Convergence of moderately interacting particle systems to a diffusion-convection equation

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Abstract

We give a probabilistic interpretation of the solution of a diffusion-convection equation. To do so, we define a martingale problem in which the drift coefficient is nonlinear and unbounded for small times whereas the diffusion coefficient is constant. We check that the time marginals of any solution are given by the solution of the diffusion-convection equation. Then we prove existence and uniqueness for the martingale problem and obtain the solution as the propagation of chaos limit of a sequence of moderately interacting particle systems.
Keywords: nonlinear martingale problem, propagation of chaos, particle systems, moderate interaction, diffusion-convection equation

According to Escobedo, Vasquez and Zuazua [2], for $q \geq 2$, the partial differential equation

$$\frac{\partial u}{\partial t} + |u|^{q-1}\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} \tag{0.1}$$

posed in the domain $(t,x) \in (0,+\infty) \times \mathbb{R}$ with initial condition δ_0 (for any C^∞ bounded function ϕ, $\lim_{t\to 0}\int_{\mathbb{R}} \phi(t,x)u(t,x)dx = \phi(0)$) admits a unique positive solution v_q in $C((0,+\infty), L^1(\mathbb{R})) \cap C^\infty((0,+\infty)\times\mathbb{R})$. In this paper we are interested in giving a probabilistic interpretation of this solution.

Since the solution satisfies $\forall t > 0,\ \int_{\mathbb{R}} v_q(t,x)dx = 1$, it is sensible to construct a probability measure P on $C([0,+\infty),\mathbb{R})$ with time marginals $(P_t)_{t\geq 0}$ such that $P_0 = \delta_0$ and for any $t > 0$, $v_q(t,.)$ is a density of P_t with respect to Lebesgue measure. To do so, we associate a nonlinear martingale problem with the partial differential equation. We say that $P \in \mathcal{P}(C([0,+\infty),\mathbb{R}))$ with time marginals $(P_t)_{t\geq 0}$ absolutely continuous with respect to Lebesgue measure for $t > 0$ solves the nonlinear martingale problem if $P_0 = \delta_0$ and for any $\phi \in C_b^2(\mathbb{R})$

$$\phi(X_t) - \phi(X_0) - \int_0^t \left(\frac{1}{2}\frac{d^2\phi}{dx^2}(X_s) + \frac{1}{q}(p(s,X_s))^{q-1}\frac{d\phi}{dx}(X_s)\right) ds \text{ is a } P\text{-martingale}$$

where for any $t > 0$, $p(t,.)$ is a density of P_t. In [4], Méléard and Roelly generalize results given by Oelschläger in [6] and prove existence and uniqueness for similar nonlinear martingale

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problems in which δ_0 and $\frac{1}{q}(p(s,X_s))^{q-1}$ are replaced by $m \in \mathcal{P}(\mathbb{R})$ and $F(X_s, p(s,X_s))$ where $F : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is bounded and satisfies the following Lipschitz assumption

$$\forall x,x' \in \mathbb{R},\ \forall y,y' \in \mathbb{R},\ |F(x,y)-F(x',y')| + |yF(x,y) - y'F(x',y')| \leq K_F(|x-x'|+|y-y'|)$$

They obtain existence by a limit theorem. Indeed they prove propagation of chaos to a solution of the martingale problem for the following sequence of moderately interacting particle systems

$$X_t^{i,n} = X_0^i + B_t^i + \int_0^t F(X_s^{i,n}, V^n * \mu_s^n(X_s^{i,n}))ds,\ t\geq 0,\ 1\leq i\leq n \tag{0.2}$$

where $B^i,\ i\in\mathbb{N}^*$ are independent $\mathbb{R}$-valued Brownian motions, $X_0^i,\ i\in\mathbb{N}^*$ are initial values i.i.d. with law m independent of the Brownian motions, $\mu^n = \frac{1}{n}\sum_{j=1}^n \delta_{X^{j,n}}$ denotes the empirical measure and $V^n(x) = n^\beta V^1(n^\beta x)$ for $\beta \in (0,1)$ and V^1 a probability density which satisfies some regularity assumptions.
The function $x \to x^{q-1}/q$ does not satisfy the assumptions made by Méléard and Roelly on F and it is not possible to adapt directly their results. Combining estimates given by Roynette and Vallois [9] (theorem [EVZ] (2) p484 and theorem I.1 p484) and by Escobedo and Zuazua [3] (proposition 1 (ii) (2.3) p127), we get

$$\forall q\geq 2,\ \exists k_q,\ \forall t>0,\ \|v_q(t,.)\|_{L^\infty} \leq \frac{k_q}{(t\wedge 1)^{\frac{1}{q}}} \tag{0.3}$$

This enables us to construct a function F_q on $(0,+\infty)\times\mathbb{R}$ such that $t\to \|F_q(t,.)\|_{L^\infty}$ is integrable,

$$\forall t>0, \forall x\in\mathbb{R},\ F_q(t, v_q(t,x)) = \frac{1}{q}(v_q(t,x))^{q-1}$$

and for any $\epsilon > 0$, the functions $x\to F_q(s,x)$ (resp $x \to H_q(s,x) = xF_q(s,x)$) are bounded and Lipschitz (resp Lipschitz) uniformly for $s\in[\epsilon,+\infty)$. Let (M_q) denote the martingale problem in which $\frac{1}{q}(p(s,X_s))^{q-1}$ is replaced by $F_q(s,p(s,X_s))$. If P solves (M_q) it is easy to see that the flow $t\to P_t$ is a weak solution of the partial differential equation

$$\frac{\partial P_t}{\partial t} = \frac{1}{2}\frac{\partial^2 P_t}{\partial x^2} - \frac{\partial}{\partial x}(F_q(t,p(t,.))P_t) \tag{0.4}$$

In the first part of this paper we prove that $t\to v_q(t,x)dx$ is the unique solution of this equation in a well chosen space. In the second part, we show that (M_q) admits a unique solution P^q. Moreover, for any $t>0$, $v_q(t,.)$ is a density of P_t^q. Hence P^q is a probabilistic representation of v_q. Uniqueness is an easy consequence of the first part. Unlike in Méléard and Roelly [4], existence is proved directly. In the last part, adapting arguments of Oelschläger [6] and Méléard and Roelly [4], we prove the propagation of chaos to P^q for the particle systems

$$X_t^{i,n} = B_t^i + \int_0^t F_q(s, V^n * \mu_s^n(X_s^{i,n}))ds,\ t\geq 0,\ 1\leq i\leq n$$

This propagation of chaos result provides a constructive way of approximating v_q. To our knowledge, it is the first result for an unbounded drift coefficient in the case of moderate interaction.

Since we do not control $F_q(t,x)$ and $H_q(t,x)$ when $t\to 0$, many proofs are based on time-shifts meant for getting away from 0.

Notations and hypotheses

Let $\Omega = C([0,+\infty),\mathbb{R})$ endowed with the topology of uniform convergence on compact sets and

with the corresponding Borel σ-field, $\Omega^T = C([0,T],\mathbb{R})$ endowed with the topology of uniform convergence and X be the canonical process. For a Borel space E, $\mathcal{P}(E)$ is the space of probability measures on E endowed with the topology of weak convergence.
If $P \in \mathcal{P}(\Omega)$, $(P_t)_{t\geq 0}$ is the set of time marginals of P.

$$\tilde{\mathcal{P}}(\Omega) = \{P \in \mathcal{P}(\Omega);\ \forall t>0,\ P_t \text{ is absolutely continuous with respect to Lebesgue measure}\}$$

$$\begin{aligned}\tilde{C}_0([0,+\infty),\mathcal{P}(\mathbb{R})) = \{\mu \in C([0,+\infty),\mathcal{P}(\mathbb{R}));\ \mu(0)=\delta_0 \\ \forall t>0,\ \mu(t) \text{ is absolutely continuous with respect to Lebesgue measure}\}\end{aligned}$$

If $P \in \tilde{\mathcal{P}}(\Omega)$ (resp $\mu \in \tilde{C}_0([0,+\infty),\mathcal{P}(\mathbb{R}))$), there is a measurable function $p(s,x)$ (resp $m(s,x)$) on $(0,+\infty)\times\mathbb{R}$ such that for any $s>0$, $p(s,.)$ (resp $m(s,.)$) is a density of P_s (resp $\mu(s)$) with respect to Lebesgue measure. See for example Meyer [5] pages 193-194. Such a function is called a measurable version of the densities.

For $t>0$, G_t denotes the heat kernel on $\mathbb{R}$: $G_t(x) = \frac{1}{\sqrt{2\pi t}}\exp(-\frac{x^2}{2t})$.
The following estimate will be very useful :

$$\|\frac{\partial G_t}{\partial x}\|_{L^1} \leq \frac{1}{\sqrt{t}} \tag{0.5}$$

Let $\mathcal{F}$ denote the Fourier transform.
For $r>0$, $H^r(\mathbb{R})$ is the sobolev space $\{f \in L^2(\mathbb{R});\ \int_{\mathbb{R}}(1+|\lambda|^{2r})|\mathcal{F}(f)(\lambda)|^2 d\lambda < +\infty\}$.

Let V^1 be a bounded and Lipschitz probability density on $\mathbb{R}$ such that $\int_{\mathbb{R}}|x|V^1(x)dx < +\infty$ and $V^1 = W^1 * W^1$ with W^1 a probability density belonging to $H^r(\mathbb{R})$ for some $r>0$. Remark that necessarily $V^1 \in H^r(\mathbb{R})$. For example, the function G_1 satisfies these assumptions.

We now define precisely the functions H_q and F_q. For the constant k_q given by (0.3), let h_q be the odd function such that

$$h_q(x) = \begin{cases} \frac{x^q}{q} \quad \text{if} \quad 0\leq x\leq k_q \\ (q-1)k_q^{q-2}\left(\frac{(x-k_q)^2}{2} - \frac{(x-k_q)^3}{6}\right) + k_q^{q-1}(x-k_q) + \frac{k_q^q}{q} \quad \text{if} \quad k_q < x < k_q+1 \\ (\frac{(q-1)k_q^{q-2}}{2} + k_q^{q-1})(x-k_q-1) + \frac{(q-1)k_q^{q-2}}{3} + k_q^{q-1} + \frac{k_q^q}{q} \quad \text{if} \quad x \geq k_q+1 \end{cases}$$

In the following lemma, we group a few obvious properties of h_q.

