Multifractal Formalism for Selfsimilar Functions under the action of Nonlinear Dynamical Systems

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Abstract. We study functions which are selfsimilar under the action of some nonlinear dynamical systems: we compute the exact pointwise Hölder regularity, then we determine the spectrum of singularities and the Besov's "smoothness" index, and finally we prove the multifractal formalism. The main tool in our computation is the wavelet analysis.
Key words: Hölder exponent, Hausdorff dimension, spectrum of singularities, wavelets, multifractal formalism, selfsimilar functions.

1 Introduction

Numerous experimental studies have shown the phenomenon of intermittency for the velocity of fully developped turbulence: the velocity ϑ has an irregular behavior, i.e has local Hölder exponents in a certain interval $[\alpha_{min}, \alpha_{max}]$ and each α in this range occurs in a set E^α (for $0 < \alpha \leq 1$, $E^\alpha = \{x : |\vartheta(x+h,t) - \vartheta(x,t)| \sim |h|^\alpha \text{ as } |h| \mapsto 0\}$, with Hausdorff dimension $d(\alpha)$ called the spectrum of singularities. We say that ϑ is a multifractal function.
In [11], Frisch and Parisi conjectured a formula which relates $d(\alpha)$ with the scaling exponent $\zeta(p)$ of the structure functions of order p

$$S_p(h) = \int_{\mathbf{R}^m} \mid \vartheta(x+h,t) - \vartheta(x,t) \mid^p \, dx \sim |h|^{\zeta(p)} \quad \text{as } |h| \mapsto 0$$

by a Legendre transform

$$d(\alpha) = \inf_p (\alpha p - \zeta(p) + m) \ .$$

This conjecture is called the Multifractal Formalism.

Most examples of multifractal functions follow some selfsimilarity conditions: locally the graph of the function F is a contraction of the global graph modulo an error g. This means that F satisfies a functional equation of the type

$$F(x) = \sum_{i=1}^{d} \lambda_i F(S_i^{-1}(x)) + g(x) \tag{1}$$

where the S_i are contractions on a bounded set and $|\lambda_i| < 1$.

Our purpose is to determine the spectrum of singularities for such functions and to check when they satisfy the multifractal formalism. (Note that some particular cases have been

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studied: Arneodo, Bacry and Muzy (see [3]) for the primitives of multinomial measures on $\Omega =]0,1[$ with the "separated open set condition $S_i(\overline{\Omega}) \cap S_j(\overline{\Omega}) = \emptyset$ pour $i \neq j$"; the function φ used in the construction of wavelets bases (see [9]); and Jaffard (see [15]) when the S_i are linear and satisfy a separation condition and g is smooth).
We will describe our results in cases where strict asumptions do not hold (see [5], [4] and [6]): for instance we will prove that the multifractal formalism holds in one dimension when the S_i are no more linear, and 2 dimension when the S_i are analytic mappings of $z = x + iy$.

We denote by T_j the new contractions and we suppose that they are defined in the interval $I = [0,1]$ (respectively in a bounded open domain Ω of the complex plane $\mathbb{C}$). We suppose also that they are C^{k+1} (respectively complex analytic) and satisfy:

- $$T_i(I) \subset I \quad \forall \quad i \tag{2}$$
  $$T_i(I) \cap T_j(I) = \emptyset \quad \text{if} \quad i \neq j \tag{3}$$

- there exist constants χ and ρ such that
  $$0 < \chi \leq |T_j'(x)| \leq \rho < 1 \quad \forall\, j = 1, \ldots, d \quad \text{and} \quad x \in I \tag{4}$$
  and
  $$\mid T_j^{(l)}(x) \mid \leq \chi^l \quad \forall\, j = 1, \ldots, d; l = 2, \ldots, k+1 \quad \text{and} \quad x \in I\ . \tag{5}$$

We also assume that g is a C^k function with derivatives of order less than k having fast decay, and that F is not uniformly C^k in a certain closed subset of $]0,1[$.
The conditions for the 2 dimensional case are identical to those of the 1 dimensional case and from now on, the letters x and I can be replaced by z and the closure of Ω and along this paper the results will be analogous for the two cases.

For the convenience of the notations, we will take $d = 2$ although the statements and proofs extend to the general case without any modifications.

It follows from Hutchinson (see [12]) that there exists a unique non-empty compact set K satisfying
$$K = T_1(K) \cup T_2(K)\ .$$

For $i = (i_1, \ldots, i_n) \in \{1,2\}^n$, we denote by I_i the branch $T_{i_1} \circ \ldots \circ T_{i_n}(I)$. Then we have

$$\begin{aligned} K &= \{x \in I\ :\ (T_{i_1} \circ \ldots \circ T_{i_n})^{-1}(x) \in I_1 \cup I_2\,, \quad \forall\,(i_1, \ldots, i_n) \in \{1,2\}^n\} \\ &= \bigcap_{n=0}^{\infty} (\bigcup_{|i|=n} I_i) \quad \text{(with the convention that for } n = 0\,,\ I_i = I)\ . \end{aligned}$$

It follows from the separation hypothesis (3) that there is a natural bijection π from the tree $\{1,2\}^{\mathbb{N}}$ to K, given by

$$\begin{aligned} \pi(i_1, \ldots, i_n, \ldots) &= \lim_{n \mapsto \infty} T_{i_1} \circ \ldots \circ T_{i_n}(t) \\ &= \bigcap_{n=1}^{\infty} I_{(i_1, \ldots, i_n)}\ . \end{aligned}$$

The value of $\pi(i_1,\ldots,i_n,\ldots)$ is independent of the initial value t, and we call the sequence $(i_1,\ldots,i_n,\ldots)$ the code of $\pi(i_1,\ldots,i_n,\ldots)$.

We will now give precise definitions of the notions we introduced.

A function $F:\mathrm{R}^m \to \mathrm{R}$ belongs to $C^\alpha(x_0)$ for $\alpha>0$ if there exists a polynomial P of degree at most α and a constant C such that

$$|F(x)-P(x-x_0)| \le C|x-x_0|^\alpha. \tag{6}$$

The Hölder exponent of F at x is defined by

$$\alpha(x)=\sup\{\beta: \quad F\in C^\beta(x)\}\ . \tag{7}$$

The spectrum of singularities of F is the function $d(\alpha)$ which associates to each α the Hausdorff dimension of the set E^α of points x where $\alpha(x)=\alpha$ (conventionally the dimension of the empty set is $-\infty$).

A function F belongs to $C^\alpha(\mathrm{R}^m)$ if (6) holds for any x in R^m with uniform constant C.

Now define

$$\zeta(q)=\liminf_{|h|\mapsto 0}\frac{\log\int|F(x+h)-F(x)|^q\,dx}{\log|h|}; \tag{8}$$

and let us recall that $\zeta(p)$ is related to Sobolev's or Besov's "smoothness" index.

Let $s\ge 0$; if s is not an integer, $s=[s]+\sigma$ with $[s]$ the largest integer in s; let $p\ge 1$; F belongs to the space of Nikol'skij $H^{s,p}(\mathrm{R}^m)$ if $F\in L^p$ and for any multi-index γ such that $|\gamma|=[s]$ and $|h|$ small enough

$$\int|\partial^\gamma F(x+h)-\partial^\gamma F(x)|^p\,dx\le C|h|^{\sigma p}\ . \tag{9}$$

Consider

$$\xi(p)=\sup\{s: F\in H^{s/p,p}(\mathrm{R}^m)\}\ ;$$

thus if $p\ge 1$ and $\zeta(p)<p$ then $\zeta(p)=\xi(p)$.

If $\zeta(p)\ge p$, then formula (8) must be modified as follows in order to be consistent with (9): if it is equal to p, one should use the same formula but with the gradient of F, and so on until $\zeta(p)$ falls between two integers multiplied by p.

$\xi(p)$ is also related to Besov's "smoothness" index. Let us recall that if ψ is a $C^k(\mathrm{R}^m)$ radial function with all moments of order less than k vanishing and all derivatives of order less than k are well localized and k large enough depending on the properties of F we want to analyze; then the wavelet transform of F at the position $b\in\mathrm{R}^m$ and the scale $a>0$ is

$$C_{a,b}(F)=\frac{1}{a^m}\int_{\mathbf{R}^m}F(t)\psi\left(\frac{t-b}{a}\right)dt\ ; \tag{10}$$

Now, a function F belongs to the Besov space $B_p^{s,\infty}(\mathrm{R}^m)$ if (see [17]) its wavelet transform satisfies for a small enough

$$\int|C_{a,b}(F)|^p db\le Ca^{sp}. \tag{11}$$

It follows from the imbeddings (cf [1])

$$\forall \ \epsilon > 0 \ \ H^{s+\epsilon,p}(\mathrm{R}^m) \hookrightarrow B_p^{s,\infty}(\mathrm{R}^m) \hookrightarrow H^{s-\epsilon,p}(\mathrm{R}^m) \tag{12}$$

for $p \geq 1$ and $s > 0$, that for $p \geq 1$

$$\xi(p) = \sup\{s \ : \ F \in B_p^{s/p,\infty}(\mathrm{R}^m)\} \ . \tag{13}$$

Since for $0 < p < 1$, the "Besov-Hardy" spaces $B_p^{s,\infty}(\mathrm{R}^m)$ are also defined as in (11), we will consider

$$\eta(p) = \sup\{s \ : \ F \in B_p^{s/p,\infty}(\mathrm{R}^m)\} \ . \tag{14}$$

It is also well known (see [16]) that the Hölder regularity can be characterized in terms of conditions on the wavelet transform. In fact we have:

- $F \in C^\alpha(\mathrm{R}^m)$ if and only if

$$|C_{a,b}(F)| \leq Ca^\alpha .$$

- If $F \in C^\alpha(x_0)$, then

$$|C_{a,b}(F)| \leq Ca^\alpha \left(1 + \frac{|b - x_0|}{a}\right)^\alpha . \tag{15}$$

- If (15) holds and if $F \in C^\varepsilon(\mathrm{R}^m)$ for an $\varepsilon > 0$, then there exists a polynomial P such that, if $| \ x - x_0 \ | \leq 1/2$,

$$|F(x) - P(x - x_0)| \leq C|x - x_0|^\alpha \log\left(\frac{2}{|x - x_0|}\right) \tag{16}$$

  and so $F \in C^{\alpha - \varepsilon'}(x_0) \quad \forall \varepsilon' > 0$.

It should be noticed that the previous characterizations given for the continuous wavelet transform hold in the case of an orthonormal basis of wavelets (see [17] and [13]).

The following equivalent formulas (the so-called multifractal formalism for functions) have been proposed for the computation of the spectrum of singularities $d(\alpha)$ (see [11], [14] and [2])

$$d(\alpha) = \inf(\alpha p - \zeta(p) + m) \quad , \quad d(\alpha) = \inf(\alpha p - \xi(p) + m) \tag{17}$$

and

$$d(\alpha) = \inf(\alpha p - \eta(p) + m). \tag{18}$$

In the next section, we will study the existence and uniqueness of the solution of equation (1) for the general contractions T_1 and T_2 and we compute its uniform regularity.

In the third section, we determine the Hölder exponent $\alpha(x)$ of F at any point x .

In the fourth section, we compute the spectrum of singularities.

Finally, in the fifth section, we prove the validity of the multifractal formalism.

2 Existence, uniqueness and global Hölder regularity

We begin by the study of the existence and uniqueness for the solution of the equation

$$F(x) = \sum_{i=1}^{2} \lambda_i F(T_i^{-1}(x)) \ + \ g(x) \ . \tag{19}$$

Iterating (19), we obtain for any integer N:

$$\begin{aligned} F(x) \quad &= \quad \sum_{n=0}^{N-1} \sum_{(i_1,\ldots,i_n)} \lambda_{i_1} \ldots \lambda_{i_n} \; g\left(T_{i_n}^{-1} \circ \ldots \circ T_{i_1}^{-1}(x)\right) \\ &\quad + \sum_{(i_1,\ldots,i_N)} \lambda_{i_1} \ldots \lambda_{i_N} \; F\left(T_{i_N}^{-1} \circ \ldots \circ T_{i_1}^{-1}(x)\right) \ . \end{aligned} \tag{20}$$

The fundamental idea is that locally our non-linear contractions T_i can be uniformly approximated by similarities in the following sense:

Lemma 1 *There exists a constant $\mathcal{D} \geq 1$ such that*

$$\mathcal{D}^{-1} \mid I_i \mid^{-1} \leq |(T_i^{-1})'(x)| \leq \mathcal{D} \mid I_i \mid^{-1} \qquad \forall x \in I_i \ , i \in \{1,2\}^n \; and \, n \in \mathbb{N}$$

where $|I_i|$ denotes the diameter of I_i and $T_i = T_{i_1} \circ \ldots \circ T_{i_n}$.

