Existence of solution for a density-dependent magneto hydrodynamic equation[1]

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Abstract

We prove a global-in-time existence result of a weak solution for a magnetohydrodynamic (MHD) problem set in a bounded domain of $\mathbb{R}^3$. The fluid is supposed to be incompressible but with an unhomogeneous density, viscosity and electrical conductivity. The displacement currents are neglected in the time dependent Maxwell equations. The model describes in particular the flow of two immiscible fluids in presence of a magnetic field.

Résumé

Nous prouvons un résultat d'existence globale en temps de solutions faibles pour un problème de magnétohydrodynamique (MHD) dans un domaine borné de $\mathbb{R}^3$. Le fluide considéré est incompressible mais sa densité, sa viscosité et sa conductivité électrique sont variables. Nous négligeons les courants de déplacement dans les équations de Maxwell. Le modèle proposé décrit en particulier le comportement de deux fluides non miscibles en présence d'un champ magnétique.

1 Introduction

In this article, we prove the existence of a weak solution for the transient incompressible density-dependent Navier-Stokes equations coupled with the Maxwell's system where we neglect the so-called displacement currents (namely the term $\partial_t(\epsilon E)$ in the Maxwell-Ampère equation $-\partial_t(\epsilon E)+\mathrm{curl}\,\frac{B}{\mu} = j$) and also coupled with Ohm's law in a rather complete form $j = \sigma(E+u\times B)$ taking into account the Hall effect. The mathematical model we shall deal hereafter is therefore the following system, that we write here somewhat formally but that will be made precise in the next section :

$$
\begin{aligned}
\partial_t\rho + \mathrm{div}\,(\rho u) &= 0, && (1.1)\\
\partial_t(\rho u) + \mathrm{div}\,(\rho u\otimes u) - \mathrm{div}\,(2\eta d(u)) + \nabla p &= \rho f + \mathrm{curl}\, B\times B, && (1.2)\\
\mathrm{div}\, u &= 0, && (1.3)\\
\partial_t B + \mathrm{curl}\,(\frac{1}{\sigma}\mathrm{curl}\, B) &= \mathrm{curl}\,(u\times B), && (1.4)\\
\mathrm{div}\, B &= 0, && (1.5)
\end{aligned}
$$

[1]AMS subject classification : 35Q30, 35Q60, 76D05, 76W05.

together with some *ad hoc* boundary conditions and initial data (see below). The unknowns are the density ρ, the velocity u, the magnetic field B, the pressure p. We denote by $d(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ the shear rate tensor, f a given exterior body force, σ the electrical conductivity of the fluid and η its viscosity, both conductivity and viscosity being functions of the density $\sigma = \sigma(\rho)$, $\eta = \eta(\rho)$. In the sequel, we shall refer to this system by "the density-dependent MHD equations".

The density-dependent MHD equations describe in particular the motion of several conducting incompressible immiscible fluids (without surface tension) in presence of a magnetic field.

From a physical viewpoint, the assumption on the displacement currents is valid as soon as the materials are sufficiently conducting (see R. Moreau [13] for example). This occurs in particular in molten metals. From a mathematical viewpoint, this hypothesis makes the problem easier since it transforms the hyperbolic Maxwell's system into a parabolic equation.

Many works have already been devoted to the study of MHD systems for *one* fluid with constant density. We now give a brief overview on those we are aware of.

Existence and uniqueness results are established by G. Duvaut and J.-L. Lions in [5] for the case of the time-dependent MHD equations (without displacement current) posed on a simply-connected bounded domain in the framework of Bingham fluids. These results are completed by M. Sermange and R. Temam in [20] for classical Newtonian fluids. They show that the classical properties of the Navier-Stokes equations can be extended to the MHD system. More precisely, they prove in the bidimensional case the existence and the uniqueness of a global weak solution which is strong for regular data. When the space dimension is three, they prove that a global weak solution exists and that for more regular data, a strong solution exists and is unique for small times. At last, they study the large time behaviour and the Haussdorf dimension of a functional invariant set. Some of these results are also presented by R. Temam in [23] and by J.-M. Ghidaglia in [9].

The stationary MHD equations are treated by M.D. Gunzburger, A.J. Meir and J.S. Peterson in [10]. They prove the existence of a solution and its uniqueness in particular cases. Nonhomogeneous boundary conditions for u and B are used in this work and the authors propose two types of boundary conditions for the electromagnetic field (see the next section for more details). Lastly, a complete numerical analysis by the finite element method is presented. They prove in particular that any finite element spaces of $H^1(\Omega)$ is relevant to approximate the magnetic field as soon as a traditional pair of spaces (say Q2/discontinuous P1 for instance) satisfying the Ladyzenskaia-Babuska-Brezzi inf-sup condition is used for the velocity and the pressure.

The case of multiply-connected bounded sets is studied by J.-M. Domingez de la Rasilla in [4] for the stationary equation and by K. Kerieff in the time-dependent problem [11]. A numerical analysis by the finite element method is also proposed in [4].

E. Sanchez-Palancia has treated in [18] and [19] an MHD problem in an exterior domain both in the stationary and the time-dependent cases (without displacement currents) .

J. Rappaz and R. Touzani have studied the MHD equations in a particular bidimensional non connected domain which occurs in industrial applications such as electromagnetic casting. They establish existence results in [16] (summarized in [15]) and give a numerical analysis of the problem in [17].

In all the above studies, the density of the fluid is supposed to be constant. Here, we are interested in fluids with nonhomogeneous density (which covers the case of several fluids with different constant densities) and we intend to extend to the coupled case the results known so far on the density-dependent Navier-Stokes equations. Let us now recall these results.

Global existence and regularity results have been established by A.V. Kazhikov, S.N. Antontsev and A.V. Monakhev in [1] in the bidimensional case. They suppose that the viscosity is constant in the whole domain and that the initial density is bounded from below by a positive constant.

A. Nouri and F. Poupaud consider in [14] the transport equation for both the density and the viscosity and they use the concept of renormalized solutions of R.J. DiPerna and P.-L. Lions. This allows them to prove the existence of a global weak solution for several fluids with various viscosities and various densities bounded from below by a positive constant.

But to date, the most complete study of the density-dependent Navier-Stokes equations is due to P.-L. Lions in [12] and our study is largely inspired by his work. In this approach, the viscosity is a function of the density. The initial density is assumed to be nonnegative, but not necessarily bounded from below by a positive constant, which also allows one to consider free surface problems. The main result proved in [12] in this setting is the global existence of a weak solution. Moreover, as long as a strong solution exists, then any weak solution is equal to it (see [12] and also B. Desjardins [3] for a proof of existence of a strong solution under particular assumptions).

Our paper is organized as follows. We recall in Section 2 the density-dependent MHD equations along with the definition of various functions spaces. The initial and boundary conditions are also detailed as well as convenient hypotheses on the data. Section 3 will be devoted to the proof of the existence theorem, which basically follows the same pattern as the proof for the uncoupled case in [12]. We shall explain there the main mathematical difficulties raised by our problem. In a first step we establish existence, uniqueness and regularity results for a linear problem. We use these results in a second step in order to prove by a fixed point argument the existence of a smooth solution for a regularized MHD problem. Finally, in a third step, a fundamental compactness theorem proved in [12] allows us to pass to the limit in the regularized problem, which concludes the proof.

Some interesting connected questions are not treated in this paper (we refer the reader to a forthcoming work [8] where we shall address some of them).

It must be first mentioned that various several other MHD models may be considered. Let us just give three of them.

We could consider a fully static model consisting in a coupling between the steady-state Navier-Stokes equations and the stationary Maxwell's system. This is a difficult problem since existence questions related to the density-dependent stationary Navier-Stokes equations are still open even in the absence of electromagnetism, and it is not clear why we may hope to have more compactness in the coupled case.

Another possible model which raises serious mathematical difficulties is the coupling between the time-dependent Navier-Stokes equations and the complete Maxwell's system (including displacement currents). Due to the hyperbolic nature of the Maxwell equations this is a problem that remains open today even in the case of one homogeneous fluid.

Finally, a model which is to some extent easier to deal with (at least in the case of "small" initial data) but that exhibits other kinds of mathematical difficulties than the ones we face in this article, consists in a coupling between the time-dependent Navier-Stokes equations and the stationary Maxwell's system (see [8]).

Let us also notice that other density-dependent problems could be considered with the same approach : for example the Boussinesq equations presented for one homogeneous fluid by C. Bernardi, B. Métivet and B. Pernaud-Thomas in [2].

2 The equations and their functional setting

2.1 The density-dependent MHD equations

Let Ω be a simply-connected, fixed bounded domain in $\mathbb{R}^3$ enclosed in a $\mathcal{C}^\infty$ boundary $\partial\Omega$. We shall denote by n the outward-pointing normal to Ω.

The density-dependent MHD problem we shall consider is to find two vector-valued functions, the velocity u and the magnetic field B, and two scalar functions, the density ρ and the pressure p, defined on $\Omega \times [0, T]$, such that

$$\partial_t \rho + \mathrm{div}\,(\rho u) = 0 \quad \text{ on } \Omega, \tag{2.1}$$

$$\partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) - \mathrm{div}\,(2\eta d(u)) + \nabla p = \rho\, f + \mathrm{curl}\, B \times B \quad \text{on } \Omega, \tag{2.2}$$

$$\mathrm{div}\, u = 0 \quad \text{ on } \Omega, \tag{2.3}$$

$$\partial_t B + \mathrm{curl}\,(\frac{1}{\sigma}\mathrm{curl}\, B) = \mathrm{curl}\,(u \times B) \quad \text{ on } \Omega, \tag{2.4}$$

$$\mathrm{div}\, B = 0 \quad \text{ on } \Omega. \tag{2.5}$$

We recall that $d(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the shear rate tensor, f is a given exterior body force, σ the electrical conductivity of the fluid and η its viscosity. We assume that they are both function of ρ :

$$\sigma = \sigma(\rho) \text{ and } \eta = \eta(\rho).$$

This dependence of η and σ on ρ enables us to consider the density-dependent equations as a model of a multi-phase flow consisting of several immiscible fluids with various conductivities and viscosities.