Lemma 0.1 *The function h_q is strictly increasing. For any $q>2$, h_q is C^2 with bounded first and second derivatives. The function h_2 is C^1 with a bounded derivative and h_2' is continuously differentiable with a bounded derivative on $(-\infty,0)\cup(0,+\infty)$. Last, for any $q\geq 2$, h_q satisfies $h_q(0) = h_q'(0) = 0$.*

We define H_q and F_q on $(0,+\infty)\times\mathbb{R}$ by

$$H_q(t,x) = \frac{1}{t\wedge 1}h_q((t\wedge 1)^{\frac{1}{q}}x) \qquad F_q(t,x) = \begin{cases} 0 \text{ if } x=0 \\ \frac{H_q(t,x)}{x} \text{ otherwise} \end{cases}$$

Let B_0 and B_1 be bounds for h'_q and h''_q. We state some properties of F_q and H_q. Let $t > 0$.

$$\text{if} \quad |x| \le \frac{k_q}{(t\wedge 1)^{\frac{1}{q}}}, \quad H_q(t,x) = \frac{x|x|^{q-1}}{q} \quad \text{and} \quad F_q(t,x) = \frac{|x|^{q-1}}{q} \tag{0.6}$$

$$\forall x \neq 0,\ |F_q(t,x)| = \left| \frac{h_q((t\wedge 1)^{\frac{1}{q}} x)}{(t\wedge 1)x} \right| \le \frac{B_0 (t\wedge 1)^{\frac{1}{q}} |x|}{(t\wedge 1)|x|} = \frac{B_0}{(t\wedge 1)^{\frac{q-1}{q}}} \tag{0.7}$$

$$|H_q(t,x)| \le \frac{B_0 |x|}{(t\wedge 1)^{\frac{q-1}{q}}} \tag{0.8}$$

$$\forall x \neq 0,\ \left| \frac{\partial F_q}{\partial x}(t,x) \right| = \left| \frac{h'_q((t\wedge 1)^{\frac{1}{q}} x)}{(t\wedge 1)^{\frac{q-1}{q}} x} - \frac{h_q((t\wedge 1)^{\frac{1}{q}} x)}{(t\wedge 1)x^2} \right| \le \frac{3B_1}{2(t\wedge 1)^{\frac{q-2}{q}}} \tag{0.9}$$

$$\left| \frac{\partial H_q}{\partial x}(t,x) \right| = \left| \frac{h'_q((t\wedge 1)^{\frac{1}{q}} x)}{(t\wedge 1)^{\frac{q-1}{q}}} \right| \le \frac{B_0}{(t\wedge 1)^{\frac{q-1}{q}}} \tag{0.10}$$

1 An existence and uniqueness result for the partial differential equation (0.4)

1.1 The result

Definition 1.1 *The map $\mu \in \tilde{C}_0([0,+\infty), \mathcal{P}(\mathbb{R}))$ is a weak solution of (E_q) if for any $0 < t_0 < t$ and any function $\phi \in C_b^{1,2}([t_0,t]\times\mathbb{R})$,*

$$\begin{aligned} \int_{\mathbb{R}} \phi(t,x) m(t,x) dx = \int_{\mathbb{R}} \phi(t_0,x) m(t_0,x) dx \\ + \int_{(t_0,t]\times\mathbb{R}} \left(\frac{\partial \phi}{\partial s}(s,x) + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}(s,x) + F_q(s, m(s,x)) \frac{\partial \phi}{\partial x}(s,x) \right) m(s,x) ds dx \end{aligned} \tag{1.1}$$

where m is a mesurable version of the densities for μ.

Clearly, this definition does not depend on the choice of the measurable version of the densities. (E_q) is linked to an evolution equation. Indeed we prove that if μ is a solution, then m satisfies

$$\forall t_0 > 0,\ \forall t \ge t_0,\ m(t,x) = G_{t-t_0} * m(t_0,.)(x) - \int_{t_0}^{t} \frac{\partial G_{t-s}}{\partial x} * H_q(s, m(s,.))(x) ds \quad \text{a.e.} \tag{1.2}$$

Let f be a C^2 function with compact support in $\mathbb{R}$. We set $\phi(s,x) = G_{t-s} * f(x)$. The function ϕ belongs to $C_b^{1,2}([t_0,t]\times\mathbb{R})$ and satisfies

$$\forall s \in [t_0,t], \forall x \in \mathbb{R},\ \frac{\partial \phi}{\partial s}(s,x) + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2} = 0$$

Applying (1.1), we get

$$\int_{\mathbb{R}} f(x) m(t,x) dx = \int_{\mathbb{R}} (G_{t-t_0} * f)(x) m(t_0,x) dx + \int_{(t_0,t]\times\mathbb{R}} H_q(s, m(s,x)) \left(\frac{\partial G_{t-s}}{\partial x} * f \right)(x) ds dx$$

Inequalities (0.5) and (0.8) imply

$$\int_{(t_0,t]\times\mathbb{R}\times\mathbb{R}} \left| H_q(s,m(s,x)) \frac{\partial G_{t-s}}{\partial x}(x-y)f(y) \right| dydxds \leq \|f\|_{L^\infty} \frac{B_0}{(t_0\wedge 1)^{\frac{q-1}{q}}} \int_{t_0}^t \frac{ds}{\sqrt{t-s}} < +\infty$$

Therefore, by Fubini's theorem, we obtain

$$\int_{\mathbb{R}} f(x)m(t,x)dx = \int_{\mathbb{R}} f(x) \left(G_{t-t_0} * m(t_0,.)(x) - \int_{t_0}^t \frac{\partial G_{t-s}}{\partial x} * H_q(s,m(s,.))(x)ds \right) dx$$

Hence (1.2) holds. The map $t \to G_{t-t_0} * m(t_0,.)$ is clearly continuous in $L^1(\mathbb{R})$ for $t \geq t_0$. Using (1.2), (0.5) and (0.8), it is quite easy to deduce that $s \to m(t_0+s,.) \in C([0,+\infty), L^1(\mathbb{R}))$. As t_0 is arbitrary, $s \to m(s) \in C((0,+\infty), L^1(\mathbb{R}))$.

We define $V^q \in \tilde{C}_0([0,+\infty), \mathcal{P}(\mathbb{R}))$ by $V_q(0) = \delta_0$ and $\forall t > 0, V^q(t) = v_q(t,x)dx$. The function $v_q(t,x)$ is a measurable version of the densities for V_q.

Theorem 1.2 *For any $q \geq 2$, the map V_q is the unique weak solution of (E_q).*

To prove uniqueness, we need comparison results for the evolution equation (1.2) that we group in the following proposition. The next subsection is devoted to the proof of this proposition which requires some technical estimates. As the convergence $\lim_{t\to 0} \mu(t) = \delta_0$ is weak, it is not possible to get rid of these estimates.

Proposition 1.3 *Let $t_0 > 0$ and $u_0 \in L^1(\mathbb{R})$. Then the equation $(D^q_{t_0,u_0})$*

$$u(t) = G_t * u_0 - \int_0^t \frac{\partial G_{t-s}}{\partial x} * H_q(t_0+s, u(s))ds \tag{1.3}$$

admits a unique solution u in $C([0,+\infty), L^1(\mathbb{R}))$. This solution belongs to $C^1((0,+\infty), L^2(\mathbb{R})) \cap C((0,+\infty), H^2(\mathbb{R}))$. If v denotes the solution of $(D^q_{t_0,v_0})$

$$\forall t \geq 0,\ \|u(t) - v(t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1} \tag{1.4}$$

Moreover if $\int_{\mathbb{R}} u_0(x)dx = \int_{\mathbb{R}} v_0(x)dx$ and $\forall x \in \mathbb{R}$, $\int_{-\infty}^x u_0(y)dy \leq \int_{-\infty}^x v_0(y)dy$ then

$$\forall t \geq 0,\ \forall x \in \mathbb{R},\ \int_{-\infty}^x u(t,y)dy \leq \int_{-\infty}^x v(t,y)dy \tag{1.5}$$

Proof of Theorem 1.2 : We first check that V_q is a solution of (E_q). By (0.1) and (0.6),

$$\forall s > 0, \forall x \in \mathbb{R},\ \frac{\partial v_q}{\partial s}(s,x) + \frac{\partial}{\partial x}(F_q(s,v_q(s,x))v_q(s,x)) = \frac{1}{2}\frac{\partial^2 v_q}{\partial x^2}(s,x)$$

Let $0 < t_0 < t$ and ϕ be a $C^{1,2}$ function with compact support in $[t_0,t] \times \mathbb{R}$. As $\frac{\partial v_q}{\partial s}$, $\frac{\partial^2 v_q}{\partial x^2}$ and $\frac{\partial}{\partial x}(F_q(s,v_q(s,x))v_q(s,x))$ are bounded on the support of ϕ, using Fubini's theorem and the integration by parts formula, we obtain

$$\int_{\mathbb{R}} \phi(t,x)v_q(t,x)dx = \int_{\mathbb{R}} \phi(t_0,x)v_q(t_0,x)dx + \int_{(t_0,t]\times\mathbb{R}} \left(\frac{\partial \phi}{\partial s}(s,x) + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}(s,x) + F_q(s,v_q(s,x))\frac{\partial \phi}{\partial x}(s,x) \right) v_q(s,x)dsdx \tag{1.6}$$

If $\phi \in C_b^{1,2}([t_0,t]\times\mathbb{R})$, by truncation, we approximate ϕ by $C^{1,2}$ functions with compact support in $[t_0,t]\times\mathbb{R}$. As, by (0.7), $\forall s\in[t_0,t]$, $\|F_q(s,v_q(s,.))v_q(s,.)\|_{L^1}\leq B_0/(t_0\wedge 1)^{\frac{q-1}{q}}$, (1.6) still holds for ϕ. Hence V_q is a solution of (E_q).

The proof for uniqueness was inspired by [2] (proof of Theorem 3). Let μ be a solution of (E_q) and m a measurable version of the densities for μ. Equation (1.2) with $t_0=\frac{1}{n}$ implies that the map $t\rightarrow m(\frac{1}{n}+t,.)$ is the solution of $(D^q_{\frac{1}{n},m(\frac{1}{n},.)})$. Similarly, since V_q is a weak solution of (E_q), the map $t\rightarrow v_q(\frac{1}{n}+t,.)$ is the solution of $(D^q_{\frac{1}{n},v_q(\frac{1}{n},.)})$. We are going to compare v_q and m thanks to (1.4) and (1.5).

