Instantaneous travel times computation for macroscopic traffic flow models [2]

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August 1996

1 Introduction.

The first aim of this paper is to give a rigorous background to the definition of travel times in the context of macroscopic models, with dynamic assignment as their main application field. The second aim is to provide simple computational rules for the estimation of these travel times. Although implementation in relation with first order models will be emphasized in this paper, nearly all results carry over without modification to higher order models, as long as they satisfy to some basic requirements.

The motivating problem behind the estimation of travel times being dynamic assignment, a distinction must be drawn at this point, depending on whether we consider *reactive* or *predictive* assignment, according to the terminology introduced in [PA 90]. Further distinctions can be made, relating to the nature of traffic flow. A survey of the literature concerning all possible computational methods for travel times is out of the scope of the present paper and the reader is refered to [RB 94] and [BLL 96]. Roughly, as far as travel times are concerned, one can distinguish between:

- *the experienced travel time*, abbreviated as ETT, i.e. the travel time of the user who has just completed the considered trip (for instance INTEGRATION [VA 95]),

- *the predictive travel time*, abbreviated as PTT, usually integrated into the flow model ([FBSTW 93], CONTRAM ([LGT 89], [LTB 78]), [RB 94]), which is an estimate of the travel time that shall be experienced by the user who begins his trip,

- *the instantaneous travel time*, abbreviated as ITT, which is an index of the current state of the network, translated into a travel time that is provided to users in real time in order to allow them to choose between various possible paths. ITTs constitute an essential ingredient of reactive dynamic assignment schemes, hence their importance.

As far as traffic flow conditions are concerned, the most important single factor is whether the traffic flow is considered as *uninterrupted* or is liable to be *interrupted*. This is of course by and large a time scale matter, since the typical traffic flow interruption results from traffic management operations. The time scale in the interrupted case is therefore smaller than say the cycle length in the presence of traffic lights. Standard practice in this case is the averaging of travel times over a cycle, or the use of queuing formulas ([WFH 94], [JMH 94]).

Nevertheless, other causes may determine traffic interruptions: notably downstream congestion and incidents. It is characteristic of these interruptions that their duration cannot be known beforehand, and that they are related to traffic supply restrictions. This leads to difficulties. Indeed, the natural estimate of the ITT of say a link $[a, b]$ at time t would be $\int_a^b \frac{dx}{V(x,t)}$, with $V(x, t)$ the speed at location x and time t. This integral becomes infinite if the speed $V(x, t)$ becomes locally nil. For this reason,

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[2] The present paper provides some background material for the joint LICIT-CERMICS STRADA project as well as two papers submitted jointly by the author and C. BUISSON and J.B. LESORT to the EURO 96 Conference the IEEE-CESA'96 Conference and accepted both for publication

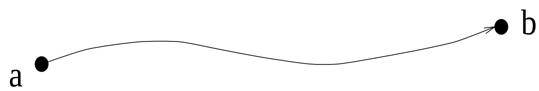
several models use a minimal > 0 speed (DYNASMART [JMH 94], METACOR [EL 96]), thus preventing the divergence of the above integral. Such an approximation, usefull and generally relevant for freeway networks, is not really satisfying for urban network traffic flow modelling.

In this paper, we shall first give rigorous definitions of the three kinds of travel time, *experienced travel time* ETT, *predictive travel time*, PTT, and *instantaneous travel time*, ITT, for a link and whithin the framework of macroscopic traffic modelling. Then, the issue of computation of the two most usefull kinds (for reactive dynamic assignment), the link ETT and ITT, will be adressed, first in a distributed setting, then in a discretized setting. Last, various extensions will be studied, notably the adaptation to partial densities, intersections and path travel times. One quite separate, but important topic will be left for future investigations: the temporal aggregation of travel times, i.e. their smoothing in the presence of periodical travel time fluctuations. Indeed, the two problems: how to estimate travel times, and how to aggregate them temporally in order to provide users with slow-varying and consistent information, are basically different. The first one is a problem of *traffic data collection and elaboration*, related to the state estimation level of the traffic flow modelling process, whereas the second one is a matter of *traffic control and user response to information*, therefore related to the control level of the traffic flow modelling process. Travel time estimation and travel time temporal aggregation, though both part of the dynamic assignment process, are not situated at the same level, and the travel time estimation is definitely the most basic of the two and must therefore be addressed first.

2 Link travel times: a general framework.

2.1 Definition of predictive and experienced travel times.

We consider now the *macroscopic approximation* of *interrupted traffic flow*. The starting point of our analysis of possible travel time expressions is the single link, say $[a, b]$, with its associated traffic flow modelled with the help of the usual macroscopic variables $Q(x, t)$ (flow), $K(x, t)$ (density) and $V(x, t)$ (speed), which are considered as *functions* of the position x and the time t. The existence of these functions expresses the *continuum hypothesis*, and constitutes only an approximation. A discussion of this hypothesis and of the position of macroscopic models in the whole scale of traffic flow models can be found in [LE 95-1].



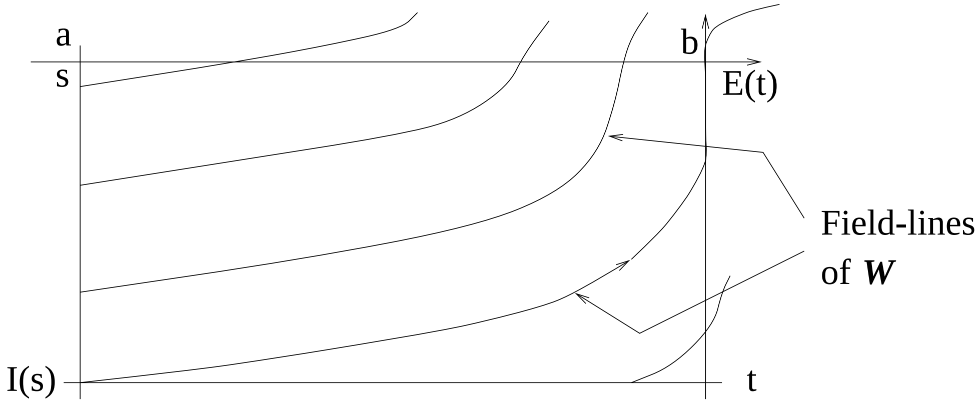
Essential for the sequel will be the *speed field* $\mathcal{W}(x, t) = (V(x, t), 1)$, defined on the $[a, b] \times \mathbb{R}$ band of the (x, t) space *times* time plane, and assumed to be integrable, with existence and unicity of the corresponding field-lines. Under reasonable assumptions on the link traffic supply and demand for instance (see [LE 95-2] for a definition of these concepts), and if a first-order LWR model is considered ([LW 55],[RI 56]), the field $\mathcal{W}$ can be shown trivially to be piecewise differentiable, with discontinuities situated on continuous lines which are piecewise differentialble manifolds in the (x, t) plane. Let us denote $\mathcal{Z}$ a vehicle trajectory associated to such a field-line of $\mathcal{W}$: $\mathcal{Z}(x_0, t_0; t)$ is the position at time t of the user whose position at time t_0 is x_0. $\mathcal{Z}(x_0, t_0; t)$ is the solution at time t of:

$$\text{(1)} \qquad \begin{aligned} \dot{x}(t) &= V(x(t), t) \\ x(t_0) &= x_0 \quad . \end{aligned}$$

The corresponding field-line would be $(\mathcal{Z}(x_0, t_0; t), t)$. We admit, and there will be no difficulty at this point, that if (x_0, t_0) is located in the discontinuity set of V, $V(x_0+, t_0+)$ is well-defined and can be used as an initial condition for the computation of $\mathcal{Z}$ according to (1) (remember that $V \geq 0$). Since the field-lines $(\mathcal{Z}(x_0, t_0; t), t)$ do not intersect in the (x, t)-plane, the field $\mathcal{W}$ being nonzero everywhere, this kind of description is intrinsically in agreement with the FIFO hypothesis: in accordance with this representation, vehicles exit the link in the precise order they have entered it. Considering a vehicle entering the link at time t, we may define its exit time $E(t)$ as

$$\text{(2)} \qquad E(t) \overset{def}{=} \inf_s \{s \,/\, \mathcal{Z}(a, t; s) > b\} \quad .$$

This definition is justified by the fact that the exit speed $V(b,.)$ may well be equal to 0 for finite time intervals; this is precisely the expression of the hypothesis that the traffic flow can be interrupted. The following figure, on which field-lines of the speed-field $\mathcal{W}$ have been depicted, illustrates such a situation.



The function E is increasing, piecewise continuous and admits an "inverse" I. Precisely $I(t)$ is defined as the time at which a vehicle about to leave the link at time t has entered it. Hence:

(3) $$\mathcal{Z}(a, I(t); t) = b \quad .$$

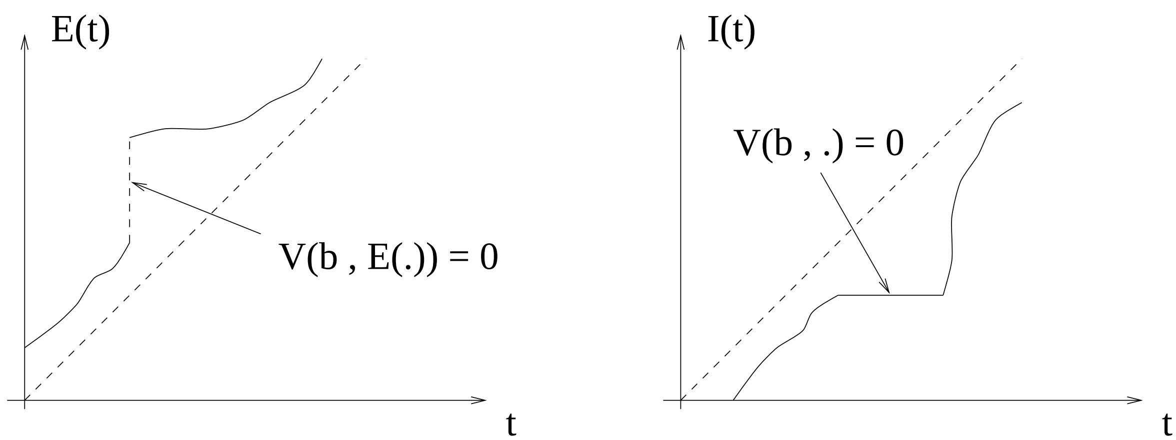
Whenever $V(b,t) = 0$ on some time interval, $I(t)$ is constant on the same interval. This means that physically, it is the same vehicle that is about to leave during the whole duration of such an interval. It follows that:

$$I(E(t)) = t$$

unconditionally, and that:

$$E(I(t)) \begin{cases} = & t \quad \text{if } V(b,t) > 0 \\ \geq & t \quad \text{if } V(b,t) = 0 \end{cases} \quad .$$

The following figure illustrates again this point. The graphs of E and I are symetrical relatively to their bisecting line.



We can now define two fundamental quantities:

- *the experienced travel time* (of the user about to exit the link at time t):

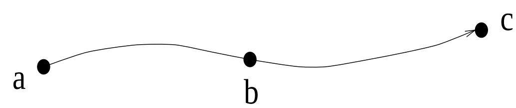
(4) $$ETT(a,b;t) \stackrel{def}{=} t - I(t) \quad ,$$

- *the predictive travel time* of users entering the link at time t:

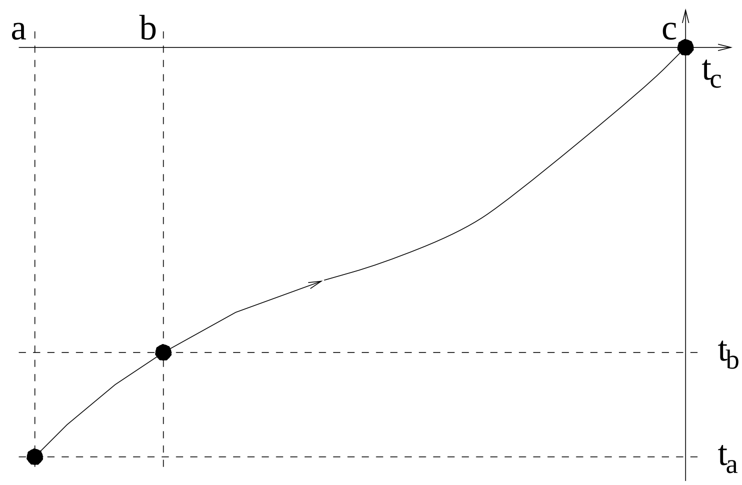
(5) $$PTT(a,b;t) \stackrel{def}{=} E(t) - t \quad .$$

The experienced travel time grows as t (i.e. its derivative equals 1) if $V(b,t) = 0$, since then $I(t)$ is constant. This is an important property we shall use later on.

Let us now consider three points a, b, c in that order on a line, as depicted on the following figure:



and let us consider the trajectory of a vehicle passing through these three points.



The vehicle passes through points a, b, c at times t_a, t_b, t_c. It follows:

$$\begin{array}{rcccl} t_c - t_b & = & ETT(b,c;t_c) & = & PTT(b,c;t_b) \\ t_b - t_a & = & ETT(a,b;t_b) & = & PTT(a,b;t_a) \\ t_c - t_a & = & ETT(a,c;t_c) & = & PTT(a,c;t_a) \end{array}$$

and of course

$$t_c - t_a = (t_c - t_b) + (t_b - t_a) \quad .$$

It follows

$$PTT(a,c;t_a) = PTT(a,b;t_a) + PTT(b,c;t_b)$$

with

$$t_b = t_a + PTT(a,b;t_a) \quad ,$$

and

$$ETT(a,c;t_c) = ETT(a,b;t_b) + ETT(b,c;t_c)$$

with

$$t_b = t_c - ETT(b,c;t_c) \quad .$$

By taking $t = t_a$ and $t = t_c$ respectively, the following functional equations result:

$$(6) \qquad \left[\begin{array}{l} PTT(a,c;t) = PTT(a,b;t) + PTT(b,c;t + PTT(a,b;t)) \\ \\ ETT(a,c;t) = ETT(b,c;t) + ETT(a,b;t - ETT(b,c;t)) \quad . \end{array} \right.$$

These equations express the combination rules that must apply to the predictive and experienced travel times. Obviously these travel times are not additive quantities! Hence the difficulties related to spatial aggregation.

