in A^n}\sum_{j=0}^{n}r'(n,p)|\lambda_{(\omega_1,\dots,\omega_j)}|^p t^{jp}|h|^p s^{-n}t^{-n}\\ &= |h|^p\sum_{n=0}^{N}r'(n,p)s^{-n}t^{-n}\sum_{j=0}^{n}t^{jp}a^{n-j}(\sum_{\omega\in A}\lambda_\omega^p)^j,\end{aligned}$$

hence if (34) holds then the previous term will be bounded by

$$|h|^p\sum_{n=0}^{N}r'(n,p)s^{-n}t^{-n}t^{np}(\sum_{\omega\in A}\lambda_\omega^p)^n\ ;$$

and if $\sigma\tau(p)+1+\sigma<p$, then it will be estimated by $C|h|^{\sigma\tau(p)+1+\sigma-\varepsilon}$.

We will now estimate the term $\sum_{\omega\in A^{N+1}}\int_{\Re_\omega}|F(x+h)-F(x)|^p\,dx$; for that, remark that $\omega(N,x+h)=\omega(N,x)=(\omega_1,\dots,\omega_N)$, so

$$\begin{aligned}|F(x+h)-F(x)| &\leq |\sum_{l=0}^{N-1}\lambda_{(\omega_1,\dots,\omega_l)}\left(g(S^{-1}_{(\omega_1,\dots,\omega_l)}(x+h))-g(S^{-1}_{(\omega_1,\dots,\omega_l)}(x))\right)|\\ &+ |\lambda_{(\omega_1,\dots,\omega_N)}|\left(|F(S^{-1}_{(\omega_1,\dots,\omega_N)}(x+h))|+|F(S^{-1}_{(\omega_1,\dots,\omega_N)}(x))|\right).\end{aligned}$$

From the fact that g is C^1 and F is bounded, the previous quantity will be bounded by

$$C|h|\sum_{l=0}^{N-1}|\lambda_{(\omega_1,\dots,\omega_l)}|t^l+C|\lambda_{(\omega_1,\dots,\omega_N)}|.$$

Thanks to the asumption $|\lambda|_{min}>1/t$, we get

$$|F(x+h)-F(x)|\leq C|h|\,N\,|\lambda_{(\omega_1,\dots,\omega_N)}|\,t^N+C|\lambda_{(\omega_1,\dots,\omega_N)}|\ ;$$

hence

$$|F(x+h)-F(x)|^p\leq C'N^p|\lambda_{(\omega_1,\dots,\omega_N)}|^p.$$

Hence

$$\begin{aligned}\sum_{\omega\in A^{N+1}}\int_{\Re_\omega}|F(x+h)-F(x)|^p\,dx &\leq CN^p s^{-N}t^{-N}\sum_{\omega\in A^{N+1}}|\lambda_{(\omega_1,\dots,\omega_N)}|^p\\ &\leq C|h|^{\sigma\tau(p)+1+\sigma-\epsilon}.\end{aligned}$$

Let us now estimate the term $\sum_{n=0}^{N}\sum_{\omega\in A^n}\int_{\Delta'_{h,\omega}}|F(x+h)-F(x)|^p\,dx$. By analogous arguments, for $x\in\Delta'_{h,\omega}$ and $\omega\in A^n$, $|F(x+h)-F(x)|^p$ will be bounded by

$$C_p\left(|\lambda_{(\omega,\omega_{n+1}(x+h))}|^p|g(S^{-1}_{(\omega,\omega_{n+1}(x+h))}(x+h))|^p+\sum_{j=0}^{n}r'(n,p)|\lambda_{(\omega_1,\dots,\omega_j)}|^p t^{jp}|h|^p\right)\ ;$$

thus

$$
\begin{aligned}
&\sum_{n=0}^{N}\sum_{\omega\in A^n}\int_{\Delta'_{h,\omega}}|F(x+h)-F(x)|^p\,dx\\
&\leq C\sum_{n=0}^{N}\sum_{\omega\in A^n}\int_{\Delta'_{h,\omega}}|\lambda_{(\omega,\omega_{n+1}(x+h))}|^p|g(S^{-1}_{(\omega,\omega_{n+1}(x+h))}(x+h))|^p\,dx \qquad (37)\\
&+C|h|\sum_{n=0}^{N}s^{-n}\sum_{\omega\in A^n}\sum_{j=0}^{n}r'(n,p)|\lambda_{(\omega_1,\dots,\omega_j)}|^p t^{jp}|h|^p \qquad (38)
\end{aligned}
$$

The term (38) is smaller than $C|h|^{\sigma\tau(p)+1+\sigma-\epsilon}$. Now, for the term (37), we have

$$
\begin{aligned}
&\int_{\Delta'_{h,\omega}}|\lambda_{(\omega,\omega_{n+1}(x+h))}|^p|g(S^{-1}_{(\omega,\omega_{n+1}(x+h))}(x+h))|^p\,dx\\
&\leq Cs^{-n}t^{-n}|\lambda_\omega|^p\int_{\Delta_{h,n}}|g\left(S^{-1}_{\omega_{n+1}(S_\omega(y)+h)}(y+(s^nh_1,t^nh_2))\right)|^p\,dy
\end{aligned}
$$

with

$$
\Delta_{h,n}=S^{-1}_{\omega}(\Delta'_{h,\omega})=\{y\in\Delta_\emptyset\,:\,y+(s^nh_1,t^nh_2)\in\bigcup_{\omega'\in A}\Delta_{\omega'}\}\ .
$$