Let $r>0$.
If $\int_{-r}^{r}m(\frac{1}{n},x)dx \geq \int_{-r}^{r}v_q(\frac{1}{n},x)dx$, we define $v^{n,0}(x)=1_{\{x\in[-r,r]\}}v_q(\frac{1}{n},x)$ and for s such that $\int_{-s}^{s}m(\frac{1}{n},x)dx=\int_{-r}^{r}v_q(\frac{1}{n},x)dx$ we set $m^{n,0}(x)=1_{\{x\in[-s,s]\}}m(\frac{1}{n},x)$. Otherwise, we make the symmetrical construction. In this way,

$$\forall x\in\mathbb{R},\ \int_{-\infty}^{x}v^{n,0}(y-2r)dy\leq\int_{-\infty}^{x}m^{n,0}(y)dy\leq\int_{-\infty}^{x}v^{n,0}(y+2r)dy$$

If v^n and m^n denote the solutions of $(D^q_{\frac{1}{n},v^{n,0}})$ and $(D^q_{\frac{1}{n},m^{n,0}})$, using (1.5), we deduce

$$\forall t\geq 0,\forall x\in\mathbb{R},\ \int_{-\infty}^{x}v^{n}(t,y-2r)dy\leq\int_{-\infty}^{x}m^{n}(t,y)dy\leq\int_{-\infty}^{x}v^{n}(t,y+2r)dy \qquad (1.7)$$

As μ and V^q belong to $\tilde{C}_o([0,+\infty),\mathcal{P}(\mathbb{R}))$, $\lim_{n\rightarrow+\infty}V_q(\frac{1}{n})=\lim_{n\rightarrow+\infty}\mu(\frac{1}{n})=\delta_0$.
Hence $\|v^{n,0}-v_q(\frac{1}{n})\|_{L^1}=\|m^{n,0}-m(\frac{1}{n})\|_{L^1}\rightarrow_{n\rightarrow+\infty}0$.
With equation (1.4), this implies

$$\forall t\geq 0,\ \lim_{n\rightarrow+\infty}\|v^n(t)-v_q(t+\frac{1}{n})\|_{L^1}=\lim_{n\rightarrow+\infty}\|m^n(t)-m(t+\frac{1}{n})\|_{L^1}=0$$

Since $\|m^n(t)-m(t)\|_{L^1}\leq\|m^n(t)-m(t+\frac{1}{n})\|_{L^1}+\|m(t+\frac{1}{n})-m(t)\|_{L^1}$, with the continuity of $s\rightarrow m(s)$ on $(0,+\infty)$, we conclude

$$\forall t>0,\ m(t)=\lim_{n\rightarrow+\infty}m^n(t)\quad\text{in}\ \ L^1(\mathbb{R})$$

And the same holds for v_q and v^n. Taking the limit $n\rightarrow+\infty$ in (1.7), we get

$$\forall t>0,\forall x\in\mathbb{R},\ \int_{-\infty}^{x}v_q(t,y-2r)dy\leq\int_{-\infty}^{x}m(t,y)dy\leq\int_{-\infty}^{x}v_q(t,y+2r)dy$$

As r is arbitrary, $\forall t>0,\ \|v_q(t)-m(t)\|_{L^1}=0$. Hence $\mu=V^q$. ∎

1.2 Proof of Proposition 1.3

Existence and uniqueness for $(D^q_{t_0,u_0})$ (equation (1.3)) can be proved easily by a fixed-point method. But to show (1.4) and (1.5), it is necessary to obtain regularity properties of the fixed-points, which requires some technical estimates.

The main ideas come from the articles of Escobedo, Vasquez and Zuazua [2] and Escobedo and Zuazua [3]. These authors often refer to "classical results" in their arguments which are thus quite sketchy. It seems that the ideas are classical in the theory of quasilinear equations but it was not possible to find any precise proof. That is why we detail the particular case that we are interested in.

We begin with a lemma which prepares the application of Picard's fixed-point theorem. Let $w \in L^1(\mathbb{R})$ and $t_1 > 0$. On $C([0,T], L^1(\mathbb{R}))$ we define the map $\phi_{t_1,w}$ by

$$\phi_{t_1,w}(v)(t) = G_t * w - \int_0^t \frac{\partial G_{t-s}}{\partial x} * H_q(t_1 + s, v(s))ds$$

Lemma 1.4 *Let $t_0 > 0$. If $T > 0$ is small enough (depending on t_0), then for any $t_1 \geq t_0$ and any $w \in L^1(\mathbb{R})$*

(i) *The map $\phi_{t_1,w}$ is a contraction on $C([0,T], L^1(\mathbb{R}))$.*

(ii) *There is a constant C_0 depending only on w such that if $v \in C([0,T], L^1(\mathbb{R}))$ satisfies*

$$\forall t \in (0,T],\ v(t) \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R}) \ \text{ and } \ \|v(t)\|_{L^p} \leq \frac{C_0}{\sqrt{t}} \ \text{ for } p = 2, +\infty \tag{1.8}$$

then $\phi_{t_1,w}(v)$ satisfies (1.8)

(iii) *For any $\alpha \in (0,T]$, there is a constant C_1 depending only on α and w such that if v satisfies (1.8) and*

$$\forall t \in (\alpha,T],\ v(t) \in H^1(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R}) \ \text{ and } \ \|\frac{\partial v(t)}{\partial x}\|_{L^p} \leq \frac{C_1}{\sqrt{t-\alpha}} \ \text{ for } p = 2, +\infty \tag{1.9}$$

then $\phi_{t_1,w}(v)$ satisfies (1.9).
($W^{1,\infty}(\mathbb{R})$ denotes the Sobolev space of L^∞ functions with first derivative in L^∞.)

(iv) *For any $0 < \alpha < \beta \leq T$, there is a constant C_2 depending only on α, β, t_0 and w such that if v satisfies (1.8), (1.9) and*

$$\forall t \in (\beta,T],\ v(t) \in H^2(\mathbb{R}) \ \text{ and } \ \|\frac{\partial^2 v(t)}{\partial x^2}\|_{L^2} \leq \frac{C_2}{\sqrt{t-\beta}} \tag{1.10}$$

then $\phi_{t_1,w}(v)$ satisfies (1.10).

Proof : **(i)** Clearly $t \to G_t * w$ is continuous in $L^1(\mathbb{R})$. With $\sup_{t\in[0,T]} \|v(t)\|_{L^1} < +\infty$, it is not

difficult to obtain that $\phi_{t_1,w}(v) \in C([0,T], L^1(\mathbb{R}))$.
Let $v, v' \in C([0,T], L^1(\mathbb{R}))$. Using (0.5) and (0.10), we have for any $t \in [0,T]$,

$$\begin{aligned}\|\phi_{t_1,w}(v)(t) - \phi_{t_1,w}(v')(t)\|_{L^1} &\le \int_0^t \|\frac{\partial G_{t-s}}{\partial x}\|_{L^1} \|H_q(t_1+s, v(s)) - H_q(t_1+s, v'(s))\|_{L^1} ds \\ &\le \frac{2\sqrt{T} B_0}{(t_0 \wedge 1)^{\frac{q-1}{q}}} \sup_{s \in [0,T]} \|v(s) - v'(s)\|_{L^1}\end{aligned}$$

Hence if $T \le \frac{(t_0 \wedge 1)^{\frac{2q-2}{q}}}{16 B_0^2}$, then $\phi_{t_1,w}$ is a contraction on $C([0,T], L^1(\mathbb{R}))$.

(ii) Let $v \in C([0,T], L^1(\mathbb{R}))$ which satisfies (1.8). Using (0.5) and (0.8) we get for $p = 2, +\infty$,

$$\begin{aligned}\|\phi_{t_1,w}(v)(t)\|_{L^p} &\le \|G_t\|_{L^p} \|w\|_{L^1} + \int_0^t \|\frac{\partial G_{t-s}}{\partial x}\|_{L^1} \|H_q(t_1+s, v(s))\|_{L^p} ds \\ &\le \|G_t\|_{L^p} \|w\|_{L^1} + \int_0^t \frac{B_0 C_0}{(t_0 \wedge 1)^{\frac{q-1}{q}} \sqrt{s}\sqrt{t-s}} ds\end{aligned}$$

Hence

$$\begin{aligned}\|\phi_{t_1,w}(v)(t)\|_{L^2} &\le \frac{1}{\sqrt{t}} \left(\frac{\|w\|_{L^1} T^{\frac{1}{4}}}{(4\pi)^{\frac{1}{4}}} + \frac{B_0 C_0 \pi \sqrt{T}}{(t_0 \wedge 1)^{\frac{q-1}{q}}} \right) \\ \|\phi_{t_1,w}(v)(t)\|_{L^\infty} &\le \frac{1}{\sqrt{t}} \left(\frac{\|w\|_{L^1}}{\sqrt{2\pi}} + \frac{B_0 C_0 \pi \sqrt{T}}{(t_0 \wedge 1)^{\frac{q-1}{q}}} \right)\end{aligned}$$

We set $C_0 = \left(\frac{4}{\pi}\right)^{\frac{1}{4}} \|w\|_{L^1}$. If $T \le \frac{(t_0 \wedge 1)^{\frac{2q-2}{q}}}{4\pi^2 B_0^2} \wedge 1$, then (1.8) holds for $\phi_{t_1,w}(v)$.

(iii) Let $T \le \frac{(t_0 \wedge 1)^{\frac{2q-2}{q}}}{4\pi^2 B_0^2} \wedge 1$, $\alpha \in (0,T]$ and $v \in C([0,T], L^1(\mathbb{R}))$ which satisfies (1.8) and (1.9). With the definition of $\phi_{t_1,w}(v)(\alpha)$ and Fubini's theorem, we obtain

$$\forall t \in [0, T-\alpha],\ \phi_{t_1,w}(v)(t+\alpha) = G_t * \phi_{t_1,w}(v)(\alpha) - \int_0^t \frac{\partial G_{t-s}}{\partial x} * H_q(t_1+\alpha+s, v(\alpha+s)) ds$$

Let $s \in (0, T-\alpha]$. As $v(\alpha+s) \in H^1(\mathbb{R})$ and the function $x \to H_q(t_1+\alpha+s, x)$ is C^1 and satisfies $H_q(t_1+\alpha+s, 0) = 0$, $H_q(t_1+\alpha+s, v(\alpha+s)) \in H^1(\mathbb{R})$ and

$$\frac{\partial}{\partial x} H_q(t_1+\alpha+s, v(\alpha+s)) = \frac{h'_q(((t_1+\alpha+s)\wedge 1)^{\frac{1}{q}} v(\alpha+s))}{((t_1+\alpha+s)\wedge 1)^{\frac{q-1}{q}}} \frac{\partial v(\alpha+s)}{\partial x}$$

(see for example Corollary VIII.10 p.131 in [1]). We deduce that for $t \in (0, T-\alpha]$,

$$\begin{aligned}\frac{\partial \phi_{t_1,w}(v)(t+\alpha)}{\partial x} &= \frac{\partial G_t}{\partial x} * \phi_{t_1,w}(v)(\alpha) \\ &\quad - \int_0^t \frac{\partial G_{t-s}}{\partial x} * \left(\frac{h'_q(((t_1+\alpha+s)\wedge 1)^{\frac{1}{q}} v(\alpha+s))}{((t_1+\alpha+s)\wedge 1)^{\frac{q-1}{q}}} \frac{\partial v(\alpha+s)}{\partial x} \right) ds\end{aligned} \tag{1.11}$$

For $p=2$ or $p=+\infty$, using (1.9) and (ii), we obtain

$$\begin{aligned}
\|\frac{\partial \phi_{t_1,w}(v)(t+\alpha)}{\partial x}\|_{L^p} &\le \|\frac{\partial G_t}{\partial x}\|_{L^1}\|\phi_{t_1,w}(v)(\alpha)\|_{L^p} + \int_0^t \|\frac{\partial G_{t-s}}{\partial x}\|_{L^1}\frac{B_0}{(t_0\wedge 1)^{\frac{q-1}{q}}}\|\frac{\partial v(\alpha+s)}{\partial x}\|_{L^p} ds \\
&\le \frac{C_0}{\sqrt{t\alpha}} + \int_0^t \frac{B_0 C_1}{(t_0\wedge 1)^{\frac{q-1}{q}}\sqrt{t-s}\sqrt{s}} ds \\
&\le \frac{1}{\sqrt{t}}\Big(\frac{C_0}{\sqrt{\alpha}} + \frac{B_0 C_1 \pi\sqrt{T}}{(t_0\wedge 1)^{\frac{q-1}{q}}}\Big)
\end{aligned}$$

We set $C_1 = \frac{2C_0}{\sqrt{\alpha}}$. Since we have supposed that $T \le \frac{(t_0\wedge 1)^{\frac{2q-2}{q}}}{4\pi^2 B_0^2}$, $\phi_{t_1,w}(v)$ satisfies (1.9).