2.2 Analytical computation of predictive and experienced travel times.

1. Analytical computation of predictive travel times can only be carried out easily in the case of a first-order macroscopic traffic flow models of the LWR type:

$$\frac{\partial K}{\partial t} + \frac{\partial}{\partial x} Q_e(K,x) = 0$$

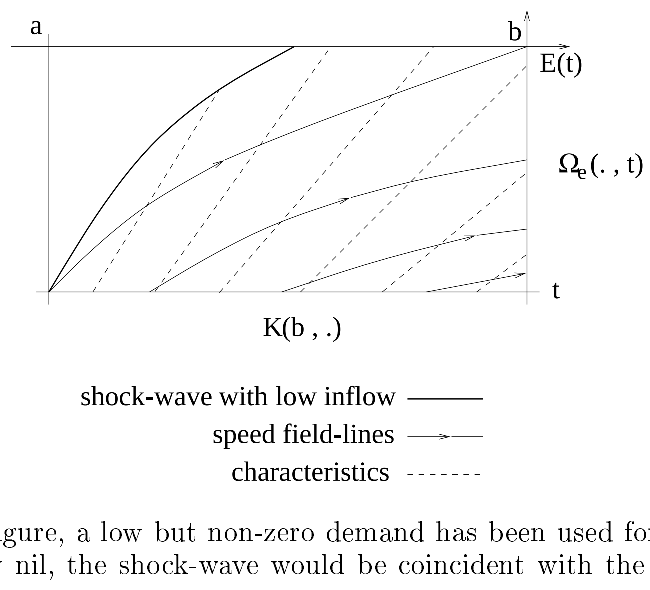
with Q_e the equilibrium flow-density relationship assumed to be a concave piecewise differentiable function of the density K and piecewise continuous function of the position x. For the computation of $PTT(a,b;t)$ at time t whithin the framework of the LWR model, only the initial condition $K(.,t)$ on the link and the downstream traffic supply at point b, $\Sigma(b,r)$ (see [LE 95-2] for a definition of this notion), for times r ranging from t to $E(t)$ are needed. Indeed, according to the characteristics method, the fastest propagation of the data at the entry point, i.e. the upstream demand $\Delta(a,r)$ for $r \geq t$, occurs when

$$\Delta(a,r) = 0 \quad \forall r \geq t \quad .$$

The propagation of the entry point data follows then a line $(\eta(r), r)$ defined by the following equation:

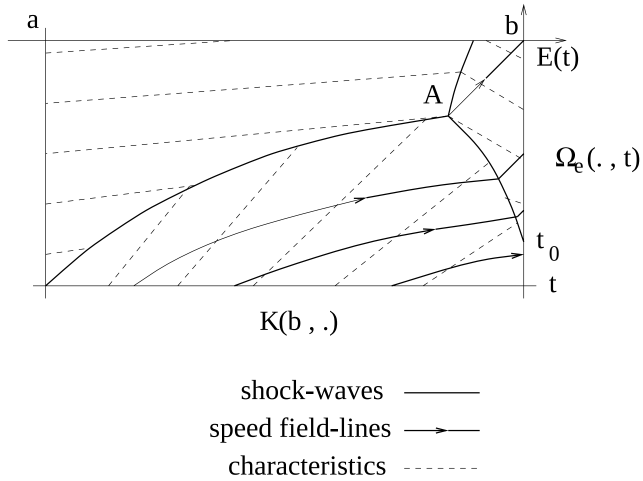
(7)
$$\left| \begin{array}{l} \frac{d\eta}{dr} = \frac{\partial Q_e}{\partial K}(K(\eta(r)+, r), \eta(r)+) \\ \\ \eta(r)_{|r=t} = a \quad , \end{array} \right.$$

where the values $K(\eta(r)+, r)$ depend on the initial data or the downstream supply only. Since $V_e(K,x) \geq \frac{\partial Q_e}{\partial K}(K,x)$, $\forall x$, with equality iff $K = 0$ (by concavity of $Q_e(.,x)$), it follows that the field-line of $\mathcal{W}$ associated to the trajectory of the vehicle entered on the link at time t cannot cross the line $(\eta(r), r)$, $r \in [t, +\infty)$, and is in fact situated on the right-hand side of this line, depending therefore only on the initial data and the future downstream supply. To compute analytically $PTT(t)$, given the initial condition and the future downstream supply, it sufficies to solve (7) with nil upstream demand by the method of characteristics. Further, if the downstream supply is sufficient, i.e. the link demand $\Delta_e(K(b,r), b)$ is less than the supply $\Sigma_e(b,r)$ at all times $r \geq t$, then $PTT(t)$ depends only on the initial condition $K(x,t)$, $x \in [a,b]$. The following figure illustrates this argument, in a situation in which the downstream traffic supply is sufficient.



To construct the above figure, a low but non-zero demand has been used for illustrative purposes. If the demand were exactly nil, the shock-wave would be coincident with the trajectory originating at point (a,t).

If the downstream supply is not sufficient at all times, the picture is somewhat different, as is illustrated by the following figure, illustrating a case in which the downstream supply is less than the link demand from time say t_0 on.



On this figure, the upstream demand has been assumed nil (this is the extreme possibility), hence the trajectory originating at (a,t) and the shockwave originating at the same point are coincident till point A. From point A on, traffic conditions imposed by the value of the downstream supply prevail along the trajectory originating at (a,t).

2. The analytical computation of $ETT(a,b;t)$ can be effectuated by solving the following partial differential equation:

$$\left| \begin{array}{l} V\frac{\partial T}{\partial x} + \frac{\partial T}{\partial t} = 1 \\ \\ T(a,t) = 0 \qquad (\forall t) \end{array} \right. . \tag{8}$$

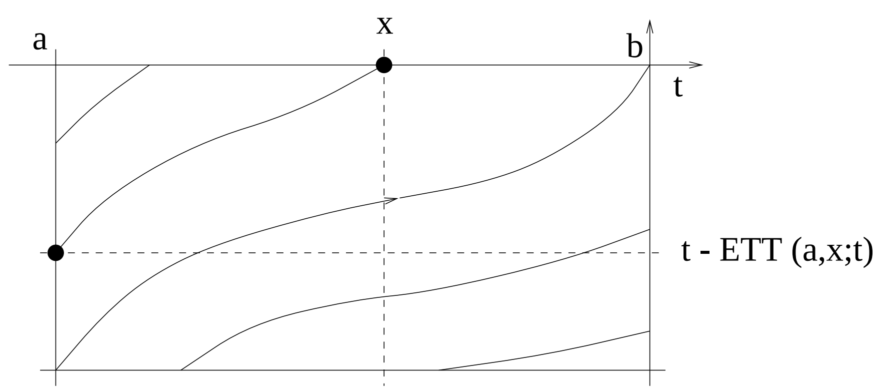
Indeed, $ETT(a,b;t) = t - I(t)$, hence if one considers the field-line of $\mathcal{W}$ originating from point (a,s), ending at point (x,t),

$$ETT(a,x;t) = t - s \quad \text{iff } \mathcal{Z}(a,s;t) = x \quad ,$$

which shows that

$$\frac{dETT}{dt} = 1$$

along a field-line of $\mathcal{W}$. T is equal to t along trajectories, hence $T(x,t) = ETT(a,x;t)$.



Since the operator

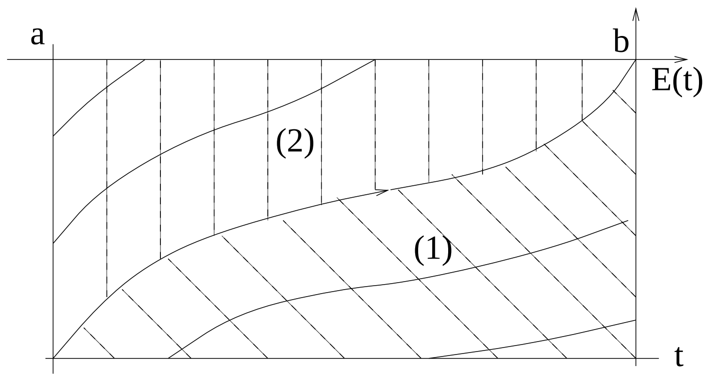
$$V\frac{\partial}{\partial x} + \frac{\partial}{\partial t}$$

represents the time-derivative along the field-lines of $\mathcal{W}$, the formula (8) results trivially.

3. One last point: by integrating the conservation equation

$$\frac{\partial K}{\partial t} + \frac{\partial Q}{\partial x} = 0$$

over areas (1) and (2) depicted hereafter (and bounded diagonally by a trajectory),



the following relationships result:

$$N(t) \stackrel{def}{=} \int_a^b K(\xi, t) d\xi = \int_{I(t)}^t Q(a, s) ds = \int_t^{E(t)} Q(b, s) ds \quad ,$$

relating the link inflow and outflow to the number $N(t)$ of vehicles contained in the link at a given time.

2.3 Definition of instantaneous travel times.

Although experienced travel times may be used for reactive assignment (this is the case in the INTEGRATION model), they do not necessarily constitute the prime choice, since an experienced travel time reflects more that what has just happened than what is about to happen.

Another possibility is to define directly an *instantaneous travel time* $ITT(a, b; t)$ for the link in order to satisfy some set of properties. A natural definition is given in [RB 94] and can be stated as: "The instantaneous travel time is the travel time that would result if prevailing traffic conditions remained unchanged". In the present setting, this would imply that the actual speed-field $\mathcal{W}$ at time t should be extrapolated for future values of time say $\tau \geq t$ as a time-constant field

$$\mathcal{W}^i(x, \tau) \stackrel{def}{=} (V(x, t), 1) \qquad (\forall \tau \geq t) \quad .$$

The corresponding exit-time $E^i(t)$ computed according to this extrapolation $\mathcal{W}^i$ of the speed-field satisfies trivially

$$E^i(t) - t = \int_a^b d\xi / V(\xi, t)$$

since the field $\mathcal{W}^i$ is time-invariant. The corresponding ITT would be given by:

(9) $$ITT(a, b; t) \stackrel{def}{=} \int_a^b \frac{d\xi}{V(\xi, t)}$$

and even more generally

(10) $$ITT(\text{path}; t) \stackrel{def}{=} \int_{\text{path}} \frac{d\xi}{V(\xi, t)}$$

since (9) is evidently additive, as a consequence of the time-invariance of the extrapolated field $\mathcal{W}^i$. Regrettably, formulas (9) and (10) can only be used if the speed does not become equal to 0. Such a constraint can be reasonably applied in macroscopic models for motorway traffic, and has been applied in some cases. But it is obviously inadequate for urban traffic, which can be interrupted, and for the modelling of which relatively small time-scales must be considered, allowing for no smoothing of short-time (less than a cycle) traffic flow variations.

Therefore, some less obvious definition of ITTs must be introduced in the case of interupted traffic. The properties we retained for our definition of an ITT are the following.

- In the case of uncongested traffic, specifically at low density and high speed, the ITT should be approximated by (10). For a link $[a, b]$, two possibilities can be envisionned, either the estimation of the function $(x, t) \longrightarrow ITT(a, x; t)$ (*forward ITT*) or the estimation of the function $(x, t) \longrightarrow ITT(x, b; t)$ (*backward ITT*). The *forward ITT*, at low density and high speed, should therefore be given by

$$\frac{\partial}{\partial x} ITT(a, x; t) \approx 1/V(x, t) \quad , \tag{11}$$

  and the *backward ITT*, at low density and high speed, should be given by

$$\frac{\partial}{\partial x} ITT(x, b; t) \approx -1/V(x, t) \quad . \tag{12}$$

- In the case of strongly congested traffic (high density and low speed) some different rule must be applied for the estimation of the ITT. A possible rule is to consider that the ITT should behave as the ETT in such traffic conditions, hence

$$\begin{aligned} &\tfrac{\partial}{\partial t} ITT(x, b; t) \approx 1 \quad , \\ &\tfrac{\partial}{\partial t} ITT(a, x; t) \approx 1 \quad . \end{aligned} \tag{13}$$

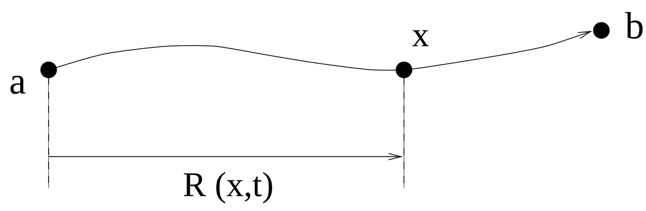
  The idea here is that, in the case of interrupted traffic flow, (13) yields the simplest and possibly only possible prediction of the interruption duration, i.e. its actual duration, especially in the case of an incident in which this duration may not by definition be known beforehand.

- Finally, if the traffic flow is stationary, but nonzero, the ITT should converge towards the actual travel time, say ATT i.e.

$$ATT(\text{path}; t) \stackrel{def}{=} \int_{\text{path}} \frac{d\xi}{V(\xi)} \tag{14}$$

  which is of course (10) again, but applied to a stationary traffic flow.

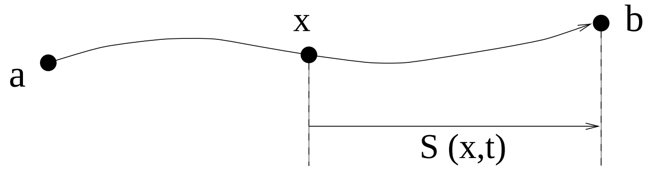
The simplest models satisfying to the above properties are the following partial differential equations (15) and (16), which are obtained by interpolating linearly (13) with (11) in the case of forward ITTs and with (12) in the case of backward ITTs. More precisely, denoting V_{max} the maximum speed, (15) and (16) reduce to (11) and (12) respectively if $V = 0$, and to (13) if $V = V_{max}$. Let us denote $R(x, t) \stackrel{def}{=} ITT(a, x; t)$ the *forward ITT* from a to x at time t, as illustrated by the following figure.



The forward ITT R will satisfy the following equation:

$$\left| \begin{array}{l} V \frac{\partial R}{\partial x} + (1 - \frac{V}{V_{max}}) \frac{\partial R}{\partial t} = 1 \\ R(a, t) = 0 \qquad (\forall t) \quad . \end{array} \right. \tag{15}$$

Let us denote now $S(x, t) \stackrel{def}{=} ITT(x, b; t)$ the *backward instantaneous travel time* from x to b estimated at time t), as illustrated by the following figure.



The following equation results :

(16)
$$\left| \begin{array}{l} -V\frac{\partial S}{\partial x} + (1 - \frac{V}{V_{max}})\frac{\partial S}{\partial t} = 1 \\ S(b,t) = 0 \qquad (\forall t) \quad . \end{array} \right.$$

It must be emphasized at this point that the functions R and S do not have any simple relation to each other as might be misleadingly suggested by their defining formulas. Indeed, the forward and backward instantaneous travel times have quite different properties, as will be shown in the next section, and should be considered as analogous but absolutely independent notions. For this reason, we shall use in the sequel the notations R and S, and if we really wish to introduce the notation ITT in connexion with these quantities, we shall distinguish between $ITTf$ (forward ITT) and $ITTb$ (backward ITT).

3 Analytical computation of instantaneous travel times

In this section, we address specifically the problem of computing the analytical solutions of (15) and (16). This means that by instantaneous travel times we mean those quantities defined in the preceding section, i.e. the forward and backward $ITTf$ and $ITTb$ as represented by the functions R and S. In the sequel, we shall assume V_{max} to be independant of the position x, a hypothesis which will enable us to derive methods for the analytical computation of instantaneous travel times.