For the convenience of the reader, let us briefly recall where these equations come from. First of all, we assume that the magnetic permeability μ is constant over the domain, and we set it to 1. The system (2.1)-(2.3) is the density dependent Navier-Stokes equation. Let us recall the Maxwell-Ampère equation where the displacement currents are neglected :

$$\mathrm{curl}\, B = j, \tag{2.6}$$

the Maxwell-Faraday equation :

$$\partial_t B + \mathrm{curl}\, E = 0, \tag{2.7}$$

and the Ohm's law :

$$j = \sigma(E + u \times B), \tag{2.8}$$

where j is the current density and E the electric field. Using (2.6), we see that the Lorentz force $j \times B$ acting on the fluid is $\operatorname{curl} B \times B$ which explains the second term of the right-hand-side of (2.2). Eliminating j and E between (2.6), (2.7) and (2.8) we obtain the first equation of (2.4). As soon as B satisfying (2.4) is obtained, we may recover E through (2.7) and a gauge condition on $\operatorname{div} E$.

We require ρ and B to satisfy the initial conditions

$$\rho|_{t=0} = \rho_0 \text{ on } \Omega, \tag{2.9}$$

$$B|_{t=0} = B_0 \text{ on } \Omega. \tag{2.10}$$

If ρ_0 vanishes on some part of Ω we cannot directly impose an initial condition on u. That is why the initial condition is imposed on ρu in [12]. Though we shall suppose in this article that ρ_0 does not vanish, we use the same approach, having in mind future developpements of the present work :

$$\rho u|_{t=0} = m_0 \text{ on } \Omega. \tag{2.11}$$

On $\partial\Omega$, we impose the homogeneous no-slip boundary condition :

$$u|_{\partial\Omega} = 0. \tag{2.12}$$

For the sake of simplicity, we suppose that the boundary $\partial\Omega$ is fixed and perfectly conducting. Using Ohm's law (2.8) and Maxwell-Ampère equation (2.6), we deduce the boundary condition for B :

$$(B.n)|_{\partial\Omega} = 0, \tag{2.13}$$

$$(\operatorname{curl} B \times n)|_{\partial\Omega} = 0. \tag{2.14}$$

Let us notice that our arguments and results may be extended to treat the quite general case

$$(B.n)|_{\partial\Omega} = q \text{ and } (E \times n)|_{\partial\Omega} = k, \tag{2.15}$$

with q and k arbitrarily fixed, independent of time, or even depending in a convenient way on the time.

Remark 2.1 M.D. Gunzburger and coworkers give in [10] a complete study of the general case ($q \neq 0$ and $k \neq 0$) for the stationary MHD equations (with ρ constant) and propose another set of electromagnetic boundary condition, namely :

$$(B \times n)|_{\partial\Omega} = q \text{ and } (E.n)|_{\partial\Omega} = k.$$

2.2 Function spaces

For $m \geq 0$, we denote as usual by $H^m(\Omega)$ the Sobolev space

$$H^m(\Omega) = \{u \in L^2(\Omega); D^\gamma u \in L^2(\Omega), \forall \gamma, |\gamma| \leq m\}$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is a multi-index and $|\gamma| = \gamma_1 + \gamma_2 + \gamma_3$. The norm associated with $H^m(\Omega)$ that we will use is :

$$||u||_{H^m(\Omega)} = \left(\sum_{|\gamma|=0}^{m} ||D^\gamma u||^2_{L^2(\Omega)} \right)^{1/2}.$$

For $m \geq 1$, $H^m_0(\Omega)$ is the subspace of $H^m(\Omega)$ consisting of functions vanishing on $\partial\Omega$. For any space X, we shall denote $(X)^3$ by $\mathbb{X}$ (e.g. $(L^2(\Omega))^3$ by $\mathbb{L}^2(\Omega)$, $(H^m(\Omega))^3$ by $\mathbb{H}^m(\Omega)$,...).

Let $T > 0$ and let X be a Banach space. $L^p(0, T; X)$, $1 \leq p \leq \infty$ is the space of classes of L^p functions from $[0, T]$ into X. We recall that this is a Banach space for the norm

$$\left(\int_0^T ||u(t)||^p_X \, dt \right)^{1/p} \text{ if } 1 \leq p < \infty, \qquad \operatorname*{ess\,sup}_{t \in [0,T]} ||u(t)||_X \text{ if } p = \infty.$$

We denote by $\mathcal{C}^\infty_c(\Omega)$ (resp. $\mathcal{C}^\infty_c(\overline{\Omega})$) the space of real functions infinitely differentiable with a compact support in Ω (resp. $\overline{\Omega}$). We introduce the spaces

$$\mathcal{V} = \{v \in (\mathcal{C}^\infty_c(\Omega))^3, \operatorname{div} v = 0\},$$

$$V = \{v \in \mathbb{H}^1_0(\Omega), \operatorname{div} v = 0\},$$

$$\mathcal{W} = \{C \in (\mathcal{C}^\infty_c(\overline{\Omega}))^3, \operatorname{div} C = 0, C.n|_{\partial\Omega} = 0\},$$

$$W = \{C \in \mathbb{H}^1(\Omega), \operatorname{div} C = 0, C.n|_{\partial\Omega} = 0\},$$

$$H = \{v \in \mathbb{L}^2(\Omega), \operatorname{div} v = 0, v.n|_{\partial\Omega} = 0\}.$$

The space V (resp. W) is the closure of $\mathcal{V}$ (resp. $\mathcal{W}$) in $\mathbb{H}^1_0(\Omega)$ (resp. $\mathbb{H}^1(\Omega)$). H is the closure of $\mathcal{V}$ (and $\mathcal{W}$) in $\mathbb{L}^2(\Omega)$. Let us remark that $u.n$ makes sense in $H^{-1/2}(\partial\Omega)$ as soon as $u \in \mathbb{L}^2(\Omega)$ satisfies $\operatorname{div} u = 0$. For $v \in V$ and $C \in W$ we denote

$$||v||_V = \left(\int_\Omega |\nabla v|^2 \, dx \right)^{1/2},$$

$$||C||_W = \left(\int_\Omega |\operatorname{curl} C|^2 \, dx \right)^{1/2}.$$

One can establish that $||.||_V$ (resp. $||.||_W$) defines a norm (resp. W) which is equivalent to that induced by $\mathbb{H}^1(\Omega)$ on V (resp. W) (cf. G. Duvaut and J.-L. Lions [6]). The fact that Ω is simply-connected is essential for this point.

We shall make frequent use of the following formulas of vector analysis : for all vector fields Φ and Ψ we have

$$\int_\Omega \mathrm{curl}\,\Phi.\Psi\,dx = \int_\Omega \Phi.\mathrm{curl}\,\Psi\,dx + \int_{\partial\Omega} n\times\Phi.\Psi\,dx, \tag{2.16}$$

whenever these integrals make sense. Moreover, for all velocity fields u and densities ρ, we have

$$\mathrm{div}\,(\rho u\otimes u) = u\mathrm{div}\,(\rho u) + \rho(u.\nabla)u$$

in the sense of distributions on Ω.

2.3 Regularity of the data

In the same fashion as in [12], the initial data for the hydrodynamic variables are required to have the following properties :

$$\rho_0 \in L^\infty(\Omega), \tag{2.17}$$

$$m_0 \in \mathbb{L}^2(\Omega), \tag{2.18}$$

$$\frac{|m_0|^2}{\rho_0} \in L^1(\Omega). \tag{2.19}$$

However, while in [12] for the Navier-Stokes equations, the only assumption on the initial density is $\rho_0 \geq 0$, which in particular covers the case when there is some vacuum ($\rho_0 = 0$) on some part of the domain Ω at $t = 0$, we are obliged to assume here, because of the coupling with the magnetic field (see Remark 3.4), that

$$\rho_0 > 0 \quad \text{a.e. in } \Omega. \tag{2.20}$$

Moreover, we shall suppose in the sequel – unless otherwise mentioned – that

$$f \in L^2(0,T;\mathbb{L}^2(\Omega)) \tag{2.21}$$

and that η and σ are continuous functions on $[0,+\infty)$ such that

$$0 < \eta_1 \leq \eta(\xi) \leq \eta_2 \quad \text{for } \xi\in[0,\infty), \tag{2.22}$$

$$0 < \sigma_1 \leq \sigma(\xi) \leq \sigma_2 \quad \text{for } \xi\in(0,\infty). \tag{2.23}$$

Finally, we assume that

$$B_0 \in H. \tag{2.24}$$

3 Existence of a weak solution

This section is devoted to the statement and proof of our main result. We need first

Definition 3.1 For $T > 0$, we shall say that (ρ, u, B) is a *weak solution* on $\Omega \times [0, T]$ of the problem (2.1)-(2.14) with the assumptions (2.17)-(2.24) if

$$\rho \in L^\infty(\Omega \times (0,T)) \cap \mathcal{C}(0,T; L^p(\Omega)), \quad \forall p \geq 1, \tag{3.1}$$

$$u \in L^2(0,T;V), \tag{3.2}$$

$$\rho|u|^2 \in L^\infty(0,T;L^1(\Omega)), \tag{3.3}$$

$$B \in L^2(0,T;W) \cap L^\infty(0,T;H) \cap \mathcal{C}([0,T],H_w)^2 \tag{3.4}$$

and (ρ, u, B) are such that (2.1) holds in the sense of distributions in $\Omega \times (0,T)$ and

$$\begin{aligned} &\iint_{\Omega\times(0,\infty)} -\rho u.\partial_t \phi - \rho u \otimes u.\nabla\phi + 2\eta d(u).d(\phi)\, dxdt = \\ &\qquad \iint_{\Omega\times(0,\infty)} \left(\rho f + (\mathrm{curl}\, B) \times B\right).\phi \, dxdt + \int_\Omega m_0.\phi(x,0)\, dx, \end{aligned} \tag{3.5}$$

$$\begin{aligned} \iint_{\Omega\times(0,\infty)} -B\partial_t\phi + \frac{1}{\sigma}\mathrm{curl}\, B.\mathrm{curl}\,\phi\, dxdt = &\iint_{\Omega\times(0,\infty)} \mathrm{curl}\,(u\times B).\phi\, dxdt \\ &+ \int_\Omega B_0.\phi(x,0)\, dx, \end{aligned} \tag{3.6}$$

for all $\phi \in \mathcal{C}_c^\infty(\Omega \times [0,\infty))^3$.