And by integrating the previous integral respectively on

$$
\{y\in\Delta_{h,n}\,:\,S^{-1}_{\omega_{n+1}(S_\omega(y)+h)}(y+(s^nh_1,t^nh_2))\in[0,\frac{1}{2}]\times[0,\frac{1}{2}]\}\ ;
$$

$$
\{y\in\Delta_{h,n}\,:\,S^{-1}_{\omega_{n+1}(S_\omega(y)+h)}(y+(s^nh_1,t^nh_2))\in[0,\frac{1}{2}]\times[\frac{1}{2},1]\}\ ;
$$

$$
\{y\in\Delta_{h,n}\,:\,S^{-1}_{\omega_{n+1}(S_\omega(y)+h)}(y+(s^nh_1,t^nh_2))\in[\frac{1}{2},1]\times[0,\frac{1}{2}]\}\ ;
$$

and

$$
\{y\in\Delta_{h,n}\,:\,S^{-1}_{\omega_{n+1}(S_\omega(y)+h)}(y+(s^nh_1,t^nh_2))\in[\frac{1}{2},1]\times[\frac{1}{2},1]\}\ ;
$$

we can easily show that

$$
\int_{\Delta'_{h,\omega}}|\lambda_{(\omega,\omega_{n+1}(x+h))}|^p|g(S^{-1}_{(\omega,\omega_{n+1}(x+h))}(x+h))|^p\,dx\leq Cs^{-n}t^{-n}|\lambda_\omega|^p t^{np}|h|^p\,;
$$

and thus the term (37) is bounded by $C|h|^{\sigma\tau(p)+1+\sigma}$.

Lemmas 5 and 6 yield the following proposition

Proposition 7 *Assume that $|\lambda|_{min}>1/t$. Let $p_0>0$ such that $\sum_{\omega\in A}(t\lambda_\omega)^{p_0}=st$ (i.e $\sigma\tau(p_0)+1+\sigma=p_0$); then for $p>p_0$, we have $\sigma\tau(p)+1+\sigma<p$ and so*

$$
\zeta(p)=\sigma\tau(p)+1+\sigma\ .
$$

If we define $f(\alpha)$ by

$$f(\alpha) = \inf_{p>p_0} (\alpha p - \zeta(p) + 2) \tag{39}$$

then for $\alpha = \sigma\tau'(q)$ with $q > p_0$, the infimum in the previous Legendre transform is attained for q, and

$$\begin{aligned} f(\alpha) &= \alpha q - \sigma\tau(q) + 1 - \sigma \\ &> \alpha q - \sigma\tau(q) \\ &\geq \inf_p(\alpha p - \sigma\tau(p)). \end{aligned}$$

Thus, in the second case,

$$d(\alpha) < f(\alpha).$$

So the multifractal formalism doesn't hold.

In the first case, since the cardinality of A is smaller than s, then for any $0 < p \leq 1$, $t^{-1}s^{-1}t^p \sum_{\omega\in A} \lambda_\omega^p < 1$; hence $p_0 > 1$ and so for $\alpha = \sigma\tau'(q)$ with $q > p_0$, we get

$$\begin{aligned} \inf_{p>p_0}(\alpha p - \zeta(p) + 2) &\geq \sigma d(\alpha) + 1 - \sigma \\ &> d(\alpha) \end{aligned}$$

because $d(\alpha) \leq dimK = \frac{\log a}{\log s} < 1$. So the multifractal formalism fails too.

Remark: Even the version of the multifractal formalism studied by Daubechies and Lagarias in [6], which says that $\zeta(p)$ is the Legendre transform of $d(\alpha) - 2$, fails for the previous two cases. Indeed, for the first case, since $d(\alpha) = \inf_q(q\sigma^{-1}\alpha - \tau(q))$ and $\tau(q)$ is concave and continue, then $\sigma\tau(q) = \inf_\alpha(q\alpha - \sigma d(\alpha))$; thus for $p > 1$ such that $\sigma\tau(p) + 1 + \sigma < p$, (36) implies that

$$\zeta(p) \leq \inf_\alpha(q\alpha - \sigma d(\alpha) + 1 + \sigma),$$

but $d(\alpha) < 1$, so

$$\zeta(p) < \inf_\alpha(q\alpha - d(\alpha) + 2)\ .$$

Identically for the second case.

6 The Anisotropic Multifractal Formalism

We have shown that the multifractal formalism fails for the two previous cases: the Euclidean norm used in the definitions of the pointwise regularity does not interact in a good way with the anisotropic contractions.
We propose instead an "homogenous norm" that was used by Calderón and Torchinsky on one side (see [4] and [5]), Folland and Stein on the other one (see [8]) to develop a theory of anisotropic $\mathcal{H}^p$ spaces.
We begin by defining the homogenous norm and describing some of its basic properties, for proofs and more details we refer to [8].

For $r > 0$, consider the dilation group of anisotropic linear transformation of R^m

$$A_r(x_1, \ldots, x_m) = (r^{d_1}x_1, \ldots, r^{d_m}x_m)$$

where $1 = d_1 \leq d_2 \leq \ldots \leq d_m$. We define the homogenous norm of $x \in \mathrm{R}^m$ by: $\rho(x) = 0$ if $x = 0$, and for $x \neq 0$, $\rho(x)$ is the unique value of r for which $|A_r^{-1}(x)| = 1$, $|x|$ denoting the Euclidean norm of x. The function ρ is continuous and homogenous in the sense that

$$\rho(A_r(x)) = r\rho(x). \tag{40}$$

Remark that in the isotropic case $d_1 = 1 = d_2 = \ldots = d_m$ and $\rho(x)$ coincides with the Euclidean norm.