3.1 Analytical computation of forward instantaneous travel times

3.1.1 Basic ideas

We shall describe in this subsection how to compute the function R which is defined by (15) and represents the forward instantaneous travel time: $R(x,t) = ITTf(a,x;t)$. Let us associate to (15) the field

(17)
$$\mathcal{U}(x,t) \stackrel{def}{=} (V(x,t), 1 - \frac{V(x,t)}{V_{max}}) \quad ,$$

and the parameter

$$u \stackrel{def}{=} t + \frac{x}{V_{max}} \quad ,$$

which has the same dimension as t. It follows trivially, by

$$(V, 1 - \frac{V}{V_{max}}).(\frac{1}{V_{max}}, 1) = 1$$

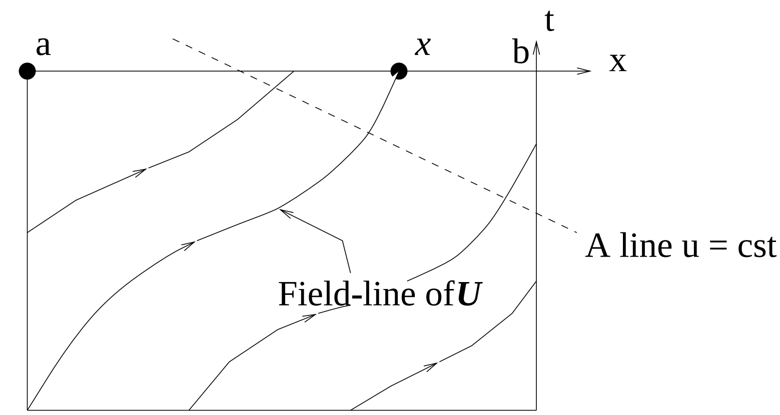
that

$$< \mathcal{U}, du >= 1$$

with $< ., . >$ the usual canonical bracket between fields and differentials. Hence u is a proper parameter for the field-lines of $\mathcal{U}$, which we denote $(\mathcal{X}^f(u), \mathcal{T}^f(u))$ and define by:

(18)
$$\left[\begin{array}{l} \frac{d\mathcal{X}^f}{du} = V(\mathcal{X}^f, \mathcal{T}^f)(u) \\ \frac{d\mathcal{T}^f}{du} = 1 - V(\mathcal{X}^f, \mathcal{T}^f)(u)/V_{max} \quad . \end{array} \right.$$

The superscript f stands for *forward*. The field $\mathcal{U}$ and its field-lines are illustrated hereafter

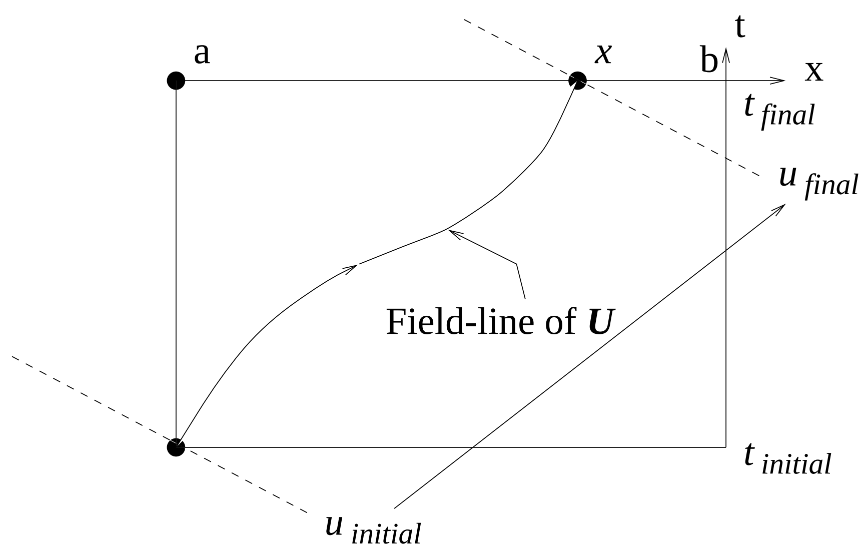


The relationship $V\frac{\partial R}{\partial x} + (1 - \frac{V}{V_{max}})\frac{\partial R}{\partial t} = 1$ can be rewritten as

$$\frac{d}{du}R(\mathcal{X}^f, \mathcal{T}^f)(u) = 1 \tag{19}$$

which, with $< \mathcal{U}, du >, = 1$ means that $dR = du$ along a field-line of $\mathcal{U}$. Therefore, the analytical computation procedure of R can be summarized by the following steps:

- compute the field-lines $(\mathcal{X}^f, \mathcal{T}^f)(u)$ of $\mathcal{U}$ by (18),
- impose the boundary condition $R(a, .) = 0$,
- deduce R by applying (19) i.e. $dR = du$ along field-lines $(\mathcal{X}^f, \mathcal{T}^f)(u)$. This last procedure is illustrated by the following figure.



In the above example, it results trivially that:

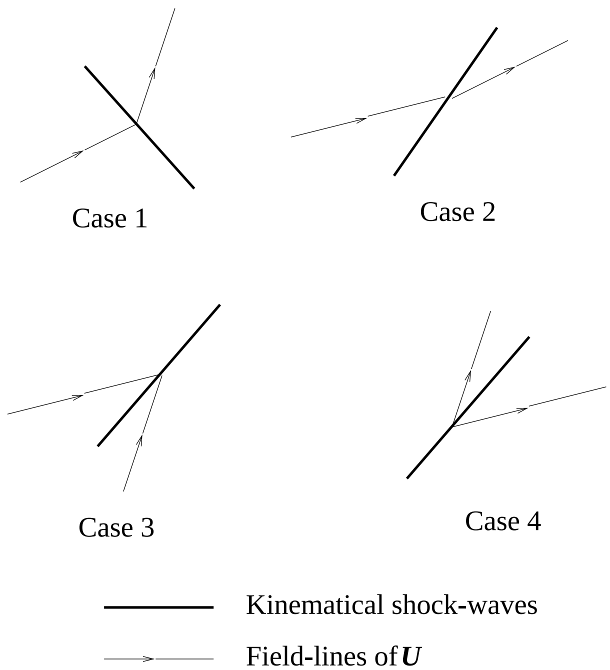
$$\begin{aligned} R(x, t_{final}) &= u_{final} - u_{initial} \\ &= t_{final} - t_{initial} + (x - a)/V_{max} \quad . \end{aligned}$$

3.1.2 Impact of kinematical waves

The above procedure must be detailed when kinematical shock-waves are present. Let $\mathcal{K}$ be such a kinematical shock-wave. It is always possible to define an upstream (left-hand) and downstream (right-hand) side of $\mathcal{K}$, represented by the symbols $-$ and $+$, since the propagation speed of kinematical shock-waves is finite. At some point $(x, t) \in \mathcal{K}$, the field $\mathcal{U}$ admits two distinct values $\mathcal{U}(x-, t)$ (upstream value) and $\mathcal{U}(x+, t)$ (dowstream value). Four situations are possible,

1. a field-line of $\mathcal{U}$ crosses $\mathcal{K}$ at (x,t) with the slope of $\mathcal{K}$ negative,
2. a field-line of $\mathcal{U}$ crosses $\mathcal{K}$ at (x,t) with the slope of $\mathcal{K}$ positive,
3. two field-lines of $\mathcal{U}$ converge towards $\mathcal{K}$ at (x,t),
4. two field-lines of $\mathcal{U}$ diverge from $\mathcal{K}$ at (x,t),

which are illustrated by the following figure:



Situations 1 and 2 are trivial: R is continuous at (x,t) and only its derivative relative to u admits a discontinuity at that point. In situation 3, R admits a discontinuity at point (x,t): $R(x+,t)$ is determined by the field-line converging from the right towards $\mathcal{K}$, and $R(x-,t)$ is determined by the field-line converging from the left towards $\mathcal{K}$.

Let us examine now case 4, which is the only non-trivial one. The speed V admits a discontinuity at point (x,t) of $\mathcal{K}$, let $V(x+,t)$ and $V(x-,t)$ be the values of the speed downstream and upstream of the point (x,t). As the field-lines of $\mathcal{U}$ diverge from $\mathcal{K}$, it follows that the slope of $\mathcal{K}$ must be positive, and that $V(x-,t) \leq V(x+,t)$, since the value of the angle of $(V, 1-V/V_{max})$ with the x axis decreases as V increases. Hence by linearity of $V \longrightarrow (V, 1-V/V_{max})$, there exists a unique speed $V^f_{\mathcal{K}}(x,t) \in [V(x-,t), V(x+,t)]$ such that $(V^f_{\mathcal{K}}(x,t), 1 - V^f_{\mathcal{K}}(x,t)/V_{max})$ be in the tangent space $T_{(x,t)}\mathcal{K}$ of $\mathcal{K}$ at point (x,t). As long as $\mathcal{K}$ is smooth, $V^f_{\mathcal{K}}$ is continuous (or of the same regularity as $\mathcal{K}$). Hence, $\mathcal{K}$ can be defined as the solution of:

$$\left[\begin{array}{l} \frac{d\mathcal{X}^f}{du} = V^f_{\mathcal{K}}(\mathcal{X}^f, \mathcal{T}^f)(u) \\ \\ \frac{d\mathcal{T}^f}{du} = 1 - V^f_{\mathcal{K}}(\mathcal{X}^f, \mathcal{T}^f)(u)/V_{max} \end{array}\right. \quad ,$$

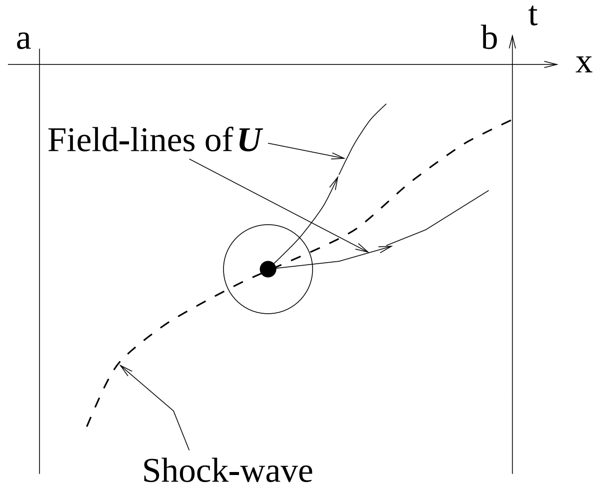
or, extending the definition of $\mathcal{U}$ as

$$\mathcal{U}^*(x,t) \stackrel{def}{=} [(V(x-,t), 1 - V(x-,t)/V_{max}), (V(x+,t), 1 - V(x+,t)/V_{max})] \quad ,$$

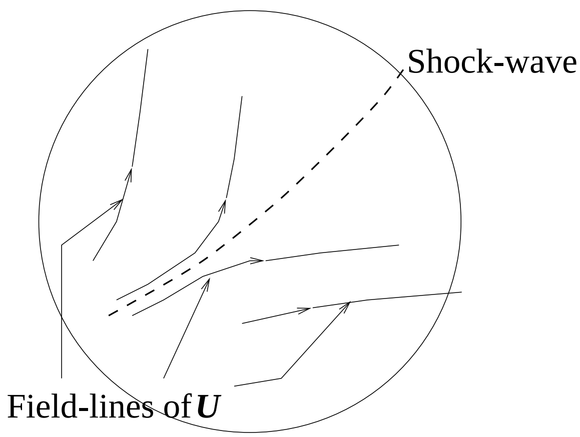
a unique definition results for both ordinary field-lines of $\mathcal{U}$ and kinematical shock-waves $\mathcal{K}$:

$$\frac{d}{du}(\mathcal{X}^f, \mathcal{T}^f) \in \mathcal{U}^* \quad .$$

The result is that in case 4 the forward ITT R is continuous at $\mathcal{K}$ and can indeed be calculated along $\mathcal{K}$ by applying the same rule as along ordinary field-lines, i.e. $dR = du$. To understand physically the preceding remarks, one might visualize a enlargement of the vicinity of the point (x, t) of $\mathcal{K}$, as depicted on the following figure:



In this vicinity, the field $\mathcal{U}$ might be considered to vary with a very sharp gradient from $\mathcal{U}(x-, t)$ to $\mathcal{U}(x+, t)$ while crossing $\mathcal{K}$, and to take precisely the intermediate value $(V_{\mathcal{K}}^f(x,t), 1 - V_{\mathcal{K}}^f(x,t)/V_{max})$, with $(x, t) \in \mathcal{K}$, resulting in field-lines diverging from the central field-line $\mathcal{K}$, as depicted on the following figure.



Two final remarks:

1. Everything that has been said about case 4 can be transposed trivialy to case 3, essentially by replacing the divergence of field-lines by their convergence towards the shock-wave $\mathcal{K}$.

2. If the underlying macroscopic model is a first order model of the LWR kind, and if the solution set is restricted to entropy solutions thereof [AN 90], then case 4 is in fact excluded, since it implies that the velocity be greater downstream of the shock-wave than upstream, which is excluded by the entropy conditions at the locus of the shock.

3.2 Analytical computation of backward instantaneous travel times

The object of this subsection is to show how to compute the function S which is defined by (16) and represents the backward instantaneous travel time: $S(x, t) = ITTb(x, b; t)$. This whole subsection is "symetrical" of the preceding subsection on the analytical computation of forward ITTs. For this reason, explanations and comments will be given only as necessary, to emphasize the differences with the preceding subsection.