Then we have

Theorem 1
Under the regularity assumptions on the data (2.17)-(2.24), there exists a weak solution (ρ, u, B) of the density dependent MHD equations (2.1)-(2.5), with initial conditions (2.9)- (2.11) and boundary conditions (2.12)-(2.14), satisfying (3.1)-(3.4).

Furthermore,

$$\left.\begin{array}{l} meas\{x \in \Omega / \alpha \leq \rho(x,t) \leq \beta\} \\ \textit{is independent of } t \geq 0 \textit{ for all } 0 \leq \alpha \leq \beta < \infty. \bullet \end{array}\right\} \tag{3.7}$$

[2] $B \in \mathcal{C}([0,T],H_w)$ means $\forall C \in H$, $t \to \int_\Omega B(t).C\, dx$ is a continuous scalar function.

Remark 3.1 Let us note that initial conditions (2.9) and (2.10) make sense in view of the assumption of continuity made on ρ and B in Definition 3.1. But we did not assume any continuity on ρu and therefore, the sense of the initial condition (2.11) is not clear. Roughly speaking, ρu converges to m_0 up to a "gradient-like" distribution when $t \to 0$. We refer to [12] for a precise explanation of this technical point.

Nevertheless, if we suppose that $0 < \rho_1 \leq \rho_0(x)$ a.e in Ω (instead of (2.20)) and if $\frac{m_0}{\rho_0}$ is divergence free, then we can prove that $u \in \mathcal{C}([0,T], H_w)$ (like in R. Temam [21]), which gives sense to (2.11).

Remark 3.2 In the case of a multi-phase incompressible flow of K immiscible fluids we have $\rho|_{t=0} = \rho^k$ on Ω_k, $k = 1, .., K$, where ρ^k is the density of the k^{th} phase and $(\Omega_k)_{k=1..K}$ is a partition of Ω. The property (3.7) means nothing but the mass conservation of each phase.

Note that this property holds of course for the density dependent Navier-Stokes equations without electromagnetism.

Remark 3.3 It is important to note that, like for the standard Navier-Stokes equation and *a fortiori* for the density-dependent equation with given forces treated in [12], we do not know if a weak solution is unique. We do not know either if a strong solution always exists. However, it is an extension of our work to show that the same regularity results holding under restrictive assumptions in the case of the density dependent equations, that we mentioned in the introduction, may be extended to our case.

Remark 3.4 It would be interesting to allow, like in [12], the initial density to be zero somewhere in Ω (think for instance of a conducting fluid with a free surface). Our proof could easily be extended to this case if we endowed the vacuum with a conductivity $\sigma_1 > 0$. But this hypothesis would not be very convincing from a physical viewpoint since, in the set $\{\rho = 0\}$, the magnetic field B would not be a solution of the Maxwell equations in the vacuum. This is why we are obliged to suppose here that the initial density does not vanish.

Before we turn to the proof of Theorem 1, let us briefly describe our strategy of proof and say a few words on how we circumvent the mathematical difficulties raised by the problem (2.1)-(2.5).

System (2.1)-(2.5) couples two equations of parabolic type with the transport equation (2.1). It is intuitively clear (and it is indeed the case) that the parabolic equation (2.4) is the easiest one to treat. This is why it is somewhat natural that the same results as in the standard density-dependent case also hold true here.

The idea to prove the existence of a solution is to introduce a regularized problem (namely (3.50)-(3.59) in Section 3.2 below) for which the solution, denote by u^ε, is regular enough to allow one to define (2.1) as a classical transport equation.

At the same time, the magnetic field evolves according to the parabolic equation (2.4), linear with respect to B, which provides at any time a force term in the right-hand side of (2.2).

Showing the existence of a solution to this regularized problem is the purpose of our first two steps. We linearize the problem in Subsection 3.1 and then use a fixed point argument in Subsection 3.2. Proving the theorem then amounts to passing to the limit in the regularized problem (when $\varepsilon \to 0$). In this third step, we make use of a powerful compactness result due to P.-L. Lions (Theorem 2 below).

In comparison with the case studied in [12], the new difficulty is that we have to check that the force term $\operatorname{curl} B \times B$ does not introduce any perturbation on the estimates on the velocity u and the density ρ. Moreover, we have to recover some compactness on B through the parabolic equation (2.4-a) in order to pass to the limit in the nonlinear terms $\operatorname{curl} B \times B$ (and $\operatorname{curl}(u \times B)$).

3.1 First step : a linear coupled problem

In this section, we prove a preliminary result which will be useful in section 3.2. The problem presented below is a linearized MHD system with prescribed density and will be solved by classical arguments. Let us notice that there are several possibilities to linearize the initial system (see Remark 3.5).

For ρ, w and h arbitrarily fixed such that

$$\rho \in \mathcal{C}([0,T], \mathcal{C}^k(\overline{\Omega})), \forall k \geq 0, \text{ such that } 0 < \rho_1 \leq \rho(x,t) \leq \rho_2, \tag{3.8}$$

$$\partial_t \rho \in L^2(0,T; \mathcal{C}^k(\overline{\Omega})), \forall k \geq 0, \tag{3.9}$$

$$w \in L^2(0,T; \mathbb{L}^\infty(\Omega)), \text{ with } \operatorname{div} w = 0 \text{ and } \partial_t \rho + \operatorname{div}(\rho w) = 0, \tag{3.10}$$

$$h \in L^2(0,T; \mathbb{L}^\infty(\Omega) \cap \mathbb{W}^{1,3}(\Omega)) \text{ with } \operatorname{div} h = 0, \tag{3.11}$$

the problem is to find two vector-valued functions u and B and a scalar function p defined on $\Omega \times [0,T]$, such that

$$\begin{aligned}
\rho\partial_t u + \rho(w.\nabla)u - \mathrm{div}\,(2\eta d(u)) + \nabla p &= \rho f + \mathrm{curl}\,B \times h, && (3.12)\\
\mathrm{div}\,u &= 0, && (3.13)\\
\partial_t B + \mathrm{curl}\,(\frac{1}{\sigma}\mathrm{curl}\,B) &= \mathrm{curl}\,(u \times h), && (3.14)\\
\mathrm{div}\,B &= 0, && (3.15)
\end{aligned}$$

with

$$u = 0 \text{ on } \partial\Omega, \tag{3.16}$$

$$B.n = 0 \text{ and } \mathrm{curl}\,B \times n = 0 \text{ on } \partial\Omega, \tag{3.17}$$

and

$$u|_{t=0} = u_0, \tag{3.18}$$

$$B|_{t=0} = B_0. \tag{3.19}$$

In this subsection, we require the viscosity and the conductivity to have the following regularity properties :

$$\eta \in \mathcal{C}^\infty([0,\infty)) \text{ such that } 0 < \eta_1 \le \eta(\xi) \le \eta_2, \tag{3.20}$$

$$\sigma \in \mathcal{C}^\infty([0,\infty)) \text{ such that } 0 < \sigma_1 \le \sigma(\xi) \le \sigma_2, \tag{3.21}$$

and, for the moment, we only suppose that :

$$f \in L^2(0,T;\mathbb{H}^{-1}(\Omega)), \tag{3.22}$$

$$u_0, B_0 \in H. \tag{3.23}$$

Although we shall use a strong solution of this problem in the sequel, it will be useful for the proof of the following proposition to define a notion of weak solution : we shall say that (u,B) is a weak solution of (3.12)-(3.19) if this pair is a solution of the problem (P) defined by

To find $u \in L^2(0,T;V)$ and $B \in L^2(0,T;W)$ satisfying the initial conditions (3.18) and (3.19) and such that

$$\int_\Omega \rho(\partial_t u + (w.\nabla)u).v\,dx + \int_\Omega 2\eta d(u).d(v)\,dx = <\rho f, v> + \int_\Omega \mathrm{curl}\,B \times h.v\,dx \tag{3.24}$$

$$\int_\Omega \partial_t B.C\,dx + \int_\Omega \frac{1}{\sigma}\mathrm{curl}\,B.\mathrm{curl}\,C\,dx = \int_\Omega \mathrm{curl}\,(u\times h).C\,dx \tag{3.25}$$

for all $v \in V$ and for all $C \in W$.

Let us notice that we have made use of the regularity (3.10) of w to define this problem.

Proposition 1

1. *Under the assumptions (3.8)-(3.11) and (3.20)-(3.23), there exists a unique pair* $(u, B) \in L^2(0,T;V) \times L^2(0,T;W)$ *weak solution of the problem (3.12)-(3.19) and a distribution* $p \in \mathcal{D}(\Omega \times (0,T))$*, unique up to an additive constant, satisfying (3.12). Moreover,* u *and* B *belong to* $\mathcal{C}(0,T;H)$ *.*

2. *If we suppose* $f \in L^2(0,T;\mathbb{L}^2(\Omega))$*,* $u_0 \in V$ *and* $B_0 \in W$*, we have moreover :*

$$u \in L^2(0,T;\mathbb{H}^2(\Omega)) \cap \mathcal{C}(0,T;V), \tag{3.26}$$

$$B \in L^2(0,T;\mathbb{H}^2(\Omega)) \cap \mathcal{C}(0,T;W), \tag{3.27}$$

$$\partial_t u \in L^2(0,T;H), \tag{3.28}$$

$$\partial_t B \in L^2(0,T;H), \tag{3.29}$$

$$p \in L^2(0,T;H^1(\Omega)).\bullet \tag{3.30}$$

Proof.