Lemma 7 *There exist positive constants C_1, C_2 and γ such that*

$$C_1|x| \leq \rho(x) \leq C_2|x|^{1/d_m} \quad \textit{whenever} \quad \rho(x) \leq 1 \tag{41}$$

$$C_1^{1/d_m}|x|^{1/d_m} \leq \rho(x) \leq C_2^{d_m}|x| \quad \textit{whenever} \quad \rho(x) \geq 1 \tag{42}$$

$$\rho(x+y) \leq \gamma(\rho(x) + \rho(y)) \quad \forall x, y \in \mathrm{R}^m \tag{43}$$

$$|\rho(x+y) - \rho(x)| \leq \gamma\rho(y) \quad \forall x, y \in \mathrm{R}^m\,;\, \rho(y) \leq \rho(x)/2 \tag{44}$$

$$(1+\rho(y))^{-s} \leq \gamma^s(1+\rho(x))^{-s}(1+\rho(x-y))^s \quad \forall x, y \in \mathrm{R}^m \textit{ and } s > 0. \tag{45}$$

The ρ-Mean Value Theorem: there exist $C > 0$ and $\nu > 0$ such that for all function f of class $C^{(1)}$ on R^m and all $x, y \in \mathrm{R}^m$

$$|f(x+y) - f(x)| \leq C \sum_{j=1}^{m} \rho(y)^{d_j} \sup_{\rho(h) \leq \nu\rho(y)} |\partial_{x_j} f(x+h)|.$$

We adopt the following multiindex notation for higher order derivatives. For $I = (i_1, \ldots, i_m) \in \mathrm{N}^m$, we set $\partial^I = \partial_{x_1}^{i_1} \ldots \partial_{x_m}^{i_m}$ and $x^I = x_1^{i_1} \ldots x_m^{i_m}$. Further we set $|I| = i_1 + \ldots + i_m$ and $d(I) = d_1 i_1 + \ldots + d_m i_m$. Thus $|I|$ is the order of the differential operator ∂^I, while $d(I)$ is its degree of homogeneity, or, as we shall say, its homogenous degree. We shall denote by Δ the additive sub-semigroup of R generated by $0, d_1, \ldots$ and d_m. In other words, Δ is the set of all numbers $d(I)$ as I ranges over N^m. We observe that $\mathrm{N} \subset \Delta$ since $d_1 = 1$.
Let now P be a polynomial i.e $P = \sum_I a_I x^I$, $a_I \in \mathrm{R}$. We define its homogenous degree to be $max\{d(I) \,:\, a_I \neq 0\}$. There is a version of Taylor's theorem with remainder for the homogenous norm ρ.

The ρ-Taylor Inequality: suppose $\delta \in \Delta$ $(\delta > 0)$, and $k = [\delta]$. There is a constant $C_\delta > 0$ such that for all functions f of class $C^{(k+1)}$ on R^m and all $x, y \in \mathrm{R}^m$,

$$|f(x+y) - P_x(y)| \leq C_\delta \sum_{|I| \leq k+1\,,\, d(I) > \delta} \rho(y)^{d(I)} \sup_{\rho(h) \leq \nu^{k+1}\rho(y)} |\partial^I f(x+h)|$$

where P_x is the Taylor polynomial of f at x of homogenous degree δ.

We will now replace the Euclidean norm by the homogenous one in the terminology that appear in the formulation of the multifractal formalism, in order to be adapted to the geometric anisotropy. Then we will show that the anisotropic wavelet transform gives "good" characterizations for the new pointwise regularity for a large class of anisotropic selfsimilar functions.

Definition 3 *Let $\alpha > 0$, $\alpha \notin \Delta$ and $x_0 \in \mathrm{R}^m$; by definition a function $F : \mathrm{R}^m \to \mathrm{R}$ belongs to $C_\rho^\alpha(x_0)$ if there exists a polynomial P of homogenous degree smaller than α such that*

$$|F(x) - P(x - x_0)| \leq C\rho(x - x_0)^\alpha. \tag{46}$$

The ρ-Hölder exponent of F at x is defined by

$$\alpha_\rho(x) = \sup\{\beta : \quad F \in C_\rho^\beta(x)\}. \tag{47}$$

And we say that F belongs to $C_\rho^\alpha(\mathrm{R}^m)$ if (46) holds for any x in R^m with uniform constant C.

Let us first define the anisotropic wavelet transform.
Let ψ in the L.Schwartz class $S(\mathrm{R}^m)$, supported in $|x| \leq 1$, and with vanishing moments; and let φ be another function of $S(\mathrm{R}^m)$ whose Fourier transform $\hat{\varphi}$ has compact support disjoint of the origin and has the property that for all $x \neq 0$,

$$\int_0^\infty \hat{\varphi}(A_r x)\hat{\psi}(A_r x)\, dr/r = 1. \tag{48}$$

Let $\bar{x} = (2^{d_1}, \ldots, 2^{d_m})$; $\bar{\varphi}(x) = \varphi(x - \bar{x})$; $\quad \bar{\psi}(x) = \psi(x + \bar{x})$, and for $a > 0$ set

$$\bar{\varphi_a}(x) = \frac{1}{a^Q}\bar{\varphi}(A_a^{-1}x) \text{where } Q = d_1 + \ldots + d_m.$$

Given a tempered distribution F, $a > 0$ and $b \in \mathrm{R}^m$, the anisotropic wavelet transform of F is defined by

$$C_\rho(a,b)(F) = (F * \bar{\varphi_a})(x) = \frac{1}{a^Q}\int F(x)\bar{\varphi}(A_a^{-1}(x - b))\, dx. \tag{49}$$

F is reconstructed from its anisotropic wavelet transform by (see [4])

$$F(x) = \int_{a>0}\int_{\mathbf{R}^m} C_\rho(a,b)(F)\bar{\psi_a}(x - b)\, db\, da/a \tag{50}$$

i.e

$$F(x) = \frac{1}{a^{Q+1}}\int_{a>0}\int_{\mathbf{R}^m} C_\rho(a,b)(F)\bar{\psi}(A_a^{-1}(x - b))\, db\, da$$

One of the fondamental properties of the anisotropic wavelet transform is that it characterizes the ρ-Hölder regularity by conditions analogous to those of the classic wavelet transform for the isotropic case.