3.2.1 Basic ideas

Let us associate to the defining equation (16) of S the field $\mathcal{V}$

$$\mathcal{V}(x,t) \stackrel{def}{=} (-V(x,t), 1 - \frac{V(x,t)}{V_{max}}) \tag{20}$$

and the parameter

$$v \stackrel{def}{=} t - \frac{x}{V_{max}} \quad ,$$

which has the same dimension as t. $\mathcal{V}$ and v are the analogues of $\mathcal{U}$ and u. It follows, since

$$(-V, 1 - \frac{V}{V_{max}}).(-\frac{1}{V_{max}}, 1) = 1 \quad ,$$

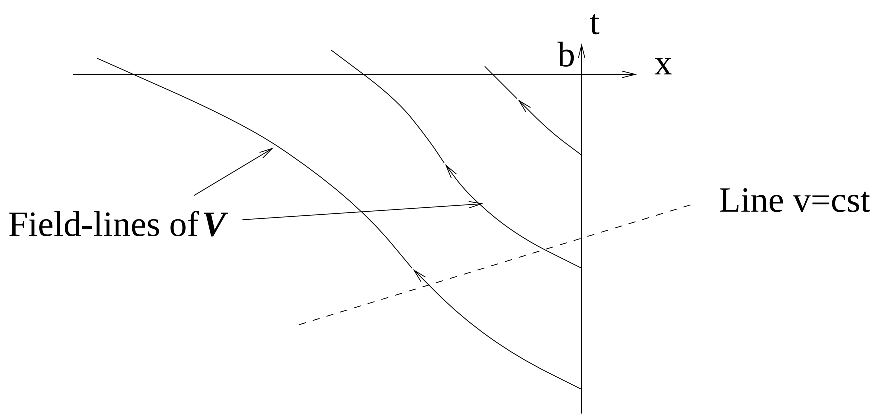
that

$$< \mathcal{V}, dv >= 1 \quad ,$$

(with $< .,. >$ again the usual canonical bracket between fields and differentials). Hence v is a proper parameter for the field-lines of $\mathcal{V}$, which we denote $(\mathcal{X}^b(v), \mathcal{T}^b(v))$ and define by:

$$\begin{cases} \frac{d\mathcal{X}^b}{dv} = -V(\mathcal{X}^b, \mathcal{T}^b)(v) \\ \frac{d\mathcal{T}^b}{dv} = 1 - V(\mathcal{X}^b, \mathcal{T}^b)(v)/V_{max} \quad . \end{cases} \tag{21}$$

The superscript b stands for *backward*. The field $\mathcal{V}$ and its field-lines are illustrated hereafter



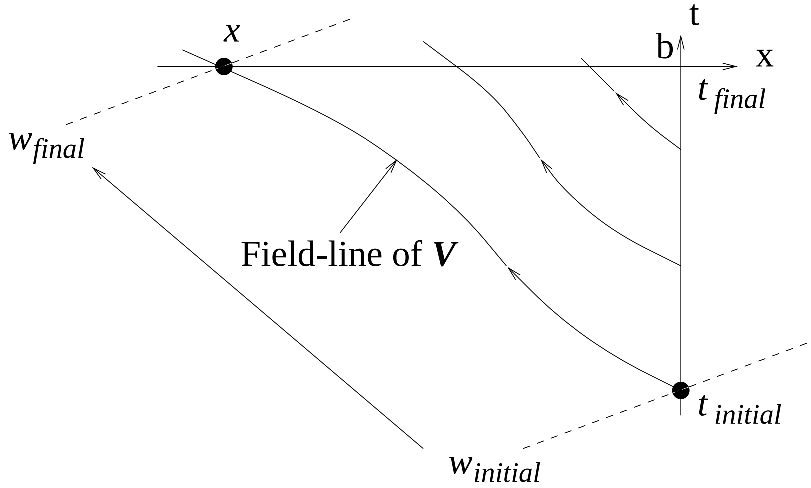
The relationship $-V\frac{\partial S}{\partial x} + (1 - \frac{V}{V_{max}})\frac{\partial S}{\partial t} = 1$ can be rewritten as

$$\frac{d}{dv} S(\mathcal{X}^b, \mathcal{T}^b)(v) = 1 \tag{22}$$

which, with $< \mathcal{V}, dv >= 1$ means that $dS = dv$ along a field-line of $\mathcal{V}$. Therefore, the analytical computation procedure of S can be summarized by the following steps, which are nearly identical to those defined for the computation of R:

- compute the field-lines $(\mathcal{X}^b, \mathcal{T}^b)(v)$ of $\mathcal{V}$ by (21),
- impose the boundary condition $S(b, .) = 0$,
- deduce S by applying (22) i.e. $dS = dv$ along field-lines $(\mathcal{X}^b, \mathcal{T}^b)(v)$.

The second step should be modified if $V(b,t) = 0$ for some interval $t \in [t_0, t_1]$, becoming $S(b, t_0) = 0$, $S(b,t) = t - t_0$, for all $t \in [t_0, t_1]$, at least if one wishes to retain the continuity of S at point b. Nevertheless this modification of the boundary condition has actually no other impact or usefulness. The last step of the procedure can be illustrated by the following figure.



It is trivial to check on the above example that

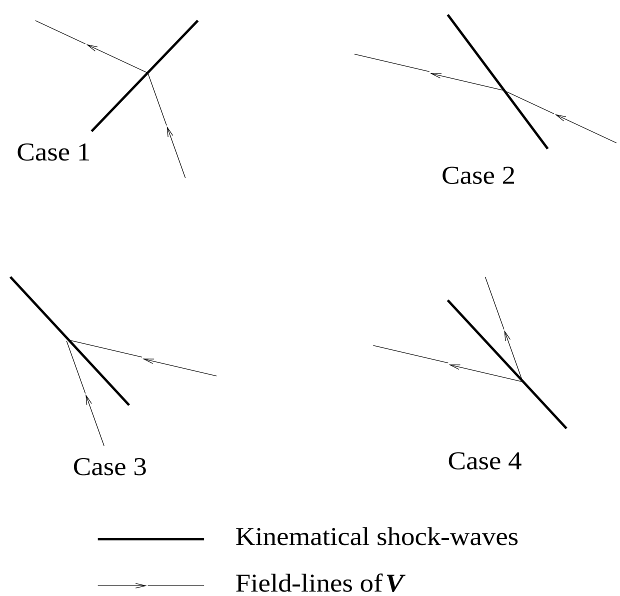
$$
\begin{aligned}
S(x,t) &= w_{final} - w_{initial} \\
&= t_{final} - t_{initial} + (b-x)/V_{max} \quad .
\end{aligned}
$$

3.2.2 Impact of kinematical waves

Let us analyze now the impact of kinematical shock-waves on the procedure. The method is essentially the same as for forward ITTs. Let $\mathcal{K}$ be such a kinematical shock-wave. As the propagation speed of kinematical shock-waves is finite, we define again an upstream (left-hand) and downstream (right-hand) side of $\mathcal{K}$, represented by the symbols $-$ and $+$. Let (x,t) be a point $\in \mathcal{K}$; at such a point the field $\mathcal{V}$ admits the two distinct values $\mathcal{V}(x-,t)$ (upstream value) and $\mathcal{V}(x+,t)$ (dowstream value). Four situations are possible,

1. a field-line of $\mathcal{V}$ crosses $\mathcal{K}$ at (x,t) with positive slope for $\mathcal{K}$,
2. a field-line of $\mathcal{V}$ crosses $\mathcal{K}$ at (x,t) with negative slope for $\mathcal{K}$,
3. two field-lines of $\mathcal{V}$ converge towards $\mathcal{K}$ at (x,t),
4. two field-lines of $\mathcal{V}$ diverge from $\mathcal{K}$ at (x,t),

which are illustrated by the following figure:



Situations 1 and 2 are trivial: S is continuous at (x,t) and only its derivative relative to u admits a discontinuity at that point. In situation 3, S admits a discontinuity at point (x,t): $S(x+,t)$ is determined by the field-line converging from the right towards $\mathcal{K}$, and $S(x-,t)$ is determined by the field-line converging from the left towards $\mathcal{K}$. Furthermore case 3 implies a velocity that is greater downstream than upstream of the shockwave, hence contradicts the entropy condition if the underlying model is a first order one of the LWR type.

Let us examine now case 4, which is the only non-trivial one. The speed V admits a discontinuity at point (x,t) of $\mathcal{K}$, let $V(x+,t)$ and $V(x-,t)$ be the values of the speed downstream and upstream of the point (x,t). As the field-lines of $\mathcal{V}$ diverge from $\mathcal{K}$, it follows that the slope of $\mathcal{K}$ must be negative, and that $V(x-,t) \geq V(x+,t)$, since the angle of $(-V, 1-V/V_{max})$ with the x axis increases as V increases. Hence by linearity of $V \longrightarrow (-V, 1-V/V_{max})$, there exists a unique speed $V^b_{\mathcal{K}}(x,t) \in [V(x+,t), V(x-,t)]$ such that $(-V^b_{\mathcal{K}}(x,t), 1-V^b_{\mathcal{K}}(x,t)/V_{max})$ be in the tangent space $T_{(x,t)}\mathcal{K}$ of $\mathcal{K}$ at point (x,t). As long as $\mathcal{K}$ is smooth, $V^b_{\mathcal{K}}$ is continuous (or of the same regularity as $\mathcal{K}$). Hence, $\mathcal{K}$ can be defined as the solution of:

$$\left[\begin{array}{l} \frac{d\mathcal{X}^b}{dv} = -V^b_{\mathcal{K}}(\mathcal{X}^b, \mathcal{T}^b)(v) \\ \\ \frac{d\mathcal{T}^b}{dv} = 1 - V^b_{\mathcal{K}}(\mathcal{X}^b, \mathcal{T}^b)(v)/V_{max} \end{array}\right. \quad ,$$

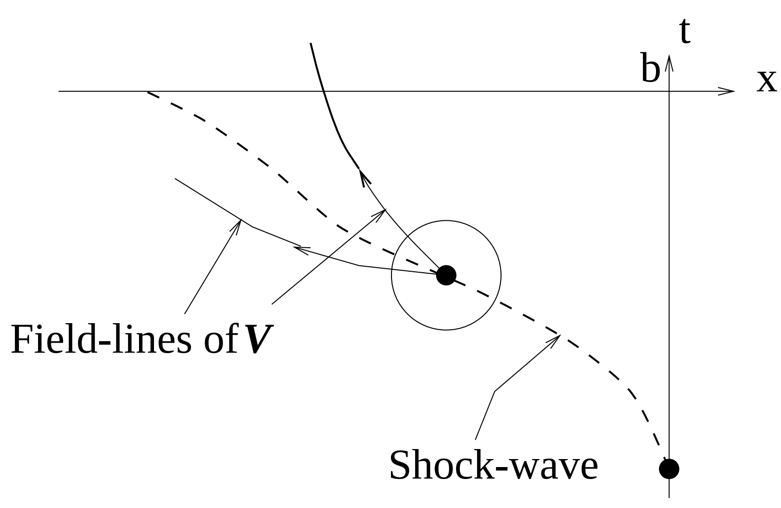
or, extending the definition of $\mathcal{V}$ as

$$\mathcal{V}^*(x,t) \stackrel{def}{=} [(-V(x-,t), 1-V(x-,t)/V_{max}), (-V(x+,t), 1-V(x+,t)/V_{max})] \quad ,$$

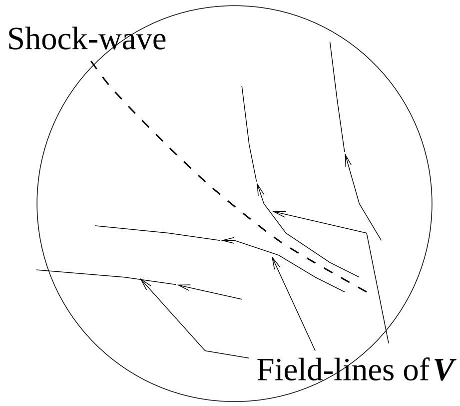
a unique definition results for both ordinary field-lines of $\mathcal{V}$ and kinematical shock-waves $\mathcal{K}$:

$$\frac{d}{dv}(\mathcal{X}^b, \mathcal{T}^b) \in \mathcal{V}^* \quad .$$

The result is that in case 4 the backward ITT S is continuous at $\mathcal{K}$ and can indeed be calculated along $\mathcal{K}$ by applying the same rule as along ordinary field-lines, i.e. $dS = dv$. Let us visualize a enlargement of the vicinity of the point (x,t) of $\mathcal{K}$, as depicted in the following figure:



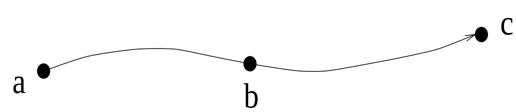
In this vicinity, the field $\mathcal{V}$ might be considered to vary very fast from $\mathcal{V}(x-,t)$ to $\mathcal{V}(x+,t)$ while crossing $\mathcal{K}$, and to take the intermediate value $(-V^b_{\mathcal{K}}(x,t), 1-V^b_{\mathcal{K}}(x,t)/V_{max})$, resulting in field-lines diverging from the central field-line $\mathcal{K}$, as illustrated hereafter.



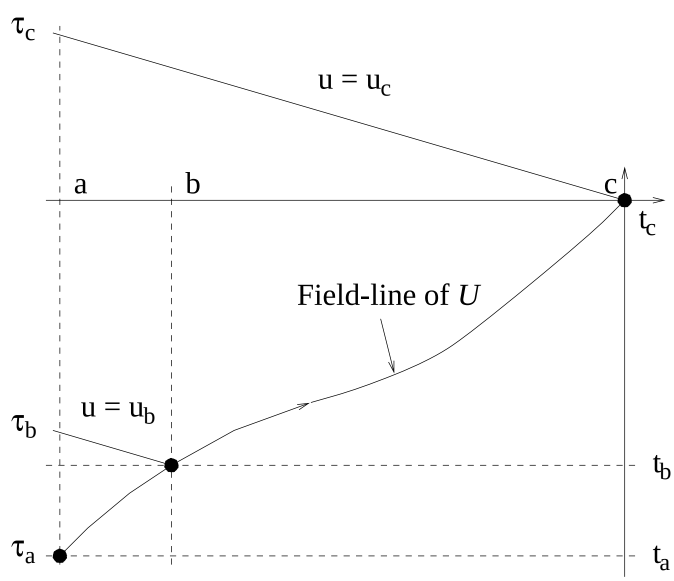
One final remark: everything that has been said about case 4 can be transposed trivialy to case 3, essentially by replacing the divergence of field-lines by their convergence towards the shock-wave $\mathcal{K}$. Nevertheless, case 3 is here of very limited import as has been noted above.

3.3 Composition of instantaneous travel times

To derive the composition rule of ITTs, let us again consider three points on a line:



and let us consider first the forward ITT: $ITTf$. The field-line of $\mathcal{U}$ originating at point (a, t_a) passes through points (b, t_b) and (c, t_c).



Let x_a, x_b, x_c be the coordinates of points a, b, c. The lines

$$t + \frac{x}{V_{max}} = u = u_i \stackrel{def}{=} t_i + \frac{x_i}{V_{max}}$$

for $i = a, b, c$ intersect the line $x = x_a$ at points τ_i given by:

$$\tau_i = t_i + \frac{x_i - x_a}{V_{max}} \quad .$$

Now we deduce from (19) that:

$$\left| \begin{array}{lllll} \tau_c - \tau_a & = & u_c - u_a & = & ITTf(a,c;t_c) \\ \tau_b - \tau_a & = & u_b - u_a & = & ITTf(a,b;t_b) \\ \tau_c - \tau_b & = & u_c - u_b & = & ITTf(b,c;t_c) \end{array} \right.$$

Considering that

$$\tau_c - \tau_b = u_c - u_b = t_c - t_b + \frac{x_c - x_b}{V_{max}} \quad ,$$

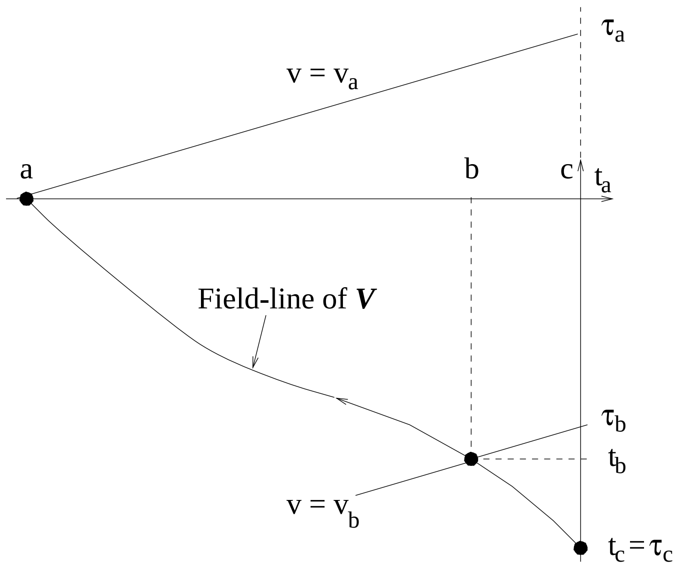
it follows that

$$t_b = t_c + ITTf(b,c;t_c) - \frac{x_c - x_b}{V_{max}} \quad .$$

Finally, replacing t_c by t and using the trivial relationship $\tau_c - \tau_a = (\tau_c - \tau_b) + (\tau_b - \tau_a)$, one gets:

(23) $$ITTf(a,c;t) = ITTf(b,c;t) + ITTf(a,b;t + ITTf(b,c;t) - \frac{x_c - x_b}{V_{max}}) \quad .$$

The derivation of the composition rule of backward ITTs, i.e. $ITTb$, follows the same steps.