(iv) Let $T \le \frac{(t_0\wedge 1)^{\frac{2q-2}{q}}}{4\pi^2 B_0^2}\wedge 1$, $0<\alpha<\beta\le T$ and $v\in C([0,T],L^1(\mathbb{R}))$ which satisfies (1.8), (1.9) and (1.10). Let $s\in(0,T-\beta]$. If $q>2$, since h'_q is C^1 satisfies $h'_q(0)=0$ and $v(\beta+s)\in H^1(\mathbb{R})$, $h'_q(((t_1+\beta+s)\wedge 1)^{\frac{1}{q}} v(\beta+s))\in H^1(\mathbb{R})$ and

$$\frac{\partial}{\partial x} h'_q(((t_1+\beta+s)\wedge 1)^{\frac{1}{q}} v(\beta+s)) = ((t_1+\beta+s)\wedge 1)^{\frac{1}{q}} h''_q(((t_1+\beta+s)\wedge 1)^{\frac{1}{q}} v(\beta+s))\frac{\partial v(\beta+s)}{\partial x}$$

If $q=2$ the conclusion still holds with the convention $h''_2(0)=0$ since h'_2 is Lipschitz, C^1 outside of 0 and satisfies $h'_2(0)=0$.
On the other hand, $\frac{\partial v(\beta+s)}{\partial x}\in H^1(\mathbb{R})$. Hence, by the formula giving the derivative of a product in $H^1(\mathbb{R})$, $h'_q(((t_1+\beta+s)\wedge 1)^{\frac{1}{q}} v(\beta+s))\frac{\partial v(\beta+s)}{\partial x}\in H^1(\mathbb{R})$ with derivative

$$((t_1+\beta+s)\wedge 1)^{\frac{1}{q}} h''_q(((t_1+\beta+s)\wedge 1)^{\frac{1}{q}} v(\beta+s))\Big(\frac{\partial v(\beta+s)}{\partial x}\Big)^2 + h'_q(((t_1+\beta+s)\wedge 1)^{\frac{1}{q}} v(\beta+s))\frac{\partial^2 v(\beta+s)}{\partial x^2}$$

(See Corollary VIII.9 p.131 in [1]). Let $g(s)$ denote the last expression. Differenciating (1.11) with β replacing α, we obtain

$$\frac{\partial^2 \phi_{t_1,w}(v)(t+\beta)}{\partial x^2} = \frac{\partial G_t}{\partial x} * \frac{\partial \phi_{t_1,w}(v)(\beta)}{\partial x} - \int_0^t \frac{\partial G_{t-s}}{\partial x} * \frac{g(s)}{((t_1+\beta+s)\wedge 1)^{\frac{q-1}{q}}} ds \qquad (1.12)$$

With (1.9) and (1.10), we bound $\|g(s)\|_{L^2}$.

$$\begin{aligned}
\|g(s)\|_{L^2} &\le B_1((t_1+\beta+s)\wedge 1)^{\frac{1}{q}}\|\frac{\partial v(\beta+s)}{\partial x}\|_{L^\infty}\|\frac{\partial v(\beta+s)}{\partial x}\|_{L^2} + B_0\|\frac{\partial^2 v(\beta+s)}{\partial x^2}\|_{L^2} \\
&\le \frac{B_1 C_1^2((t_1+\beta+s)\wedge 1)^{\frac{1}{q}}}{\beta-\alpha} + \frac{B_0 C_2}{\sqrt{s}}
\end{aligned}$$

With (1.12) we deduce,

$$\|\frac{\partial^2 \phi_{t_1,w}(v)(t+\beta)}{\partial x^2}\|_{L^2} \le \frac{1}{\sqrt{t}}\Big(\frac{C_1}{\sqrt{\beta-\alpha}} + \frac{2B_1 C_1^2 T}{(t_0\wedge 1)^{\frac{q-2}{q}}(\beta-\alpha)} + \frac{B_0 C_2 \pi\sqrt{T}}{(t_0\wedge 1)^{\frac{q-1}{q}}}\Big)$$

We set

$$C_2 = 3\Big(\frac{C_1}{\sqrt{\beta-\alpha}} \vee \frac{2B_1 C_1^2}{(t_0\wedge 1)^{\frac{q-2}{q}}(\beta-\alpha)}\Big)$$

If $T \leq \frac{(t_0 \wedge 1)^{\frac{2q-2}{q}}}{9\pi^2 B_0^2} \wedge 1$, then $\phi_{t_1,w}(v)$ satisfies (1.10). Moreover, (i), (ii) and (iii) hold. ∎

The next lemma gives existence of a unique fixed-point for $\phi_{t_1,w}$ and states regularity properties of this fixed-point.

Lemma 1.5 *Let $t_0 > 0$, $t_1 \geq t_0$ and $w \in L^1(\mathbb{R})$. Then, for T given by Lemma 1.4, $\phi_{t_1,w}$ admits a unique fixed-point in $C([0,T], L^1(\mathbb{R}))$.*
This fixed-point belongs to $C((0,T), H^2(\mathbb{R})) \cap C^1((0,T), L^2(\mathbb{R}))$ and satisfies

$$\forall t \in (0,T), \frac{\partial u(t)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(t)}{\partial x^2} - \frac{\partial}{\partial x} H_q(t_1 + t, u(t)) \text{ in } L^2(\mathbb{R}) \tag{1.13}$$

We obtain the regularity in t thanks to results on semigroups of linear operators given by Pazy [7] (Theorem 3.1 p.110 and Corollary 3.3 p.113) that we group in the following theorem.

Theorem 1.6 *Let $(A, \mathcal{D}(A))$ be the infinitesimal generator of an analytic semigroup $T(t)$ (see [7] p.60) on a Banach space X, $x \in X$ and $f \in L^1([0,T], X)$. We set*

$$v(t) = T(t)x + \int_0^t T(t-s)f(s)ds$$

(i) *If $f \in L^p([0,T], X)$ for $p > 1$, then v is Hölder continuous with exponent $\frac{p-1}{p}$ on $[\epsilon, T]$ for any $\epsilon \in (0,T]$.*

(ii) *If f is locally Hölder continuous on $(0,T]$, then*

- $v \in C^1((0,T), X)$
- $\forall t \in (0,T)$, $v(t) \in \mathcal{D}(A)$ *and* $t \to Av(t)$ *is continuous on* $(0,T)$
- $\forall t \in (0,T)$, $\frac{dv(t)}{dt} = Av(t) + f(t)$

Proof of Lemma 1.5 : By Lemma 1.4 (i) and Picard's fixed-point theorem, $\phi_{t_1,w}$ admits a unique fixed-point u in $C([0,T], L^1(\mathbb{R}))$.
We define a sequence of fixed-point iterations by setting

$$v^0 = 0 \text{ and } \forall n \in \mathbb{N},\ v^{n+1} = \phi_{t_1,w}(v^n)$$

Since v^0 satisfies (1.8), (1.9) and (1.10) for any $0 < \alpha < \beta \leq T$, by Lemma 1.4 (ii) (iii) and (iv), for any $n \in \mathbb{N}$, v^n satisfies (1.8), (1.9) and (1.10) for any $0 < \alpha < \beta \leq T$. As

$\forall t \in [0,T]$, $v^n(t) \to u(t)$ in the distribution sense, we obtain that $u(t)$ satisfies (1.8), (1.9) and (1.10) for any $0 < \alpha < \beta \le T$. Hence

$$\forall t \in (0,T],\ u(t) \in W^{1,\infty}(\mathbb{R}) \cap H^2(\mathbb{R})$$

$$\forall \gamma \in (0,T],\ \sup_{t\in[\gamma,T]} \|u(t)\|_{L^p} < +\infty \text{ and } \sup_{t\in[\gamma,T]} \|\frac{\partial u(t)}{\partial x}\|_{L^p} < +\infty \text{ for } p = 2, +\infty \tag{1.14}$$

$$\forall \gamma \in (0,T],\ \sup_{t\in[\gamma,T]} \|\frac{\partial^2 u(t)}{\partial x^2}\|_{L^2} < +\infty \tag{1.15}$$

Let us deduce the regularity properties in t and (1.13). Let $\epsilon \in (0,T]$. By the proof of Lemma 1.4, we know that

$$\forall t \in [0, T-\frac{\epsilon}{2}],\ u(t+\frac{\epsilon}{2}) = G_t * u(\frac{\epsilon}{2}) + \int_0^t G_{t-s} * \left(-\frac{\partial}{\partial x} H_q(t_1 + \frac{\epsilon}{2} + s, u(\frac{\epsilon}{2}+s))\right) ds$$
$$\frac{\partial u(t+\frac{\epsilon}{2})}{\partial x} = G_t * \frac{\partial u(\frac{\epsilon}{2})}{\partial x} + \int_0^t G_{t-s} * \left(-\frac{\partial^2}{\partial x^2} H_q(t_1 + \frac{\epsilon}{2} + s, u(\frac{\epsilon}{2}+s))\right) ds \tag{1.16}$$

with for any $s \in (0,T]$

$$\frac{\partial}{\partial x} H_q(t_1+s, u(s)) = \frac{h_q'(((t_1+s)\wedge 1)^{\frac{1}{q}} u(s))}{((t_1+s)\wedge 1)^{\frac{q-1}{q}}} \frac{\partial u(s)}{\partial x}$$
$$\frac{\partial^2}{\partial x^2} H_q(t_1+s, u(s)) = \frac{h_q''(((t_1+s)\wedge 1)^{\frac{1}{q}} u(s))}{((t_1+s)\wedge 1)^{\frac{q-2}{q}}} \left(\frac{\partial u(s)}{\partial x}\right)^2 + \frac{h_q'(((t_1+s)\wedge 1)^{\frac{1}{q}} u(s))}{((t_1+s)\wedge 1)^{\frac{q-1}{q}}} \frac{\partial^2 u(s)}{\partial x^2}$$