Proposition 8 *1. $F \in C_\rho^s(\mathrm{R}^m)$ if and only if*

$$|C_\rho(a,b)(F)| \leq Ca^s. \tag{51}$$

2. If $F \in C_\rho^s(x_0)$ then

$$|C_\rho(a,b)(F)| \leq Ca^s\left(1 + \frac{\rho(b - x_0)}{a}\right)^s. \tag{52}$$

3. *If (52) holds and if $F \in C^{\beta}_{\rho}(\mathrm{R}^m)$ for $\beta > 0$, there exists a polynomial P such that if $\rho(x - x_0) \leq 1/2$,*

$$|F(x) - P(x - x_0)| \leq C\rho(x - x_0)^s \log\left(\frac{1}{\rho(x - x_0)}\right). \tag{53}$$

Proof:

1. If $F \in C^s_\rho(\mathrm{R}^m)$

$$\begin{aligned} |C_\rho(a,b)(F)| &= \frac{1}{a^Q}|\int F(x)\bar{\varphi}(A_a^{-1}(x-b))\,dx| \\ &= \frac{1}{a^Q}|\int F(x)\varphi(A_a^{-1}(x-b) - \bar{x})\,dx| \\ &= \frac{1}{a^Q}|\int F(x + A_a\bar{x})\varphi(A_a^{-1}(x-b))\,dx|\ ; \end{aligned}$$

Since $\hat{\varphi}$ has compact support disjoint of the origin then

$$\begin{aligned} |C_\rho(a,b)(F)| &= \frac{1}{a^Q}|\int (F(x + A_a\bar{x}) - P(x + A_a\bar{x} - b))\varphi(A_a^{-1}(x-b))\,dx| \\ &\leq \frac{1}{a^Q}\int C\ \rho(x + A_a\bar{x} - b)^s|\varphi(A_a^{-1}(x-b))|\,dx \end{aligned}$$

which by property (43) will be bounded by

$$\begin{aligned} &C\frac{1}{a^Q}\int \gamma^s(\rho(A_a\bar{x}) + \rho(x-b))^s|\varphi(A_a^{-1}(x-b))|\,dx \\ &\leq C\frac{1}{a^Q}\int (\rho(A_a\bar{x})^s + \rho(x-b)^s)|\varphi(A_a^{-1}(x-b))|\,dx \\ &\leq Ca^s + C\frac{a^s}{a^Q}\int \rho(A_a^{-1}(x-b))^s|\varphi(A_a^{-1}(x-b))|\,dx \\ &\leq 2Ca^s. \end{aligned}$$

Conversely, F is expressed in terms of $C_\rho(a,b)(F)$ as in (50). Let

$$W(a,x) = \frac{1}{a^Q}\int_{\mathbf{R}^m} C_\rho(a,b)(F)\bar{\psi}(A_a^{-1}(x-b))\,db.$$

If (51) holds then

$$|W(a,x)| \leq Ca^s \tag{54}$$

and

$$|\partial^I W(a,x)| \leq Ca^{s-d(I)}. \tag{55}$$

Let $x_0 \in \mathrm{R}^m$, and set $\delta = max\{d(I)\ :\ d(I) < s\}$ and $PW_a(x - x_0)$ the Taylor polynomial of $W(a,.)$ at x_0 of homogenous degree δ

$$PW_a(x - x_0) = \sum_{I\,:\,d(I)\leq\delta} \frac{\partial^I W(a,x_o)}{I!}(x - x_0)^I$$

and $P(x-x_0)$ the one of F, then

$$\begin{aligned}
|F(x)-P(x-x_0)| &= |\int_{a>0}(W(a,x)-PW_a(x-x_0))\ da/a| \\
&\le \int_0^{\rho(x-x_0)}(|W(a,x)|+|PW_a(x-x_0)|)\ da/a \\
&\quad + \int_{\rho(x-x_0)}^{\infty}|W(a,x)-PW_a(x-x_0)|\, da/a.
\end{aligned}$$

It follows from (54) and (55) that the first term is bounded by

$$\int_0^{\rho(x-x_0)}\left(Ca^s+\sum_{I\,:\,d(I)<s}Ca^{s-d(I)}|(x-x_0)^I|\right)\ da/a$$

but from the definition of ρ

$$|(x-x_0)^I|\le\rho(x-x_0)^{d(I)}$$

hence the previous term is estimated by

$$\begin{aligned}
&C\int_0^{\rho(x-x_0)}\left(a^{s-1}+\sum_{I\,:\,d(I)<s}Ca^{s-d(I)-1}\rho(x-x_0)^{d(I)}\right)\ da \\
&\le C\rho(x-x_0)^s.
\end{aligned}$$