1) We solve (P) by the Faedo-Galerkin method : since V (resp. W) are separable there exists a sequence of linearly independent elements $v_1, v_2, ..., v_n, ...$ (resp. $C_1, C_2, ..., C_n, ...$) which is total in V (resp. in W). For all n we define an approximated solution (u_n, B_n) as follows :

$$u_n = \sum_{i=1}^{n} \alpha_i(t) v_i, \tag{3.31}$$

$$B_n = \sum_{i=1}^{n} \beta_i(t) C_i, \tag{3.32}$$

where α_i and β_i, i=1,...,n, are scalar functions defined on $[0,T]$ solutions of

$$\left\{ \begin{array}{ll} \displaystyle\sum_{i=1}^{n} & \displaystyle\left(\int_\Omega \rho v_i.v_j \, dx\right) \alpha_i' + \left(\int_\Omega (\rho(w.\nabla)v_i.v_j + 2\eta d(v_i).d(v_j)) \, dx\right) \alpha_i \\ & \displaystyle -\left(\int_\Omega \mathrm{curl}\, C_i \times h.v_j \, dx\right) \beta_i = < \rho f(t), v_j >, \forall j = 1, ..., n. \\ \displaystyle\sum_{i=1}^{n} & \displaystyle\left(\int_\Omega C_i.C_j \, dx\right) \beta_i' + \left(\int_\Omega \frac{1}{\sigma} \mathrm{curl}\, C_i.\mathrm{curl}\, C_j \, dx\right) \beta_i \\ & \displaystyle -\left(\int_\Omega \mathrm{curl}\,(v_i \times h).C_j \, dx\right) \alpha_i = 0, \forall j = 1, ..., n. \end{array} \right. \tag{3.33}$$

with for $i = 1, ..., n$:

$$\left\{ \begin{array}{lcl} \alpha_i(0) & = & \alpha_{i0} \\ \beta_i(0) & = & \beta_{i0}. \end{array} \right. \tag{3.34}$$

$(\alpha_{i0})_{i=1..n}$ (resp. $(\beta_{i0})_{i=1..n}$) are the coordinates of the orthogonal projection in H of u_0 (resp. B_0) on the space spanned by $v_1, ..., v_n$ (resp. $C_1, ..., C_n$).

The matrix $\left(\int_\Omega \rho v_i.v_j\, dx\right)_{i,j=1..n}$ (resp. $\left(\int_\Omega C_i.C_j\, dx\right)_{i,j=1..n}$) is nonsingular since the family $(\sqrt{\rho} v_i)_{i=1..n}$ with $\rho > 0$ (resp. $(C_i)_{i=1..n}$) is free. Thus, the Cauchy-Lipschitz theorem implies that the linear differential system (3.33) with coefficients in $L^2(0,T)$ together with the initial conditions (3.34) defines uniquely the functions α_i and β_i on the whole interval $[0,T]$. Then, we obtain with (3.31) and (3.32)

$$u_n \in \mathcal{C}(0,T;V), B_n \in \mathcal{C}(0,T;W).$$

Moreover, with the regularity of w, h, ρ, f, η and σ coming from (3.8)-(3.11) and (3.20)-(3.23), we have :

$$u_n' \in L^2(0,T;V), B_n' \in L^2(0,T;W).$$

In view of this regularity we have :

$$\int_\Omega \rho \partial_t u_n.u_n\, dx = \frac{1}{2}\frac{d}{dt}\int_\Omega \rho |u_n|^2\, dx - \int_\Omega \partial_t\rho |u_n|^2\, dx$$

and

$$\int_\Omega \partial_t B_n.B_n\, dx = \frac{1}{2}\frac{d}{dt}\int_\Omega |B_n|^2\, dx.$$

We multiply the first (resp. second) equations of (3.33) by α_i (resp. β_i) and we add them for $i = 1$ to n. This yields :

$$\left\{ \begin{array}{l} \displaystyle \frac{1}{2}\frac{d}{dt}\int_\Omega \rho|u_n|^2\, dx + \int_\Omega 2\eta |d(u_n)|^2\, dx =< \rho f, u_n > + \int_\Omega \mathrm{curl}\, B_n \times h.u_n\, dx \\ \displaystyle \frac{1}{2}\frac{d}{dt}\int_\Omega |B_n|^2\, dx + \int_\Omega \frac{1}{\sigma}|\mathrm{curl}\, B_n|^2\, dx = \int_\Omega \mathrm{curl}\,(u_n \times h).B_n\, dx. \end{array} \right. \tag{3.35}$$

(we have used

$$-\int_\Omega \partial_t \rho |u_n|^2\, dx + \int_\Omega \rho w.\nabla u_n.u_n\, dx = -\int_\Omega (\partial_t\rho + \mathrm{div}\,(\rho w))|u_n|^2\, dx = 0.)$$

With (3.35) and (2.16), we obtain the "energy equation" :

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|u_n|^2 + |B_n|^2\, dx + \int_\Omega 2\eta|d(u_n)|^2 + \frac{1}{\sigma}|\mathrm{curl}\, B_n|^2\, dx =< \rho f, u_n > .$$

Remark 3.5 Note that the way we have linearized the terms curl $B \times B$ and curl $(u \times B)$ is especially chosen among all the different manners in which the system may be linearized, in order to easily obtain the above *a priori* estimates.

Let us notice that

$$\int_\Omega 2\eta |d(u_n)|^2 \, dx \geq \frac{\eta_1}{2} \int_\Omega |\nabla u_n + \nabla u_n^T|^2 \, dx = \eta_1 \int_\Omega |\nabla u_n|^2 \, dx$$

since div $u_n = 0$. So we have :

$$\frac{d}{dt} \int_\Omega \rho |u_n|^2 + |B_n|^2 \, dx + \eta_1 \int_\Omega |\nabla u_n|^2 \, dx + \frac{2}{\sigma_2} \int_\Omega |\text{curl}\, B_n|^2 \, dx$$
$$\leq \frac{1}{\eta_1} ||\rho||_{\mathcal{C}^1(\overline{\Omega})} ||f||_{H^{-1}(\Omega)}.$$

Using $0 < \rho_1 \leq \rho$, we deduce by Gronwall's lemma that :

$$u_n \text{ is bounded in } L^2(0,T;V) \cap L^\infty(0,T;H),$$

$$B_n \text{ is bounded in } L^2(0,T;W) \cap L^\infty(0,T;H).$$

So, there exists $u \in L^2(0,T;V) \cap L^\infty(0,T;H)$ such that u_n converges to u (up to the extraction of subsequences) for the weak-star topology of $L^\infty(0,T;H)$ and for the weak topology of $L^2(0,T;V)$. In the same way, there exists $B \in L^2(0,T;W) \cap L^\infty(0,T;H)$ such that B_n converges to B for the weak-star topology of $L^\infty(0,T;H)$ and for the weak topology of $L^2(0,T;W)$. Clearly, the pair (u,B) is a solution of (P).

Let us assume now that (u_1, B_1) and (u_2, B_2) are two solutions of (P) and let $(\widetilde{u}, \widetilde{B}) = (u_1 - u_2, B_1 - B_2)$. We easily check from (3.24) and (3.25) that

$$\frac{d}{dt} \int_\Omega (\rho |\widetilde{u}|^2 + |\widetilde{B}|^2) \leq 0.$$

Thus $(\widetilde{u}, \widetilde{B}) = (0,0)$, and the uniqueness of the solution of (P) is proved.

It is classical to show that for the solution (u,B) of problem (P), there exists a distribution p such that (3.12) is satisfied for (u,B,p) in the distribution sense in $\Omega \times [0,T]$ (see e.g. R. Temam [21], [22]).

Moreover, $\partial_t u$ and $\partial_t B$ belong to $L^2(0,T;H^-1)$ (at least). Therefore, since u and B belong to $L^2(0,T;H^1)$, we deduce that u and B belong to $\mathcal{C}(0,T;H)$ (see R. Temam [21]).

2) The additional assumptions of regularity for f, u_0 and B_0 enable us to obtain another estimate for the approximate solution (u_n, B_n) built by the Faedo-Galerkin method.