Proof:
We have

$$\mid I_i \mid = \sup_{x,y \in I} |T_i(x) - T_i(y)| \, ;$$

Using the mean value theorem, we get

$$|x-y| \ \inf_{u \in I} |T_i'(u)| \leq |T_i(x) - T_i(y)| \leq |x-y| \ \sup_{u \in I} |T_i'(u)|.$$

Let

$$m_i = \inf_{t \in I_i} |(T_i^{-1})'(t)| \quad \text{and} \quad M_i = \sup_{t \in I_i} |(T_i^{-1})'(t)| \ ;$$

Then

$$\inf_{u \in I} |T_i'(u)| = M_i^{-1} \quad \text{and} \quad \sup_{u \in I} |T_i'(u)| = m_i^{-1} \ ;$$

Hence

$$M_i^{-1} \mid I \mid \leq \mid I_i \mid \leq m_i^{-1} \mid I \mid \ . \tag{21}$$

For the complex case, we can assume that $|\Omega| = 1$, so in the two cases

$$M_i^{-1} \leq \mid I_i \mid \leq m_i^{-1} \ . \tag{22}$$

Let now $\phi : I_1 \cup I_2 \to \mathrm{R}$ be the function defined by $\phi(x) = \log |(T_j^{-1})'(x)|$ for $x \in I_j$. Since T_1 and T_2 are C^2, then ϕ is uniformly Lipschitz on each I_j, with uniform Lipschitz constant $C_\phi \leq 1$ because of (4) and (5).

Let

$$S_N\phi(x) = \sum_{n=0}^{N-1} \sum_{|i|=n} \phi(T_i^{-1}(x)) \ .$$

For x and $y \in I_{\mathbf{i}}$ with $\mathbf{i} = (\mathbf{i}_1, \ldots, \mathbf{i}_N)$

$$\begin{aligned}
| \ S_N\phi(x) - S_N\phi(y) \ | \ &\leq \ \sum_{n=0}^{N-1} |\phi(T^{-1}_{(\mathbf{i}_1,\ldots,\mathbf{i}_n)}(x)) - \phi(T^{-1}_{(\mathbf{i}_1,\ldots,\mathbf{i}_n)}(y))| \\
&\leq \ C_\phi \sum_{n=0}^{N-1} |T^{-1}_{(\mathbf{i}_1,\ldots,\mathbf{i}_n)}(x) - T^{-1}_{(\mathbf{i}_1,\ldots,\mathbf{i}_n)}(y)| \\
&\leq \ C_\phi \sum_{n=0}^{N-1} |I_{\mathbf{i_{n+1}}\ldots\mathbf{i_N}}| \\
&\leq \ C_\phi \sum_{n=0}^{N-1} \rho^{N-n} \quad \text{(because of (4))} \\
&\leq \ \frac{C_\phi \rho}{1-\rho} < \infty \, .
\end{aligned}$$

Hence

$$| \ S_N\phi(x) - S_N\phi(y) \ | \leq \frac{C_\phi \rho}{1-\rho} \quad \forall \quad N \in \mathbb{N}^*, | \ \mathbf{i} \ | = N \, \text{and} \, x, y \in I_{\mathbf{i}} \ . \tag{23}$$

But

$$\begin{aligned}
S_N\phi(x) - S_N\phi(y) \ &= \ \sum_{n=0}^{N-1} \phi(T^{-1}_{(\mathbf{i}_1,\ldots,\mathbf{i}_n)}(x)) - \phi(T^{-1}_{(\mathbf{i}_1,\ldots,\mathbf{i}_n)}(y)) \\
&= \ \sum_{n=0}^{N-1} \log \left(|(T^{-1}_{\mathbf{i_{n+1}}})'(T^{-1}_{(\mathbf{i_1},\ldots,\mathbf{i_n})}(x))| \right) - \log \left(|(T^{-1}_{\mathbf{i_{n+1}}})'(T^{-1}_{(\mathbf{i_1},\ldots,\mathbf{i_n})}(y))| \right) \\
&= \ \log \left(\prod_{n=0}^{N-1} |(T^{-1}_{\mathbf{i_{n+1}}})'(T^{-1}_{(\mathbf{i_1},\ldots,\mathbf{i_n})}(x))| \right) - \log \left(\prod_{n=0}^{N-1} |(T^{-1}_{\mathbf{i_{n+1}}})'(T^{-1}_{(\mathbf{i_1},\ldots,\mathbf{i_n})}(y))| \right) \\
&= \ \log \left(|(T_{\mathbf{i}}^{-1})'(x)| \right) - \log \left(|(T_{\mathbf{i}}^{-1})'(y)| \right) \\
&= \ \log \frac{|(T_{\mathbf{i}}^{-1})'(x)|}{|(T_{\mathbf{i}}^{-1})'(y)|} \ .
\end{aligned}$$

Thus, since $m_{\mathbf{i}}$ and $M_{\mathbf{i}}$ are reached, we deduce from (23) that

$$\frac{M_{\mathbf{i}}}{m_{\mathbf{i}}} \leq e^{\frac{C_\phi \rho}{1-\rho}} \ . \tag{24}$$

Finally (22) and (24) imply Lemma 1 with $\mathcal{D} = e^{\frac{C_\phi \rho}{1-\rho}}$.

As a consequence of Lemma 1 and the mean value theorem, we have the following Lemma (we call it the Distortion Lemma)

Lemma 2 *There exists a positif number $\mathcal{D}$ such that for any branchs $i=(i_1,\ldots,i_n)$ and $j=(j_1,\ldots,j_m)$*

$$\mathcal{D}^{-1} \mid I_i \mid \mid I_j \mid \leq \mid I_{ij} \mid \leq \mathcal{D} \mid I_i \mid \mid I_j \mid \ .$$

Let now

$$B_j = \{i \ : \ 2^{-j} \leq \mid I_i \mid < 2.2^{-j}\} \ ;$$

$$\alpha_{min} = \liminf_{j\to\infty} \inf_{i\in B_j} \frac{\log \mid \lambda_i \mid}{\log \mid I_i \mid} \quad \text{and} \quad \alpha_{max} = \limsup_{j\to\infty} \sup_{i\in B_j} \frac{\log \mid \lambda_i \mid}{\log \mid I_i \mid} \ ,$$

We will prove the following theorem

Theorem 1 *Let $\mathcal{D}$ be the best constant in Distortion Lemma. Assume that $\sum_{j=1}^{2} \mid \lambda_j \mid\mid I_j \mid < \mathcal{D}^{-1}$ (respectively $\sum_{j=1}^{2} \mid \lambda_j \mid\mid \Omega_j \mid^2 < \mathcal{D}^{-2}$ for the complex case). Then the selfsimilar equation* (19) *has a unique solution in $L^1(\mathrm{R})$ (respectively in $L^1(\mathrm{R}^2)$) given by the series*

$$F(x) = \sum_{n=0}^{\infty} \sum_{(i_1,\ldots,i_n)} \lambda_{i_1} \ldots \lambda_{i_n} \ g\left(T_{i_n}^{-1} \ldots T_{i_1}^{-1}(x)\right) . \tag{25}$$

If furthermore $0 < \alpha_{min} \leq k$, then $F \in C^{\alpha_{min}-\varepsilon}(\mathrm{R})$, $\forall \varepsilon > 0$ (respectively $F \in C^{\alpha_{min}-\varepsilon}(\mathrm{R}^2)$ $\forall \varepsilon > 0$).

Proof:
Since $|\lambda_1| < 1$, $|\lambda_2| < 1$ and g is bounded, then series (25) is well defined. It is easily seen that it verifies (19). Remark that for $x \in K$,

$$F(x) = \sum_{n=0}^{\infty} \lambda_{i_1(x)} \ldots \lambda_{i_n(x)} \ \ g\left(T_{i_n(x)}^{-1} \ldots T_{i_1(x)}^{-1}(x)\right)$$

where $(i_1(x), \ldots, i_n(x), \ldots)$ is the code of x.

For $i = (i_1, \ldots, i_n)$, *set* $\lambda_i = \lambda_{i_1} \ldots \lambda_{i_n}$ and $T_i = T_{i_1} \circ \ldots \circ T_{i_n}$; thus

$$\begin{aligned}
\|F\|_{L^1(\mathbf{R})} &\leq \sum_{n=0}^{\infty} \sum_{i=(i_1,\ldots,i_n)} \mid \lambda_i \mid \int \mid g(T_i^{-1}(x)) \mid \ dx \\
&\leq \sum_{n=0}^{\infty} \sum_{i=(i_1,\ldots,i_n)} \mid \lambda_i \mid \int_{I_i} \mid g(T_i^{-1}(x)) \mid \ dx \\
&\leq C \sum_{n=0}^{\infty} \sum_{i=(i_1,\ldots,i_n)} \mid \lambda_i \mid\mid I_i \mid \ .
\end{aligned}$$

(for the complex case, $\mid I_i \mid$ is replaced by $\mid \Omega_i \mid^2$).

By Distortion Lemma

$$\mathcal{D}^{-(n-1)} \mid I_{i_1} \mid \ldots \mid I_{i_n} \mid \leq \mid I_i \mid \leq \mathcal{D}^{n-1} \mid I_{i_1} \mid \ldots \mid I_{i_n} \mid \ .$$

Hence

$$\|F\|_{L^1(\mathbf{R})} \leq C \sum_{n=0}^{\infty} \mathcal{D}^n (\sum_{j=1}^{2} \mid \lambda_j \mid\mid I_j \mid)^n < \infty \, .$$

For the uniqueness of the solutions of (19) in $L^1(\mathrm{R})$, remark that if it has two solutions, it follows from the fact that (20) holds for any N that their difference $\tilde{F}$ is a distribution supported by K and verifies the homogeneous equation

$$\tilde{F} = \sum_{j=1}^{2} \lambda_j \tilde{F} \circ T_j^{-1} \, . \tag{26}$$

But

$$\| \tilde{F} \circ T_j^{-1} \|_{L^1(\mathbf{R})} = \int |\tilde{F}(x)||T_j'(x)| \, dx$$

which, by Lemma 1, is bounded by $\mathcal{D}|I_j| \, \| \tilde{F} \|_{L^1(\mathbf{R})}$; hence if $\sum_{j=1}^{2} \mid \lambda_j \mid\mid I_j \mid < \mathcal{D}^{-1}$, then $\tilde{F}$ has zero norm in $L^1(\mathrm{R})$.

For the complex case, for $z = x + iy$ (with i such that $i^2 = -1$), we write

$$T_j(z) = U_j(x, y) + iV_j(x, y)$$

and we set

$$\Phi(x, y) = (U_j(x, y), V_j(x, y))$$

and

$$\tilde{F} \circ T_j^{-1}(z) = f(\Phi^{-1}(x, y)) \, . \tag{27}$$

Since T_j is complex analytic then $\Phi = (U_j, V_j)$ is C^1 on R^2 and has the properties that

$$\partial_x U_j = \partial_y V_j \, , \;\; \partial_y U_j = -\partial_x V_j \tag{28}$$

$$T_j'(z) = \partial_x T_j(z) = \partial_x U_j(x, y) + i\partial_x V_j(x, y) \tag{29}$$

and

$$T_j'(z) = -i\partial_y T_j(z) = -i(\partial_y U_j(x, y) + i\partial_y V_j(x, y)) \, ; \tag{30}$$

hence

$$\begin{aligned}
\| \tilde{F} \circ T_j^{-1} \|_{L^1(\mathbf{R}^2)} &= \| f \circ \Phi^{-1} \|_{L^1(\mathbf{R}^2)} \\
&= \int |f(x, y)| \; | \partial_x U_j(x, y) \partial_y V_j(x, y) - \partial_y U_j(x, y) \partial_x V_j(x, y) \mid \; dx \, dy \\
&= \int |f(x, y)| \; [(\partial_x U_j(x, y))^2 + (\partial_x V_j(x, y))^2] \, dx \, dy \\
&= \int |f(x, y)| \; |T_j'(x + iy)|^2 \, dx \, dy
\end{aligned}$$

and we conclude as above. The argument(27) and the properties (28), (29) and (30) will be always used for the complex case to compute integrals.