Let $l=[\delta]$, since $W(a,.)$ is of class $C^{(l+1)}$, then using the ρ-Taylor inequality, the second term will be bounded by

$$\begin{aligned}
&\int_{\rho(x-x_0)}^{\infty}C_\delta\sum_{|J|\le l+1\,,\,d(J)>\delta}\rho(x-x_0)^{d(J)}\sup_{\rho(h)\le\nu^{l+1}\rho(x-x_0)}|\partial^J W(a,.)(x_0+h)|\, da/a \\
&\le C_\delta\sum_{|J|\le l+1\,,\,d(J)>\delta}\rho(x-x_0)^{d(J)}\int_{\rho(x-x_0)}^{\infty}a^{s-d(J)}\, da/a \\
&\le C\rho(x-x_0)^s.
\end{aligned}$$

2. If $F\in C^s_\rho(x_0)$, then

$$\begin{aligned}
|C_\rho(a,b)(F)| &= \frac{1}{a^Q}|\int(F(x+A_a\bar{x})-P(x+A_a\bar{x}-x_0))\varphi(A_a^{-1}(x-b))\,dx| \\
&\le \frac{1}{a^Q}\int C\ \rho(x+A_a\bar{x}-x_0)^s|\varphi(A_a^{-1}(x-b))|\,dx \\
&\le C\frac{1}{a^Q}\int(\rho(A_a\bar{x})^s+\rho(x-x_0)^s)|\varphi(A_a^{-1}(x-b))|\,dx \\
&\le Ca^s+C\frac{1}{a^Q}\gamma^s\int(\rho(b-x_0)^s+\rho(x-b)^s)|\varphi(A_a^{-1}(x-b))|\,dx \\
&\le Ca^s+C\rho(b-x_0)^s+Ca^s \\
&\le Ca^s\left(1+\frac{\rho(b-x_0)}{a}\right)^s.
\end{aligned}$$

3. Conversely if (52) holds and if $F \in C^\beta_\rho(\mathrm{R}^m)$ for an $\beta > 0$, then

$$\begin{aligned}
|W(a,x)| &\leq \frac{1}{a^Q}\int Ca^s\left(1+\frac{\rho(b-x_0)}{a}\right)^s |\bar\psi(A_a^{-1}(x-b))|\,db\\
&\leq \frac{1}{a^Q}\int Ca^s\left(1+\frac{\rho(x-b)}{a}\right)^s |\bar\psi(A_a^{-1}(x-b))|\,db\\
&\quad + C\frac{1}{a^Q}\rho(x-x_0)^s\int |\bar\psi(A_a^{-1}(x-b))|\,db.
\end{aligned}$$

Hence

$$|W(a,x)| \leq Ca^s\left(1+\frac{\rho(x-x_0)}{a}\right)^s \tag{56}$$

and similarly

$$|\partial^I W(a,x)| \leq Ca^{s-d(I)}\left(1+\frac{\rho(x-x_0)}{a}\right)^s. \tag{57}$$

Thus

$$\begin{aligned}
|F(x)-P(x-x_0)| &\leq \int_0^{\rho(x-x_0)^{s/\beta}} |W(a,x)|\,da/a\\
&\quad + \int_{\rho(x-x_0)^{s/\beta}}^{\rho(x-x_0)} |W(a,x)|\,da/a\\
&\quad + \int_0^{\rho(x-x_0)} |PW_a(x-x_0)|\,da/a\\
&\quad + \int_{\rho(x-x_0)}^{\infty} |W(a,x)-PW_a(x-x_0)|\,da/a.
\end{aligned}$$

Using (54) (with s remplaced by β), the first term will be bounded by

$$C\int_0^{\rho(x-x_0)^{s/\beta}} a^\beta\,da/a$$

so by $C\rho(x-x_0)^s$.

(56) implies that the second term is estimated by

$$\begin{aligned}
&\int_{\rho(x-x_0)^{s/\beta}}^{\rho(x-x_0)} C(a^s+\rho(x-x_0)^s)\,da/a\\
&\leq 2C\rho(x-x_0)^s\int_{\rho(x-x_0)^{s/\beta}}^{\rho(x-x_0)} da/a\\
&\leq C\rho(x-x_0)^s\log\frac{1}{\rho(x-x_0)}
\end{aligned}$$

also, from (57) the third term will be bounded by

$$\begin{aligned}
&\int_0^{\rho(x-x_0)} C\sum_{I\,:\,d(I)<s} a^{s-d(I)}\left(1+\frac{0}{a}\right)^s \rho(x-x_0)^{d(I)}\,da/a\\
&\leq C\rho(x-x_0)^s
\end{aligned}$$

and we use the Taylor inequality to estimate the fourth term by

$$\begin{aligned}
&\int_{\rho(x-x_0)}^{\infty} C_\delta \sum_{|J|\le l+1\,,\,d(J)>\delta} \rho(x-x_0)^{d(J)} \sup_{\rho(h)\le \nu^{l+1}\rho(x-x_0)} |\partial^J W(a,.)(x_0+h)|\,da/a \\
&\le C_\delta \sum_{|J|\le l+1\,,\,d(J)>\delta} \rho(x-x_0)^{d(J)} \int_{\rho(x-x_0)}^{\infty} a^{s-d(J)} \sup_{\rho(h)\le \nu^{l+1}\rho(x-x_0)} \left(1+\frac{\rho(h)}{a}\right)^s da/a \\
&\le C \sum_{|J|\le l+1\,,\,d(J)>\delta} \rho(x-x_0)^{d(J)} \int_{\rho(x-x_0)}^{\infty} a^{s-d(J)}\,da/a \\
&\le C\rho(x-x_0)^s.
\end{aligned}$$

Whence (53).

The proof of Proposition 8 is now achieved.

Now we will make similar modifications for the Besov spaces.