Applying (1.14) and (1.15) with $\gamma = \frac{\epsilon}{2}$, we deduce that $\|\frac{\partial}{\partial x} H_q(t_1 + \frac{\epsilon}{2} + s, u(\frac{\epsilon}{2} + s))\|_{L^2}$ and $\|\frac{\partial^2}{\partial x^2} H_q(t_1 + \frac{\epsilon}{2} + s, u(\frac{\epsilon}{2} + s))\|_{L^2}$ are bounded on $[0, T-\frac{\epsilon}{2}]$.
Hence the maps $s \to \frac{\partial}{\partial x} H_q(t_1 + \frac{\epsilon}{2} + s, u(\frac{\epsilon}{2}+s))$ and $t \to \frac{\partial^2}{\partial x^2} H_q(t_1 + \frac{\epsilon}{2} + s, u(\frac{\epsilon}{2}+s))$ belong to $L^2([0,T-\frac{\epsilon}{2}], L^2(\mathbb{R}))$.
The heat semigroup is analytic in $L^2(\mathbb{R})$ with infinitesimal generator $(\frac{\partial^2}{\partial x^2}, H^2(\mathbb{R}))$ (see [7] p.208-212). Hence applying Theorem 1.6 (i) to (1.16), we conclude that the maps $t \to u(t)$ and $t \to \frac{\partial u(t)}{\partial x}$ are Hölder continuous with exponent $\frac{1}{2}$ on $[\epsilon, T]$.
We deduce that the map $t \to \frac{\partial}{\partial x} H_q(t_1 + t, u(t))$ is Hölder continuous with exponent $\frac{1}{2}$ on $[\epsilon, T]$. Indeed for $t, t' \in [\epsilon, T]$,

$$\begin{aligned}
&\|\frac{\partial}{\partial x} H_q(t_1+t', u(t')) - \frac{\partial}{\partial x} H_q(t_1+t, u(t))\|_{L^2} \\
&\le \frac{1}{((t_1+t)\wedge 1)^{\frac{q-1}{q}}} \|h_q'(((t_1+t)\wedge 1)^{\frac{1}{q}} u(t))\|_{L^\infty} \|\frac{\partial u(t')}{\partial x} - \frac{\partial u(t)}{\partial x}\|_{L^2} \\
&+ \left| \frac{1}{((t_1+t)\wedge 1)^{\frac{q-1}{q}}} - \frac{1}{((t_1+t')\wedge 1)^{\frac{q-1}{q}}} \right| \|h_q'(((t_1+t)\wedge 1)^{\frac{1}{q}} u(t))\|_{L^\infty} \|\frac{\partial u(t')}{\partial x}\|_{L^2} \\
&+ \frac{1}{((t_1+t')\wedge 1)^{\frac{q-1}{q}}} \|h_q'(((t_1+t')\wedge 1)^{\frac{1}{q}} u(t')) - h_q'(((t_1+t)\wedge 1)^{\frac{1}{q}} u(t))\|_{L^2} \|\frac{\partial u(t')}{\partial x}\|_{L^\infty} \\
&\le \frac{B_0}{(t_0\wedge 1)^{\frac{q-1}{q}}} \|\frac{\partial u(t')}{\partial x} - \frac{\partial u(t)}{\partial x}\|_{L^2} + B_0 C(t_0)|t'-t| \|\frac{\partial u(t')}{\partial x}\|_{L^2} \\
&+ B_1 \|\frac{\partial u(t')}{\partial x}\|_{L^\infty} \left(C(t_0)|t'-t| \|u(t)\|_{L^2} + \frac{1}{(t_0\wedge 1)^{\frac{q-2}{q}}} \|u(t') - u(t)\|_{L^2} \right)
\end{aligned}$$

Applying Theorem 1.6 (ii) to (1.16) with ϵ replacing $\frac{\epsilon}{2}$, we conclude that $t \to u(t) \in C^1((\epsilon, T), L^2(\mathbb{R}))$, $t \to \frac{\partial^2 u(t)}{\partial x^2} \in C((\epsilon, T), L^2(\mathbb{R}))$ and

$$\forall t \in (\epsilon, T), \frac{\partial u(t)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(t)}{\partial x^2} - \frac{\partial}{\partial x} H_q(t_1 + t, u(t)) \text{ in } L^2(\mathbb{R})$$

Since ϵ is arbitrary, we have obtained the desired result. ∎

We are now ready to prove Proposition 1.3. The proof is divided in three steps. In the first, we prove existence and uniqueness for $(D^q_{t_0,u_0})$ (see (1.3)). The second is dedicated to the contraction property (1.4) and the third to the comparison property (1.5). The comparison property is obtained as a consequence of maximum principle results given by Protter and Weinberger in [8] (Lemma 2 p.166 and Theorem 2 p.168) and that we group in the following theorem.

Theorem 1.7 *Let E be a connected open set of the (t,x)-plane and $E_{t_1} = \{(t,x) \in E,\ t \leq t_1\}$. Let u satisfy*

$$\forall (t,x) \in E,\ a(t,x)\frac{\partial^2 u}{\partial x^2}(t,x) + b(t,x)\frac{\partial u}{\partial x}(t,x) - \frac{\partial u}{\partial t}(t,x) \geq 0$$

with a and b bounded and $a \geq C$ for a constant $C > 0$.

- **(i)** *if $\forall (t,x) \in E,\ u(t,x) \leq M$ and $u(t_0,x_0) = M$ for $(t_0,x_0) \in E$, then $u = M$ on any segment which contains (t_0,x_0) and is contained in the intersection of the line $(t = t_0)$ with E.*

- **(ii)** *if $\forall (t,x) \in E_{t_1},\ u(t,x) \leq M$ and $u(t_1,x_1) = M$ for $(t_1,x_1) \in E_{t_1}$, then $u = M$ on any segment which contains (t_1,x_1) and is contained in the intersection of E_{t_1} with the line $(x = x_1)$*

Proof of Proposition 1.3 :

Existence and uniqueness for $(D^q_{t_0,u_0})$

Let $u_0 \in L^1(\mathbb{R})$, $t_0 > 0$ and u^0 denote the unique fixed-point of ϕ_{t_0,u_0} in $C([0,T], L^1(\mathbb{R}))$ given by Lemma 1.5. If u^n is constructed, let u^{n+1} be the unique fixed-point of $\phi_{t_0+(n+1)T, u^n(T)}$. We set $u(t) = u^n(t - nT)$ if $t \in [nT, (n+1)T]$. Then u belongs to $C([0,+\infty), L^1(\mathbb{R}))$, solves $(D^q_{t_0,u_0})$ and satisfies the regularity properties presented in Lemma 1.5 outside of the points $nT, n \in \mathbb{N}$. Since the restriction of the map $t \to u((n+\frac{1}{2})T + t)$ to $[0,T]$ is a fixed-point of $\phi_{t_0+(n+\frac{1}{2})T, u((n+\frac{1}{2})T)}$, by Lemma 1.5, u also satisfies the regularity properties at the points $nT, n \in \mathbb{N}^*$. Hence

$$u \in C([0,+\infty), L^1(\mathbb{R})) \cap C^1((0,+\infty), L^2(\mathbb{R})) \cap C((0,+\infty), H^2(\mathbb{R}))$$

$$\forall t > 0,\ \frac{\partial u(t)}{\partial t} = \frac{1}{2}\frac{\partial^2 u(t)}{\partial x^2} - \frac{\partial}{\partial x} H_q(t_0 + t, u(t)) \text{ in } L^2(\mathbb{R}) \tag{1.17}$$

Uniqueness for $(D^q_{t_0,u_0})$ is an easy consequence of uniqueness for the fixed-points.

The contraction property (1.4)

Let $t_0 > 0$, $u_0, v_0 \in L^1(\mathbb{R})$ and u, v denote the solutions of $(D^q_{t_0,u_0})$ and $(D^q_{t_0,v_0})$. We set

$w = u - v$.
Let ψ be a convex C_b^2 function on $\mathbb{R}$ which satisfies $\psi(0) = \psi'(0) = 0$. As $t \to w(t)$ is in $C([0,+\infty), L^1(\mathbb{R})) \cap C^1((0,+\infty), L^2(\mathbb{R}))$, it is easy to obtain that the map $t \to \psi(w(t))$ belongs to $C^1((0,+\infty), L^1(\mathbb{R}))$ with derivative $\psi'(w(t))\frac{\partial w(t)}{\partial t}$ (where $\frac{\partial w(t)}{\partial t}$ denotes the derivative of $t \to w(t)$ considered as a $L^2(\mathbb{R})$-valued map). Let $t > 0$ and $\epsilon \in (0,t]$. We have

$$\int_{\mathbb{R}} \psi(w(t))dx = \int_{\mathbb{R}} \psi(w(\epsilon))dx + \int_\epsilon^t \int_{\mathbb{R}} \psi'(w(s))\Big(\frac{1}{2}\frac{\partial^2 w(s)}{\partial x^2} - \frac{\partial}{\partial x}(H_q(t_0+s,u(s)) - H_q(t_0+s,v(s)))\Big)dxds$$

If $s > 0$, $w(s) \in H^2(\mathbb{R})$. As ψ' is C^1 and satisfies $\psi'(0) = 0$, $\psi'(w(s)) \in H^1(\mathbb{R})$. The integration by parts formula in $H^1(\mathbb{R})$ and the convexity of ψ imply

$$\int_{\mathbb{R}} \psi'(w(s))\frac{\partial^2 w(s)}{\partial x^2}dx = -\int_{\mathbb{R}} \psi''(w(s))\Big(\frac{\partial w(s)}{\partial x}\Big)^2 dx \leq 0$$

Hence

$$\int_{\mathbb{R}} \psi(w(t))dx \leq \int_{\mathbb{R}} \psi(w(\epsilon))dx - \int_\epsilon^t \int_{\mathbb{R}} \psi'(w(s))\frac{\partial}{\partial x}(H_q(t_0+s,u(s)) - H_q(t_0+s,v(s)))dxds \tag{1.18}$$

To obtain the contraction property, we are going to approximate the function $x \to |x|$ by the convex C_b^2 functions ψ_n defined by

$$\begin{aligned} \psi_n''(x) &= \begin{cases} 0 \quad \text{if} \quad |x| \geq \frac{1}{n} \\ \frac{3n}{2}(1-(nx)^2) \quad \text{if} \quad |x| \leq \frac{1}{n} \end{cases} \\ \psi_n'(x) &= \int_0^x \psi_n''(y)dy \\ \psi_n(x) &= \int_0^x \psi_n'(y)dy \end{aligned}$$

As $x \to H_q(t_0+s,x)$ is strictly increasing,

$$\forall x,y \in \mathbb{R}, \lim_{n\to+\infty} \psi_n'(x-y) = \lim_{n\to+\infty} \psi_n'(H_q(t_0+s,x) - H_q(t_0+s,y))$$

By Lebesgue's theorem, this property implies

$$\lim_{n\to+\infty} \int_\epsilon^t \int_{\mathbb{R}} \psi_n'(w(s))\frac{\partial}{\partial x}(H_q(t_0+s,u(s)) - H_q(t_0+s,v(s)))dxds =$$
$$\lim_{n\to+\infty} \int_\epsilon^t \int_{\mathbb{R}} \psi_n'\Big(H_q(t_0+s,u(s)) - H_q(t_0+s,v(s))\Big)\frac{\partial}{\partial x}(H_q(t_0+s,u(s)) - H_q(t_0+s,v(s)))dxds$$

But if $s > 0$, as $u(s), v(s) \in H^2(\mathbb{R})$, we can suppose that $u(s)$ and $v(s)$ are C^1 functions and satisfy $\lim_{|x|\to+\infty} |u(s,x)| = \lim_{|x|\to+\infty} |v(s,x)| = 0$. Therefore

$$\forall n, \int_{\mathbb{R}} \psi_n'\Big(H_q(t_0+s,u(s)) - H_q(t_0+s,v(s))\Big)\frac{\partial}{\partial x}(H_q(t_0+s,u(s)) - H_q(t_0+s,v(s)))dx = 0$$

Hence $\lim_{n\to+\infty} \int_\epsilon^t \int_{\mathbb{R}} \psi_n'(w(s))\frac{\partial}{\partial x}(H_q(t_0+s,u(s)) - H_q(t_0+s,v(s)))dxds = 0$. Using (1.18) for ψ_n and taking the limit $n \to +\infty$, we obtain $\|w(t)\|_{L^1} \leq \|w(\epsilon)\|_{L^1}$. Letting $\epsilon \to 0$, we conclude

$$\forall t > 0, \ \|u(t) - v(t)\|_{L^1} \leq \|u_0 - v_0\|_{L^1}$$

If $v_0 = 0$, then $\forall t > 0$, $v(t) = 0$ and the last inequality provides $\|u(t)\|_{L^1} \leq \|u_0\|_{L^1}$.