Let us now prove that F is $C^{\alpha_{min} - \varepsilon}(\mathrm{R})$ for any $\varepsilon > 0$. For that we will use the Littlewood-Paley characterization.

We split F as a sum

$$F(x) = \sum_{j\geq 0} F_j(x) \quad \text{where} \quad F_j(x) = \sum_{i\in B_j} \lambda_i g(T_i^{-1}(x)) \ .$$

Let ψ be a function in the Schwartz class such that

$$\hat{\psi}(\xi) = 0 \quad \text{for} \quad |\,\xi\,| \leq 1 \quad \text{and} \quad |\,\xi\,| \geq 8$$

$$\hat{\psi}(\xi) = 1 \quad \text{for} \quad 2 \leq |\,\xi\,| \leq 4 \ .$$

Let $\psi_l(x) = 2^l \psi(2^l x)$, $\omega_{l,j} = F_j * \psi_l$ and $h_{i,l} = (g \circ T_i^{-1}) * \psi_l$.
We recall that a function F belongs to $C^r(\mathrm{R})$ if and only if

$$|\, F * \psi_l(x) \,| \leq C 2^{-rl} \ \ \forall x \in \mathrm{R} \ .$$

We have

$$|h_{i,l}(x)| = |\int g(T_i^{-1}(y)) \ \ \psi_l(x-y)\, dy| \ .$$

Denote by $P_k g_x(h)$ the Taylor developpement of order k of g at x:

$$P_k g_x(h) = \sum_{q\leq k} \frac{g^{(q)}(x)}{q!} h^q \ .$$

It follows from the cancellation of ψ that

$$|h_{i,l}(x)| = 2^l |\int \left(g(T_i^{-1}(y)) - P_{k-1}(g \circ T_i^{-1})_x(y-x) \right) \ \ \psi(2^l(x-y))\, dy| \ .$$

g being a C^k function with derivatives of order less than k well localized, then

$$|h_{i,l}(x)| \leq 2^l \int |\psi(2^l(x-y))| \left(\sup_{u\in[x,y]} |(g \circ T_i^{-1})^{(k)}(u)| \right) \ |x-y|^k\, dy \ .$$

Lemma 1 implies that for $i \in B_j$ and $u \in I_i$

$$\mathcal{D}^{-1} 2^{j-1} \leq \mathcal{D}^{-1} |I_i|^{-1} \leq |(T_i^{-1})'(u)| \leq \mathcal{D} \mid I_i \mid^{-1} \leq \mathcal{D} 2^j \ . \tag{31}$$

Besides, (4) and (5) imply that $|(T_i^{-1})^{(p)}(u)| \leq C 2^j$ for all $p \leq k$; thus thanks to the localization of g and all its derivatives of order less than k, we get

$$\begin{aligned}
|(g \circ T_i^{-1})^{(k)}(u)| \ &\leq \ C_N \frac{2^{kj}}{(1 + |T_i^{-1}(u)|)^N} \\
&\leq \ C_N \frac{2^{kj}}{(1 + \mathcal{D}^{-1} 2^{j-1} |u - x_i|)^N} \quad (\text{with } x_i = T_i(0)) \\
&\leq \ C_N \frac{2^{kj}}{(1 + \mathcal{D}^{-1} 2^{j-1} |x - x_i|)^N} (1 + 2^j |x-u|)^N \\
&\leq \ C_N \frac{2^{kj}}{(1 + \mathcal{D}^{-1} 2^{j-1} |x - x_i|)^N} (1 + 2^j |x-y|)^N \ .
\end{aligned}$$

Therefore

$$| h_{i,l}(x) | \leq C_N \frac{2^{kj}}{(1+\mathcal{D}^{-1}2^{j-1}|x-x_i|)^N} 2^l \int |\psi(2^l(x-y))| \; (1+2^j|x-y|)^N \; |x-y|^k \, dy \; .$$

Hence for $j \leq l$

$$| h_{i,l}(x) | \leq C_N \frac{2^{kj}2^{-kl}}{(1+\mathcal{D}^{-1}2^{j-1}|x-x_i|)^N} \; .$$

Whence, for $j \leq l$

$$| \omega_{l,j}(x) | \leq C_N \sum_{i \in B_j} | \lambda_i | \frac{2^{k(j-l)}}{(1+\mathcal{D}^{-1}2^{j-1} \mid x-x_i \mid)^N} \; .$$

Lemma 3 *For N large enough, there exists a number $C>0$ such that for any $x \in \mathrm{R}$*

$$\sum_{i \in B_j} \frac{1}{(1+\mathcal{D}^{-1}2^{j-1} \mid x-x_i \mid)^N} \leq C \; .$$

Lemma 3 is a consequence of the following one.

Lemma 4 *Let $x \in K$ and L large enough and set $B_j(x) = \{ \; i \in B_j \;\; : \;\; \mid x-x_i \mid \leq L2^{-j}\}$. The cardinality of $B_j(x)$ is bounded independantely of x and j by CL (CL^2 for the complex case).*

Proof:
Thanks to the separation condition (3), we can suppose that the intervals I_i for $i \in B_j(x)$ are disjoint and are all included in the interval of lenght $\sim L2^{-j}$, centered on x. Thus

$$2^{-j} \; card \, B_j(x) \leq CL2^{-j}.$$

Hence Lemma 4.

Using Lemma 3, we obtain for $j \leq l$

$$\begin{aligned} | \omega_{l,j}(x) | &\leq C(\sup_{i \in B_j} | \lambda_i |).2^{k(j-l)} \\ &\leq C2^{(-\alpha_{min}+\varepsilon)j}2^{k(j-l)} \; . \end{aligned}$$

Thus

$$\begin{aligned} \sum_{0 \leq j \leq l} | \omega_{l,j}(x) | &\leq C2^{-kl} \sum_{0 \leq j \leq l} 2^{(-\alpha_{min}+\varepsilon+k)j} \\ &\leq C2^{(-\alpha_{min}+\varepsilon)l} \; . \end{aligned}$$

On the other hand, for $j > l$

$$\begin{aligned} | \omega_{l,j}(x) | &\leq C \sup | F_j(x) | \\ &\leq C \sup_{i \in B_j} | \lambda_i | \\ &\leq C2^{(-\alpha_{min}+\varepsilon)j} \; , \end{aligned}$$

thus
$$\sum_{j>l} \mid \omega_{l,j}(x) \mid \leq C 2^{(-\alpha_{min}+\varepsilon) l} \ .$$
Whence for any $\varepsilon > 0$
$$\mid F * \psi_l(x) \mid \leq C 2^{-(\alpha_{min}-\varepsilon) l} \ . \tag{32}$$
And so $F \in C^{\alpha_{min}-\varepsilon}(\mathrm{R})$ for any $\varepsilon > 0$.

For the complex case, the same arguments give us $F \in C^{\alpha_{min}-\varepsilon}(\mathrm{R}^2)$, $\forall \varepsilon > 0$.

3 Pointwise Hölder regularity

We want now to estimate the Hölder regularity of F at every point.

Proposition 1 *If $x \notin K$ then F is C^k in a neighbourhood of x.*

Proof:
Let $x_0 \notin K$, if $x_0 \notin I$ then $F = g$ in a neighbourhood of x_0;
If $x_0 \in I$ then there exist N and $i = (i_1, \ldots, i_N) \in \{1,2\}^N$ such that $x_0 \in I_i \setminus \bigcup_{i' \in \{1,2\}} I_{ii'}$;
in this neighbourhood of x_0, $F(x) = \sum_{n=0}^{N} \lambda_{i_1} \cdots \lambda_{i_n} \ g((T_{i_1} \circ \cdots \circ T_{i_n})^{-1}(x)) \in C^k(x)$.

Now we give the value of the pointwise regularity $\alpha(x)$ for any point x of K.

Theorem 2 *Suppose that $\alpha_{min} > 0$. Let $x \in K$; then*
$$\alpha(x) = \liminf_{n \to \infty} \frac{\log |\lambda_{i_1(x)} \ldots \lambda_{i_n(x)}|}{\log |I_{i_1(x) \ldots i_n(x)}|} \ . \tag{33}$$

Proof:
We divide the proof into two steps:

3.1 Upper bound for pointwise Hölder regularity

We shall first prove the upper bound for $\alpha(x)$ using the wavelet transform size characterization (15) (the so-called two-microlocal condition) and an asumption on the uniform regularity ($F \in C^\epsilon(\mathrm{R})$ for an $\epsilon > 0$).

Let $\psi(x)$ be a wavelet, set
$$\psi_{a,b}(x) = \frac{1}{a} \psi(\frac{x-b}{a})$$
and let $C_{a,b}(F)$ and $\omega_{a,b}(g)$ be respectivly the wavelet transform of F and g. From the functional equation (20) satiafied by F, we get
$$C_{a,b}(F) = \sum_{n=0}^{N-1} \sum_{|i|=n} \lambda_i \int \psi_{a,b}(t) g(T_i^{-1}(t)) \, dt \ + \ \sum_{|i|=N} \lambda_i \int \psi_{a,b}(t) F(T_i^{-1}(t)) \, dt$$
thus
$$\begin{aligned} C_{a,b}(F) = \ & \sum_{n=0}^{N-1} \sum_{|i|=n} \lambda_i \int \psi_{a,b}(T_i(t)) g(t) T_i'(t) \, dt \\ & + \sum_{|i|=N} \lambda_i \int \psi_{a,b}(T_i(t)) F(t) T_i'(t) \, dt \ . \end{aligned} \tag{34}$$

To estimate the size of the wavelet transform, we will give asymptotic developpements for the composition of a wavelet by contraction T_i. These developpements will be well adapted with the wavelet analysis.

Lemma 5 *Let ψ be a real even compactly supported wavelet with enough smoothness and vanishing moments. Let $b \in I_i$ and $a > 0$ small enough, then*

$$\begin{aligned} \psi_{a,b}(T_i(t)) &= |(T_i^{-1})'(b)| \;\; \psi_{a|(T_i^{-1})'(b)|, T_i^{-1}(b)}(t) \\ &+ \sum_{p=1}^{k-1} \sum_{l=2p}^{k-1+p} a^{l-p} \, A_i^{(p,l)}(b) \;\, \psi^{(p,l)}_{a|(T_i^{-1})'(b)|, T_i^{-1}(b)}(t) \\ &+ (R^i_{a,b})^{(k)}(t) \end{aligned} \tag{35}$$

where $\psi^{(p,l)}(t) = t^l \psi^{(p)}(t)$ is a compactly supported wavelet;

$$A_i^{(p,l)}(b) = \frac{1}{p!} \;\; [(T_i^{-1})'(b)]^{1+l} \quad \sum\nolimits_{2 \le q_1, \ldots, q_p \le k \atop q_1 + \ldots + q_p = l} \; \prod_{m=1}^{p} \frac{T_i^{(q_m)}(T_i^{-1}(b))}{q_m!} \; ,$$

$$|A_i^{(p,l)}(b)| \le |(T_i^{-1})'(b)| \; ;$$

and $(R^i_{a,b})^{(k)}(t)$ is a function supported in $|\, t - T_i^{-1}(b) \,| \le Ca|(T_i^{-1})'(b)|$ such that

$$|(R^i_{a,b})^{(k)}(t)| \le Ca^{k-1} \quad \forall t \tag{36}$$

and

$$\|(R^i_{a,b})^{(k)}(t)\|_{L^1(\mathbb{R})} \le Ca^k |(T_i^{-1})'(b)| \; . \tag{37}$$

For the complex case, the analogous of Lemma 5 is

Lemma 6 *Let ψ be a radial wavelet of* R^2 *with compact support and enough smoothness and cancellation. Let $b = b_1 + ib_2 \in \Omega_j$ and $a > 0$ small enough and denote by $\psi_{a,b}(z)$ the function*

$$\psi_{a,b}(z) = \frac{1}{a^2} \psi \left(\frac{(u,v) - (b_1, b_2)}{a} \right) \quad \textit{for} \quad z = u + iv \; .$$

Then

$$\begin{aligned} \psi_{a,b}(T_j(z)) &= |(T_j^{-1})'(b)|^2 \;\; \psi_{a|(T_j^{-1})'(b)|, T_j^{-1}(b)}(z) \\ &+ \sum_{\alpha} \sum_{p} \sum_{l} a^{l-p} A_j^{(\alpha,l)}(b) \;\, \psi^{(\alpha,l)}_{a|(T_j^{-1})'(b)|, T_j^{-1}(b)}(z) \\ &+ (R^j_{a,b})^{(k)}(z) \end{aligned} \tag{38}$$

where the $\psi^{(\alpha,l)}(z)$ are compactly supported wavelets; $|A_j^{(\alpha,l)}(b)| \le |(T_j^{-1})'(b)|^2$; and $(R^j_{a,b})^{(k)}(z)$ is a function supported in $|\, z - T_j^{-1}(b) \,| \le Ca|(T_j^{-1})'(b)|$ such that

$$|(R^j_{a,b})^{(k)}(z)| \le Ca^{k-2} \tag{39}$$

and

$$\|(R^j_{a,b})^{(k)}(z)\|_{L^1(\mathbb{R}^2)} \leq Ca^k |(T_j^{-1})'(b)|^2 \; . \tag{40}$$

Proofs:
We will only give the proof for any k for the one-dimensional case and for $k=1$ and $k=2$ for the complex case, the proof for $k \geq 3$ is similar.