Definition 4 *Suppose $s \in \mathrm{R}$ and $p > 0$, we say that F belongs to the homogeneous anisotropic Besov space $B^{s,\infty}_{\rho,p}(\mathrm{R}^m)$ if for a small enough*

$$\int |C_\rho(a,b)(F)|^p\,db \le Ca^{sp}; \tag{58}$$

and we set

$$\eta_\rho(p) = \sup\{\tau \;:\; F \in B^{\tau/p,\infty}_{\rho,p}\}\ . \tag{59}$$

This definition does not depend on the choice of the wavelet φ nor ψ: let Φ and Ψ be two other functions satisfying the same properties, since the supports of $\widehat{\varphi}$ and $\widehat{\Phi}$ are disjoint of the origin then there exist two positive numbers α and β such that $\varphi_t * \Phi_l = 0$ for $t/l \notin [\alpha,\beta]$. Thanks to the property (48)

$$\begin{aligned}
F * \bar{\Phi}_l &= F * \bar{\Phi}_l * \int_0^\infty \varphi_t * \psi_t\,dt/t \\
&= F * \bar{\Phi}_l * \int_0^\infty \varphi_{lt} * \psi_{lt}\,dt/t \\
&= \int_\alpha^\beta F * \varphi_{lt} * \bar{\Phi}_l * \psi_{lt}\,dt/t \\
&= \int_\alpha^\beta F * \bar{\varphi}_{lt}(. - A_{lt}\bar{x}) * \bar{\Phi}_l * \psi_{lt}\,dt/t\,;
\end{aligned}$$

hence

$$\|F * \bar{\Phi}_l\|_{L^p(\mathbf{R}^m)} \le \|\bar{\Phi}\|_{L^1}\|\psi\|_{L^1} \int_\alpha^\beta \|F * \bar{\varphi}_{lt}\|_{L^p(\mathbf{R}^m)}\,dt/t\ ,$$

whence using the Young inequality in the multiplicative group R^*_+, we obtain

$$\begin{aligned}
\|l^{-s}\|F * \bar{\Phi}_l\|_{L^p(\mathbf{R}^m)}\|_{L^\infty(\mathbf{R}^*_+)} &\le C\int_\alpha^\beta t^s\,dt/t\ \ \|l^{-s}\|F * \bar{\varphi}_l\|_{L^p(\mathbf{R}^m)}\|_{L^\infty(\mathbf{R}^*_+)} \\
&\le C\|l^{-s}\|F * \bar{\varphi}_l\|_{L^p(\mathbf{R}^m)}\|_{L^\infty(\mathbf{R}^*_+)}.
\end{aligned}$$

We also modify the definition of the Hausdorff dimension and Hausdorff measure in order to be adapted to the anisotropy as follows (see [17]).

Definition 5 *Let $E \subset \mathrm{R}^m$ and R_ε the set of all coverings of E by sets of ρ-diameter at most ε. Let*

$$M_\rho(\varepsilon, d) \;=\; \inf_{r \in R_\varepsilon} \sum_{E_i \in r} (\rho - \mathrm{diam}\, E_i)^d$$

then, by definition the d-dimensional ρ-Hausdorff measure of E is

$$(d - \rho - \mathrm{Mes})(E) \;=\; \limsup_{\varepsilon \to 0} M_\rho(\varepsilon, d) \;.$$

The ρ-Hausdorff dimension of E is

$$D \;=\; \inf\{d \;:\; (d - \rho - \mathrm{Mes})(E) = 0\} = \;\sup\{d \;:\; (d - \rho - \mathrm{Mes})(E) = +\infty\}\;.$$

Finally we call the anisotropic multifractal formalism the property that the ρ-Hausdorff dimension $d_\rho(\alpha)$ of the set of points x where $\alpha_\rho(x) = \alpha$ is equal to the Legendre transform of $\eta_\rho(q) - Q$

$$d_\rho(\alpha) = \inf(\alpha q - \eta_\rho(q) + Q). \tag{60}$$

Now we will introduce the class of anisotropic selfsimilar functions for which the anisotropic multifractal formalism will be valid. This new class will contain the family of "selfsimilar" functions of the two previous cases.

7 Validity of the Anisotropic Multifractal Formalism for Anisotropic Selfsimilar Functions

Let Ω be a bounded open set of R^m; $k > 0$; $1 = d_1 \leq d_2 \leq \cdots \leq d_m$; $\mu_1 < 1, \ldots, \mu_L < 1$ and $V_1 \in \mathrm{R}^m, \ldots, V_L \in \mathrm{R}^m$. Set $S_i(x) = A_{\mu_i}(x) + V_i$. Assume that (2) and (3) hold and let g be a C^k_ρ function such that all its derivatives of order less than k have fast decay. We will call a $(d_1, \ldots, d_m)$-k-selfsimilar function, a function F satisfying:

$$F(x) = \sum_{i=1}^{L} \lambda_i \; F(S_i^{-1}(x)) \;\; + \;\; g(x) \tag{61}$$

such that F is not uniformly C^k_ρ in a certain non empty closed subset of Ω.

The "selfsimilar" functions given in the first part of this paper are $(1, \frac{\log t}{\log s})$-1-selfsimilar.

Let

$$\alpha_{min} = \inf_{j=1,\ldots,L} \frac{\log \mid \lambda_j \mid}{\log \mu_j} \quad \text{and} \quad \alpha_{max} = \sup_{j=1,\ldots,L} \frac{\log \mid \lambda_j \mid}{\log \mu_j}. \tag{62}$$

We will study the existence of the solutions of (61) in the anisotropic $\mathcal{H}^p_\rho(\mathrm{R}^m)$ spaces defined by Calderón as follows.