We multiply each first equation of (3.33) by α_i' and we add them for $i = 1$ to n :

$$\int_\Omega \rho|\partial_t u_n|^2\,dx + \int_\Omega \rho w.\nabla u_n.\partial_t u_n\,dx \quad + \int_\Omega 2\eta d(u_n).\partial_t d(u_n)\,dx = \int_\Omega (\rho f + \text{curl}\,B_n \times h).\partial_t u_n\,dx.$$

Thus :

$$\rho_1 \int_\Omega |\partial_t u_n|^2\,dx \quad + \frac{d}{dt}\int_\Omega \eta|d(u_n)|^2\,dx \leq \int_\Omega |\partial_t\eta||d(u_n)|^2\,dx + \int_\Omega \rho|w||\nabla u_n||\partial_t u_n| + |\text{curl}\,B_n||h||\partial_t u_n| + \rho|f||\partial_t u_n|\,dx.$$

Hence, using the Cauchy-Schwarz inequality, we find

$$\begin{aligned} &\frac{\rho_1}{2}\int_\Omega |\partial_t u_n|^2\,dx + \frac{d}{dt}\int_\Omega \eta|d(u_n)|^2\,dx \leq \\ &\qquad \leq \alpha_1(t)\int_\Omega \eta|d(u_n)|^2\,dx + \beta_1(t)\int_\Omega \frac{1}{\sigma}|\text{curl}\,B_n|^2\,dx + \gamma_1(t) \end{aligned} \tag{3.36}$$

with

$$\alpha_1(t) = \frac{1}{\eta_1}||\partial_t\eta||_{L^\infty(\Omega)} + \frac{3\rho_2^2}{\rho_1\eta_1}||w||^2_{\mathbb{L}^\infty(\Omega)},$$

$$\beta_1(t) = \frac{3\sigma_2}{2\rho_1}||h||^2_{\mathbb{L}^\infty(\Omega)},$$

$$\gamma_1(t) = \frac{3\rho_2^2}{2\rho_1}||f||^2_{\mathbb{L}^2(\Omega)}.$$

As well, we multiply the second equations of (3.33) by β_i' and we add them from $i = 1$ to n :

$$\int_\Omega |\partial_t B_n|^2\,dx + \int_\Omega \frac{1}{\sigma}\text{curl}\,B_n.\partial_t\text{curl}\,B_n\,dx = \int_\Omega \text{curl}\,(u_n \times h).\partial_t B_n\,dx.$$

Thus

$$\begin{aligned} &\int_\Omega |\partial_t B_n|^2\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega \frac{1}{\sigma}|\text{curl}\,B_n|^2\,dx \leq \\ &\qquad \leq \int_\Omega \left|\partial_t\left(\frac{1}{\sigma}\right)\right||\text{curl}\,B_n|^2 + |\partial_t B_n||h||\nabla u_n| + |\partial_t B_n||\nabla h||u_n|\,dx. \end{aligned}$$

Using again the Cauchy-Schwarz inequality and $||u||_{L^6(\Omega)} \leq c_0||\nabla u||_{L^2(\Omega)}$, we find :

$$\begin{aligned}\int_\Omega |\partial_t B_n|^2\, dx + \frac{d}{dt}\int_\Omega \frac{1}{\sigma}|\text{curl}\, B_n|^2\, dx \leq \\ \leq \alpha_2(t)\int_\Omega \eta|d(u_n)|^2\, dx + \beta_2(t)\int_\Omega \frac{1}{\sigma}|\text{curl}\, B_n|^2\, dx\end{aligned} \tag{3.37}$$

with

$$\alpha_2(t) = \frac{4}{\eta_1}||h||^2_{L^\infty(\Omega)} + \frac{4c_0^2}{\eta_1}||\nabla h||^2_{L^3(\Omega)},$$

$$\beta_2(t) = 2\sigma_2||\partial_t\left(\frac{1}{\sigma}\right)||_{L^\infty(\Omega)}.$$

Then, we add (3.36) and (3.37), which yields in particular :

$$A'(t) \leq \gamma_0(t)A(t) + \gamma_1(t),$$

with

$$A(t) = \int_\Omega \eta|d(u_n)|^2\, dx + \int_\Omega \frac{1}{\sigma}|\text{curl}\, B_n|^2\, dx$$

and

$$\gamma_0(t) = \alpha_1(t) + \alpha_2(t) + \beta_1(t) + \beta_2(t).$$

The hypotheses (3.8)-(3.11) and (3.20)-(3.23) imply that $\gamma_0 \in L^1(0,T)$. Moreover $\gamma_1 \in L^1(0,T)$ since $f \in L^2(0,T;\mathbb{L}^2(\Omega))$. Therefore, using that $u_0 \in V$ and $B_0 \in W$, Gronwall's lemma implies that $\sup_{t\in[0,T]} A(t)$ is bounded, hence :

$$u_n \text{ is bounded in } L^\infty(0,T;V),$$

$$B_n \text{ is bounded in } L^\infty(0,T;W).$$

We deduce by integrating (3.36) and (3.37) that :

$$\partial_t u_n \text{ is bounded in } L^2(0,T;H),$$

$$\partial_t B_n \text{ is bounded in } L^2(0,T;H).$$

By a passage to the limit, these last two properties show that :

$$\partial_t u \in L^2(0,T;H), \tag{3.38}$$

$$\partial_t B \in L^2(0,T;H). \tag{3.39}$$

Let us now prove (3.26) and (3.27). We have

$$-\text{div}\,(2\eta d(u)) + \nabla p = \rho f - \rho\partial_t u - \rho w.\nabla u + \text{curl}\, B \times h,$$

which we write

$$-\triangle u + \nabla \tilde{p} = \phi$$

with

$$\tilde{p} = \frac{p}{\eta}$$

and

$$\phi = \frac{1}{\eta}\left(\rho f + 2\nabla\eta . d(u) - \rho\partial_t u - \rho w.\nabla u + \operatorname{curl} B \times h - p\frac{\nabla\eta}{\eta^2}\right).$$

Thus we have

$$\begin{cases} -\triangle u + \nabla\tilde{p} &= \phi \quad \text{on } \Omega \\ \operatorname{div} u &= 0 \quad \text{on } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases}$$

With the above assumptions on the data and (3.38) we have $\phi \in L^2(0,T;\mathbb{L}^2(\Omega))$. Therefore, by classical regularity results on the Stokes problem (see R. Temam [21] for instance) we have :

$$u \in L^2(0,T;\mathbb{H}^2(\Omega)), \tag{3.40}$$

$$p \in L^2(0,T;H^1(\Omega)).$$

As well, we have :

$$\operatorname{curl}\left(\frac{1}{\sigma}\operatorname{curl} B\right) = -\partial_t B + \operatorname{curl}(u \times h)$$

which leads to

$$\begin{cases} -\triangle B &= \psi \quad \text{on } \Omega \\ B.n &= 0 \quad \text{on } \partial\Omega \\ \operatorname{curl} B \times n &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.41}$$

with

$$\psi = \sigma\left(-\nabla\left(\frac{1}{\sigma}\right) \times \operatorname{curl} B - \partial_t B + h.\nabla u - u.\nabla h\right).$$

The assumptions on the data and (3.39) give $\psi \in L^2(0,T;\mathbb{L}^2(\Omega))$. Then, we use a regularity result due to V. Georgescu [7] on boundary problems of the type (3.41) which implies that :

$$B \in L^2(0,T;\mathbb{H}^2(\Omega)). \tag{3.42}$$

Finally, (3.38) and (3.40) together imply that

$$u \in \mathcal{C}(0,T;V),$$

while (3.39) and (3.42) imply likewise that

$$B \in \mathcal{C}(0,T;W).$$

3.2 Second step : an approximated nonlinear problem

In this section, we solve a regularized MHD problem by using the Schauder fixed point theorem and the results of step 1.

3.2.1 Regularization

Let $u \in L^2(0,T;V)$, we define $r_\varepsilon(u)$ as in [12]. Let us recall that $r_\varepsilon(u) \in L^2(0,T;\mathcal{C}^\infty(\Omega)^3)$, $\operatorname{div} r_\varepsilon(u) = 0$ and $r_\varepsilon(u)$ vanishes near $\partial\Omega$. Moreover we have :

$$\lim_{\varepsilon \to 0} r_\varepsilon(u) = u \quad \text{in } \mathbb{L}^p(\Omega) \quad (1 \leq p < \infty) \tag{3.43}$$

and let us note that $r_\varepsilon(u) \in \mathbb{L}^\infty(\Omega)$.

For $B \in L^2(0,T;W)$, we build a regularization $s_\varepsilon(B)$ as follows : we extend B to $\mathbb{R}^3$ by 0. We next define $s_\varepsilon(B) = B * \omega_\varepsilon$ (ω_ε is a regularizing kernel). Let us notice that $s_\varepsilon(B) \in L^2(0,T;\mathcal{C}^\infty(\Omega))$ and $\operatorname{div} s_\varepsilon(B) = 0$ (since $B.n = 0$ on $\partial\Omega$) but $s_\varepsilon(B).n \neq 0$ on $\partial\Omega$. We have in particular :

$$\lim_{\varepsilon \to 0} s_\varepsilon(B) = B \quad \text{in } \mathbb{L}^p(\Omega) \quad (1 \leq p < \infty) \tag{3.44}$$

We set $f_\varepsilon = (f\, 1_{(d > 2\varepsilon)}) * \omega_\varepsilon$ where $d = \operatorname{dist}(x, \partial\Omega)$.

Without loss of generality, we may assume that $\eta(\xi)$ is constant for ξ large enough (since ρ remains in $[0, ||\rho_0||_{L^\infty(\Omega)}]$). We denote by $\eta^\varepsilon \in \mathcal{C}^\infty([0,\infty))$ a function bounded away from 0, and such that $\sup_{[0,\infty)} |\eta^\varepsilon - \eta| \leq \varepsilon$. Moreover, $\eta^\varepsilon(\xi)$ is supposed to be constant for ξ large enough. Then, we define $\eta_\varepsilon = \overline{\eta}(\rho) * \omega_\varepsilon|_\Omega$ with $\overline{\eta}(\rho) = \eta^\varepsilon(\rho)$ in Ω and $= 1$ in Ω^c.

We define σ_ε from σ like η_ε from η.

The initial data m_0 and ρ_0 are regularized like in [12]. Let us just recall that

$$\varepsilon \leq \rho_0^\varepsilon \leq \rho_2, \tag{3.45}$$

$$\lim_{\varepsilon \to 0} \rho_0^\varepsilon = \rho_0 \quad \text{in } L^p(\Omega) \quad (1 \leq p < \infty), \tag{3.46}$$

$$\lim_{\varepsilon \to 0} m_0^\varepsilon = m_0 \quad \text{in } \mathbb{L}^2(\Omega), \quad \lim_{\varepsilon \to 0} \frac{m_0^\varepsilon}{\sqrt{\rho_0^\varepsilon}} = \frac{m_0}{\sqrt{\rho_0}} \quad \text{in } \mathbb{L}^2(\Omega). \tag{3.47}$$

Moreover, we have the following decomposition

$$m_0^\varepsilon = \rho_0^\varepsilon u_0^\varepsilon + \nabla q_0^\varepsilon \tag{3.48}$$

where $u_0^\varepsilon \in \mathcal{C}_0^\infty(\Omega)$ and $\operatorname{div} u_0^\varepsilon = 0$ in Ω (see [12]).