The support of $\psi_{a,b}(T_i(t))$ is given by $\mid T_i(t) - b \mid \leq C'a$. Since $b \in I_i$, then the mean value theorem and Lemma 1 imply that $\mid t - T_i^{-1}(b) \mid \leq Ca|(T_i^{-1})'(b)|$.

For the real case and for $k=1$,

$$\begin{aligned}
\psi_{a,b}(T_i(t)) &= \frac{1}{a}\psi\left(\frac{T_i(t) - T_i(T_i^{-1}(b))}{a}\right) \\
&= \frac{1}{a}\psi\left(\frac{T_i'(T_i^{-1}(b))(t - T_i^{-1}(b)) + O_i^{(2)}(|t - T_i^{-1}(b)|^2)}{a}\right)
\end{aligned}$$

where $O_i^{(2)}(|t - T_i^{-1}(b)|^2) \leq |t - T_i^{-1}(b)|^2 \ \sup |T_i^{''}| = O(a^2)$ (i.e $\leq Ca^2$).

Hence

$$\begin{aligned}
\psi_{a,b}(T_i(t)) &= \frac{1}{a}\psi\left(\frac{T_i'(T_i^{-1}(b))(t - T_i^{-1}(b))}{a} + O(a)\right) \\
&= \frac{1}{a}\psi\left(\frac{T_i'(T_i^{-1}(b))(t - T_i^{-1}(b))}{a}\right) \; + \; \frac{1}{a}O_\psi^{(1)}(O(a))
\end{aligned}$$

where $O_\psi^{(1)}(u) \leq |u| \sup |\psi'|$; thus since ψ is even then

$$\psi_{a,b}(T_i(t)) = |(T_i^{-1})'(b)| \, \psi_{a|(T_i^{-1})'(b)|, T_i^{-1}(b)}(t) \; + \; (R^i_{a,b})^{(1)}(t)$$

with $(R^i_{a,b})^{(1)}(t) = O(1)$.

Since $(R^i_{a,b})^{(1)}(t)$ is supported in $\mid t - T_i^{-1}(b) \mid \leq Ca|(T_i^{-1})'(b)|$, then

$$\|(R^i_{a,b})^{(1)}(t)\|_{L^1(\mathbb{R})} \leq Ca|(T_i^{-1})'(b)|.$$

Hence Lemma 5 for $k=1$.

Now, for $k \geq 2$ and in the real case

$$\begin{aligned}
\psi_{a,b}(T_i(t)) &= \frac{1}{a}\psi\left(\frac{T_i(t) - T_i(T_i^{-1}(b))}{a}\right) \\
&= \frac{1}{a}\psi\left(\frac{1}{a}\left[\sum_{q=1}^{k} \frac{1}{q!} T_i^{(q)}(T_i^{-1}(b)) \; (t - T_i^{-1}(b))^q + O_i^{(k+1)}(|t - T_i^{-1}(b)|^{k+1})\right]\right) \; ,
\end{aligned}$$

where $O_i^{(k+1)}(|t - T_i^{-1}(b)|^{k+1}) \leq |t - T_i^{-1}(b)|^{k+1} \ \sup |T_i^{(k+1)}| = O(a^{k+1})$. Thus

$$\psi_{a,b}(T_i(t)) = \frac{1}{a}\psi\left(\frac{1}{a}\sum_{q=1}^{k} \frac{1}{q!} T_i^q(T_i^{-1}(b)) \; (t - T_i^{-1}(b))^q + O(a^k)\right)$$

$$
\begin{aligned}
&= \frac{1}{a}\psi\left(\frac{t-T_i^{-1}(b)}{a(T_i^{-1})'(b)}\right) \\
&\quad +\frac{1}{a}\sum_{p=1}^{k-1}\frac{1}{p!}\psi^{(p)}\left(\frac{t-T_i^{-1}(b)}{a(T_i^{-1})'(b)}\right)\left[\frac{1}{a}\sum_{q=2}^{k}\frac{1}{q!}T_i^{(q)}(T_i^{-1}(b))\ (t-T_i^{-1}(b))^q+O(a^k)\right]^p \\
&\quad +\frac{1}{a}\ O(a^k).
\end{aligned}
$$

Remark that

$$
\begin{aligned}
&\frac{1}{a}\left[\frac{1}{a}\sum_{q=2}^{k}\frac{1}{q!}T_i^{(q)}(T_i^{-1}(b))\ (t-T_i^{-1}(b))^q+O(a^k)\right]^p \\
&= \frac{1}{a}\frac{1}{a^p}\ \left[\sum_{q=2}^{k}\frac{1}{q!}T_i^{(q)}(T_i^{-1}(b))\ (t-T_i^{-1}(b))^q\right]^p\ +\ O(a^{k-1}) \\
&= \frac{1}{a}\frac{1}{a^p}\sum_{2\le q_1,\ldots,q_p\le k}\ \prod_{m=1}^{p}T_i^{(q_m)}(T_i^{-1}(b))\ \frac{(t-T_i^{-1}(b))^{q_m}}{q_m!}\ +\ O(a^{k-1}) \\
&= \frac{1}{a}\frac{1}{a^p}\sum_{\substack{2\le q_1,\ldots,q_p\le k\\ q_1+\ldots+q_p<k+p}}\ \prod_{m=1}^{p}T_i^{(q_m)}(T_i^{-1}(b))\ \frac{(t-T_i^{-1}(b))^{q_m}}{q_m!} \\
&\quad +\frac{1}{a}\frac{1}{a^p}\sum_{\substack{2\le q_1,\ldots,q_p\le k\\ q_1+\ldots+q_p\ge k+p}}\ \prod_{m=1}^{p}T_i^{(q_m)}(T_i^{-1}(b))\ \frac{(t-T_i^{-1}(b))^{q_m}}{q_m!} \\
&\quad +O(a^{k-1})
\end{aligned}
$$

and that for any $1\le p\le k-1$ and any $2\le q_1,\ldots,q_p\le k$ such that $q_1+\ldots+q_p\ge k+p$,

$$
\frac{1}{a}\frac{1}{a^p}\prod_{m=1}^{p}\ |T_i^{(q_m)}(T_i^{-1}(b))|\ \frac{|t-T_i^{-1}(b)|^{q_m}}{q_m!}\ \le\ C\frac{1}{a}\frac{1}{a^p}a^{q_1+\ldots+q_p}\le C\frac{a^{k+p}}{a^{p+1}}\le Ca^{k-1}\ .
$$

Hence $\psi_{a,b}(T_i(t))$ is equal to

$$
\begin{aligned}
&\frac{1}{a}\psi\left(\frac{t-T_i^{-1}(b)}{a(T_i^{-1})'(b)}\right) \\
&+\sum_{p=1}^{k-1}\frac{1}{a}\sum_{l=2p}^{k-1+p}\frac{1}{a^p}(t-T_i^{-1}(b))^l\psi^{(p)}\left(\frac{t-T_i^{-1}(b)}{a(T_i^{-1})'(b)}\right)\frac{1}{p!}\sum_{\substack{2\le q_1,\ldots,q_p\le k\\ q_1+\ldots+q_p=l}}\ \prod_{m=1}^{p}\frac{T_i^{(q_m)}(T_i^{-1}(b))}{q_m!} \\
&+(R_{a,b}^i)^{(k)}(t)
\end{aligned}
$$

with $(R_{a,b}^i)^{(k)}(t)$ a function supported in $|\ t-T_i^{-1}(b)\ |\le Ca|(T_i^{-1})'(b)|$ and bounded by Ca^{k-1}, thus

$$
\|(R_{a,b}^i)^{(k)}(t)\|_{L^1(\mathbf{R})}\le Ca^k|(T_i^{-1})'(b)|\ .
$$

For $p = 1, \ldots, k-1$ and $2p \leq l \leq k-1+p$, set

$$\psi^{(p,l)}(t) = t^l \psi^{(p)}(t)$$

and

$$A_i^{(p,l)}(b) = \frac{1}{p!} \; [(T_i^{-1})'(b)]^{1+l} \quad \sum_{\substack{2 \leq q_1, \ldots, q_p \leq k \\ q_1 + \ldots + q_p = l}} \prod_{m=1}^{p} \frac{T_i^{(q_m)}(T_i^{-1}(b))}{q_m!} \; .$$

Hence (38). And thanks to (4) and (5)

$$|A_i^{(l,p)}(b)| \leq |(T_i^{-1})'(b)| \; .$$

Whence Lemma 5.

Now for the complex case, set $d = T_j^{-1}(b) = d_1 + id_2$, $\delta = (d_1, d_2)$, $\xi = (u, v)$ and let $D\Phi = D(U_j, V_j)$ be the differential of Φ (recall that $\Phi(x,y) = (U_j(x,y), V_j(x,y))$ and that $U_j(x,y) + iV_j(x,y) = T_j(x+iy)$), then

$$\begin{aligned}
\psi_{a,b}(T_j(z)) &= \frac{1}{a^2} \; \psi \left(\frac{\Phi(u,v) - (b_1, b_2)}{a} \right) \\
&= \frac{1}{a^2} \; \psi \left(\frac{\Phi(\xi) - \Phi(\delta)}{a} \right) \\
&= \frac{1}{a^2} \; \psi \left(\frac{D\Phi(\delta).(\xi - \delta) + O_j^{(2)}(\|\xi - \delta\|^2)}{a} \right) \\
&= \frac{1}{a^2} \; \psi \left(\frac{D\Phi(\delta).(\xi - \delta) + O(a^2)}{a} \right) \\
&= \frac{1}{a^2} \; \psi \left(\frac{D\Phi(\delta).(\xi - \delta)}{a} \right) + \frac{1}{a^2} \; O_\psi^{(1)}(a) \\
&= \frac{1}{a^2} \; \psi \left(\frac{D\Phi(\delta).(\xi - \delta)}{a} \right) + (R_{a,b}^j)^{(1)}(z)
\end{aligned}$$

with $(R_{a,b}^j)^{(1)}(z) = O(1/a)$, so

$$\|(R_{a,b}^j)^{(1)}(z)\|_{L^1(\mathbb{R}^2)} \leq Ca|(T_j^{-1})'(b)|^2 \; .$$

It is easy to show that properties (28), (29) and (30) imply

$$\frac{|z - T_j^{-1}(b)|}{|(T_j^{-1})'(b)|} = |T_j'(d)| \; \|\xi - \delta\| = \|D\Phi(\delta).(\xi - \delta)\| \; ;$$

Thus, since ψ is radial then

$$\psi_{a,b}(T_j(z)) = |(T_j^{-1})'(b)|^2 \;\; \psi_{a|(T_j^{-1})'(b)|, T_j^{-1}(b)}(z) + (R_{a,b}^j)^{(1)}(z) \; .$$

Hence Lemma 6 for $k = 1$.