Definition 6 *Let for $a > 0$ the maximal function associated with F be*

$$M_a(x) = \sup_{\rho(b) \leq ra} |C_\rho(a, x + b)(F)|.$$

For $0 < p \leq \infty$

$$F \in \mathcal{H}^p_\rho(\mathrm{R}^m) \quad \textit{if} \quad M_a(x) \in L^p(\mathrm{R}^m).$$

This property of $M_a(x)$ is independent of the choice of φ and a. We define the $\mathcal{H}^p_\rho(\mathrm{R}^m)$ norm of F as the norm of $M_a(x)$ in $L^p(\mathrm{R}^m)$.

A k-atom ϕ is a bounded function with compact support and with vanishing moments of all orders less than or equal to k. The p-norm of the atom ϕ is defined as

$$\|\phi\|_p = \inf M|B|^{1/p} \tag{63}$$

where M is a bound for $|\phi|$ and $|B|$ is the measure of a ball $B = \{x : \rho(x - x_0) \leq r\}$ (which is an ellipse in the euclidean space) containing the support of ϕ. If $0 < p \leq 1$ and $k \geq Q/p - 1$, then $\phi \in \mathcal{H}^p_\rho(\mathrm{R}^m)$ and

$$\|\phi\|_{\mathcal{H}^p_\rho(\mathbf{R}^m)} \leq C\|\phi\|_p \tag{64}$$

where the constant C depends on k and the choice of the norm in $\mathcal{H}^p_\rho(\mathrm{R}^m)$.

In the case where the solution will be a function, we will compute its global and pointwise anisotropic Hölder regularity and then we will show that the anisotropic multifractal formalism holds.

We are now ready to state our main results.

Proposition 9 *Suppose that $\sum_{j=1}^L \mid \lambda_j \mid \mu_j^Q < 1$; in this case (61) has a unique distribution solution, which is an L^1 function and given by the series*

$$\sum_{n=0}^{\infty} \sum_{(i_1,\dots,i_n)} \lambda_{i_1} \dots \lambda_{i_n} g\left(S_{i_n}^{-1} \dots S_{i_1}^{-1}(x)\right). \tag{65}$$

If furthermore $0 < \alpha_{min} < k$, this function belongs to $C_\rho^{\alpha_{min}}(\mathrm{R}^m)$.

Suppose that $\sum_{j=1}^L \mid \lambda_j \mid \mu_j^Q \geq 1$; in that case (61) may have several distribution solutions; let $p < 1$ such that $\sum_{j=1}^L \mid \lambda_j \mid^p \mu_j^Q < 1$; if g is C^k_ρ with $k > Q/p - 1$, and if the moments of g of order less than k vanish, (65) converges in the anisotropic Hardy real space $\mathcal{H}^p_\rho(\mathrm{R}^m)$, so that (61) has at least one solution in that space of distributions. Furthermore, these results are optimal.

Proposition 10 *Let K be the unique non-empty compact set satisfying $K = \bigcup_{j=1}^L S_j(K)$.*
If $x \notin K$, F is C^k_ρ in a neighbourhood of x.

Proposition 11 *Suppose that $\alpha_{min} > 0$. Let T be the tree constructed in the "time-frequency half-space": the root is conventionally the point $(0,1) \in \mathrm{R}^m \times \mathrm{R}^+$, this root is linked to the L first nodes, which are the $(S_j(0), \mu_j)$, each point $(S_j(0), \mu_j)$ is linked to the $(S_jS_k(0),\ \mu_j\mu_k), \dots$*
Let $x \in K$ and $B_j(x)$ be the set of branches $(i_1, ..., i_n)$ of the tree such that

$$\rho(S_{i_1} \cdots S_{i_n}(0) - x) \leq \mu_{i_1} \dots \mu_{i_n}$$

and

$$2^{-j} \leq \mu_{i_1} \cdots \mu_{i_n} < 2^{-(j-1)}.$$

Then

$$\alpha_\rho(x) = \liminf_{j\to\infty} \inf_{i \in B_j(x)} \frac{Log|\lambda_i|}{Log\mu_i}. \tag{66}$$

Proposition 12 *Define a function τ by the equation $\sum_{j=1}^{L}|\lambda_j|^a\mu_j^{-\tau(a)} = 1$. Let $\alpha < k$ and define $d_\rho(\alpha)$ as the ρ-Hausdorff dimension of the set of points x where $\alpha_\rho(x) = \alpha$. Then $d_\rho(\alpha)$ equals $-\infty$ outside $[\alpha_{min}, \alpha_{max}]$, and on this interval,*

$$d_\rho(\alpha) = \inf_a(a\alpha - \tau(a)). \tag{67}$$

Proposition 13 *Let F be a $(d_1, \ldots, d_m)$-k-selfsimilar and let q such that $\tau(q) \leq kq - Q$. Then*

$$\eta_\rho(q) = \tau(q) + Q.$$

Theorem 2 *Let F be a $(d_1, \ldots, d_m)$-k-selfsimilar. If $\alpha_{min} > 0$, the function $d_\rho(\alpha)$ equals $-\infty$ outside the interval $[\alpha_{min}, \alpha_{max}]$ and is analytic and concave on this interval. Its maximal value $d_{\rho,max}$ satisfies*

$$\sum \mu_i^{d_{\rho,max}} = 1.$$

Let α_0 be the value for which this maximum is attained.

If g is C_ρ^∞; if $\alpha \leq \alpha_0$, $d_\rho(\alpha)$ can be obtained by computing the Legendre transform of $\eta_\rho(q) - Q$.