The comparison property (1.5)

Let $u_0 \in L^1(\mathbb{R})$ and u be the solution of $(D^q_{t_0,u_0})$. We define $U(t,x) = \int_{-\infty}^x u(t,y)dy$. Let us prove that

$$\forall (t,x) \in (0,+\infty) \times \mathbb{R},\ \frac{\partial U}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 U}{\partial x^2}(t,x) - H_q\Big(t_0+t, \frac{\partial U}{\partial x}(t,x)\Big) \tag{1.19}$$

As $u \in C((0,+\infty), H^2(\mathbb{R}))$, for any $t > 0$, the function $x \to U(t,x)$ is C^2 and satisfies

$$\lim_{|x|\to+\infty} \left|\frac{\partial U}{\partial x}(t,x)\right| = \lim_{|x|\to+\infty} \left|\frac{\partial^2 U}{\partial x^2}(t,x)\right| = 0$$

Moreover, the functions $t \to \frac{\partial U}{\partial x}(t,x)$ and $t \to \frac{\partial^2 U}{\partial x^2}(t,x)$ are continuous on $(0,+\infty)$ and bounded on compact sets of $(0,+\infty)$ uniformly for $x \in \mathbb{R}$.

Let $x \in \mathbb{R},\ t,t' > 0$ and $n \in \mathbb{N}$. By (1.17), we have

$$U(t',x) - U(t',-n) - U(t,x) + U(t,-n) =$$
$$\int_t^{t'} \left(\frac{1}{2}\Big(\frac{\partial^2 U}{\partial x^2}(s,x) - \frac{\partial^2 U}{\partial x^2}(s,-n)\Big) - \Big(H_q\Big(t_0+s, \frac{\partial U}{\partial x}(s,x)\Big) - H_q\Big(t_0+s, \frac{\partial U}{\partial x}(s,-n)\Big)\Big)\right) ds$$

Taking the limit $n \to +\infty$, we obtain by Lebesgue's theorem,

$$U(t',x) - U(t,x) = \int_t^{t'} \left(\frac{1}{2}\frac{\partial^2 U}{\partial x^2}(s,x) - H_q\Big(t_0+s, \frac{\partial U}{\partial x}(s,x)\Big)\right) ds \tag{1.20}$$

The continuity of $s \to \frac{1}{2}\frac{\partial^2 U}{\partial x^2}(s,x) - H_q(t_0+s, \frac{\partial U}{\partial x}(s,x))$ allows to conclude that U satisfies (1.19). If we let $x \to +\infty$ in (1.20), we get the mass conservation : $\forall t,t' > 0,\ \int_{\mathbb{R}} u(t',y)dy = \int_{\mathbb{R}} u(t,y)dy$ and as $u \in C([0,+\infty), L^1(\mathbb{R}))$, we deduce $\forall t > 0,\ \int_{\mathbb{R}} u(t,y)dy = \int_{\mathbb{R}} u_0(y)dy$.

Let $v_0 \in L^1(\mathbb{R})$ be such that $\int_{\mathbb{R}} u_0(x)dx = \int_{\mathbb{R}} v_0(x)dx$ and $\forall x \in \mathbb{R},\ \int_{-\infty}^x u_0(y)dy \le \int_{-\infty}^x v_0(y)dy$. Let v be the solution of $(D^q_{t_0,v_0})$. We set $V(t,x) = \int_{-\infty}^x v(t,y)dy$ and $W = U - V$. To prove the comparison property, we are going to apply theorem 1.7 to W. By (1.19),

$$\forall (t,x) \in (0,+\infty) \times \mathbb{R},\ \frac{\partial W}{\partial t}(t,x) = \frac{1}{2}\frac{\partial^2 W}{\partial x^2}(t,x) - G_q\Big(t, \frac{\partial U}{\partial x}(t,x), \frac{\partial V}{\partial x}(t,x)\Big)\frac{\partial W}{\partial x}(t,x)$$

where $G_q(t,x,y) = \frac{H_q(t_0+t,x) - H_q(t_0+t,y)}{x-y} 1_{\{x \neq y\}}$. By (0.10), G_q is bounded by $\frac{B_0}{(t_0 \wedge 1)^{\frac{q-1}{q}}}$.

As for any $s \ge 0$, the function $x \to W(s,x)$ is continuous and satisfies $\lim_{|x|\to+\infty} W(s,x) = 0$ (for $x \to +\infty$ it is a consequence of the mass conservation), $M(s) = \sup\{W(s,x),\ x \in \mathbb{R}\}$ is finite. Since $s \to u(s) - v(s)$ belongs to $C([0,+\infty), L^1(\mathbb{R}))$ the functions $s \to W(s,x)$ are continuous uniformly in $x \in \mathbb{R}$. Hence $s \to M(s)$ is continuous.

Let $t > 0$ and $M_t = \sup\{M(s),\ s \le t\}$. We are going to prove that $M_t = 0$. There is $s_0 \in [0,t]$ such that $M_t = M(s_0)$.

- if $s_0 = 0$. By the choice of u_0 and v_0, $M(0) = 0$. Hence $M_t = 0$.
- if $s_0 > 0$. We meet two cases.

. $\forall x \in \mathbb{R},\ W(s_0,x) < M(s_0)$. As $\lim_{|x|\to+\infty} W(s_0,x) = 0$, $M_t = M(s_0) = 0$

. $\exists x_0 \in \mathbb{R},\ M(s_0) = W(s_0,x_0)$. Then we apply Theorem 1.7 with $u = W$, $M = M_t$, $a = \frac{1}{2}$, and $b(s,x) = -G_q(s, \frac{\partial U}{\partial x}(s,x), \frac{\partial V}{\partial x}(s,x))$. If $s_0 \in (0,t)$, then for $E = (0,t) \times \mathbb{R}$, Theorem 1.7 (i) implies $\forall x \in \mathbb{R},\ W(s_0,x) = M(s_0) = M_t$. When we take the limit $x \to +\infty$, we conclude $M_t = 0$. If $s_0 = t$, then for $E_t = (0,t] \times \mathbb{R}$, Theorem 1.7 (ii) implies that $W(\frac{t}{2}, x_0) = M_t$ and we conclude like previously. ∎

2 The nonlinear martingale problem

Definition 2.1 *We say that $P \in \tilde{P}(\Omega)$ solves the nonlinear martingale problem (M_q) if $P_0 = \delta_0$ and for any $\phi \in C_b^2(\mathbb{R})$,*

$$\phi(X_t) - \phi(0) - \int_0^t \left(\frac{1}{2} \frac{d^2\phi}{dx^2}(X_s) + F_q(s, p(s, X_s)) \frac{d\phi}{dx}(X_s) \right) ds \quad \text{is a } P\text{-martingale} \tag{2.1}$$

where $p(s,x)$ is measurable version of the densities for P.

This definition does not depend on the choice of the measurable version. Indeed, if $p'(s,x)$ is another such version then

$$\forall t \geq 0, \ \int_0^t F_q(s, p(s, X_s)) \frac{d\phi}{dx}(X_s) ds = \int_0^t F_q(s, p'(s, X_s)) \frac{d\phi}{dx}(X_s) ds, \ P \text{ almost surely}$$

Theorem 2.2 *For any $q \geq 2$, the nonlinear martingale problem (M_q) admits a unique solution and $v_q(s,x)$ is a measurable version of the densities for this solution.*

Proof : In the proof for existence like in the proof for uniqueness, we are confronted to the lack of control of $F_q(s,x)$ when $s \to 0$. That is why we use time-shifts on the sample-paths.

Uniqueness

Let P and P' be two solutions. We first prove that $v_q(t,x)$ is a measurable version of the densities for P and P'. The map $t \to P_t$ belongs to $\tilde{C}_0([0, +\infty), \mathcal{P}(\mathbb{R}))$. By Paul Lévy's characterization, $X_t - \int_0^t F_q(s, p(s, X_s)) ds$ is a Brownian motion under P. Taking expectations in Itô's formula, we obtain that $t \to P_t$ is a weak solution of (E_q) (see equation (1.1)). Theorem 1.2 then implies that v_q is a measurable version of the densities for P. The same is true for P'.

We introduce the shift $y \in \Omega \to D_n(y) = y(\frac{1}{n} + .) \in \Omega$. Let $P^n = P \circ D_n^{-1}$, $P'^n = P' \circ D_n^{-1}$. Both P^n and P'^n solve the martingale problem :

$$\left\{ \begin{array}{l} Q_0 = v_q(\frac{1}{n}, x) dx \ \text{ and } \ \phi(X_t) - \phi(X_0) - \int_0^t \left(\frac{1}{2} \frac{d^2\phi}{dx^2}(X_s) + F_q(\frac{1}{n} + s, v_q(\frac{1}{n} + s, X_s)) \frac{d\phi}{dx}(X_s) \right) ds \\ \text{is a } Q\text{-martingale for any } \ \phi \in C_b^2(\mathbb{R}) \end{array} \right. \tag{2.2}$$

Since $x \to F_q(\frac{1}{n} + s, v_q(\frac{1}{n} + s, x))$ is bounded uniformly in s (see (0.7)), by Girsanov's theorem, this martingale problem admits a unique solution and $P^n = P'^n$.
As for any $y \in \Omega$, $\lim_{n \to +\infty} D_n(y) = y$, P^n and P'^n converge weakly to P and P'. Therefore

$$P = P'$$

Existence

The natural idea would consist in constructing a solution to the martingale problem : $Q_0 = \delta_0$

$$\forall \phi \in C_b^2(\mathbb{R}), \ \phi(X_t) - \phi(0) - \int_0^t \left(\frac{1}{2} \frac{d^2\phi}{dx^2}(X_s) + F_q(s, v_q(s, X_s)) \frac{d\phi}{dx}(X_s) \right) ds \quad \text{is a } Q\text{-martingale}$$

and proving that this solution belongs to $\tilde{\mathcal{P}}(\Omega)$ and admits v_q as a measurable version for its densities. But the drift coefficient $F_q(s, v_q(s,.))$ is not bounded and to our knowledge, there is no classical existence result for such a martingale problem. That is why we introduce P^n the solution of the martingale problem (2.2). We first prove that $v_q(\frac{1}{n}+t, x)$ is a measurable version of the densities for P^n.
By Girsanov's theorem, since the drift coefficient $F_q(\frac{1}{n}+s, v_q(\frac{1}{n}+s, X_s))$ is bounded, $P^n \in \tilde{\mathcal{P}}(\Omega)$. Let $p^n(t,x)$ be a measurable version of the densities for P^n, $t>0$ and $\phi \in C_b^{1,2}([0,t]\times\mathbb{R})$. Taking expectations in Itô's formula, we obtain

$$\int_{\mathbb{R}} \phi(t,x)p^n(t,x)dx = \int_{\mathbb{R}} \phi(0,x)v_q(\frac{1}{n},x)dx$$
$$+\int_{(0,t]\times\mathbb{R}} \left(\frac{\partial \phi}{\partial s}(s,x) + \frac{1}{2}\frac{\partial^2 \phi}{\partial x^2}(s,x) + F_q(\frac{1}{n}+s, v_q(\frac{1}{n}+s,x))\frac{\partial \phi}{\partial x}(s,x)\right) p^n(s,x)dsdx$$