For $k = 2$,

$$\psi_{a,b}(T_j(z)) \;\; = \;\; \frac{1}{a^2} \; \psi \left(\frac{D\phi(\delta).(\xi - \delta) + 1/2 \, D^2\phi(\delta).(\xi - \delta)^2 + O_j^{(3)}(\|\xi - \delta\|^3)}{a} \right)$$

where

$$O_j^{(3)}(\|\xi-\delta\|^3) \leq \|\xi-\delta\|^3 \sup \|D^3\Phi\| = O(a^3).$$

Hence,

$$\begin{aligned}
\psi_{a,b}(T_j(z)) &= \frac{1}{a^2}\,\psi\left(\frac{D\Phi(\delta).(\xi-\delta)}{a}\right) \\
&+\frac{1}{a^2}\,\langle\nabla\psi\left(\frac{D\Phi(\delta).(\xi-\delta)}{a}\right)\;,\;\frac{D^2\Phi(\delta).(\xi-\delta)^2}{2a}+O(a^2)\rangle \\
&+O(1).
\end{aligned}$$

Thanks to the fact that ψ is radial, we obtain

$$\begin{aligned}
\psi_{a,b}(T_j(z)) &= \tfrac{1}{a^2}\,\psi\left(\tfrac{\xi-\delta}{a|(T_j^{-1})'(b)|}\right) \\
&+1/2\;\tfrac{1}{a^2}\partial_x\psi\left(\tfrac{\xi-\delta}{a|(T_j^{-1})'(b)|}\right)\;\tfrac{1}{a}\,\partial_x^2 U_j(\delta).(u-d_1)^2 \\
&+\tfrac{1}{a^2}\partial_x\psi\left(\tfrac{\xi-\delta}{a|(T_j^{-1})'(b)|}\right)\;\tfrac{1}{a}\,\partial_x\partial_y U_j(\delta).(u-d_1)(v-d_2) \\
&+1/2\;\tfrac{1}{a^2}\partial_x\psi\left(\tfrac{\xi-\delta}{a|(T_j^{-1})'(b)|}\right)\;\tfrac{1}{a}\,\partial_y^2 U_j(\delta).(v-d_2)^2 \\
&+1/2\;\tfrac{1}{a^2}\partial_y\psi\left(\tfrac{\xi-\delta}{a|(T_j^{-1})'(b)|}\right)\;\tfrac{1}{a}\,\partial_x^2 V_j(\delta).(u-d_1)^2 \\
&+\tfrac{1}{a^2}\partial_y\psi\left(\tfrac{\xi-\delta}{a|(T_j^{-1})'(b)|}\right)\;\tfrac{1}{a}\,\partial_x\partial_y V_j(\delta).(u-d_1)(v-d_2) \\
&+1/2\;\tfrac{1}{a^2}\partial_y\psi\left(\tfrac{\xi-\delta}{a|(T_j^{-1})'(b)|}\right)\;\tfrac{1}{a}\,\partial_y^2 V_j(\delta).(v-d_2)^2 \\
&+O(1).
\end{aligned}$$

Whence Lemma 6 for $k=2$.

Now, let $C_{a,b}^{(p,l)}(F)$ and $\omega_{a,b}^{(p,l)}(g)$ be respectively the $\psi^{(p,l)}$-wavelet transform of F and g. Using Lemma 5 and equation (34), the ψ-wavelet transform of F will satisfy the following equation

$$
\begin{aligned}
C_{a,b}(F) = \quad & \sum_{n=0}^{N-1} \sum_{|i|=n} \lambda_i \, |(T_i^{-1})'(b)| \; \omega_{a|(T_i^{-1})'(b)|, T_i^{-1}(b)}(gT_i') \\
& + \sum_{|i|=N} \lambda_i \, |(T_i^{-1})'(b)| \; C_{a|(T_i^{-1})'(b)|, T_i^{-1}(b)}(FT_i') \\
& + \sum_{p=1}^{k-1} \sum_{l=2p}^{k-1+p} a^{l-p} \sum_{n=0}^{N-1} \sum_{|i|=n} \lambda_i \; A_i^{(p,l)}(b) \; \omega^{(p,l)}_{a|(T_i^{-1})'(b)|, T_i^{-1}(b)}(gT_i') \\
& + \sum_{p=1}^{k-1} \sum_{l=2p}^{k-1+p} a^{l-p} \sum_{|i|=N} \lambda_i \; A_i^{(p,l)}(b) \; C^{(p,l)}_{a|(T_i^{-1})'(b)|, T_i^{-1}(b)}(FT_i') \\
& + \sum_{n=0}^{N-1} \sum_{|i|=n} \lambda_i \int (R^i_{a,b})^{(k)}(t) \, g(t) \, T_i'(t) \, dt \\
& + \sum_{|i|=N} \lambda_i \int (R^i_{a,b})^{(k)}(t) \, F(t) \, T_i'(t) \, dt
\end{aligned}
\tag{41}
$$

For the complex case, using Lemma 6, we get an equation similar to the previous one ($|(T_i^{-1})'(b)|$ will be replaced by $|(T_i^{-1})'(b)|^2$, T_i' by $|T_i'|^2$ and the indexes (p,l) by the (α,p,l)).

We are now ready to estimate the size of the wavelet transform near each point of K. Define

$$\Lambda_j(x) = \sup_{i \in B_j(x)} | \lambda_i |$$

and

$$L_j(x) = \sum_{l=1}^{j} \Lambda_l(x) 2^{-A(j-l)} \quad \text{with} \quad A > \alpha_{max} \; .$$

And let us first estimate the order of the magnitude of the λ_i for $i \in B_j$.

Lemma 7

$$
\begin{aligned}
\liminf_{j\to\infty} \frac{\log L_j(x)}{-j \log 2} = \liminf_{j\to\infty} \frac{\log \Lambda_j(x)}{-j \log 2} \quad & = \quad \liminf_{j\to\infty} \inf_{i \in B_j(x)} \frac{\log |\lambda_i|}{\log \mid I_i \mid} \\
& = \quad \liminf_{n\to\infty} \frac{\log |\lambda_{i_1(x)} \ldots \lambda_{i_n(x)}|}{\log |I_{i_1(x)\ldots i_n(x)}|}
\end{aligned}
$$

and $\forall x \in \mathrm{R}$ *and* $i \in B_j$

$$| \lambda_i | \leq C L_j(x) (1 + 2^j \mid x - x_i \mid)^A \; . \tag{42}$$

Proof:
Clearly because $L_j(x) \geq \Lambda_j(x)$, it suffices to show that

$$\liminf_{j\to\infty} \frac{\log L_j(x)}{-j\log 2} \geq \liminf_{j\to\infty} \frac{\log \Lambda_j(x)}{-j\log 2} \ ;$$

this holds because $2^{-Al} \leq \Lambda_l(x)$. Now, inequality (42) is trivial for $i \in B_j(x)$ because $\lambda_i \leq \Lambda_j(x) \leq L_j(x)$. And for $i \notin B_j(x)$, let $\bar{i}$ be the largest subbranch such that $\bar{i} \in B_j(x)$, clearly $|x - x_i| \sim |I_{\bar{i}}|$ and $|\lambda_i| \leq \Lambda_l(x)$ with l such that $|I_{\bar{i}}| \sim 2^{-l}$, (because all the $|\lambda_j|$ are < 1), so that

$$|\lambda_i| \leq L_j(x)\frac{\Lambda_l(x)}{L_j(x)} \leq L_j(x)2^{A(j-l)} \leq L_j(x)(C2^j|x-x_i|)^A$$

hence Lemma 7.

We shall now prove the following proposition:

Proposition 2 *Let $x \in K$, $J \in \mathrm{N}$ large enough such that $\Lambda_J(x) \geq \frac{1}{2}L_J(x)$, then there exists a branch $j^0 = (j_1^0, ..., j_n^0)$ in $B_J(x)$, $b \in I_{j^0}$ and $a \sim 2^{-J}$ such that*

$$|b - x| \leq Ca$$

and

$$|\ C_{a,b}(F) - O(a^k)\ | \geq C\Lambda_J(x)\ .$$

Proof:
Let j^0 be a branch of $B_J(x)$ for which $\Lambda_J(x) = \sup_{i\in B_J(x)} |\lambda_i|$ is reached (j^0 exists because of Lemma 4) and $a \sim 2^{-J}$.

We can write equation (41) differently

$$\begin{aligned}
C_{a,b}(F) \quad = \quad & \sum_{j=0}^{J-1}\sum_{i\in B_j} \lambda_i\ |(T_i^{-1})'(b)|\ \omega_{a|(T_i^{-1})'(b)|,T_i^{-1}(b)}(gT_i') && (43)\\
& + \sum_{i\in B_J} \lambda_i\ |(T_i^{-1})'(b)|\ C_{a|(T_i^{-1})'(b)|,T_i^{-1}(b)}(FT_i') && (44)\\
& + \sum_{p=1}^{k-1}\sum_{l=2p}^{k-1+p} a^{l-p}\sum_{j=0}^{J-1}\sum_{i\in B_j} \lambda_i\ A_i^{(p,l)}(b)\ \omega^{(p,l)}_{a|(T_i^{-1})'(b)|,T_i^{-1}(b)}(gT_i') && (45)\\
& + \sum_{p=1}^{k-1}\sum_{l=2p}^{k-1+p} a^{l-p}\sum_{i\in B_J} \lambda_i\ A_i^{(p,l)}(b)\ C^{(p,l)}_{a|(T_i^{-1})'(b)|,T_i^{-1}(b)}(FT_i') && (46)\\
& + O(a^k)\ ;
\end{aligned}$$

$O(a^k)$ comes from the estimation of

$$\begin{aligned}
& \sum_{j=0}^{J-1}\sum_{i\in B_j} \lambda_i \int (R^i_{a,b})^{(k)}(t)\, g(t)\, T_i'(t)\, dt && (47)\\
& + \sum_{i\in B_J} \lambda_i \int (R^i_{a,b})^{(k)}(t)\, F(t)\, T_i'(t)\, dt\ . && (48)
\end{aligned}$$

For the term (47), we use the localization of g to bound it by

$$\sum_{j=0}^{J-1} \sum_{i \in B_j} |\lambda_i| \int |(R^i_{a,b})^{(k)}(t)| \, (1+|t|)^{-M} \, |T_i'(t)| \, dt$$
$$\leq C \sum_{j=0}^{J-1} \sum_{i \in B_j} |\lambda_i| \int |(R^i_{a,b})^{(k)}(t)| (1+\mathcal{D}^{-1}2^{j-1} \mid b-x_i \mid)^{-M} \; (1+|t-T_i^{-1}(b)|)^M 2^{-j} \, dt$$

which by Lemma 7 and for M large enough, will be bounded by

$$\sum_{j=0}^{J-1} \sum_{i \in B_j} L_j(b)(1+\mathcal{D}^{-1}2^{j-1} \mid b-x_i \mid)^{A-M} \int_{|t-T_i^{-1}(b)| \leq Ca|(T_i^{-1})'(b)|} Ca^{k-1}2^{-j}(1+Ca2^j)^M \, dt$$
$$\leq Ca^{k-1} \sum_{j=0}^{J-1} L_j(b) \sum_{i \in B_j} (1+\mathcal{D}^{-1}2^{j-1} \mid b-x_i \mid)^{A-M} a$$

and thanks to Lemma 3), it will be bounded by $Ca^k \sum_{j=0}^{J-1} L_j(b)$ so by Ca^k.