At last, $B_0 \in H$ is regularized as follows : we extend B_0 on $\mathbb{R}^3$ by 0 and we define $B_0^\varepsilon = (B_0\, 1_{(d>2\varepsilon)}) * \omega_\varepsilon$. Note that B_0 vanishes near $\partial\Omega$ and that we have :

$$\lim_{\varepsilon\to 0} B_0^\varepsilon = B_0 \quad \text{in } \mathbb{L}^p(\Omega) \quad (1 \le p < \infty). \tag{3.49}$$

3.2.2 Approximated problem

Our goal is to solve the following problem :

$$\partial_t \rho + \text{div}\,(r_\varepsilon(u)\rho) = 0, \tag{3.50}$$

$$\partial_t(\rho u) + \text{div}\,(\rho r_\varepsilon(u) \otimes u) - \text{div}\,(2\eta_\varepsilon d(u)) + \nabla p = \rho f_\varepsilon + \text{curl}\, B \times s_\varepsilon(B), \tag{3.51}$$

$$\partial_t B + \text{curl}\,(\frac{1}{\sigma_\varepsilon}\text{curl}\, B) = \text{curl}\,(u \times s_\varepsilon(B)), \tag{3.52}$$

$$\text{div}\, u = 0, \tag{3.53}$$

$$\text{div}\, B = 0, \tag{3.54}$$

all equations being on Ω, with the boundary conditions

$$u = 0 \text{ on } \partial\Omega, \tag{3.55}$$

$$B.n = 0 \text{ and curl}\, B \times n = 0 \text{ on } \partial\Omega, \tag{3.56}$$

and the initial conditions

$$\rho|_{t=0} = \rho_0^\varepsilon, \tag{3.57}$$

$$u|_{t=0} = u_0^\varepsilon, \tag{3.58}$$

$$B|_{t=0} = B_0^\varepsilon. \tag{3.59}$$

Proposition 2
The above regularized problem (3.50)-(3.59) has a solution $(\rho, u, B) \in \mathcal{C}^\infty(\overline{\Omega}\times [0, +\infty))^3$ $\bullet$

Proof.
1) First, we prove by a fixed point argument that the regularized problem has a solution in $\mathcal{C}(\overline{\Omega} \times [0,T]) \times L^2(0,T;V) \times L^2(0,T;W)$.

Let us consider the convex set C_ε in $\mathcal{C}(\overline{\Omega}\times[0,T])\times L^2(0,T;V)\times L^2(0,T;W)$ defined by

$$\begin{aligned} C_\varepsilon \quad = \quad & \{(\overline{\rho}, \overline{u}, \overline{B}) \in \mathcal{C}(\overline{\Omega} \times [0,T]) \times L^2(0,T;V) \times L^2(0,T;W), \text{ such that} \\ & \varepsilon \le \overline{\rho} \le \rho_2 \text{ in } \overline{\Omega} \times [0,T], ||\overline{u}||_{L^2(0,T;V)} \le R_0, ||\overline{B}||_{L^2(0,T;W)} \le R_0\} \end{aligned}$$

where R_0 is a constant to be determined.

For $(\overline{\rho}, \overline{u}, \overline{B}) \in C_\varepsilon$ we define $F(\overline{\rho}, \overline{u}, \overline{B}) = (\rho, u, B)$ as follows : first of all, we solve

$$\left\{ \begin{array}{rcll} \partial_t \rho + \text{div}\,(\rho r_\varepsilon(\overline{u})) & = & 0 & \text{in } \Omega \times (0,T), \\ \rho|_{t=0} & = & \rho_0^\varepsilon & \text{in } \Omega. \end{array} \right. \tag{3.60}$$

This is a classical transport equation since, by construction, $r_\varepsilon(\overline{u})$ is regular, divergence free and vanishes near $\partial\Omega$. Thus ρ is given by

$$\rho(x,t) = \rho_0^\varepsilon(X(0;x,t)), \quad \forall (x,t) \in \overline{\Omega} \times [0,T],$$

where X is the solution of the ordinary differential equation

$$\left\{ \begin{array}{rcl} \dfrac{dX}{ds} & = & r_\varepsilon(\overline{u})(X(s;x,t),s) \\ X(t;x,t) & = & x. \end{array} \right.$$

We deduce from (3.45) that $\varepsilon \leq \rho \leq \rho_2$ in $\overline{\Omega} \times [0,T]$. Thus $\rho \in \mathcal{C}([0,T]; \mathcal{C}^k(\overline{\Omega}))$ for all $k \geq 0$ and is bounded in this space uniformly in $(\overline{\rho}, \overline{u})$. Furthermore, we deduce from (3.60) that $\partial_t \rho$ is bounded in $L^2(0,T; \mathcal{C}^k(\overline{\Omega}))$ for all $k \geq 0$ uniformly in $(\overline{\rho}, \overline{u})$. Therefore the set of ρ (such that $(\rho, u, B) = F(\overline{\rho}, \overline{u}, \overline{B})$ for $(\overline{\rho}, \overline{u}, \overline{B}) \in C_\varepsilon$) is compact in $\mathcal{C}(\overline{\Omega} \times [0,T])$.

Next, we set $w = r_\varepsilon(\overline{u})$ and $h = s_\varepsilon(\overline{B})$ and we invoke Proposition 1 to define (u, B) as the unique solution of :

$$\partial_t(\rho u) + \text{div}\,(\rho r_\varepsilon(\overline{u}) \otimes u) - \text{div}\,(2\eta_\varepsilon d(u)) + \nabla p = \rho f_\varepsilon + \text{curl}\, B \times s_\varepsilon(\overline{B}) \tag{3.61}$$

$$\partial_t B + \text{curl}\,(\frac{1}{\sigma_\varepsilon} \text{curl}\, B) = \text{curl}\,(u \times s_\varepsilon(\overline{B})) \tag{3.62}$$

$$\text{div}\, u = 0 \tag{3.63}$$

$$\text{div}\, B = 0 \tag{3.64}$$

with the boundary conditions (3.55)-(3.56) and the initial conditions (3.58)-(3.59). We recall that $u \in L^2(0,T; \mathbb{H}^2(\Omega)) \cap \mathcal{C}(0,T;V)$ and $B \in L^2(0,T; \mathbb{H}^2(\Omega)) \cap \mathcal{C}(0,T;W)$ which justifies the manipulations hereafter.

Now, let us choose R_0 in such a way that (ρ, u, B) is in C_ε. We multiply (3.61) by u and we integrate :

$$\frac{1}{2}\frac{d}{dt} \int_\Omega \rho |u|^2\, dx + \int_\Omega 2\eta_\varepsilon |d(u)|^2\, dx = \int_\Omega \rho f_\varepsilon . u\, dx + \int_\Omega \text{curl}\, B \times s_\varepsilon(\overline{B}). u\, dx$$

As well we multiply (3.62) by B and we integrate :

$$\frac{1}{2}\frac{d}{dt} \int_\Omega |B|^2\, dx + \int_\Omega \frac{1}{\sigma_\varepsilon} |\text{curl}\, B|^2\, dx = \int_\Omega \text{curl}\,(u \times s_\varepsilon(\overline{B})). B\, dx.$$

We add these equations using (2.16) to obtain the energy identity :

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho|u|^2+|B|^2\,dx+\int_\Omega 2\eta_\varepsilon|d(u)|^2+\frac{1}{\sigma_\varepsilon}|\mathrm{curl}\,B|^2\,dx=\int_\Omega \rho f_\varepsilon.u\,dx \quad (3.65)$$

Then, the Cauchy-Schwarz inequality and $||u||_{L^2(\Omega)}\leq c(\Omega)||\nabla u||_{L^2(\Omega)}$ leads to :

$$\frac{d}{dt}\int_\Omega \rho|u|^2+|B|^2\,dx+\int_\Omega \frac{\eta_1}{2}|\nabla u|^2+\frac{2}{\sigma_2}|\mathrm{curl}\,B|^2\,dx\leq \frac{2\rho_2 c(\Omega)^2}{\eta_1}||f_\varepsilon||^2_{L^2(\Omega)}$$

Finally, using $0<\varepsilon\leq\rho$ we obtain by Gronwall's lemma :

$$\sup_{t\in[0,T]}||u(t)||_{L^2(\Omega)}+\sup_{t\in[0,T]}||B(t)||_{L^2(\Omega)}+||u||_{L^2(0,T;V)}+||B||_{L^2(0,T;W)}\leq c_0$$

where c_0 is a constant which is independent of R_0, $\overline{u}$, $\overline{B}$. Hence, with $R_0=c_0$, we have $F(\overline{\rho},\overline{u},\overline{B})\in C_\varepsilon$.

In order to apply the Schauder theorem, we still have to prove that the mapping F is compact on C. Replacing w by $r_\varepsilon(\overline{u})$ and h by $s_\varepsilon(\overline{B})$ in the proof of Proposition 1, part 2 , we see that :

$$\partial_t B \text{ and } \partial_t u \text{ are bounded in } L^2(0,T;\mathbb{L}^2(\Omega)), \text{ and}$$

$$B \text{ and } u \text{ are bounded in } L^2(0,T;\mathbb{H}^2(\Omega)).$$

We deduce that the set of u (resp. B) built above is relatively compact in $L^2(0,T;\mathbb{H}^1(\Omega))$. Since V and W are closed subsets of $\mathbb{H}^1(\Omega)$, the set of u (resp. B) is relatively compact in $L^2(0,T;V)$ (resp. in $L^2(0,T;W)$). Let us recall that the set of ρ is compact in $\mathcal{C}(\overline{\Omega}\times[0,T])$. Hence the mapping F is compact on C and has a fixed point (ρ,u,B) which is a solution of (3.50)-(3.59).