We denote

$$v_i \stackrel{def}{=} t_i - \frac{x_i}{V_{max}}$$

for $i = a, b, c$, and call τ_i the intersections of the lines

$$t - \frac{x}{V_{max}} \stackrel{def}{=} v = v_i$$

with the line $x = x_c$, for $i = a, b, c$. The τ_i are given by

$$\tau_i = t_i - \frac{x_i - x_c}{V_{max}} \quad .$$

Now we deduce from (22) that:

$$\left| \begin{array}{lllll} \tau_a - \tau_c & = & v_a - v_c & = & ITTb(a,c;t_a) \\ \tau_b - \tau_c & = & v_b - v_c & = & ITTb(b,c;t_b) \\ \tau_a - \tau_b & = & v_a - v_b & = & ITTb(a,b;t_a) \end{array} \right.$$

From

$$t_b - t_a = \tau_b - \tau_a + \frac{x_b - x_a}{V_{max}}$$

we deduce

$$t_b = t_a - ITTb(a,b;t_a) + \frac{x_b - x_a}{V_{max}} \quad .$$

Replacing t_a by t and using again $\tau_a - \tau_c = (\tau_a - \tau_b) + (\tau_b - \tau_c)$ it follows that:

$$ITTb(a,c;t) = ITTb(a,b;t) + ITTb(b,c;t + \frac{x_b - x_a}{V_{max}} - ITTb(a,b;t)) \quad . \tag{24}$$

Remarks. The ITTs may "converge" towards $\int_. \frac{d\xi}{V(\xi,.)}$ and be nearly additive under two conditions. The first one is that the flow be nearly stationary, hence $\frac{\partial}{\partial t} \approx 0$, leaving $V \frac{\partial}{\partial x} \approx 1$. The ETT "converges" towards $\int_. \frac{d\xi}{V(\xi,.)}$ under the same condition. The second condition is that the product $(1 - \frac{V}{V_{max}}) \frac{\partial}{\partial t} \approx 0$, i.e. $V \approx V_{max}$. This condition is specific of ITTs. The above formulas (23) and (24) show this near-additivity clearly. Indeed, if $ITTb(b,c;t) \approx \frac{x_c - x_b}{V_{max}}$ or if $ITTb(a,b;t) \approx \frac{x_b - x_a}{V_{max}}$, then both formulas simplify as:

$$ITT(a,c;t) \approx ITT(a,b;t) + ITT(b,c;t) \quad .$$

4 Semidiscretized models.

This very short section is devoted to semidiscretized models, mainly as a tribute to the historical importance of the subject for theoretical studies on assignment. It must be noted that more and more doubts on the relevance of this type of models express themselves in the literature, as for instance in [DA 95], [HA 96] (and the references therein). For this reason, our review of this subject will be extremely cursory, and limited to the illustration of the PTT concept. By semidiscretized models we mean models continuous in time and discretized in space, with the link as space discretization unit. As indicated above, the link $PTT(t)$ is essentially a function of the link state $K(.,t)$ at time t and the downstream traffic flow supply $\Sigma(b,s)$ for $s \in [t, E(t)]$. In fluid traffic conditions (downstream traffic flow supply always sufficient to accomodate the traffic demand of the link), and at the zero-th order approximation, one might consider $PTT(t)$ as a function of $N(t)$. This is the basis of some flow models for assignment problems ([FC 94], [FBSTW 93], [AS 96], [RHB 96]). In such models the link traffic flow dynamics are described by a model of the following kind:

$$\frac{d}{dt} N(t) = u(t) - v(t)$$

($u(t)$ the link inflow and $v(t)$ the link outflow), supplemented by a model for the $PTT(t)$, which is usually called $\tau(t)$ in this context:

$$\tau(t) = f(N(t))$$

and the FIFO condition, which in the present case does *not* result naturally from the model. This last condition implies ([AS 96]) that:

$$v(t + \tau(t)) = \frac{u(t)}{1 + \frac{d\tau}{dt}(t)} = \frac{u(t)}{\frac{dE}{dt}(t)} \tag{25}$$

by expressing that users entering the link at time t exit it at time $t + \tau(t)$. With the FIFO hypothesis, the following integral relationships result, similar to those already stated in the continuous case:

$$N(t) = \int_{I(t)}^{t} u(s) ds = \int_{t}^{E(t)} v(s) ds \quad .$$

It can be shown that the only consistent FIFO model of this kind is the one associated to a linear travel time function:

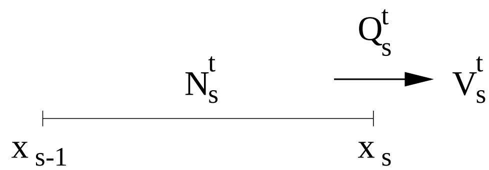
$$\tau(t) = \alpha + \beta N(t) \quad .$$

This result was suggested in [DA 95], the sufficiency of this linear form was demonstrated in [FBSTW 93], and its necessity in [LL 96]. The linear part represents the average time lost in the queue at the exit of the link, which is somewhat at odds with the hypothesis that the downstream traffic supply can be neglected. It is not known to the author of the present paper whether non-FIFO models of the above kind can be built. The analysis of such models might proove difficult since they would not admit any closed expressions such as (25) for the link outflow.

5 Fully discretized macroscopic models.

5.1 Discretization: principles

We shall consider in the sequel discretizations of the following kind: links are divided into cells say $(s) = [x_{s-1}, x_s]$, of length l_s, containing N_s^t vehicles at time $t\Delta t$, with the average cell exit flow Q_s^t during time-step $[t\Delta t, (t+1)\Delta t]$, estimated at the cell exit point x_s. (A notable exception to this kind of approach to discretized macroscopic modelling is the particle discretization approach, as in INTEGRATION).



No hypothesis is made on the macroscopic model itself, and on the exact manner in which the above quantities are computed. Nevertheless, we expect the discretization to respect some basic rules, and we shall refer to discretizations satisfying to those rules as *proper discretizations*.

The first requirement will be that the ratio

$$\alpha_s \overset{def}{=} \frac{V_{s,max}\Delta t}{l_s} \quad , \tag{26}$$

be less than 1, with of course Δt the time-step, l_s the length of cell (s) and $V_{s,max}$ the maximum speed in cell (s). Another significant ratio is

$$\beta_s^t \overset{def}{=} \frac{Q_s^t \Delta t}{N_s^t} \quad , \tag{27}$$

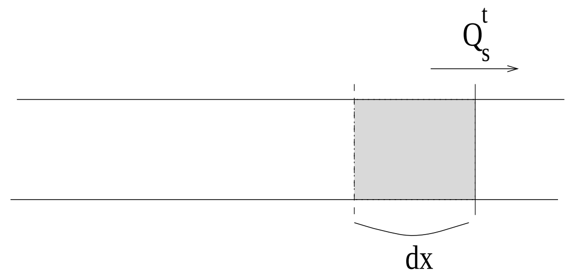
which will also be expected to be less than 1, as such a condition expresses that the number of vehicles exiting a cell during a time-step be less than the number of vehicles present in the cell at the beginning of the time-step. It is only if $\beta_s^t \leq 1$ is satisfied that it will be possible to keep track of vehicle propagation in the model. To summarize, a proper discretization in our sense satisfies to:

$$\left[\begin{array}{l} \alpha_s \leq 1 \\ \beta_s^t \leq 1 \end{array} \right. \quad .$$

For travel-time estimation, we need some relevant expression of the speed. For discretized macroscopic models as we consider here, either there exist no intrinsic estimates of speed (1st order models), or, if such estimates exist, they may yield irrealistic values of the travel times. Hence we propose to introduce a specific speed, which we shall call *cell exit speed* and define as

$$V_s^t = \frac{Q_s^t}{K_s^t} = \frac{Q_s^t l_s}{N_s^t} \tag{28}$$

with $K_s^t \overset{def}{=} N_s^t / l_s$ defined as *the mean cell density* at time $t\Delta t$. The significance of this choice is that it permits to emulate FIFO behaviour whithin each cell, which, as has been noted previously, is intrinsic to the interpretation of macroscopic models with the speed flow. As is illustrated by the following figure,



the outflow during a time step, $Q_s^t \Delta t$, will thus be equal to the mean cell density K_s^t times the distance covered during the time-step at speed V_s^t, i.e. $K_s^t V_s^t \Delta t$, which is the precise translation of the FIFO hypothesis if the density is assumed uniform in cell (s). Associated to this cell exit speed, it is possible to define the following ratio:

$$\nu_s^t = \frac{V_s^t}{V_{s,max}} \quad . \tag{29}$$

Regrettably, most models do not give any guarantee that if the discretization is proper, the inequality

$$\nu_s^t \leq 1$$

will be satisfied. Nevertheless, in the case of first-order models of the LWR kind, discretized by Godunov's method [LE 95-2], this property holds true, and will therefore be used occasionnally. Let us note finally that

$$\alpha_s \nu_s^t = \beta_s^t \quad .$$

5.2 ETT estimation: the naive approach

5.2.1 Introduction

The most straightforward approach to the numerical calculation of ETTs in the framework defined in the preceding subsection would be to hold an account of the entry-time of every vehicle in every cell, and to process vehicles inside a cell in the order defined by their entry-time, letting them exit the cell in the precise order they entered it. In fact, if the traffic flow is relatively fluid, or if there exists some lower bound on speeds (a hypothesis which has already been discussed and is usually relevant for motorway networks), this idea may prove quite efficient, as the cell traversal time will then be bounded from above. The boundedness of traversal times is an essential feature if one wishes to prevent an occasional but uncontrollable inflation of the data relative to vehicle cell entry times. To summarize, the approach described here is feasable for motorway networks, with the corresponding macroscopic model endowed with a lower bound for speed.

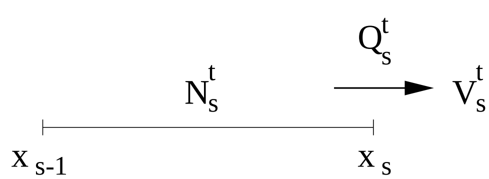
One last remark: in the context of a formally similar problem i.e. the propagation of partial densities in a discretized macroscopic freeway model, whose equations for compositions χ_d are given by [LE 95-2]:

$$V\frac{\partial \chi_d}{\partial x} + \frac{\partial \chi_d}{\partial t} = 0 \quad ,$$

Daganzo [DA 94] proposed an algorithm of the same family as the one proposed hereafter.

5.2.2 The algorithm

Let us now consider a cell (s) as depicted hereafter:



and let us define the following quantities:

$$\begin{array}{rcl} r^t_{s,0} & = & N^t_s \\ r^t_{s,1} & = & N^{t-1}_s - Q^{t-1}_s \Delta t \\ \vdots & & \vdots \\ r^t_{s,i} & = & N^{t-i}_s - \sum_{j=1}^{i} Q^{t-j}_s \Delta t \\ \vdots & & \vdots \end{array}$$

with $r^t_{s,i}$ being by definition the number of vehicles left over at time $t\Delta t$ from the vehicles present in the link at time $(t-i)\Delta t$. Further we define I^t_s as:

$$I^t_s \stackrel{def}{=} Min\{i \,/\, r^t_{s,i} \leq 0\}$$

a definition which means that if $I = I^t_s$, then $r^t_{s,I} \leq 0$ and $r^t_{s,I-1} > 0$. This definition is completely consistent, since the quantities $r^t_{s,i}$ are decreasing with i, as is shown by the following straightforward calculation:

$$\begin{array}{rcl} r^t_{s,i} - r^t_{s,i+1} & = & N^{t-i}_s - N^{t-i-1}_s + Q^{t-i-1}_s \Delta t \\ & = & Q^{t-i-1}_{s-1} \Delta t \\ & \geq & 0 \quad , \end{array}$$

since the N^{t-i}_s satisfy to the conservation equation

$$N^{t-i}_s = N^{t-i-1}_s + Q^{t-i-1}_{s-1} \Delta t - Q^{t-i-1}_s \Delta t \quad .$$

At time $t\Delta t$ the following set summarizes the information relative to the entry time of the vehicles in cell (s):

(30) $$\mathcal{R}^t_s \stackrel{def}{=} \{r^t_{s,i} \,/\, i = 0, I^t_s\} \quad .$$

To expand on previous remarks, if the cell traversal time increases, I^t_s, which measures the quantity of data relative to cell entry times that is stored for the algorithm, increases in the same measure.

Let us now consider an iteration, and show how $\mathcal{R}^{t+1}_s$ is deduced from $\mathcal{R}^t_s$. By definition of the quantities $r^t_{s,i}$,

$$\left| \begin{array}{rcl} r^{t+1}_{s,0} & = & N^{t+1}_s \\ r^{t+1}_{s,i} & = & r^t_{s,i} - Q^t_s \Delta t \quad (\forall i) \\ I^{t+1}_s & = & Min\{i \,/\, r^{t+1}_{s,i} \leq 0\} \end{array} \right.$$

(hence the $r^{t+1}_{s,i}$ should be computed only as long as $r^t_{s,i-1} - Q^t_s \Delta t > 0$). The initialization of the $\mathcal{R}^t_s$ at time say $t = 0$ is relatively straightforward. Known at time $t = 0$ are the N^0_s and the Q^0_s. It suffices to assume that the traffic situation prior to the initialization has been stationary to deduce:

$$\left| \begin{array}{rcl} r^0_{s,i} & = & N^0_s - iQ^0_s \Delta t \\ I^0_s & = & Min\{i \,/\, i \geq N^0_s / Q^0_s \Delta t\} \end{array} \right. \quad .$$

Whatever the initialization error, it will be eliminated in finite time, i.e. the time it takes for the N^0_s initial vehicles to exit the cell (s). After that time, the initial set $\mathcal{R}^0_s$ will no longer have any influence on the actual set $\mathcal{R}^t_s$.