For the term (48), we use the boundedness of F to estimate it by

$$\sum_{i \in B_J} |\lambda_i| \int_{|t-T_i^{-1}(b)| \leq Ca|(T_i^{-1})'(b)|} Ca^{k-1}2^{-j} \, dt$$
$$\leq Ca^k \sum_{i \in B_J} |\lambda_i|$$
$$\leq C'a^k$$

We shall first estimate the term (44) corresponding to the branch $i = j^0$: by asumption, F is not uniformly C^k on a non empty closed subset $\tilde{K}$ of $]0,1[$; thus thanks to the characterization of the uniform Hölder regularity by the wavelet transform, there exist $a_n \to 0$, $b_n \in \tilde{K}$ and $C_n \to +\infty$ such that

$$\mid C_{a_n,b_n}(F) \mid \geq C_n a_n^k \; . \tag{49}$$

Take $b = T_{j^0}(b_n)$ for n large enough and $a = a_n|T_{j^0}'(b_n)|$; then

$$\mid x-b \mid \leq |x-x_{j^0}| + |x_{j^0}-b| \leq L2^{-J} + |I_{J^0}| \leq C2^{-J}$$

and

$$\mid C_{a|(T_{j^0}^{-1})'(b)|, T_{j^0}^{-1}(b)}(F) \mid \geq C_n a^k |(T_{j^0}^{-1})'(b)|^k \; . \tag{50}$$

On the other hand,

$$|(T_{j^0}^{-1})'(b)| \mid C_{a|(T_{j^0}^{-1})'(b)|, T_{j^0}^{-1}(b)}(FT_{j^0}') \mid = \frac{1}{a} \mid \int \psi\left(\frac{t-T_{j^0}^{-1}(b)}{a(T_{j^0}^{-1})'(b)}\right) \; F(t) \; T_{j^0}'(t) \, dt|$$
$$= \frac{1}{a} \mid \int \psi\left(\frac{t-T_{j^0}^{-1}(b)}{a(T_{j^0}^{-1})'(b)}\right) \; F(t) \; \left(T_{j^0}'(T_{j^0}^{-1}(b)) + O^{(2)}_{j^0}(|t-T_{j^0}^{-1}(b)|)\right) \, dt|$$

with

$$O^{(2)}_{j^0}(|t-T^{-1}_{j^0}(b)|) \le (\sup |T''_{j^0}(u)|)\ |t-T^{-1}_{j^0}(b)| \le C|T'_{j^0}(b)|^2\ |t-T^{-1}_{j^0}(b)|\ .$$

Thus

$$|(T^{-1}_{j^0})'(b)|\ |\ C_{a|(T^{-1}_{j^0})'(b)|,T^{-1}_{j^0}(b)}(FT'_{j^0})\ | \ge \frac{1}{a|(T^{-1}_{j^0})'(b)|}\ |\int \psi\left(\frac{t-T^{-1}_{j^0}(b)}{a(T^{-1}_{j^0})'(b)}\right)\ F(t)\,dt|$$
$$-C\frac{|(T'_{j^0}(b)|}{a|(T^{-1}_{j^0})'(b)|}\int |\ \psi\left(\frac{t-T^{-1}_{j^0}(b)}{a(T^{-1}_{j^0})'(b)}\right)|\ \ |F(t)|\ |t-T^{-1}_{j^0}(b)|\,dt\ .$$

It follows from (50) and the fact that F is bounded that

$$|(T^{-1}_{j^0})'(b)|\ \ |\ C_{a|(T^{-1}_{j^0})'(b)|,T^{-1}_{j^0}(b)}(FT'_{j^0})\ | \ge C_n a^k |(T^{-1}_{j^0})'(b)|^k - Ca\ .$$

Therefore

$$|\lambda_{j^0}|\ |(T^{-1}_{j^0})'(b)|\ |C_{a|(T^{-1}_{j^0})'(b)|,T^{-1}_{j^0}(b)}(FT'_{j^0})| \ge \Lambda_J(x)(C_n a^k \mathcal{D} 2^{kJ} - Ca)\ . \qquad (51)$$

Now, let us estimate the righthand side of (43).
For $i \in B_j$ and $0 \le j \le J-1$

$$\begin{aligned}
&|(T^{-1}_i)'(b)|\ \ |\ \omega_{a|(T^{-1}_i)'(b)|,T^{-1}_i(b)}(gT'_i)\ | \\
&= \frac{1}{a}\ |\int \psi\left(\frac{t-T^{-1}_i(b)}{a(T^{-1}_i)'(b)}\right)\ g(t)\ T'_i(t)\,dt| \\
&= \frac{1}{a}\ |\int \psi\left(\frac{t-T^{-1}_i(b)}{a(T^{-1}_i)'(b)}\right)\ \left(g(t)\ T'_i(t) - P_{k-1}(gT'_i)_{T^{-1}_i(b)}(t-T^{-1}_i(b))\right)\ dt|.
\end{aligned}$$

Using the mean value theorem, the previous term will be bounded by

$$\frac{1}{a}\int |\psi\left(\frac{t-T^{-1}_i(b)}{a(T^{-1}_i)'(b)}\right)|\ \left(\sup_{u\in[t,T^{-1}_i(b)]} |(gT'_i)^{(k)}(u)|\right)\ |t-T^{-1}_i(b)|^k\,dt\ ,$$

it follows from the formula $(gT'_i)^{(k)}(u) = \sum_{q=0}^{k} C^q_k\ g^{(k-q)}(u)\ T^{(q+1)}_i(u)$, the asumptions (4), (5), Lemma 1 and the localization of the derivative of order k of g, that

$$\begin{aligned}
|(gT'_i)^{(k)}(u)| &\le \frac{C_N}{(1+|u|)^N}\ |T'_i(u)| \\
&\le \frac{C_N}{(1+|T^{-1}_i(b)|)^N}\ (1+|u-T^{-1}_i(b)|)^N\ |T'_i(T^{-1}_i(b))| \\
&\le \frac{C_N}{(1+\mathcal{D}^{-1}2^{j-1}|b-x_i|)^N}\ (1+|t-T^{-1}_i(b)|)^N\ |T'_i(T^{-1}_i(b))|\ .
\end{aligned}$$

So $|(T^{-1}_i)'(b)|\ \ |\ \omega_{a|(T^{-1}_i)'(b)|,T^{-1}_i(b)}(gT'_i)\ |$ is bounded by

$$\frac{C_N}{(1+\mathcal{D}^{-1}2^{j-1}|b-x_i|)^N}\frac{1}{a|(T^{-1}_i)'(b)|}\int |\psi\left(\frac{t-T^{-1}_i(b)}{a(T^{-1}_i)'(b)}\right)|\ (1+|t-T^{-1}_i(b)|)^N\ |t-T^{-1}_i(b)|^k dt$$

and because $a|(T_i^{-1})'(b)| \sim 2^{-J}2^j < 1$, the previous term will be bounded by

$$C \; \frac{a^k|(T_i^{-1})'(b)|^k}{(1+\mathcal{D}^{-1}2^{j-1}|b-x_i|)^N} \; .$$

Thus for $0 \le j \le J-1$

$$\begin{aligned}
&\sum_{i\in B_j} |\lambda_i| \;\; |(T_i^{-1})'(b)| \;\; | \, \omega_{a|(T_i^{-1})'(b)|, T_i^{-1}(b)}(gT_i') \, | \\
&\le C \sum_{i\in B_j} | \, \lambda_i \, | \; \frac{a^k|(T_i^{-1})'(b)|^k}{(1+\mathcal{D}^{-1}2^{j-1}|b-x_i|)^N} \\
&\le C \sum_{i\in B_j} \frac{| \, \lambda_i \, |}{(1+\mathcal{D}^{-1}2^{j-1} \mid b-x_i \mid)^N} a^k 2^{kj}
\end{aligned}$$

which by the second part of Lemma 7 will be bounded by

$$\begin{aligned}
&Ca^k2^{kj}L_j(b) \sum_{i\in B_j} \frac{1}{(1+\mathcal{D}^{-1}2^{j-1} \mid b-x_i \mid)^{N-A}} \\
&\le Ca^k2^{kj}L_j(b) \quad \text{(because of Lemma 3).}
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{j\le J-1}\sum_{i\in B_j} |\lambda_i| \; |(T_i^{-1})'(b)| \;\; | \, \omega_{a|(T_i^{-1})'(b)|, T_i^{-1}(b)}(gT_i') \, | \quad &\le \quad Ca^k \sum_{j\le J-1} 2^{kj}L_j(b) \\
&\le \quad Ca^k2^{kJ}L_J(b) \qquad\qquad (52)
\end{aligned}$$

because we can assume that $\alpha_{max} < k$.

Consider now the term (44) (for which we exclude $i = j^0$): the function F is not uniformly C^k on $\tilde{K}$, put $\tilde{K}_\epsilon = \tilde{K} + [-\epsilon, \epsilon]$ where ϵ is small enough so that $\tilde{K}_\epsilon$ stays in I. Thus outside $\tilde{K}_\epsilon$, F is uniformly C^k.
Recall that $b = T_{j^0}(b_n)$ and that $b_n \in \tilde{K}$, thus it follows from the separation condition (3) that $T_i^{-1}(b) \notin \tilde{K}_\epsilon$ for all $i \in B_J$, $i \neq j^0$, hence

$$\begin{aligned}
&| \, C_{a|(T_i^{-1})'(b)|, T_i^{-1}(b)}(FT_i') \, |= \frac{1}{a|(T_i^{-1})'(b)|} \, | \int \psi\left(\frac{t-T_i^{-1}(b)}{a|(T_i^{-1})'(b)|}\right) \; F(t) \; T_i'(t) \, dt| \\
&= \frac{1}{a|(T_i^{-1})'(b)|} \, | \int \psi\left(\frac{t-T_i^{-1}(b)}{a|(T_i^{-1})'(b)|}\right) \; F(t) \; \left(T_i'(T_i^{-1}(b)) + O_i^{(2)}(|t-T_i^{-1}(b)|)\right) \; dt|
\end{aligned}$$

which is bounded by

$$\begin{aligned}
&|T_i'(T_i^{-1}(b))| \; \frac{1}{a|(T_i^{-1})'(b)|} \, | \int \psi\left(\frac{t-T_i^{-1}(b)}{a|(T_i^{-1})'(b)|}\right) \; F(t) \, dt| \\
&+C|T_i'(T_i^{-1}(b))|^2 \; \frac{1}{a|(T_i^{-1})'(b)|} \; \int |\psi\left(\frac{t-T_i^{-1}(b)}{a|(T_i^{-1})'(b)|}\right)| \; |F(t)| \, |t-T_i^{-1}(b)| \, dt \\
\le \quad & C|T_i'(T_i^{-1}(b))| \; a^k \; |(T_i^{-1})'(b)|^k + C|T_i'(T_i^{-1}(b))| \; a \; .
\end{aligned}$$

Thus

$$\sum_{\substack{i\in B_J\\ i\neq j^0}} |\lambda_i|\,|(T_i^{-1})'(b)|\ |C_{a|(T_i^{-1})'(b)|,T_i^{-1}(b)}(FT_i')| \leq C(a^k 2^{kJ}+a)L_J(b)\ . \qquad (53)$$

Let us now estimate the terms (45) and (46): thanks to the smoothness, the cancellation and the localization of the wavelets $\psi^{(p,l)}$, and the property $|A_i^{(p,l)}| \leq |(T_i^{-1})'(b)|$, the previous arguments give us

$$\begin{aligned}
&\sum_{p=1}^{k-1}\sum_{l=2p}^{k-1+p} a^{l-p} \sum_{\substack{i\in B_J\\ i\neq j^0}} |\lambda_i|\ |A_i^{(p,l)}(b)|\ |C^{(p,l)}_{a|(T_i^{-1})'(b)|,T_i^{-1}(b)}(FT_i')|\\
&\leq C\sum_{p=1}^{k-1}\sum_{l=2p}^{k-1+p} a^{l-p}(a^k 2^{kJ}+a)L_J(b)\\
&\leq C\sum_{p=1}^{k-1} a^p a^k 2^{kJ} L_J(b)\\
&\leq Caa^k 2^{kJ} L_J(b)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{p=1}^{k-1}\sum_{l=2p}^{k-1+p} a^{l-p} \sum_{j=0}^{J-1}\sum_{i\in B_j} |\lambda_i|\ |A_i^{(p,l)}(b)|\ |\omega^{(p,l)}_{a|(T_i^{-1})'(b)|,T_i^{-1}(b)}(gT_i')|\\
&\leq C\sum_{p=1}^{k-1}\sum_{l=2p}^{k-1+p} a^{l-p} a^k 2^{kJ} L_J(b)\\
&\leq C\sum_{p=1}^{k-1} a^p a^k 2^{kJ} L_J(b)\\
&\leq Caa^k 2^{kJ} L_J(b)\ .
\end{aligned}$$

On the other hand, since F is bounded, then

$$\begin{aligned}
\sum_{p=1}^{k-1}\sum_{l=2p}^{k-1+p} a^{l-p}|\lambda_{j^0}|\ |A_{j^0}^{(p,l)}(b)|\ |C^{(p,l)}_{a|(T_{j^0}^{-1})'(b)|,T_{j^0}^{-1}(b)}(FT_{j^0}')| \ &\leq\ C\sum_{p=1}^{k-1}\sum_{l=2p}^{k-1+p} a^{l-p}\Lambda_J(x)\\
&\leq\ Ca\Lambda_J(x)\ .
\end{aligned}$$

Therefore

$$|\ C_{a,b}(F) - O(a^k) -\ \lambda_{j^0}\ |(T_{j^0}^{-1})'(b)|\ C_{a|(T_{j^0}^{-1})'(b)|,T_{j^0}^{-1}(b)}(FT_{j^0}')\ | \leq Ca^k 2^{kJ} L_J(b)\ .$$

Choose C_n large enough, then (51) yields

$$|C_{a,b}(F) - O(a^k)| \geq \frac{1}{2}\,|\lambda_{j^0}|\ |(T_{j^0}^{-1})'(b)|\ \ |\ C_{a|(T_{j^0}^{-1})'(b)|,T_{j^0}^{-1}(b)}(FT_{j^0}')\ |\ \ . \qquad (54)$$

Whence (51) and (54) yield Proposition 2.