Like in the proof of the evolution equation (1.2), we deduce

$$\forall t>0,\ p^n(t,x) = G_t * v_q(\frac{1}{n},.)(x) - \int_0^t \frac{\partial G_{t-s}}{\partial x} * (p^n(s,.)F_q(\frac{1}{n}+s, v_q(\frac{1}{n}+s,.)))(x)ds \ \text{ a.e.}$$

For $\mu = V_q$ and $t_0 = \frac{1}{n}$, equation (1.2) provides

$$\forall t>0,\ v_q(\frac{1}{n}+t,x) = G_t * v_q(\frac{1}{n},.)(x) - \int_0^t \frac{\partial G_{t-s}}{\partial x} * (v_q(\frac{1}{n}+s,.)F_q(\frac{1}{n}+s, v_q(\frac{1}{n}+s,.)))(x)ds \ \text{ a.e.}$$

Using (0.7) and (0.5), we obtain

$$\|p^n(t,.) - v_q(\frac{1}{n}+t,.)\|_{L^1(\mathbb{R})} \le B_0 n^{\frac{q-1}{q}} \int_0^t \frac{\|p^n(s,.) - v_q(\frac{1}{n}+s,.)\|_{L^1}}{\sqrt{t-s}} ds$$

After an iteration, we get

$$\begin{aligned}\|p^n(t,.) - v_q(\frac{1}{n}+t,.)\|_{L^1} &\le B_0^2 n^{\frac{2q-2}{q}} \int_0^t \frac{1}{\sqrt{t-s}} \int_0^s \frac{\|p^n(r,.) - v_q(\frac{1}{n}+r,.)\|_{L^1}}{\sqrt{s-r}} drds \\ &\le \pi B_0^2 n^{\frac{2q-2}{q}} \int_0^t \|p^n(r,.) - v_q(\frac{1}{n}+r,.)\|_{L^1} dr\end{aligned}$$

Gronwall's lemma implies $\forall t>0,\ \|p^n(t,.) - v_q(\frac{1}{n}+t,.)\|_{L^1} = 0$.
Hence $v_q(\frac{1}{n}+s,x)$ is a measurable version of the densities for P^n.

Let Q^n denote the image of P^n by the shift $y \in \Omega \to y((.-\frac{1}{n})\vee 0) \in \Omega$. We now prove that the sequence $(Q^n)_n$ converges weakly to the solution of (M_q). Since $Q_0^n = V_q(\frac{1}{n})$ converges weakly to δ_0 and the map $s \to \|F_q(s,.)\|_{L^\infty}$ is integrable, for any $T>0$, the images of the probability measures Q^n by the canonical restriction from Ω to Ω^T are tight. Therefore the sequence $(Q^n)_n$ is tight. Let Q^∞ be the limit of a convergent subsequence that we still index by n for convenience. Let $p \in \mathbb{N}^*$, $\phi \in C_b^2(\mathbb{R})$, $g \in C_b(\mathbb{R}^p)$, $0 < s_1 \le \ldots \le s_p \le s \le t$ and $G : \Omega \to \mathbb{R}$,

$$G(y) = \left(\phi(y(t)) - \phi(y(s)) - \int_s^t \frac{1}{2}\frac{d^2\phi}{dx^2}(y(r)) + F_q(r, v_q(r, y(r)))\frac{d\phi}{dx}(y(r))dr\right) g(y(s_1),\ldots,y(s_p))$$

Since the functions $x \to F_q(s, v_q(s,x))$ are continuous and bounded uniformly in $s \ge s_1$, the function G is continuous and bounded. Hence

$$\mathbb{E}^{Q^\infty}(G(X)) = \lim_{n\to+\infty} \mathbb{E}^{Q^n}(G(X))$$

Clearly, for any $n \geq \frac{1}{s_1}$, $\mathbb{E}^{Q^n}(G(X)) = 0$. Hence $\mathbb{E}^{Q^\infty}(G(X)) = 0$. By Lebesgue's theorem, as $s \to \|F_q(s,.)\|_{L^\infty}$ is integrable, this equality still holds when we take the limits $s_p \to 0$ and $s \to 0$. Therefore

$$\phi(X_t) - \phi(X_0) - \int_0^t \left(\frac{1}{2}\frac{d^2\phi}{dx^2}(X_r) + F_q(r, v_q(r, X_r))\frac{d\phi}{dx}(X_r) \right) dr \quad \text{is a } P^\infty\text{-martingale} \tag{2.3}$$

If $t > 0$, for any $n \geq \frac{1}{t}$, $v_q(t,.)$ is a density of $Q^n_t = P^n_{t-\frac{1}{n}}$ with respect to Lebesgue measure. Hence Q^∞_t is absolutely continuous with density $v_q(t,.)$. Since $Q^n_0 = V^q(\frac{1}{n})$ converges weakly to δ_0, $Q^\infty_0 = \delta_0$. These two properties and (2.3) imply that Q^∞ solves (M_q). Hence we have proved existence for this problem. Moreover, by uniqueness, the whole sequence $(Q^n)_n$ converges weakly to the solution of (M_q). ∎

3 The propagation of chaos result

3.1 The particle systems

We recall the definition of the moderately interacting particle systems

$$X^{i,n}_t = B^i_t + \int_0^t F_q(s, V^n * \mu^n_s(X^{i,n}_s))ds, \ t \geq 0, \ 1 \leq i \leq n$$

where $B^i, i \in \mathbb{N}^*$ are independant Brownian motions, $\mu^n_s = \frac{1}{n}\sum_{j=1}^n \delta_{X^{j,n}_s}$ and $V^n(x) = n^\beta V^1(n^\beta x)$.

Proposition 3.1 *For any $n \in \mathbb{N}^*$, there is existence and pathwise uniqueness for the particle system $(X^{1,n}, X^{2,n}, \dots, X^{n,n})$.*

Proof : In this proof, n is constant. For $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we set $|y| = \max_{i=1}^n |y_i|$. Since V^1 is Lipschitz, $V^n = n^\beta V^1(n^\beta .)$ is also Lipschitz. Let C denote its Lipschitz constant.
We set $X_t = (X^{1,n}_t, \dots, X^{n,n}_t)$, $B_t = (B^1_t, \dots, B^n_t)$ and

$$G(s,y) = \begin{pmatrix} F_q(s, \frac{1}{n}\sum_{j=1}^n V^n(y_1 - y_j)) \\ \dots \\ F_q(s, \frac{1}{n}\sum_{j=1}^n V^n(y_n - y_j)) \end{pmatrix}$$

We are interested in the stochastic differential equation

$$X_t = B_t + \int_0^t G(s, X_s)ds \tag{3.1}$$

The map G does not satisfy the classical linear growth and local Lipschitz assumptions. Therefore, to prove our claim, we construct functions indexed by $m \in \mathbb{N}^*$ which satisfy these assumptions and are equal to G on $(0, +\infty) \times [-\frac{m}{2C}, \frac{m}{2C}]^n$. We set $F^m_q(s,x) = F_q(s, -m \vee x \wedge m)$ and define G^m like G with F_q^m replacing F_q. We have $G^m(s,y) = G(s,y)$ if $|y| \leq \frac{m}{2C}$. Moreover the functions $y \in \mathbb{R}^n \to G^m(s,y)$ are bounded and Lipschitz uniformly in s. Indeed by (0.6),

$$\left(t \leq \left(\frac{k_q}{m}\right)^q\right) \Rightarrow \left(m \leq \frac{k_q}{(t \wedge 1)^{\frac{1}{q}}}\right) \Rightarrow \left(\text{if } |x| \leq m,\ F_q^m(t,x) = F_q(t,x) = \frac{|x|^{q-1}}{q}\right)$$

With (0.7) et (0.9), we obtain that $x \to F_q^m(s,x)$ is bounded by $\frac{m^{q-1}}{q} \vee \frac{B_0 m^{q-1}}{k_q^{q-1}} \vee B_0$ and Lipschitz with constant $\frac{(q-1)m^{q-2}}{q} \vee \frac{3B_1 m^{q-2}}{2k_q^{q-2}} \vee \frac{3B_1}{2}$ uniformly in s. Since

$$|\frac{1}{n}\sum_{j=1}^n V^n(z_i - z_j) - \frac{1}{n}\sum_{j=1}^n V^n(y_i - y_j)| \leq \frac{C}{n}\sum_{j=1}^n (|z_i - y_i| + |z_j - y_j|) \leq 2C|y - z|$$

we deduce that $y \to G^m(s,y)$ is bounded by $\frac{m^{q-1}}{q} \vee \frac{B_0 m^{q-1}}{k_q^{q-1}} \vee B_0$ and Lipschitz with constant $2C\left(\frac{(q-1)m^{q-2}}{q} \vee \frac{3B_1 m^{q-2}}{2k_q^{q-2}} \vee \frac{3B_1}{2}\right)$ uniformly in s.

Hence, there is existence and pathwise uniqueness for the stochastic differential equation

$$X_t^m = B_t + \int_0^t G^m(s, X_s^m)ds$$

We set $T^m = \inf\{t : |X_t^m| \geq \frac{m}{2C}\}$ and for $m \leq l$, $T^{m,l} = \inf\{t : \max(|X_t^m|, |X_t^l|) \geq \frac{m}{2C}\}$. By pathwise uniqueness for the equation indexed by m, X^m and X^l coincide on $[0, T^{m,l}]$. We deduce $T^{m,l} = T^m$. Hence X^m and X^l coincide on $[0, T^m]$. Therefore the sequence (T^m) is increasing.

$$\sup_{s\leq t}|X_s^m| \leq \sup_{s\leq t}|B_s| + \sup_{s\leq t}|\int_0^s G^m(r, X_r^m)dr|$$

As $s \to \|F_q(s,.)\|_{L^\infty}$ is integrable, we get $\mathbb{E}(\sup_{s\leq t}|X_s^m|) \leq A(t)$ where $A(t)$ does not depend on m. Using Markov's inequality, we deduce $P(\{\sup_{s\leq t}|X_s^m| \geq \frac{m}{2C}\}) \leq \frac{2CA(t)}{m}$. Hence

$$\forall t \in (0, +\infty),\ P(\{\lim_m T_m \leq t\}) = 0 \quad \text{and} \quad a.s.,\ \lim_m T_m = +\infty$$

We set $X_t = X_t^m$ on $[T_{m-1}, T_m]$ with $T_0 = 0$. Then X solves equation (3.1).