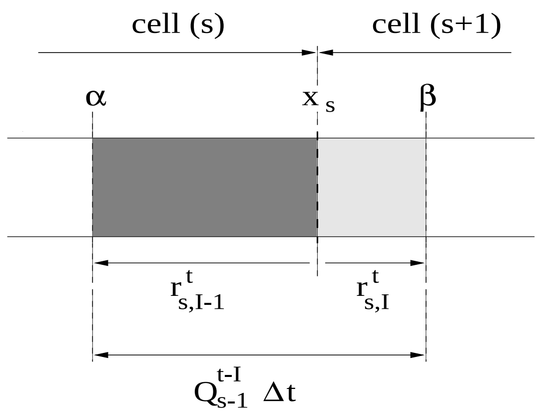
Finally, let us address the problem of estimating the ETT from the data contained in the sets $\mathcal{R}^t_s$. The clue to the result is the fact that :

$$r^t_{s,i} - r^t_{s,i+1} = Q^{t-i-1}_{s-1} \Delta t \geq 0$$

which has been proven above. This result means that $r^t_{s,i} - r^t_{s,i+1}$ represents the number of vehicles having entered the cell (s) during the time interval $[(t-i-1)\Delta t, (t-i)\Delta t]$. Now let us denote, in order to simplify notations,

$$I = I^t_s \quad .$$

By definition, $r^t_{s,I-1} > 0$ and $r^t_{s,I} \leq 0$.



Let us represent $r^t_{s,I-1}$ and $r^t_{s,I}$ as depicted on the above figure. The first vehicle that has entered the cell (s) at time $(t-I)\Delta t$ is located at location β and has exited the cell before time $t\Delta t$. The last vehicle having entered the cell at time $(t-I+1)\Delta t$ is located at point α inside cell (s). Finally, at point x_s is located a vehicle about to leave cell (s); downstream of this vehicle there are $-r^t_{s,I}$ vehicles out of the $Q^{t-I}_{s-1}\Delta t$ vehicles that have entered the cell during time interval $[(t-I)\Delta t, (t-I+1)\Delta t]$. Therefore, with the usual assumption that the inflow has been constant during that time-interval, it follows that the time of entry of the vehicle about to leave the cell at time $t\Delta t$ is

$$(t-I)\Delta t + \frac{-r^t_{s,I}}{Q^{t-I}_{s-1}} \quad .$$

It follows that ETT^t_s is $t\Delta t$ minus the above entry time, hence replacing $Q^{t-I}_{s-1}\Delta t$ by $-r^t_{s,I} + r^t_{s,I-1}$, it follows that

(31) $$ETT^t_s = (I + \frac{-r^t_{s,I}}{r^t_{s,I-1} - r^t_{s,I}})\Delta t \quad .$$

This formula entails no errors other than those resulting from the discretization and from the interpolation of ETT^t_s. Notably, there will be no error propagation from one time-step to the next. Let us note also that:

$$I\Delta t \leq ETT^t_s < (I+1)\Delta t \quad .$$

Finally, it must be noted that the preceding method gives us no clue as how to combine the cell ETTs in order to obtain link ETTs. In fact, the simplest method would be to apply directly the preceding formula (31) to a link with, of course, the cell subscript s being changed into a link subscript.

5.3 Calculation of ETTs according to (8)

Let us recall (8)

$$\left| \begin{array}{l} V\frac{\partial T}{\partial x} + \frac{\partial T}{\partial t} = 1 \\ \\ T(a,t) = 0 \qquad (\forall t) \end{array} \right. \quad .$$

The setting is the same as in the rest of this section. In order to discretize (8), we shall need to choose a cell speed; we shall again use the cell exit speed as defined by (28). The basic ideas of the discretization are the following:

- the approximate speed field say $\tilde{\mathcal{W}}$ will be defined as piecewise constant:

$$\tilde{\mathcal{W}}(\xi,\tau) \stackrel{def}{=} (V_s^t, 1) \quad \text{iff} \quad \xi \in [x_{s-1}, x_s] \;\; \text{and} \;\; \tau \in [t\Delta t, (t+1)\Delta t] \quad ,$$

- the function T is approximated by a piecewise linear function $\tilde{T}(\xi,\tau)$ taking values

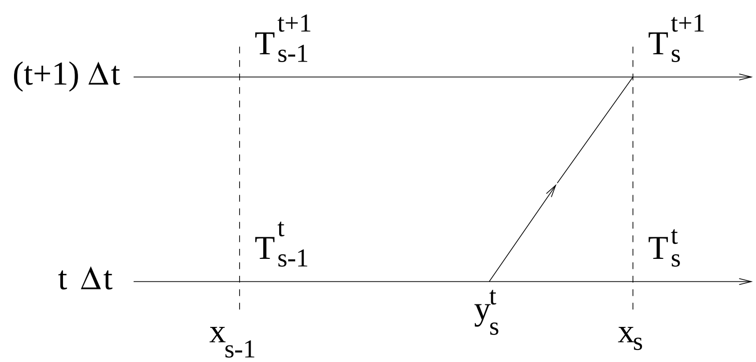
$$T_s^t \stackrel{def}{=} \tilde{T}(x_s, t\Delta t)$$

at the node points,

- The value T_s^{t+1} will be derived from $\tilde{T}(.,t\Delta t)_{|\text{cell } (s)}$ by applying

$$\frac{dETT}{dt} = 1$$

along field-lines of $\tilde{\mathcal{W}}$.



Hence let us define

$$y_s^t \stackrel{def}{=} x_s - V_s^t \Delta t$$

the first coordinate of the intersection point $(y_s^t, t\Delta t)$ of the field-line of $\tilde{\mathcal{W}}$ originating at point $(x_s, (t+1)\Delta t)$ (which is a line of direction $(V_s^t, 1)$ with the line $\tau = t\Delta t$. It follows:

$$T_s^{t+1} = \tilde{T}(y_s^t, t\Delta t) + \Delta t$$

with

$$\tilde{T}(y_s^t, t\Delta t) = \beta_s^t T_{s-1}^t + (1 - \beta_s^t) T_s^t$$

by the linearity of $\tilde{T}$. The boundary condition is expressed by:

$$T_0^t = 0 \quad .$$

Hence:

(32)
$$\begin{cases} T_s^{t+1} = \beta_s^t T_{s-1}^t + (1-\beta_s^t) T_s^t + \Delta t \\ T_0^t = 0 \quad , \end{cases}$$

with β_s^t defined by (27).

The preceding method is related to the method of characteristics, as described in [PI 88] and applicable to the convection equation yielding T. But in the present case, the method has been restricted to a single time-step, which is of course very convenient from a practical point of view, since the algorithm reduces to the simple recursive smoothing formula (32). Of course, the complete characteristics method (implying backward computation of the field-lines of $\tilde{\mathcal{W}}$ till the boundary $\xi = a$ is encountered) would be exact for $\tilde{\mathcal{W}}$ but extremely unwieldy and would require as much storage space as the naive method. In the context of traffic problems, the usual tradeoff between precision and computational efficiency must favour the latter.

Remarks

1. If $S = 1$, the formula (32) becomes:

$$\left\{ \begin{array}{l} T^{t+1} = (1-\beta^t)T^t \\ T^t = 0 \\ \beta^{t+1} = Q^t \Delta t / N^t \end{array} \right. ,$$

by dropping the subscript s. This formula was introduced on a purely heuristic basis in [BLL 95-96].

2. It is interesting to introduce the cell travel times

$$ETT_s^t \stackrel{def}{=} T_s^t - T_{s-1}^t \quad .$$

From (32) it follows:

(33)
$$\begin{array}{rcl} ETT_s^{t+1} & = & T_s^{t+1} - T_{s-1}^{t+1} \\ & = & -T_{s-1}^{t+1} + \beta_s^t T_{s-1}^t + (1-\beta_s^t)T_s^t + \Delta t \\ & = & -(T_{s-1}^{t+1} - T_{s-1}^t) + \Delta t + (1-\beta_s^t)ETT_s^t \quad . \end{array}$$

Further from (32) it is possible to deduce:

$$T_s^{t+1} - T_s^t = \Delta t - \beta_s^t ETT_s^t \quad ,$$

and, combining these last identities, it follows:

(34)
$$ETT_s^{t+1} = \beta_s^t ETT_{s-1}^t + (1-\beta_s^t)ETT_s^t \quad .$$

This last equation shows that, if initially, the cell travel times are positive, they stay so at all times. As a consequence, it follows also that at all times

$$\Delta t - (T_s^{t+1} - T_s^t) \geq 0 \quad .$$

3. Further, if $V_s^t \leq V_{s,max}$, i.e. $\nu_s^t \leq 1$, a condition satisfied for instance by the Godunov scheme applied to the LWR model, one can deduce from (32) that:

(35)
$$T_s^t \geq \sum_{i=1}^{s} \frac{l_i}{V_{i,max}} \quad .$$

Indeed, if this inequality is satisfied initially, it follows by recurrence that if (35) holds at time $t\Delta t$ for all s, then (35) holds at time $(t+1)\Delta t$ for all s:

$$\begin{array}{rcl} T_s^{t+1} & = & \beta_s^t T_{s-1}^t + (1-\beta_s^t)T_s^t + \Delta t \\ & \geq & [\beta_s^t + 1 - \beta_s^t] \displaystyle\sum_{i=1}^{s-1} \frac{l_i}{V_{i,max}} + (1-\beta_s^t)\frac{l_s}{V_{s,max}} + \Delta t \quad . \end{array}$$

Hence:

$$\begin{array}{rcl} T_s^{t+1} - \displaystyle\sum_{i=1}^{s} \frac{l_i}{V_{i,max}} & \geq & \Delta t - (1-\beta_s^t)\dfrac{l_s}{V_{s,max}} \\ & = & \Delta t(1-\nu_s^t) \geq 0 \quad . \end{array}$$

4. From (33) one deduces trivially that:

$$\begin{aligned} ETT_s^{t+1} - \frac{l_s}{V_s^t} &= -(T_{s-1}^{t+1} - T_{s-1}^t) - \frac{l_s}{V_s^t} + \Delta t + (1-\beta_s^t)(ETT_s^t - \frac{l_s}{V_s^t}) + (1-\beta_s^t)\frac{l_s}{V_s^t} \\ &= -(T_{s-1}^{t+1} - T_{s-1}^t) + (1-\beta_s^t)(ETT_s^t - \frac{l_s}{V_s^t}) \end{aligned}$$

which shows the convergence of cell travel times towards

$$\frac{l_s}{V_s^t} \approx \int_{\text{cell }(s)} \frac{d\xi}{V(\xi, t\Delta t)}$$

when the traffic flow conditions tend towards stationary conditions.

5. Let us suppose the flow has been disaggregated according to attribute value d, which might be a destination (global or local), an OD pair value, a category of users (informed, uninformed), etc .... Then the macroscopic model provides also the values of the relevant partial quantities:

$$\begin{array}{l} Q_s^{d,t},\ K_s^{d,t} \\ N_s^{d,t} = l_s K_s^{d,t} \\ V_s^{d,t} = Q_s^{d,t}/K_s^{d,t} \\ \beta_s^{d,t} = V_s^{d,t}\Delta t/l_s \end{array}$$

(partial flows, densities, number of vehicles, cell exit speeds). Note that in the special case where the model is FIFO-like, then:

$$Q_s^{d,t} = \frac{K_s^{d,t}}{K_s^t} Q_s^t$$

and the formula for the $\beta_s^{d,t}$ simplifies as $\beta_s^{d,t} = \beta_s^t$.

Formula (32) can be generalized without any difficulty to disaggregated flows, by substituting to superscript t the superscripts d, t:

$$\begin{cases} T_s^{d,t+1} = \beta_s^{d,t} T_{s-1}^{d,t} + (1-\beta_s^{d,t}) T_s^{d,t} \\ T_0^{d,t} = 0 \quad . \end{cases}$$

6. The generalization of (32) to paths is straightforward, as it suffices to divide the path into cells as if it were a link. Let us note that if link travel times are the only quantities available, then (34) can be iterated along the path, with the subscript s in (34) refering in that case to links instead of cells.

7. It is also possible to generalize (32) to exchange zones as defined in the STRADA model for intersection modelling [BLL 95-96]. Using the notations and definitions provided in that reference for network modelling, for zone exit point j of zone say z, one may define the exit speed

$$VO_{z,j}^t \stackrel{def}{=} \frac{QO_{z,j}^t}{NO_{z,j}^t} \lambda_z$$

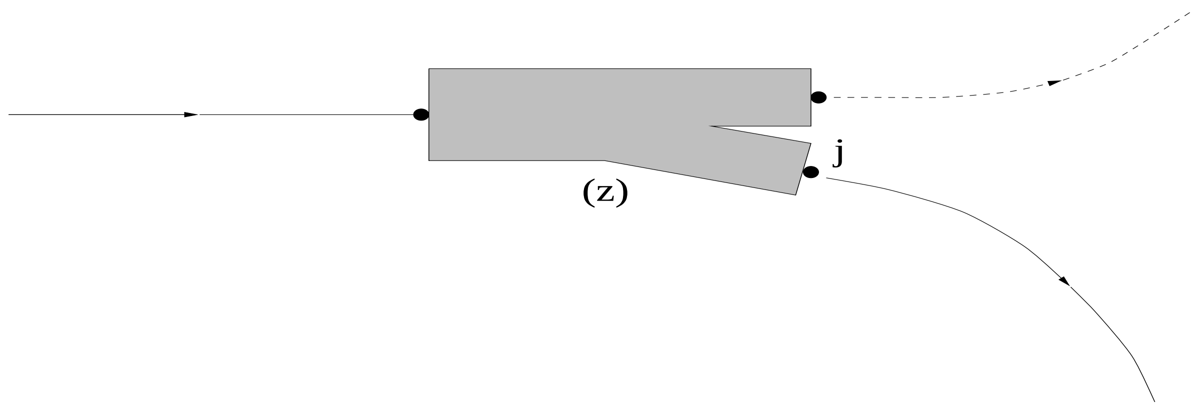
with λ_z the zone length scale, $QO_{z,j}^t$ the zone outflow through its exit point j during time-interval $[t\Delta t, (t+1)\Delta t]$, and $NO_{z,j}^t$ the number of vehicles about to exit zone z through exit point j. Finally, we define:

$$\beta_{z,j}^t \stackrel{def}{=} \frac{\Delta t\, QO_{z,j}^t}{NO_{z,j}^t}$$

and deduce trivially a single cell formula for the experienced traversal time of the zone z at time $t\Delta t$ from any entry point to exit point j:

$$T_{z,j}^{t+1} = \Delta t + (1-\beta_{z,j}^t) T_{z,j}^t \quad .$$

The zone might also be included into one or several paths, as illustrated herafter, and would then contribute to the path ETT estimation as any other ordinary cell of the path.



5.4 Discretization: forward ITT

Let us recall that the forward ITT, $ITTf(a,x;t) = R(x,t)$, is defined as the solution of the following equation (15):

$$\left| \begin{array}{l} V\frac{\partial R}{\partial x} + (1 - \frac{V}{V_{max}})\frac{\partial R}{\partial t} = 1 \\ \\ R(a,t) = 0 \qquad (\forall t) \quad . \end{array} \right.$$

The principle of the discretization is exactly the same as for the ETT T.