If g is C_ρ^k, let p_0 be defined by $\tau(p_0) + Q = kp_0$ and let α_1 be the Legendre transform at p_0 of the function $\tau(q)$ (i.e $\alpha_1 = \tau'(p_0)$); if $\alpha \leq \alpha_1$ then for $\alpha \leq \alpha_1$, $d_\rho(\alpha)$ can be obtained by computing the Legendre transform of $\eta_\rho(q) - Q$.

Notice that in order to show that F belongs to $\mathcal{H}_\rho^p$, we split F as a sum $F = \sum_{j\geq 1} F_j$ where F_j is the series (65) restricted to the indexes $i \in I_j$ such that $2^{-j} \leq \mu_i < 2^{-(j-1)}$ and that the regularity and the cancellation that we requested for g is consistent with the atomic decomposition of the F_j; thus the $\mathcal{H}_\rho^p$ norm of F_j, using (63), (64) and the fact that $a + b \leq (a^p + b^p)^{1/p}$ for $a > 0$, $b > 0$ and $0 < p \leq 1$, is bounded by

$$\begin{aligned}
\|\sum_{i\in I_j} \lambda_i g \circ S_i^{-1}\|_{\mathcal{H}_\rho^p} &\leq \left(\sum_{i\in I_j} \|\lambda_i g \circ S_i^{-1}\|_{\mathcal{H}_\rho^p}^p\right)^{1/p} \\
&\leq C\left(\sum_{i\in I_j} \|\lambda_i g \circ S_i^{-1}\|_p\right)^{1/p} \\
&\leq C\left(\sum_{i\in I_j} |\lambda_i|^p \mu_i^Q\right)^{1/p}
\end{aligned}$$

this quantity is exponentially decreasing with j, so that F belongs to $\mathcal{H}_\rho^p$.

The solution F given by the series (65) looks like an anisotropic wavelet decomposition. The proofs follow from the properties of the homogenous norm ρ, Proposition 8 and arguments similar to those of Jaffard's paper [12]; in all situations Q served as a substitute for the space dimension m: the anisotropic wavelet transform of F satisfies a functional equation similar to (61)

$$C_\rho(a,b)(F) = \sum_{j=1}^{J} \sum_{2^{-j}\leq\mu_i<2.2^{-j}} \lambda_i C_\rho\left(\frac{a}{\mu_i}, S_i^{-1}(b)\right)(g)$$

$$+ \sum_{2^{-J} \leq \mu_i < 2.2^{-J}} \lambda_i \, C_\rho(\frac{a}{\mu_i}, S_i^{-1}(b))(F) \, ,$$

hence, we prove that its order of the magnitude near the tree is large, more precisely, near $(S_{i_1} \dots S_{i_n}(0), \mu_{i_1} \dots \mu_{i_n})$, it is $\sim |\lambda_{i_1}| \dots |\lambda_{i_n}|$.
The dimensions of the singularities will be obtained by constructing invariant measures on the sets of singularities and using an adapted lemma 3.
For the proof of the anisotropic mulifractal formalism, we show that for q such that $\tau(q) \leq kq - Q$

$$\int |C_\rho(a,b)(F)|^q db \sim a^{m+\tau(q)} \, .$$

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References

[1] A. Arneodo, E. Bacry, et J. F. Muzy. Wavelet analysis of fractal signals. direct determination of the singularity spectrum of fully developped turbulence data. Preprint, 1991.

[2] A. Arneodo, Bacry E., et J. F. Muzy. Singularity spectrum of fractal signals from wavelet analysis: Exact results. Preprint, 1991.

[3] M. Ben Slimane. Multifractal formalism for selfsimilar functions under the action of nonlinear dynamical systems. Preprint, 1995.

[4] A.P. Calderón. An atomic decomposition of distributions in parabolic $\mathcal{H}^p$ spaces. *Advances In Mathematics*, 25:216–225, 1977.

[5] A.P. Calderón et A. Torchinsky. Parabolic maximal functions associated with a distribution. *Advances In Mathematics*, 24:101–171, 1977.

[6] I. Daubechies et J.C. Lagarias. On the thermodynamic formalism for functions. *Reviews in Mathematical Physics*, Vol. 6(No. 5a):1033–1070, 1994.

[7] K. Falconer. *Fractal Geometry.* John Wiley and sons, 1990.

[8] G.B. Folland et E.M. Stein. *Hardy spaces on homogeneous groups*, volume Mathematical Notes 28, *Princeton University Press and University of Tokyo Press.* Princeton, New Jersey, 1993.

[9] U. Frisch et G. Parisi. Fully developped turbulence and intermittency. *Proc. Int. Summer school Phys. Enrico Fermi*, pages pp.84–88, 1985.

[10] J. Hutchinson. Fractals and self-similarity. *Indiana Univ.Math.J*, pages 713–747, 1981.

[11] S. Jaffard. Multifractal formalism for functions. Part 1: Results valid for all functions. Preprint, 1993.

[12] S. Jaffard. Multifractal formalism for functions. Part 2: Selfsimilar functions. Preprint, 1993.

[13] S. Jaffard et Y. Meyer. Pointwise behavior of functions. Preprint, 1993.

[14] J. King. The singularity spectrum for general Sierpinski carpets. Preprint, 1992.

[15] Y. Meyer. *Ondelettes et opérateurs.* Hermann, 1990.

[16] L. Olsen. Self-affine multifractal Sierpinski sponges in R^d. Preprint.

[17] C.A. Rogers. *Hausdorff measures.* Cambridge University Press, 1970.