2) The solution (ρ,u,B) built above satisfies $\rho\in\mathcal{C}([0,T];\mathcal{C}^k(\overline{\Omega}))$, $u\in L^2(0,T;\mathbb{H}^2(\Omega))\cap\mathcal{C}(0,T;V)$, $B\in L^2(0,T;\mathbb{H}^2(\Omega))\cap\mathcal{C}(0,T;W)$, $\partial_t u$ and $\partial_t B\in L^2(0,T;H)$.

The smoothness of $r_\varepsilon(u)$, $s_\varepsilon(B)$, u_0^ε and B_0^ε allows us to apply the same regularity arguments as in part 2 of Proposition 1 which provides more regularity on (u,B) and therefore on ρ. By bootstrapping we conclude that ρ, u and B are in $\mathcal{C}^\infty(\overline{\Omega}\times[0,+\infty))$.

3.3 Third step : passage to the limit

The aim of this last section is to prove Theorem 1 by passing to the limit in the above regularized problem (3.50)-(3.59). The fundamental tool is a compactness result due to P.-L. Lions that we recall now for the reader's convenience, in the case N=3 and in a slightly particular form :

Theorem 2 (P.-L. Lions, [12])
We suppose that two sequences ρ_n and u_n are given satisfying $\rho_n \in C([0,T], L^1(\Omega))$, $0 \leq \rho_n \leq C$ a.e on $\Omega \times (0,T)$, $u_n \in L^2(0,T;\mathbb{H}^1_0(\Omega))$, $||u_n||_{L^2(0,T;\mathbb{H}^1(\Omega))} \leq C$ and $\operatorname{div} u_n = 0$ (C denotes various constants independent of n). We note $\rho_{0n} = \rho_n(0)$ and we assume :

$$\partial_t \rho_n + \operatorname{div}(\rho_n u_n) = 0 \text{ in } \mathcal{D}'(\Omega \times (0,T))$$

$$\rho_{0n} \to \rho_0 \text{ in } L^1(\Omega) \text{ and } u_n \rightharpoonup u \text{ weakly in } L^2(0,T;\mathbb{H}^1(\Omega)).$$

Then :

1) ρ_n converges in $\mathcal{C}([0,T], L^p(\Omega))$ for all $1 \leq p < \infty$ to the unique ρ bounded on $\Omega \times (0,T)$ solution of

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \text{ in } \mathcal{D}'(\Omega \times (0,T)) \\ \rho(0) = \rho_0 \text{ in } \Omega \\ \rho \in \mathcal{C}([0,T], L^1(\Omega)) \end{cases}$$

2) We assume in addition that $\rho_n |u_n|^2$ is bounded in $L^\infty(0,T;L^1(\Omega))$ and that we have for some $m \geq 1$

$$| < \partial_t(\rho_n u_n), \phi > | \leq C ||\phi||_{L^2(0,T;\mathbb{H}^m(\Omega))}$$

for all $\phi \in \mathcal{C}_0^\infty(\Omega \times (0,T))^3$ such that $\operatorname{div} \phi = 0$ on $\Omega \times (0,T)$. Then : $\sqrt{\rho_n} u_n$ converges to $\sqrt{\rho} u$ in $L^p(0,T;\mathbb{L}^r(\Omega))$ for $2 < p < \infty$, $1 \leq r < \frac{6p}{3p-4}$ and u_n converges to u in $L^\theta(0,T;\mathbb{L}^{3\theta}(\Omega))$ for $1 \leq \theta < 2$ on the set $\{\rho > 0\}$. •

We denote by $(\rho^\varepsilon, u^\varepsilon, B^\varepsilon)$ the smooth approximated solution given by Proposition 2. We have from (3.50) :

$$\partial_t \rho^\varepsilon + \operatorname{div}(r_\varepsilon(u^\varepsilon)\rho^\varepsilon) = 0. \tag{3.66}$$

Let β_n be a function of class $\mathcal{C}^1(\mathbb{R}, \mathbb{R})$. Multiplying (3.66) by $\beta_n'(\rho)$ and using $\operatorname{div} r_\varepsilon(u^\varepsilon) = 0$ we have

$$\partial_t \beta_n(\rho^\varepsilon) + r_\varepsilon(u^\varepsilon).\nabla \beta_n(\rho^\varepsilon) = 0.$$

We integrate this equation on $\Omega \times [0, T]$ and we use again that $r_\varepsilon(u^\varepsilon)$ is divergence free and vanishes on the boundary to obtain

$$\int_\Omega \beta_n(\rho^\varepsilon(x,t))\,dx = \int_\Omega \beta_n(\rho_0(x))\,dx. \tag{3.67}$$

For $0 \le \alpha \le \beta < \infty$ we choose (for n large enough) $0 \le \beta_n \le 1$ such that $\beta_n(\xi) = 0$ if $\xi \notin [\alpha, \beta]$, $\beta_n(\xi) = 1$ if $\xi \in [\alpha + 1/n, \beta - 1/n]$. Letting n go to $+\infty$ in (3.67) we deduce that (3.7) holds with ρ^ε, i.e.

$$\int_\Omega \chi_{[\alpha,\beta]}(\rho^\varepsilon(x,t))\,dx = \int_\Omega \chi_{[\alpha,\beta]}(\rho_0^\varepsilon(x))\,dx \tag{3.68}$$

where $\chi_{[\alpha,\beta]}(\xi) = 1$ on $[\alpha, \beta]$ and 0 elsewhere. In particular, with $\alpha = 0$ and $\beta = ||\rho_0||_{L^\infty(\Omega)}$ this yields to the following L^∞-estimate on ρ :

$$0 \le \rho^\varepsilon \le ||\rho_0||_{L^\infty(\Omega)}$$

Furthermore, we have the energy identity (3.65) :

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho^\varepsilon|u^\varepsilon|^2 + |B^\varepsilon|^2\,dx + \int_\Omega 2\eta_\varepsilon|d(u^\varepsilon)|^2 + \frac{1}{\sigma_\varepsilon}|\mathrm{curl}\, B^\varepsilon|^2\,dx = \int_\Omega \rho^\varepsilon f_\varepsilon.u^\varepsilon\,dx$$

which implies (using as usual Gronwall's lemma) :

$$||u^\varepsilon||_{L^2(0,T;V)} \le c \tag{3.69}$$

$$\sup_{t\in[0,T]} ||\rho^\varepsilon|u^\varepsilon|^2||_{L^1(\Omega)} \le c \tag{3.70}$$

$$||\frac{1}{\sqrt{\sigma_\varepsilon}}\mathrm{curl}\, B^\varepsilon||_{L^2(0,T;\mathbb{L}^2(\Omega))} \le c \tag{3.71}$$

$$\sup_{t\in[0,T]} ||B^\varepsilon||_{\mathbb{L}^2(\Omega)} \le c \tag{3.72}$$

where c denotes various constants independent of ε.

In view of these estimates, and using Theorem 2, our goal is now to pass to the limit in the following weak formulation of (3.51)-(3.59) :

$$\begin{aligned}\iint_{\Omega\times(0,\infty)} &-\rho^\varepsilon u^\varepsilon.\partial_t\phi - \rho^\varepsilon r_\varepsilon(u^\varepsilon)\otimes u^\varepsilon.\nabla\phi + 2\eta_\varepsilon d(u^\varepsilon).d(\phi)\,dxdt = \\ &\iint_{\Omega\times(0,\infty)} (\rho^\varepsilon f_\varepsilon + (\mathrm{curl}\, B^\varepsilon)\times s_\varepsilon(B^\varepsilon)).\phi\,dxdt + \int_\Omega m_0^\varepsilon.\phi(x,0)\,dx,\end{aligned} \tag{3.73}$$

$$\iint_{\Omega\times(0,\infty)} -B^\varepsilon \partial_t\phi + \frac{1}{\sigma_\varepsilon}\mathrm{curl}\, B^\varepsilon.\mathrm{curl}\,\phi\, dxdt = \iint_{\Omega\times(0,\infty)} \mathrm{curl}\,(u^\varepsilon \times s_\varepsilon(B^\varepsilon)).\phi\, dxdt + \int_\Omega B_0^\varepsilon.\phi(x,0)\, dx \tag{3.74}$$

Extracting subsequences if necessary and using (3.69) and (3.72), we may define u as the weak limit of u^ε in $L^2(0,T;V)$ and B as the limit of B^ε for the weak-star topology of $L^\infty(0,T;\mathbb{L}^2(\Omega))$.

Let us remark that $0 \le \sigma \le \sigma_2$ and (3.71) imply that $B \in \mathbb{H}^1(\Omega)$ and $\mathrm{curl}\, B^\varepsilon$ converges to $\mathrm{curl}\, B$ weakly in $L^2(0,T;\mathbb{L}^2(\Omega))$.

In view of (3.46) and (3.69), the first assertion of Theorem 2 implies that ρ^ε converges (up to the extraction of subsequences) to some $\rho \in \mathcal{C}([0,T];L^p(\Omega))$ with $1 \le p < \infty$ and

$$\partial_t\rho + \mathrm{div}\,(\rho u) = 0.$$

Passing to the limit in (3.68), we deduce that for $0 \le \alpha \le \beta < \infty$

$$\int_\Omega \chi_{[\alpha,\beta]}(\rho(x,t))\, dx = \int_\Omega \chi_{[\alpha,\beta]}(\rho_0(x))\, dx,$$

which proves (3.7).