- The field $\mathcal{U}$ is approximated by a piecewise constant field $\tilde{\mathcal{U}}$:

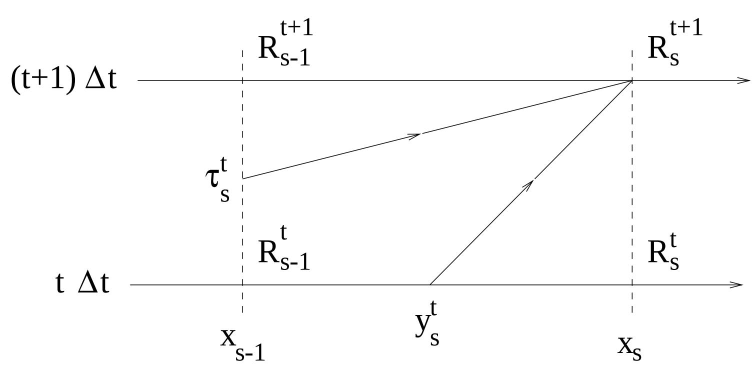
$$\tilde{\mathcal{U}}(x,t) \stackrel{def}{=} (V_s^t, 1 - \frac{V_s^t}{V_{s,max}})$$

  with $x \in [x_{s-1}, x_s]$ and $t \in [t\Delta t, (t+1)\Delta t]$.

- The function R is approximated by a continuous and piecewise linear function $\tilde{R}$ defined by the values

$$R_s^t \stackrel{def}{=} \tilde{R}(x_s, t\Delta t) \quad .$$

- The values R_s^t are calculated every time-step using the relationship $dR = du$ along the field-lines of $\tilde{\mathcal{U}}$, with $u = \tau - \xi/V_{max}$.



In the space × time plane, which shall be denoted here the (ξ, τ)-plane, the field-line of $\tilde{\mathcal{U}}$ originating at point $(x_s, (t+1)\Delta t)$ intersects the line $\xi = x_{s-1}$ at point (x_{s-1}, τ_s^t), or the line $t = t\Delta t$ at point $(y_s^t, t\Delta t)$, depending on the relative value of V_s^t.

Let us consider first intersection point $(y_s^t, t\Delta t)$. This point is given by

$$\left| \begin{array}{cc} x_s - y_s^t & V_s^t \\ \\ \Delta t & 1 - V_s^t/V_{s,max} \end{array} \right| = 0$$

hence

$$x_s - y_s^t = \frac{V_s^t \Delta t}{1 - (V_s^t / V_{s,max})} \quad .$$

It must be noted that, in the present case, it is necessary that $\nu_s^t \leq 1$, where ν_s^t has been defined as the ratio $\nu_s^t = V_s^t / V_{s,max}$. Let us also remind that:

$$\alpha_s \nu_s^t = \beta_s^t \quad .$$

Now, $x_s - y_s^t \leq l_s$ imposes

$$\frac{\nu_s^t}{1 - \nu_s^t} \leq \frac{1}{\alpha_s} \quad ,$$

i.e.:

(36)
$$\nu_s^t \leq \frac{1}{1 + \alpha_s} \quad .$$

Equality in (36) is equivalent to $\beta_s^t = 1 - \nu_s^t$. Now, applying $dR = du$ along the approximate field-line $[(y_s^t, t\Delta t), (x_s, (t+1)\Delta t)]$, it follows that:

$$R_s^{t+1} = \tilde{R}(y_s^t, t\Delta t) + u(x_s, (t+1)\Delta t) - u(y_s^t, t\Delta t) \quad .$$

By linearity of $\tilde{R}$,

$$\tilde{R}(y_s^t, t\Delta t) = R_{s-1}^t \frac{x_s - y_s^t}{l_s} + R_s^t \frac{y_s^t - x_{s-1}}{l_s}$$

hence, if $\nu_s^t \leq \frac{1}{1+\alpha_s}$ (i.e. at low speed),

$$R_s^{t+1} = \frac{\Delta t}{1 - \nu_s^t} + R_{s-1}^t \frac{x_s - y_s^t}{l_s} + R_s^t \frac{y_s^t - x_{s-1}}{l_s} \quad .$$

Let us now consider intersection point (x_{s-1}, τ_s^t). τ_s^t results from:

$$\begin{vmatrix} l_s & V_s^t \\ (t+1)\Delta t - \tau_s^t & 1 - V_s^t / V_{s,max} \end{vmatrix} = 0$$

hence:

$$(t+1)\Delta t - \tau_s^t = \Delta t \frac{1 - \nu_s^t}{\beta_s^t} \quad .$$

This quantity is less than Δt iff

$$1 - \nu_s^t \leq \beta_s^t = \alpha_s \nu_s^t$$

i.e.:

$$\nu_s^t \geq \frac{1}{1 + \alpha_s} \quad .$$

From $dR = du$ again, applied here along the approximate field-line $[(x_s, (t+1)\Delta t), (x_{s-1}, \tau_s^t)]$, we deduce that:

$$\begin{aligned} R_s^{t+1} &= \tilde{R}(x_{s-1}, \tau_s^t) + u(x_s, (t+1)\Delta t) - u(x_{s-1}, \tau_s^t) \\ &= \tilde{R}(x_{s-1}, \tau_s^t) + (t+1)\Delta t - \tau_s^t + \frac{l_s}{V_{s,max}} \\ &= \tilde{R}(x_{s-1}, \tau_s^t) + \frac{\Delta t}{\beta_s^t} \quad , \end{aligned}$$

and by linearity of $\tilde{R}$,

$$\begin{aligned}\tilde{R}(x_{s-1},\tau_s^t) &= R_{s-1}^{t+1}\frac{\tau_s^t-\Delta t}{\Delta t}+R_{s-1}^t\frac{(t+1)\Delta t-\tau_s^t}{\Delta t}\\ &= \frac{\Delta t}{\beta_s^t}+R_{s-1}^{t+1}\frac{\beta_s^t+\nu_s^t-1}{\beta_s^t}+R_{s-1}^t\frac{1-\nu_s^t}{\beta_s^t}\quad .\end{aligned}$$

Finally, if $\nu_s^t \geq \frac{1}{1+\alpha_s}$ (i.e. at high speed),

$$R_s^{t+1}=\frac{1}{\beta_s^t}[\Delta t+R_{s-1}^{t+1}(\beta_s^t+\nu_s^t-1)+R_{s-1}^t(1-\nu_s^t)]\quad .$$

The end result is:

$$(37)\qquad \left[\begin{array}{lcl} R_s^{t+1} &=& \frac{1}{1-\nu_s^t}[\Delta t+\beta_s^t R_{s-1}^t+(1-\nu_s^t-\beta_s^t)R_s^t]\\ & & \quad \text{if } \nu_s^t \leq \frac{1}{1+\alpha_s}\\ R_s^{t+1} &=& \frac{1}{\beta_s^t}[\Delta t+R_{s-1}^{t+1}(\beta_s^t+\nu_s^t-1)+R_{s-1}^t(1-\nu_s^t)]\\ & & \quad \text{if } \nu_s^t \geq \frac{1}{1+\alpha_s}\\ R_0^{t+1} &=& 0 \end{array}\right.$$

Properties of the R_s^t

1. Introducing the link travel times

$$TTf_s^t \stackrel{def}{=} R_s^t - R_{s-1}^t \quad ,$$

it follows from (37) that:

$$\left[\begin{array}{lcl} TTf_s^{t+1} &=& \frac{1}{1-\nu_s^t}[\Delta t+(1-\nu_s^t-\beta_s^t)TTf_s^t]-(R_{s-1}^{t+1}-R_{s-1}^t)\\ & & \quad \text{if } \nu_s^t \leq \frac{1}{1+\alpha_s}\\ TTf_s^{t+1} &=& \frac{1}{\beta_s^t}[\Delta t-(1-\nu_s^t)(R_{s-1}^{t+1}-R_{s-1}^t)]\\ & & \quad \text{if } \nu_s^t \geq \frac{1}{1+\alpha_s} \end{array}\right.$$

This last formula can also be rewritten as

$$\left[\begin{array}{lcl} TTf_s^{t+1}-\frac{l_s}{V_s^t} &=& (1-\frac{\beta_s^t}{1-\nu_s^t})[TTf_s^t-\frac{l_s}{V_s^t}]-(R_{s-1}^{t+1}-R_{s-1}^t)\\ & & \quad \text{if } \nu_s^t \leq \frac{1}{1+\alpha_s}\\ TTf_s^{t+1}-\frac{l_s}{V_s^t} &=& -\frac{1-\nu_s^t}{\beta_s^t}(R_{s-1}^{t+1}-R_{s-1}^t)\\ & & \quad \text{if } \nu_s^t \geq \frac{1}{1+\alpha_s} \end{array}\right.$$

which illustrates the convergence of cell travel times towards

$$\frac{l_s}{V_s} \approx \int_{\text{cell } s}\frac{d\xi}{V(\xi,.)}$$

under traffic conditions that tend towards stationary traffic conditions. The question of whether those cell travel times TTf_s^t remain ≥ 0 at all times is open. It is possible to prove the following somewhat weaker result:

2. If, for all $s=1,S$, and for some initial time $t=t_0$, the following inequality holds:

$$R_s^t \geq \sum_{i=1,s}\frac{l_s}{V_{s,max}}\quad ,$$

then it holds for all subsequent instants $t \geq t_0$. This inequality means that the forward ITT numerical estimate of the travel time is always greater than the smallest possible experienced travel time. The result is a consequence of the inequality

$$V_{s,max} \geq V_s^t$$

(i.e. $\nu_s^t \leq 1$), and of a technical result which we shall state now. Before stating this technical result, let us first introduce the following notations:

$$\begin{aligned} \epsilon_s^{t+1} &\stackrel{def}{=} R_s^{t+1} - \sum_{i=1,s} \frac{l_s}{V_s^t} \\ \delta_s^t &\stackrel{def}{=} R_s^t - \sum_{i=1,s} \frac{l_s}{V_s^t} \quad . \end{aligned}$$

We can now state the technical result:

$$\left[\begin{array}{lll} \epsilon_s^{t+1} & = & \frac{\beta_s^t}{1-\nu_s^t}\delta_{s-1}^t + (1 - \frac{\beta_s^t}{1-\nu_s^t})\delta_s^t \\ & & \text{if } \nu_s^t \leq \frac{1}{1+\alpha_s} \\ \epsilon_s^{t+1} & = & \frac{1-\nu_s^t}{\beta_s^t}\delta_{s-1}^t + (1 - \frac{1-\nu_s^t}{\beta_s^t})\epsilon_{s-1}^{t+1} \\ & & \text{if } \nu_s^t \geq \frac{1}{1+\alpha_s} \quad , \end{array} \right.$$

as a trivial consequence of (37) and of the identity

$$\Delta t = \beta_s^t \frac{l_s}{V_s^t} \quad .$$

3. The extension of (37) to partial flows is trivial. We adopt the notations and setting of the remark 5 of the preceding subsection relative to the discretization of ETTs, which we complete with the introduction of the following notation:

$$\nu_s^{d,t} \stackrel{def}{=} \frac{V_s^{d,t}}{V_{s,max}} \quad .$$

In the case of a FIFO model, we would of course have

$$\nu_s^{d,t} = \nu_s^t \quad \forall d \quad ,$$

as a trivial consequence of $V_s^{d,t} = V_s^t$ for all d. Now, (37) can be rewritten as:

$$\left[\begin{array}{lll} R_s^{d,t+1} & = & \frac{1}{1-\nu_s^{d,t}}[\Delta t + \beta_s^{d,t} R_{s-1}^{d,t} + (1 - \nu_s^{d,t} - \beta_s^{d,t}) R_s^{d,t}] \\ & & \text{if } \nu_s^{d,t} \leq \frac{1}{1+\alpha_s} \\ R_s^{d,t+1} & = & \frac{1}{\beta_s^{d,t}}[\Delta t + R_{s-1}^{d,t+1}(\beta_s^{d,t} + \nu_s^{d,t} - 1) + R_{s-1}^{d,t}(1 - \nu_s^{d,t})] \\ & & \text{if } \nu_s^{d,t} \geq \frac{1}{1+\alpha_s} \\ R_0^{d,t+1} & = & 0 \quad . \end{array} \right.$$

In a similar spirit, what was told in the case of ETTs about generalizations to zones holds also true here, by replacing in (37) subscripts s by the couple z, j of zone and zone exit point. Of course, the following definition would have to be added to those already given:

$$\nu_{z,j}^t \stackrel{def}{=} \frac{VO_{z,j}^t}{V_{z,max}} \quad ,$$

with $V_{z,max}$ the relevant desired speed parameter associated to the zone equilibrium supply and demand functions, and $VO_{z,j}^t$ the outflow speed relative to exit point j of zone z.

Finally, (37) can also be extended to paths, with the subscript s refering either to cells or links, depending on how the path is subdivided.

5.5 Discretization: backward ITT

The analysis will follow very closely what has been done in the preceding subsection for forward ITTs. The backward ITT, $ITTb(x, b; t) = S(x, t)$, is defined as the solution of the following equation (16):

$$\left| \begin{array}{l} -V\frac{\partial S}{\partial x} + (1 - \frac{V}{V_{max}})\frac{\partial S}{\partial t} = 1 \\ S(b,t) = 0 \qquad (\forall t) \quad . \end{array} \right.$$

The principle of the discretization is exactly the same as for the ETT T and the forward ITT R.

- The field $\mathcal{V}$ is approximated by a piecewise constant field $\tilde{\mathcal{V}}$:

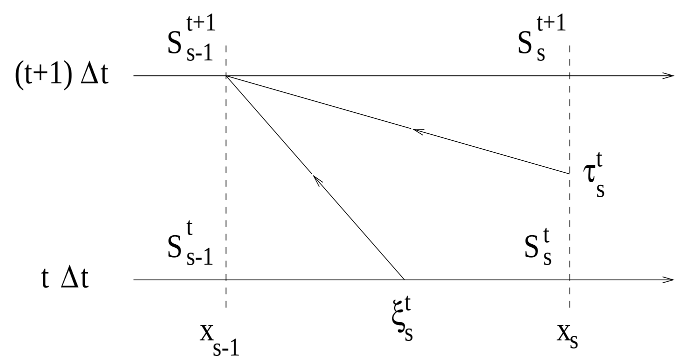
$$\tilde{\mathcal{V}}(\xi, \tau) \stackrel{def}{=} (-V_s^t, 1 - \frac{V_s^t}{V_{s,max}})$$

  with $\xi \in [x_{s-1}, x_s]$ and $\tau \in [t\Delta t, (t+1)\Delta t]$.

- The function S is approximated by a piecewise linear function $\tilde{S}$ defined by its values at node points

$$S_s^t \stackrel{def}{=} \tilde{S}(x_s, t\Delta t) \quad .$$

- The values S_s^t are calculated every time-step using the relationship $dS = dv$ along the field-lines of $\tilde{\mathcal{V}}$, with $v = \tau + \xi/V_{max}$.