3.2 Lower bound for pointwise Hölder regularity

Using definition (6), we will show the lower bound for the Hölder exponent $\alpha(x)$.

Let $x \in K$ and $a(x) = \liminf_{n\to\infty} \frac{\log|\lambda_{i_1(x)}\cdots\lambda_{i_n(x)}|}{\log|I_{i_1(x)\ldots i_n(x)}|}$; for $\beta < a(x)$ define

$$PF_x(h) = \sum_j \sum_{i\in B_j} P_{[\beta]}(g\circ T_i^{-1})_x(h) \ .$$

We have

$$| PF_x(h) | \leq \sum_{j=0}^{\infty} \sum_{i\in B_j} | \lambda_i | \frac{2^{j\beta}}{(1+\mathcal{D}^{-1}2^{j-1} \mid x-x_i \mid)^N}$$

hence thanks to Lemma 3 $PF_x(h)$ converges.

Let J such that $2^{-J} \leq |h| < 2.2^{-J}$ and $L = |h|^{-\epsilon}$ where ϵ is small enough; write

$$\begin{aligned}
F(x+h) - PF_x(h) &= \sum_{j<J} \sum_{\{i\in B_j \,:\, |x-x_i|\leq L2^{-j}\}} \lambda_i \left(g(T_i^{-1}(x+h)) - P_{[\beta]}(g\circ T_i^{-1})_x(h)\right) \\
&+ \sum_{j<J} \sum_{\{i\in B_j \,:\, |x-x_i|> L2^{-j}\}} \lambda_i \left(g(T_i^{-1}(x+h)) - P_{[\beta]}(g\circ T_i^{-1})_x(h)\right) \\
&+ \sum_{i\in B_J} \lambda_i F(T_i^{-1}(x+h)) - \sum_{j\geq J} \sum_{i\in B_j} \lambda_i P_{[\beta]}(g\circ T_i^{-1})_x(h) \ .
\end{aligned}$$

For each j of the series of the first term, there is $O(L)$ terms (because of Lemma 4), thus using the mean value theorem, the first term will be bounded by

$$\begin{aligned}
CL\sum_{j<J}\sum_{i\in B_j} L_j(x)(1+L)^A \mid h \mid^{[\beta]+1} 2^{j([\beta]+1)} &\leq C \mid h \mid^{[\beta]+1} \sum_{j<J} 2^{-\beta j} 2^{j([\beta]+1)} L^{1+A} \\
&\leq C \mid h \mid^{\beta} L^{1+A} \\
&\leq C \mid h \mid^{\beta-\epsilon(1+A)} \ .
\end{aligned}$$

Now, for i such that $\mid x - x_i \mid > L2^{-j}$, there exists a constant C such that $\mid x - x_i \mid \leq CL2^{-j}$, hence the second term is bounded by

$$\begin{aligned}
& CL\sum_{q=0}^{[\beta]}\sum_{j<J} L_j(x)\frac{(1+CL)^A}{(1+\frac{\mathcal{D}^{-1}}{2}L)^N} \mid h \mid^q 2^{qj} \\
&\leq C\sum_{q=0}^{[\beta]}\sum_{j<J} 2^{-\beta j} \mid h \mid^q 2^{qj} L^{-N+1+A} \\
&\leq C\sum_{q=0}^{[\beta]} \mid h \mid^q L^{-N+1+A} \ ;
\end{aligned}$$

choosing N large enough the previous sum will be bounded by $\mid h \mid^{\beta-\epsilon'}$ for any $\epsilon' > 0$.

It follows from the fact that F is bounded that the third term is bounded by $C\Lambda_J(x)$, hence by $C|h|^{a(x)-\varepsilon}$ because of the first part of Lemma 7 and the fact that $|h| \sim 2^{-J}$.

The fourth term is bounded by

$$C \sum_{q=0}^{[\beta]} | h |^q \sum_{j \geq J} \sum_{i \in B_j} | \lambda_i | \frac{2^{jq}}{(1 + \mathcal{D}^{-1} 2^{j-1} | x - x_i |)^N} .$$

We split this sum into the sets

$$B_{j,l} = \{ i \in B_j \ : \ 2^l <| I_i |^{-1} | x - x_i | \leq 2^{l+1} \} .$$

The cardinality of $B_{j,l}$ is $O(2^l)$ (because of Lemma 4) and for $i \in B_{j,l}$, $| \lambda_i | \leq C L_j(x) 2^{Al}$ (because of Lemma 7). Thus the fourth term is bounded by

$$\begin{aligned} C \sum_{q=0}^{[\beta]} | h |^q \sum_{j \geq J} \sum_{l} L_j(x) 2^{Al} 2^{qj} 2^{l(1-N)} \quad &\leq \quad C \sum_{q=0}^{[\beta]} | h |^q \sum_{j \geq J} 2^{(q-\beta)j} \\ &\leq \quad C \, | h |^\beta \ . \end{aligned}$$

The proof of Theorem 2 is now achieved.

4 Computation of the spectrum of singularities

To determine the spectrum of singularities, we will construct probability measures supported by the sets of singularities and then use the following Lemma (see [10]):

Lemma 8 *Let H^s be the Hausdorff measure of dimension s. Let ν be a probability measure on R^m, $A \subset \mathrm{R}^m$ and C such that $0 < C < \infty$*

- *If* $\limsup_{r \to 0} \dfrac{\nu(B(x,r))}{r^s} < C \quad \forall x \in A$ *then* $H^s(A) \geq \frac{\nu(A)}{C}$.
- *If* $\limsup_{r \to 0} \dfrac{\nu(B(x,r))}{r^s} > C \quad \forall x \in A$ *then* $H^s(A) \leq \frac{2^s}{C}$.

Let μ be the probability measure on $[0,1]$ which associates the weight $| \lambda_{i_1 \dots i_n} | \Lambda^{-n}$ (with $\Lambda = |\lambda_1| + |\lambda_2|$) for each interval $I_{i_1 \dots i_n}$. This measure is supported by K and satisfies

- (A_1) There exists a constant $C \geq 1$ such that for any branchs $i = (i_1, \dots, i_l)$ and $j = (j_1, \dots, j_p)$

  $C^{-1} \mu(I_{i_1 \dots i_l}) \mu(I_{j_1 \dots j_p}) \leq \mu(I_{i_1 \dots i_l j_1 \dots j_p}) \leq C \mu(I_{i_1 \dots i_l}) \mu(I_{j_1 \dots j_p})$ (for our case $C = 1$)

  We will denote the previous property by $\mu(I_{i,j}) \approx \mu(I_i) \, \mu(I_j)$.

On the other hand, we have

- (A_2)
  $$| I_{i,j} | \approx | I_i || I_j | \qquad \text{(because of the Distortion Lemma)}$$
- (A_3)
  $$\limsup_{n \to \infty} \frac{1}{n} \log \left(\sup_{|i|=n} | I_i | \right) < 0 \ .$$

By subadditivity argument, Brown, Michon and Peyrière (see [7], [18] and [20]); Collet, Lebowitz and Porzio (see [8]) proved that asumptions (A_1) and (A_2) imply that the sequence $C_n(x,y) := \frac{1}{n}\log\left(\sum_{|i|=n}\mu(I_i)^{x+1} \mid I_i \mid^{-y}\right)$ has a finite limit $C(x,y)$ for any $(x,y)\in \mathrm{R}^2$, that $C(x,y)$ is C^2 and $D^2C(.,.)\neq 0$. It results that the sequence

$$\tilde{C}_n(x,y) := \frac{1}{n}\log\left(\sum_{|i|=n}|\lambda_i|^{x+1} \mid I_i \mid^{-y}\right)$$

goes to

$$\tilde{C}(x,y) := C(x,y) + (x+1)\log\Lambda$$

for any $(x,y)\in\mathrm{R}^2$, besides $\tilde{C}(x,y)$ is C^2 and $D^2\tilde{C}(.,.)\neq 0$ on R^2.

Consider the set $\Delta = \{(x,y)\in\mathrm{R}^2 \ : \ \tilde{C}(x,y) < 0\}$. Δ is not empty because of (A_3). On the other hand, since $\tilde{C}(x,y)$ is convex, nonincreasing as a function of x (because the $|\lambda_j|$ are <1) and nondecreasing as a function of y, the set Δ, if it contains a point (a,b), also contains the whole quadrant $\{(a+x, b-y) \ : \ x\geq 0 \, and \, y\geq 0\}$. It results that there exists a function $\varphi : \mathrm{R}\to\overline{\mathrm{R}}$ nondecreasing and concave(thus almost everywhere differentiable) such that the interior of Δ is identical to the set $\{(x,y)\in\mathrm{R}^2 \ : \ y<\varphi(x-0)\}$. Moreover, $\tilde{C}(t,\varphi(t)) = 0, \forall t\in\mathrm{R}$. Thus the inverse function theorem yields that for t such that $D_2\tilde{C}(t,\varphi(t))\neq 0$, φ is C^1.

In [18], Michon proved that for every $(x,y)\in\mathrm{R}^2$, there exists a probability measure $\mu_{x,y}$ on the tree $\{1,2\}^{\mathbb{N}}$ such that for any branchs i and j :

$$\mu_{x,y}(i,j) \approx \mu_{x,y}(i)\ \mu_{x,y}(j) \tag{55}$$

and

$$\mu_{x,y}(i) \approx \mu(I_i)^{x+1} \mid I_i \mid^{-y} e^{-|i|C(x,y)} \ . \tag{56}$$

Thanks to the bijection π between the tree $\{1,2\}^{\mathbb{N}}$ and K, denote by $\tilde{\mu}_{x,y}$ the associated probability measure supported in K. Hence for any branchs i and j

$$\tilde{\mu}_{x,y}(I_{ij}) \approx \tilde{\mu}_{x,y}(I_i)\tilde{\mu}_{x,y}(I_j) \tag{57}$$

and

$$\tilde{\mu}_{x,y}(I_i) \approx |\lambda_i|^{x+1} \mid I_i \mid^{-y} e^{-|i|\tilde{C}(x,y)} \ . \tag{58}$$

This measure is called a Gibbs measure and was studied separately by Olsen and Rand (see [19] and [21]). The Gibbs measure will be used in Lemma 8 in order to compute the spectrum of singularities of our selfsimilar function F.

Proposition 3 *Let $\alpha < k$ and $d(\alpha)$ be the Hausdorff dimension of the set E^α of points x where $\alpha(x)=\alpha$; Then $d(\alpha)$ is concave, equals to $-\infty$ outside $[\alpha_{min},\alpha_{max}]$, and on this interval for $\alpha=\varphi'(q)$ (hence a-a $\alpha\in[\alpha_{min},\alpha_{max}]$)*

$$d(\alpha) = \inf_{p\in\mathbb{R}}(\alpha p - \varphi(p-1)) \ .$$

Proof:
Let $x \in K$ such that $\alpha(x) = \alpha$, $r > 0$, $s > 0$, $p \in \mathrm{R}$ and j such that $2^{-j} \leq r < 2.2^{-j}$.
We have

$$\frac{\tilde{\mu}_{p-1,\varphi(p-1)}(B(x,r))}{r^s} \sim \sum_{i \in B_j(x)} \frac{\tilde{\mu}_{p-1,\varphi(p-1)}(I_i)}{|I_i|^s}$$

$$\sim \sup_{i \in B_j(x)} \frac{\tilde{\mu}_{p-1,\varphi(p-1)}(I_i)}{|I_i|^s} \quad \text{(because the cardinality of } B_j(x) \leq C)$$

$$\sim \sup_{i \in B_j(x)} |\lambda_i|^p |I_i|^{-\varphi(p-1)-s} \quad \text{(because of (58) and the fact that } \tilde{C}(t,\varphi(t)) = 0) \ .$$

Suppose that $s > -\varphi(p-1) + p\alpha$ then Theorem 2 implies that

$$\limsup_{r \to 0} \frac{\tilde{\mu}_{p-1,\varphi(p-1)}\,(B(x,r))}{r^s} = +\infty$$

thus, using the second point of Lemma 8, we get $H^s(E^\alpha) = 0$.