The convergence of ρ^ε as $\varepsilon \to 0$ implies that

$$\lim_{\varepsilon\to 0} \eta_\varepsilon = \eta(\rho) \quad \text{in } \mathcal{C}([0,T];L^p(\Omega)) \quad \text{for } 1 \le p < \infty \tag{3.75}$$

$$\lim_{\varepsilon\to 0} \sigma_\varepsilon = \sigma(\rho) \quad \text{in } \mathcal{C}([0,T];L^p(\Omega)) \quad \text{for } 1 \le p < \infty \tag{3.76}$$

$$\lim_{\varepsilon\to 0} \rho^\varepsilon f_\varepsilon = \rho f \quad \text{in } L^2(\Omega\times(0,T)) \tag{3.77}$$

Next, we remark that $r_\varepsilon(u^\varepsilon)$ converges to u weakly in $L^2(0,T;V)$ and $s_\varepsilon(B^\varepsilon)$ converges to B weakly in $L^2(0,T;W)$ (with (3.43) and (3.44)).

In order to check that we may apply the second part of Theorem 2, let us prove that for some $m \ge 1$ we have

$$| < \partial_t(\rho^\varepsilon u^\varepsilon), \phi > | \le C||\phi||_{L^2(0,T;\mathbb{H}^m(\Omega))} \tag{3.78}$$

for all $\phi \in \mathcal{C}_0^\infty(\Omega\times(0,T))^3$ such that $\mathrm{div}\,\phi = 0$ on $\Omega\times(0,T)$.

First, we have

$$\begin{aligned} < \mathrm{div}\,(2\eta_\varepsilon d(u^\varepsilon)), \phi > &= |\int_0^T \int_\Omega 2\eta_\varepsilon d(u^\varepsilon).\nabla\phi\, dx\, dt| \\ &\le ||2\eta_\varepsilon d(u^\varepsilon)||_{L^2(0,T;L^2(\Omega))}||\phi||_{L^2(0,T;\mathbb{H}^1(\Omega))} \\ &\le c||\phi||_{L^2(0,T;\mathbb{H}^1(\Omega))} \end{aligned}$$

and

$$< \rho^\varepsilon f_\varepsilon, \phi > \leq \rho_2 ||f_\varepsilon||_{L^2(0,T;\mathbb{L}^2(\Omega))} ||\phi||_{L^2(0,T;\mathbb{L}^2(\Omega))} \leq c||\phi||_{L^2(0,T;\mathbb{L}^2(\Omega))}$$

where c are various constants independent of ε.

Using $||\partial_i \phi_j||_{L^3(\Omega)} \leq c||\phi_j||_{H^{3/2}(\Omega)}$ and (3.43) we have

$$\begin{aligned} &| < \text{div}\,(\rho^\varepsilon r_\varepsilon(u^\varepsilon) \otimes u^\varepsilon), \phi > | = | \int_0^T \int_\Omega \rho^\varepsilon r_\varepsilon(u^\varepsilon) \otimes u^\varepsilon . \nabla \phi \, dx\, dt| \\ &\leq c_1 ||\rho^\varepsilon |r_\varepsilon(u^\varepsilon)|^2||_{L^\infty(0,T;L^1(\Omega))} ||\sqrt{\rho^\varepsilon} u^\varepsilon||_{L^2(0,T;\mathbb{L}^6(\Omega))} ||\phi||_{L^2(0,T;\mathbb{H}^{3/2}(\Omega))} \\ &\leq c_2 ||\phi||_{L^2(0,T;\mathbb{H}^{3/2}(\Omega))} \end{aligned}$$

Finally, the inequality $||\phi||_{\mathbb{L}^\infty(\Omega)} \leq C||\phi||_{\mathbb{H}^{3/2+\alpha}(\Omega)}$ with $\alpha > 0$ and (3.44) lead to

$$\begin{aligned} &| < \text{curl}\, B^\varepsilon \times s_\varepsilon(B^\varepsilon), \phi > | = | \int_0^T \int_\Omega \text{curl}\, B^\varepsilon \times s_\varepsilon(B^\varepsilon) . \phi \, dx\, dt| \\ &\leq c_1 ||\text{curl}\, B^\varepsilon||_{L^2(0,T;\mathbb{L}^2(\Omega))} ||B^\varepsilon||_{L^\infty(0,T;\mathbb{L}^2(\Omega))} ||\phi||_{L^2(0,T;\mathbb{H}^{3/2+\alpha}(\Omega))} \\ &\leq c_2 ||\phi||_{L^2(0,T;\mathbb{H}^{3/2+\alpha}(\Omega))} \end{aligned}$$

with $\alpha > 0$. Therefore (3.78) is true for any $m > 3/2$. Part 2 of Theorem 2 and the convergence of ρ^ε then imply that $\rho^\varepsilon u^\varepsilon$ converges to ρu strongly in $L^p(0,T;\mathbb{L}^r(\Omega))$ for $2 < p < \infty$, $1 \leq r < \frac{6p}{3p-4}$ and u^ε converges to u strongly in $L^\theta(0,T;\mathbb{L}^{3\theta}(\Omega))$ for $1 \leq \theta < 2$.

Let us prove now that B^ε converges strongly to B in $L^2(0,T;H)$. First, we check that $\partial_t B^\varepsilon$ is bounded in $L^{4/3}(0,T;W')$. Indeed, for $\phi \in L^4(0,T;W)$ we have

$$\begin{aligned} &< \partial_t B^\varepsilon, \phi >= \int_0^T \int_\Omega \left(-\frac{1}{\sigma_\varepsilon} \text{curl}\, B^\varepsilon + u^\varepsilon \times s_\varepsilon(B^\varepsilon) \right) . \text{curl}\, \phi \, dx\, dt \leq \\ &\leq || \frac{1}{\sigma_\varepsilon} \text{curl}\, B^\varepsilon ||_{L^2(0,T;L^2(\Omega))} ||\phi||_{L^2(0,T;W)} + \int_0^T ||u^\varepsilon||_{L^4(\Omega)} ||s_\varepsilon(B^\varepsilon)||_{L^4(\Omega)} ||\phi||_W \, dt \end{aligned}$$

In the last term we use (3.44) and the interpolation inequality $||h||_{L^4(\Omega)} \leq ||h||_{L^6(\Omega)}^{3/4} ||h||_{L^2(\Omega)}^{1/4}$ to obtain :

$$\begin{aligned} \int_0^T ||u^\varepsilon||_{L^4(\Omega)} ||s_\varepsilon(B^\varepsilon)||_{L^4(\Omega)} ||\phi||_W \, dt \leq c ||u^\varepsilon||_{L^\infty(0,T;H)}^{1/4} ||u^\varepsilon||_{L^2(0,T;V)} \\ ||B^\varepsilon||_{L^\infty(0,T;H)}^{1/4} ||B^\varepsilon||_{L^2(0,T;W)} ||\phi||_{L^4(0,T;W)} \end{aligned}$$

Therefore $\partial_t B^\varepsilon$ is bounded in $L^{4/3}(0,T;W')$. Moreover, we know that B^ε is bounded in $L^2(0,T;W)$. Thus, up to the extraction of a subsequence, B^ε converges strongly to B in $L^2(0,T;H)$. We deduce in particular that $s_\varepsilon(B^\varepsilon)$ converges strongly to B in $L^2(0,T;H)$. Furthermore, in view of (3.72), note that B^ε is bounded in $L^\infty(0,T;H)$. Thus $B \in L^\infty(0,T;H)$.

In particular $\partial_t B \in L^1(0,T;W')$, thus B is almost everywhere equal to a function continuous from $[0,T]$ into W'. Moreover, $B \in L^\infty(0,T;H)$ and $H \subset W'$ with a continuous injection, therefore, we know that B is weakly continuous from $[0,T]$ into H (see R. Temam [21] for instance).

The weak and strong convergences obtained for B^ε and u^ε enable us to pass to the limit in the nonlinear terms

$$\iint_{\Omega\times(0,\infty)} \rho^\varepsilon r_\varepsilon(u^\varepsilon) \otimes u^\varepsilon.\nabla\phi \, dxdt,$$

$$\iint_{\Omega\times(0,\infty)} (\mathrm{curl}\, B^\varepsilon) \times s_\varepsilon(B^\varepsilon).\phi \, dxdt.$$

The weak convergence of u^ε in $L^2(0,T;V)$ and the strong convergence of B^ε in $L^2(0,T;H)$ enable us to pass to the limit in

$$\iint_{\Omega\times(0,\infty)} \mathrm{curl}\,(u^\varepsilon \times s_\varepsilon(B^\varepsilon)).\phi \, dxdt = \iint_{\Omega\times(0,\infty)} u^\varepsilon \times s_\varepsilon(B^\varepsilon).\mathrm{curl}\,\phi \, dxdt.$$

Furthermore, we have in view of (3.47) and (3.49) :

$$\lim_{\varepsilon\to 0} \int_\Omega m_0^\varepsilon.\phi(x,0)\, dx = \int_\Omega m_0.\phi(x,0)\, dx,$$

$$\lim_{\varepsilon\to 0} \int_\Omega B_0^\varepsilon.\phi(x,0)\, dx = \int_\Omega B_0.\phi(x,0)\, dx.$$

Therefore, passing to the limit in (3.73) and (3.74), we recover (3.5) and (3.6), which concludes the proof.

Remark 3.6 We can check arguing as in [12] that any solution built as above satisfies the energy inequalities :

$$\frac{d}{dt}\int_\Omega \rho|u|^2 + |B|^2\, dx + \int_\Omega \eta|\nabla u + \nabla u^T|^2 + \frac{2}{\sigma}|B|^2\, dx \le 2\int_\Omega \rho f.u\, dx.$$

and

$$\begin{aligned}\int_\Omega \rho|u|^2 + |B|^2\, dx \quad + \quad & \int_0^t \int_\Omega \eta|\nabla u + \nabla u^T|^2 + \frac{2}{\sigma}|B|^2\, dx\, ds \le \\ & \le \int_\Omega \frac{|m_0|^2}{\rho_0} + |B_0|^2\, dx + 2\int_0^t \int_\Omega \rho f.u\, dx\, ds.\end{aligned}$$

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