The field-line of $\tilde{\mathcal{V}}$ originating at point $(x_{s-1}, (t+1)\Delta t)$ of the space $\times$ time (ξ, τ) plane intersects either the line $\xi = x_s$ at point (x_s, τ_s^t), or the line $\tau = t\Delta t$ at point $(\xi_s^t, t\Delta t)$, depending on the relative value of V_s^t.

The intersection point $(\xi_s^t, t\Delta t)$ is given by

$$\begin{vmatrix} \xi_s^t - x_{s-1} & V_s^t \\ \Delta t & 1 - V_s^t/V_{s,max} \end{vmatrix} = 0$$

hence

$$\xi_s^t - x_{s-1} = \frac{V_s^t \Delta t}{1 - (V_s^t/V_{s,max})} \quad .$$

Here again, the derivation relies on the inequality

$$\nu_s^t \leq 1$$

(i.e. the relative exit speed be les than 1), a condition assumed to be true from now on till the end of the present subsection. The condition $x_s - y_s^t \leq l_s$ is equivalent to (36):

$$\nu_s^t \leq \frac{1}{1 + \alpha_s} \quad ,$$

as with the forward ITT. This is of course the low-speed case. Now, using $dS = dv$ along the approximate field-line $[(x_{s-1},(t+1)\Delta t),(\xi_s^t, t\Delta t)]$ of $\mathcal{V}$, it follows that:

$$S_{s-1}^{t+1} = \tilde{S}(\xi_s^t, t\Delta t) + v(x_{s-1},(t+1)\Delta t) - v(\xi_s^t, t\Delta t) \quad ,$$

Contrarily to the calculation of R, the calculation of the S_s^{t+1} proceeds, starting from $s = S$, backwards to $s = 0$, with the boundary condition $S_S^{t+1} = 0$. Applying the linearity of $\tilde{S}$ and the definition of v, it follows, in the case when $\nu_s^t \le \frac{1}{1+\alpha_s}$ (i.e. at low speed),

$$S_{s-1}^{t+1} = \frac{\Delta t}{1-\nu_s^t} + (1 - \frac{\beta_s^t}{1-\nu_s^t})S_{s-1}^t + \frac{\beta_s^t}{1-\nu_s^t}S_s^t \quad .$$

The intersection point (x_s, τ_s^t) results from:

$$\begin{vmatrix} l_s & V_s^t \\ (t+1)\Delta t - \tau_s^t & 1 - V_s^t/V_{s,max} \end{vmatrix} = 0$$

i.e.:

$$(t+1)\Delta t - \tau_s^t = \Delta t \frac{1-\nu_s^t}{\beta_s^t} \quad .$$

This quantity is less than Δt iff

$$\nu_s^t \ge \frac{1}{1+\alpha_s} \quad ,$$

(the high-speed case). Applying the linearity of $\tilde{S}$, the the relationship $dS = dv$ along the approximate field-line $[(x_s, \tau_s^t),(x_{s-1},(t+1)\Delta t)]$ as well as the definition of v, we deduce that:

$$\begin{aligned} S_s^{t+1} &= \tilde{S}(x_s, \tau_s^t) + v(x_{s-1},(t+1)\Delta t) - v(x_s, \tau_s^t) \\ &= \frac{\Delta t}{\beta_s^t} + S_s^{t+1}\frac{\beta_s^t + \nu_s^t - 1}{\beta_s^t} + S_s^t \frac{1-\nu_s^t}{\beta_s^t} \quad . \end{aligned}$$

in the case when $\nu_s^t \ge \frac{1}{1+\alpha_s}$ (i.e. at high speed). The results are summarized by:

(38)
$$\left[\begin{array}{lcl} S_{s-1}^{t+1} &=& \frac{1}{1-\nu_s^t}[\Delta t + \beta_s^t S_s^t + (1-\nu_s^t - \beta_s^t)S_{s-1}^t] \\ && \text{if } \nu_s^t \le \frac{1}{1+\alpha_s} \\ S_{s-1}^{t+1} &=& \frac{1}{\beta_s^t}[\Delta t + (\beta_s^t + \nu_s^t - 1)S_s^{t+1} + (1-\nu_s^t)S_s^t] \\ && \text{if } \nu_s^t \ge \frac{1}{1+\alpha_s} \\ S_S^{t+1} &=& 0 \quad . \end{array}\right.$$

Properties of the S_s^t

Again, we follow closely what has already be done for the forward ITTs.

1. Introducing the link travel times

$$TTb_s^t \stackrel{def}{=} S_{s-1}^t - S_s^t \quad ,$$

it follows from (38) that:

$$\left[\begin{array}{lcl} TTb_s^{t+1} &=& \frac{1}{1-\nu_s^t}[\Delta t + (1-\nu_s^t - \beta_s^t)TTb_s^t] - (S_s^{t+1} - S_s^t) \\ && \text{if } \nu_s^t \le \frac{1}{1+\alpha_s} \\ TTb_s^{t+1} &=& \frac{1}{\beta_s^t}[\Delta t - (1-\nu_s^t)(S_s^{t+1} - S_s^t)] \\ && \text{if } \nu_s^t \ge \frac{1}{1+\alpha_s} \end{array}\right.$$

which can be rewritten as

$$\left[\begin{array}{lcl} TTb_s^{t+1} - \frac{l_s}{V_s^t} & = & (1 - \frac{\beta_s^t}{1-\nu_s^t})[TTb_s^t - \frac{l_s}{V_s^t}] - (S_s^{t+1} - S_s^t) \\ & & \text{if } \nu_s^t \leq \frac{1}{1+\alpha_s} \\ TTb_s^{t+1} - \frac{l_s}{V_s^t} & = & -\frac{1-\nu_s^t}{\beta_s^t}(S_s^{t+1} - S_s^t) \\ & & \text{if } \nu_s^t \geq \frac{1}{1+\alpha_s} \end{array}\right.$$

which illustrates the convergence of cell travel times towards

$$\frac{l_s}{V_s} \approx \int_{\text{cell } s} \frac{d\xi}{V(\xi,.)}$$

as traffic conditions tend towards stationary traffic conditions. The question of whether those cell travel times TTb_s^t remain > 0 at all times is open, but as previously it is possible to prove the following somewhat weaker result:

2. If, for all $s = 1, S$, and for some initial time $t = t_0$, the following inequality holds:

$$S_{s-1}^t \geq \sum_{i=s,S} \frac{l_s}{V_{s,max}} \quad ,$$

then it holds for all subsequent instants $t \geq t_0$. This inequality means that the backward ITT numerical estimate of the travel time is always greater than the smallest possible experienced travel time. Introducing the following notations:

$$\epsilon_s^{t+1} \stackrel{def}{=} S_{s-1}^{t+1} - \sum_{i=s,S} \frac{l_s}{V_s^t}$$

$$\delta_s^t \stackrel{def}{=} S_{s-1}^t - \sum_{i=s,S} \frac{l_s}{V_s^t} \quad ,$$

we can state:

$$\left[\begin{array}{lcl} \epsilon_s^{t+1} & = & \frac{\beta_s^t}{1-\nu_s^t}\delta_s^t + (1 - \frac{\beta_s^t}{1-\nu_s^t})\delta_{s-1}^t \\ & & \text{if } \nu_s^t \leq \frac{1}{1+\alpha_s} \\ \epsilon_s^{t+1} & = & \frac{1-\nu_s^t}{\beta_s^t}\delta_s^t + (1 - \frac{1-\nu_s^t}{\beta_s^t})\epsilon_s^{t+1} \\ & & \text{if } \nu_s^t \geq \frac{1}{1+\alpha_s} \quad , \end{array}\right.$$

as a trivial consequence of (38) and of the identity

$$\Delta t = \beta_s^t \frac{l_s}{V_s^t} \quad .$$

The result announced above is then an easy consequence of these formulas and of the inequality

$$V_{s,max} \geq V_s^t$$

i.e. ($\nu_s^t \leq 1$).

3. The extension of (38) to partial flows is trivial. We adopt the notations and setting of the remark 3 of the preceding subsection relative to the discretization of forward ITTs. Now, (38) can be rewritten as:

$$\left[\begin{array}{lcl} S_{s-1}^{d,t+1} & = & \frac{1}{1-\nu_s^{d,t}}[\Delta t + \beta_s^{d,t} S_s^{d,t} + (1 - \nu_s^{d,t} - \beta_s^{d,t}) S_{s-1}^{d,t}] \\ & & \text{if } \nu_s^{d,t} \leq \frac{1}{1+\alpha_s} \\ S_{s-1}^{d,t+1} & = & \frac{1}{\beta_s^{d,t}}[\Delta t + S_s^{d,t+1}(\beta_s^{d,t} + \nu_s^{d,t} - 1) + S_s^{d,t}(1 - \nu_s^{d,t})] \\ & & \text{if } \nu_s^{d,t} \geq \frac{1}{1+\alpha_s} \\ S_S^{d,t+1} & = & 0 \quad . \end{array}\right.$$

The adaptations to zone travel-times are exactly the same for forward and backward ITTs and need not be commented upon in more detail in this subsection.

Finally, (38) can also be extended to paths, with the subscript s refering either to cells or links, depending on the way the path is subdivided.

6 Conclusion.

Much work remains to be done. Concerning the quantities defined in this report, one should mention error analysis, and the analysis of some nontrivial, but analytically tractable case. Research is under way concerning both those points. Other ITT estimates are of course conceivable, depending on the properties one deems important for such quantities, since the defining properties we have used imply by no way unicity of the ITTs. These alternative ITT estimates should be investigated. The problem of the time-aggregation of travel times must be adressed, especially in the case of adaptative regulation schemes lacking periodicity. The impact of the proposed estimators on traffic assignment and management schemes must be studied as well. Especially since one of the motivations behind such an "axiomatic" definition of travel times as we have given here is to provide some solid ground for assignment computations. Finally, some experimental assessment of the proposed travel time estimators should be attempted, although it would seem difficult to separate the properties of the estimators from those of the associated macroscopic traffic flow model.

References

[AN 90] Ansorge, R. *What does the entropy condition mean in traffic flow theory.* Transportation Research 24B, 133-144.

[AS 96] Astarita, V. (1996). *A continuous-time link model for dynamic network loading based on travel time function.* Published in the Proceedings of the 13th ISTTT (*Transportation and traffic theory*, ed. J.B. Lesort, Pergamon).

[BLL 95-96] Buisson, C.; Lebacque, J.P.; Lesort, J.B. (1995)
1.*Macroscopic modelling of traffic flow and assignment in mixed networks.* Computing in Civil and Building Engineering, Berlin, Germany, 12-15 July 1995, pp 1367-1374.
2. *STRADA, a discretized macroscopic model of vehicular traffic flow in complex networks based on the Godunov scheme.* Published in the Proceedings of the CESA'96 IEEE Conference.

[BLL 96] Buisson, C.; Lebacque, J.P.; Lesort, J.B. (1996) *Travel time computation for dynamic assignment modelling.* Submitted to the 1996 EURO Conference.

[DA 94] Daganzo, C.F. (1994). *The cell transmission model 1: a dynamic representation of highway traffic consistent with the hydrodynamic theory.* Transportation Research 8B. 4: 269-287. 1994.
The cell transmission model 2: network traffic simulation. Transportation Research 29B. 2: 79-93. 1995.

[DA 95] Daganzo, C.F. (1995). *Properties of link travel time functions under dynamic loads.* Transportation Research B, B29, 95-98.

[EL 96] Elloumi, N. (1996). *Modélisation et commande du trafic sur un corridor - Application de la méthode LP.* Sciences. Orsay, Université Paris XI: 179.

[FC 94] Fernandez, J. Enrique; de Cea, Joaquin. (1994). *Flow propagation description in dynamic network assignment models.* Triennal symposium in transportation analysis (TRISTAN), Capri, Italie, June 1994, pp 119-133.

[FBSTW 93] Friesz, T.L.; Bernstein, D.H.; Smith, T.E.; Tobin, R.L.; Wie, B.W. (1993). *A variational inequality formulation of the dynamic network user equilibrium problem.* Operations Research 41, 179-191.

[HA 96] Heydecker, B.G.; Addison, J.D. (1996). *An exact expression of dynamic traffic control and road pricing.* Published in the Proceedings of the 13th ISTTT (*Transportation and traffic theory*, ed. J.B. Lesort, Pergamon).

[JMH 94] Jayakrishnan, R.; Mahmassani, H. S.; Hu, T.Y. (1994). *An evaluation tool for advanced traffic information and management systems in urban networks.* Transportation Research 2 C(3): 129-147.

[LE 95-1] Lebacque, J.P. (1995). *L'échelle des modèles de traffic: du microscopique au macroscopique.* Annales des Ponts 74, 1st trim. 1995, 48-68.

[LE 95-2] Lebacque, J.P. (1995). *The Godunov scheme and what it means for first order traffic flow models.* CERMICS Report 48-95. Published in the Proceedings of the 13th ISTTT (*Transportation and traffic theory*, J.B. Lesort ed., Pergamon 1996).

[LL 96] Lebacque, J.P.; Lesort, J.B. (1996). *Simplified traffic flow models for dynamic assignment.* Groupe de Travail Modélisation du trafic CERMICS-INRETS.

[LGT 89] Leonard, D. R., Gower P., Taylor. N.B. (1989). *CONTRAM: structure of the model.* Crowthorne, Transport and Road Research Laboratory.

[LTB 78] Leonard, D. R.; Tough, J. B.; Baguley, P. C. (1978). CONTRAM: *A traffic assignment model for predicting flows and queue during peak periods.* Crowthorne, Transport and Road Research Laboratory.

[LW 55] Lighthill, M.H., Whitham, G.B. (1955). *On kinematic waves II: A theory of traffic on long crowded roads.* Proc. Royal Soc. (Lond.). A 229: 317-345.

[PA 90] Papageorgiou, M. (1990). Dynamic modeling, assignment, and route guidance in traffic networks. Transportation Research 24 B(6): 471-495.

[PI 88] Pironneau, O. (1988). *Méthode des éléments finis pour les fluides.* RMA 7. Masson.

[RB 94] Ran, B., Boyce, D.E. (1994). *Dynamic urban transportation networks models.* Springer Verlag.

[RHB 96] Ran, B., R. W. Hall, Boyce, D.E. (1996). *A link-based variational inequality model for dynamic departure time/route choice.* Transportation Research 30 B(1): 31-46.

[RI 56] Richards, P.I. (1956). *Shock-waves on the highway.* Operations Research 4: 42-51.

[VA 95] Van Aerde, M. (1995). *INTEGRATION release 2: User's guide.* Kingston, Ontario, Canada, Transportation Systems Research Group.

[WFH 94] Weymann, J., Farges J.-L., Henry, J.J. (1994). *Dynamic route guidance with queue and flow dependent travel time.* Transportation Research 2 C(3): 165-183.