Hence

$$d(\alpha) \leq \alpha p - \varphi(p-1), \forall p \in \mathrm{R}$$

i.e

$$d(\alpha) \leq \inf_{p \in \mathbf{R}} (\alpha p - \varphi(p-1)) \ .$$

And in order to prove Proposition 3, we have to find p such that $\tilde{\mu}_{p-1,\varphi(p-1)}(E^\alpha) > 0$.

Using the same proof as in [20] pages 5 and 6, we can show that for $\alpha = \varphi'(p)$ and

$$\tilde{E}_\alpha = \{x \in [0,1[\ : \quad \lim_{\substack{n \to \infty \\ |i|=n, x \in I_i}} \frac{\log|\lambda_i|}{\log \mid I_i \mid} = \alpha\} \ ;$$

we have

$$\tilde{\mu}_{p-1,\varphi(p-1)}(\tilde{E}_\alpha) > 0 \ .$$

Whence this result with the fact that $\tilde{E}_\alpha \subset E^\alpha$ yield the desired proposition.

5 Proof of the Multifractal Formalism

We shall prove that

$$d(\alpha) = inf(\alpha p - \eta(p) + 1)$$

where η is the function defined in (14). To give the order of the magnitude of $\eta(p)$, we have to estimate the size of the wavelet transform everywhere. For $i = (i_1, \ldots, i_n)$, consider

$$I_i(a) = I_i +]-a, a[$$

and

$$C_i = I_{(i_1,\ldots,i_{n-1})}(a) - I_{(i_1,\ldots,i_{n-1},i_n)}(a) \ .$$

If $i \in B_j$ and $a \leq\mid I_i \mid$ *then* $\mid I_i(a) \mid \sim \mid I_i \mid$, $\mid C_i \mid \leq \mid I_i \mid$ and inequalities (51) and (54) show that there exists $a \sim 2^{-j}$ and a point b of $I_i(a)$ for which the order of magnitude of $C_{a,b}(F)$ is $\Lambda_j(b)$.

We have also the following lemma

Lemma 9 *If $i \in B_j$, $a \sim\mid I_i \mid$ and $b \in I_i(a)$ then $|C_{a,b}(F) - O(a^k)| \leq CL_j(b)$.*
If $a \leq\mid I_i \mid$ and $b \in C_i$ then $|C_{a,b}(F) - O(a^k)| \leq Ca^k|(T^{-1}_{i_1 \ldots i_{n-1}})'(b)|^k L_j(b)$.

The proof is deduced from (20) and an argument similar to the proof of Proposition 2.

On the other hand, remark that

$$\frac{d}{db} C_{a,b} = \frac{1}{a} \tilde{C}_{a,b}$$

where $\tilde{C}_{a,b}$ is the wavelet transform due to ψ', and

$$\frac{d}{da} C_{a,b} = -\frac{1}{a} \breve{C}_{a,b} + \frac{1}{a} C_{a,b}$$

where $\breve{C}_{a,b}$ is the wavelet transform due to $x\psi'$.

We deduce from the previous lemma and remark that there exists an interval of lenght $\sim a$ on which the order of magnitude of $C_{a,b}(F)$ is $\Lambda_j(b)$. Thus if we denote by A_j the interval $[\frac{1}{2}2^{-j}, 2^{-j}]$, then for each branch $i \in B_j$ there exists an interval of lenght $\sim 2^{-j}$ in the time frequency half-space $\mathrm{R}^+ \times \mathrm{R}$, located near x_i and in frequency in the interval A_j and where

$$\mid C_{a,b}(F) - O(a^k) \mid \geq C' \mid \lambda_i \mid \ .$$

Lemma 9 and the previous remark yield

$$\begin{aligned} C' \sum_{i \in B_j} 2^{-2j} |\lambda_i|^p \quad &\leq \quad \int_{A_j \times \mathbf{R}} |C_{a,b}(F) - O(a^k)|^p da\ db &(59)\\ &\leq C \sum_{i \in B_j} 2^{-2j} |\lambda_i|^p + O\left(2^{-j} \sum_{|I_i| \geq 2.2^{-j}} \frac{2^{-kpj}|\lambda_i|^p}{\mid I_i \mid^{kp-1}}\right) \\ &\leq C 2^{-j} \left[\sum_{i \in B_j} 2^{-j}|\lambda_i|^p + O\left(\sum_{|I_i| \geq 2.2^{-j}} \frac{2^{-kpj}|\lambda_i|^p}{\mid I_i \mid^{kp-1}}\right)\right] &(60) \end{aligned}$$

where $O(.)$ is positive.

We have

$$0 = \tilde{C}(p-1, \varphi(p-1)) = \lim_{n \to \infty} \frac{1}{n} \log \left(\sum_{|i|=n} |\lambda_i|^p \mid I_i \mid^{-\varphi(p-1)} \right)$$

so we can write

$$\log \left(\sum_{|i|=n} |\lambda_i|^p \mid I_i \mid^{-\varphi(p-1)} \right) = o(\frac{1}{n})$$

thus

$$\sum_{|i|=n} |\lambda_i|^p \mid I_i \mid^{-\varphi(p-1)} = e^{o(\frac{1}{n})} \ .$$

Define

$$G(j) = \sum_{i \in B_j} |\lambda_i|^p \mid I_i \mid^{-\varphi(p-1)} \ ;$$

Asumption (4) and Lemma 1 yield that for $j \geq 0$ and $i \in B_j$

$$(j-1)\frac{\log 2}{\log \chi^{-1}} \leq \mid i \mid \leq j\frac{\log 2}{\log \rho^{-1}}$$

thus

$$C' e^{o(\frac{1}{j})} \leq G(j) \leq Cj e^{o(\frac{1}{j})}\ . \tag{61}$$

Therefore

$$\begin{aligned} 2^{-j}\sum_{i \in B_j} 2^{-j}|\lambda_i|^p \quad &\sim \quad 2^{-j}2^{-(1+\varphi(p-1))j}\sum_{i\in B_j}|\lambda_i|^p|I_i|^{-\varphi(p-1)} \\ &= 2^{-j}2^{-(1+\varphi(p-1))j}G(j) \\ &\leq Cje^{o(\frac{1}{j})}2^{-j}2^{-(1+\varphi(p-1))j} \end{aligned} \tag{62}$$

and

$$2^{-j}\sum_{i\in B_j}2^{-j}|\lambda_i|^p \geq C'e^{o(\frac{1}{j})}2^{-j}2^{-(1+\varphi(p-1))j}. \tag{63}$$

The term

$$2^{-j}O\left(\sum_{|I_i|\geq 2.2^{-j}}\frac{2^{-kpj}|\lambda_i|^p}{\mid I_i \mid^{kp-1}}\right) \tag{64}$$

is positive and bounded by

$$\begin{aligned} &C2^{-j}2^{-kpj}\sum_{|I_i|\geq 2.2^{-j}}|\lambda_i|^p|I_i|^{1-kp} \\ &= C2^{-j}2^{-kpj}\sum_{l\leq j-2}\ \sum_{2^{-l-1}\leq|I_i|<2^{-l}}|\lambda_i|^p|I_i|^{1-kp} \\ &\sim C2^{-j}2^{-kpj}\sum_{l\leq j-2}2^{-l(1-kp+\varphi(p-1))}\sum_{2^{-l-1}\leq|I_i|<2^{-l}}|\lambda_i|^p|I_i|^{-\varphi(p-1)} \\ &\leq C2^{-j}2^{-kpj}\sum_{l\leq j}2^{-l(1-kp+\varphi(p-1))}G(l) \end{aligned}$$

thus if $\varphi(p-1)+1<kp$ then (64) is bounded by $\quad Cje^{o(\frac{1}{j})}2^{-(1+\varphi(p-1))j}2^{-j}$.

Hence

$$C'2^{-j}e^{o(\frac{1}{j})}2^{-(1+\varphi(p-1))j} \leq \int_{A_j\times\mathbf{R}}|C_{a,b}-O(a^k)|^p da\ db \leq Cj2^{-j}e^{o(\frac{1}{j})}2^{-(1+\varphi(p-1))j}\ . \tag{65}$$

For the complex case, if $\varphi(p-1)+2<kp$ then

$$C'2^{-2j}e^{o(\frac{1}{j})}2^{-(1+\varphi(p-1))j} \leq \int_{A_j\times\mathbf{R}^2}|C_{a,b}-O(a^k)|^p da\ db \leq Cj2^{-2j}e^{o(\frac{1}{j})}2^{-(1+\varphi(p-1))j}\ . \tag{66}$$

Whence the following proposition:

Proposition 4 *If $\varphi(p-1)+1 < kp$ then* $\quad \eta(p) = \varphi(p-1)+1$.
For the complex case, if $\varphi(p-1)+2 < kp$ then $\quad \eta(p) = \varphi(p-1)+2$.

Proof:
Using (65) and the fact that $\lim_{j\to\infty} \frac{o(\frac{1}{j})}{j} = 0$, we obtain

$$\limsup_{a\to 0} a^{-(1+\varphi(p-1))} e^{o(\frac{1}{\log a})} \int_{\mathbb{R}} |C_{a,b} - O(a^k)|^p \, db \geq C' > 0 \tag{67}$$

and

$$\limsup_{a\to 0} \frac{a^{-(1+\varphi(p-1))}}{|\log a|} e^{o(\frac{1}{\log a})} \int_{\mathbb{R}} |C_{a,b} - O(a^k)|^p \, db \leq C < +\infty \ . \tag{68}$$

For $\quad \varphi(p-1)+1 < kp$, the limits (67) and (68) don't change if we replace $C_{a,b} - O(a^k)$ by $C_{a,b}$, similarly for the complex case. Hence Proposition 4.

Finally we have the following Theorem:

Theorem 3 *Let (T_j) be a system of d contractions defined on a bounded open domain in a one dimensional line (respectively the complex plane $\mathbb{C}$) and satisfying the asumptions (2), (3), (4) and (5). Let $\mathcal{D}$ be the best constant in Distortion Lemma. Assume that $\sum_{j=1}^d \mid \lambda_j \mid\mid I_j \mid < \mathcal{D}^{-1}$ (respectively $\sum_{j=1}^d \mid \lambda_j \mid\mid \Omega_j \mid^2 < \mathcal{D}^{-2}$) and let F be the unique (selfsimilar) function of L^1, satisfying*

$$F(x) = \sum_{i=1}^{d} \lambda_i F(T_i^{-1}(x)) \ + \ g(x) \ ;$$

Then for $x \in K$

$$\alpha(x) = \liminf_{n\to\infty} \frac{\log |\lambda_{i_1(x)} \dots \lambda_{i_n(x)}|}{\log |I_{i_1(x) \dots i_n(x)}|} \ ;$$

For $\alpha < k$, $\alpha \in [\alpha_{min}, \alpha_{max}]$

$$d(\alpha) = \inf_{p\in\mathbb{R}} (\alpha p - \varphi(p-1)) \ .$$

Set $m = 1$ for the one dimensional case and 2 for the complex case. If g is C^k then for p such that $\varphi(p-1)+m < kp$, $\eta(p) = \varphi(p-1)+m$.
Moreover, let p_0 such that $\varphi(p_0-1)+m = kp_0$ and α_0 the value of the inverse Legendre transform of $\varphi(p-1)$ at p_0, then for $\alpha < k$, $\alpha \in [\alpha_{min}, \alpha_{max}]$ such that $\alpha = \varphi'(p)$ $\quad$ and $\alpha \leq \alpha_0$ (thus a-a $\alpha \leq \alpha_0$)

$$d(\alpha) = \inf(\alpha p - \eta(p) + m) = \inf(\alpha p - \xi(p) + m) \ .$$

Whence the multifractal formalism is valid.

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