For uniqueness, if Y is a solution of (3.1) and $S^m = \inf\{t : \max(|X_t^m|, |Y_t|) \geq \frac{m}{2C}\}$, Y and X^m coincide on $[0, S^m]$ and therefore on $[0, T^m]$. ∎

3.2 Propagation of chaos

Theorem 3.2 *For any $q \geq 2$, the sequence of the laws of the particle systems $(X^{1,n}, \ldots, X^{n,n})$ is P^q-chaotic where P^q denotes the unique solution of the martingale problem (M_q).*

The particles are exchangeable. Therefore the propagation of chaos result is equivalent to the convergence in distribution of the empirical measures $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{X^{i,n}}$ considered as $\mathcal{P}(\Omega)$-valued random variables to δ_{P^q} (see for example [10] and the references cited in it). To prove this convergence, we adapt the approach of Méléard and Roelly in [4]. We begin with a tightness result. Then we check that the limit of any convergent subsequence is δ_{P^q}. In both steps we need the following fundamental technical result adapted from Oelschläger [6] (Proposition 3.2 p.290).

Lemma 3.3 *Let U^1 be a probability density in $H^a(\mathbb{R})$ for $a > 0$. We set $U^n(x) = n^b U^1(n^b x)$ for some $b \in (0,1)$. Then*

$$\forall c \in [0, a \wedge \frac{1-b}{2}],\ \forall 0 < \epsilon < T,\ \exists C,\ \forall s \in [\epsilon, T],\ \sup_n \mathbb{E}\Big(\int_{\mathbb{R}^d} (1 + |\lambda|^{2c}) |\mathcal{F}(U^n * \mu^n_s)(\lambda)|^2 d\lambda \Big) \leq C$$

Remark

Oelschläger proves the claim of Lemma 3.3 for the moderately interacting particle systems (0.2) mentionned in the introduction and for the particular choice $U^1 = W^1$, $b = \beta$. Since our particle systems satisfy

$$X^{i,n}_{\frac{\epsilon}{2}+t} = X^{i,n}_{\frac{\epsilon}{2}} + (B^i_{\frac{\epsilon}{2}+t} - B^i_{\frac{\epsilon}{2}}) + \int_0^t F_q(\frac{\epsilon}{2} + s, V^n * \mu^n_{\frac{\epsilon}{2}+s}(X^{i,n}_{\frac{\epsilon}{2}+s}))ds,\ 1 \leq i \leq n$$

and $F_q(\frac{\epsilon}{2} + s, x)$ is bounded, it is quite easy to adapt the proof to our framework.

3.2.1 The tightness result

Let π_n denote the law of the $\mathcal{P}(\Omega)$-valued variable μ^n. Since we have to control $V^n * \mu^n$, it is not enough to prove the tightness of the sequence $(\pi_n)_n$. That is why we introduce the space

$$\mathcal{H} = \mathcal{P}(\Omega) \times L^2_{loc}((0, +\infty), L^2(\mathbb{R}))$$

endowed with the topology of weak convergence on $\mathcal{P}(\Omega)$ and the metric

$$d(v, v') = \sum_{p \geq 1} 2^{-p} \left(\left(\int_{\frac{1}{p}}^{p} \|v_s - v'_s\|^2_{L^2} ds \right)^{\frac{1}{2}} \wedge 1 \right)$$

on $L^2_{loc}((0, +\infty), L^2(\mathbb{R}))$. The space $L^2_{loc}((0, +\infty), L^2(\mathbb{R}))$ is complete and separable for this metric. Let m and v denote the canonical projections from $\mathcal{H}$ to $\mathcal{P}(\Omega)$ and $L^2_{loc}((0, +\infty), L^2(\mathbb{R}))$ and $\tilde{\pi}_n$ be the law of the $\mathcal{H}$-valued random variable $(\mu^n, V^n * \mu^n)$.

Proposition 3.4 *The sequences $(\pi_n)_n$ and $(\tilde{\pi}_n)_n$ are tight.*

Proof : The tightness of the sequence $(\pi_n)_n$ is equivalent to the tightness of the laws of the variables $X^{1,n}$ (see [10]). These variables are tight since for any $T > 0$ their images by the canonical restriction from Ω to Ω^T are tight ($s \to \|F_q(s, .)\|_{L^\infty}$ is integrable).

To prove the tightness of the sequence $(\tilde{\pi}_n)_n$, it is enough to prove the tightness of the sequences

$(\tilde\pi_n \circ m^{-1})_n$ and $(\tilde\pi_n \circ v^{-1})_n$. We have just showed the tightness of the first sequence. Let us deal with the second.
From any subsequence of $(\tilde\pi_n \circ m^{-1})_n$ we extract a converging subsequence that we still index by n for simplicity. As $\mathcal{P}(\Omega)$ is a polish space, we obtain by Skorokhod's lemma an almost surely convergent sequence $(\nu^n)_n$ of $\mathcal{P}(\Omega)$-valued random variables defined on a probability space $(\tilde\Omega, \tilde P)$ such that for any n, the law of ν^n is $\tilde\pi_n \circ m^{-1} = \pi_n$. We are going to prove that $V^n * \nu^n$ converges in $L^1(\tilde\Omega, L^2_{loc}((0,+\infty), L^2(\mathbb{R})))$, which ensures that the sequence $(\tilde\pi_n \circ v^{-1})_n$ is weakly convergent.

$$\mathbb{E}(d(V^k * \nu^k, V^l * \nu^l)) \leq \sum_{p\geq 1} 2^{-p} \left(\left(\mathbb{E}\left(\int_{\frac{1}{p}}^{p} \|V^k * \nu^k_s - V^l * \nu^l_s\|^2_{L^2} ds \right) \right)^{\frac{1}{2}} \wedge 1 \right)$$

If we prove that $\forall p \geq 1,\ \lim_{k,l\to+\infty} \mathbb{E}\left(\int_{\frac{1}{p}}^{p} \|V^k * \nu^k_s - V^l * \nu^l_s\|^2_{L^2} ds \right) = 0$, it is easy to conclude by Lebesgue's theorem that $(V^n * \nu^n)_n$ is a Cauchy sequence. Using the Fourier isomorphism, we get

$$\begin{aligned} \mathbb{E}\Big(\int_{\frac{1}{p}}^{p} \|V^k * \nu^k_s - V^l * \nu^l_s\|^2_{L^2} ds \Big) &= \mathbb{E}\Big(\int_{\frac{1}{p}}^{p} \int_{|\lambda|\leq M} |\mathcal{F}(V^k * \nu^k_s)(\lambda) - \mathcal{F}(V^l * \nu^l_s)(\lambda)|^2 d\lambda ds \Big) \\ &\quad + \mathbb{E}\Big(\int_{\frac{1}{p}}^{p} \int_{|\lambda|> M} |\mathcal{F}(V^k * \nu^k_s)(\lambda) - \mathcal{F}(V^l * \nu^l_s)(\lambda)|^2 d\lambda ds \Big) \end{aligned} \tag{3.2}$$

$$|\mathcal{F}(V^k * \nu^k_s)(\lambda) - \mathcal{F}(V^l * \nu^l_s)(\lambda)|^2 \leq 2\Big(|\mathcal{F}(V^k)(\lambda) - \mathcal{F}(V^l)(\lambda)|^2 + \frac{|< \nu^k_s, e^{i\lambda .} > - < \nu^l_s, e^{i\lambda .} >|^2}{2\pi} \Big)$$

Therefore the first term of the right hand side of (3.2) is bounded by

$$2p \int_{|\lambda|\leq M} |\mathcal{F}(V^k)(\lambda) - \mathcal{F}(V^l)(\lambda)|^2 d\lambda + \frac{1}{\pi} \mathbb{E}\Big(\int_{\frac{1}{p}}^{p} \int_{|\lambda|\leq M} |< \nu^k_s, e^{i\lambda .} > - < \nu^l_s, e^{i\lambda .} >|^2 d\lambda ds \Big)$$

Since the probability measures $V^n(x)dx$ converge weakly to δ_0 and the sequence $(\nu^n)_n$ is almost surely weakly convergent, applying Lévy's theorem and Lebesgue's theorem, we obtain that for any $M \geq 0$ the first term of the right hand side of (3.2) goes to 0 when $k, l \to +\infty$.

The second term of the right hand side of (3.2) is bounded by

$$4 \sup_n \mathbb{E}\left(\int_{\frac{1}{p}}^{p} \int_{|\lambda|> M} |\mathcal{F}(V^n * \mu^n_s)(\lambda)|^2 d\lambda ds \right)$$

Applying Lemma 3.3 with $\epsilon = \frac{1}{p}$, $T = p$, $U^1 = V^1$, $a = r$, $b = \beta$ and $c = r \wedge \frac{1-\beta}{2}$ we obtain

$$\begin{aligned} \forall n,\ \mathbb{E}\left(\int_{\frac{1}{p}}^{p} \int_{|\lambda|> M} |\mathcal{F}(V^n * \mu^n_s)(\lambda)|^2 d\lambda ds \right) &\leq \mathbb{E}\left(\int_{\frac{1}{p}}^{p} \int_{|\lambda|> M} \frac{1+|\lambda|^{2c}}{1+M^{2c}} |\mathcal{F}(V^n * \mu^n_s)(\lambda)|^2 d\lambda ds \right) \\ &\leq \frac{Cp}{1+M^{2c}} \end{aligned}$$

We conclude $\lim_{k,l\to+\infty} \mathbb{E}\left(\int_{\frac{1}{p}}^{p} \|V^k * \nu^k_s - V^l * \nu^l_s\|^2_{L^2} ds \right) = 0$. ■

3.2.2 Identification of the limit

The sequence $(\pi_n)_n$ is tight. Let π_∞ be the limit of a converging subsequence $(\pi_{n_k})_k$. As the sequence $(\tilde{\pi}_n)_n$ is also tight, we can extract from $(\tilde{\pi}_{n_k})_k$ a subsequence which converges weakly to $\tilde{\pi}_\infty$ and that we index by n for simplicity. We are going to prove that $\tilde{\pi}_\infty$ a.s., m solves the nonlinear martingale problem (M_q). Since $\tilde{\pi}_\infty \circ m^{-1} = \pi_\infty$, we will conclude $\pi_\infty = \delta_{P^q}$.
We begin with a technical result which explicits the connection between m and v under $\tilde{\pi}_\infty$.

Lemma 3.5 *There is a Borel set $\mathcal{N}$ such that $\tilde{\pi}_\infty(\mathcal{N}) = 0$ and $\forall (m,v) \in \mathcal{N}^c$, for a.e. $t \geq 0$, m_t has a density equal to v_t with respect to Lebesgue measure.*

Proof of Lemma 3.5 : Let $p \in \mathbb{N}^*$, $(g_k)_{k\in\mathbb{N}}$ be a sequence dense in $L^2([\frac{1}{p}, p])$ and $(f_l)_{l\in\mathbb{N}}$ a sequence of C^1 functions with compact support on $\mathbb{R}$ that will be precised later. We set

$$G_{k,l}(m,v) = \int_{\frac{1}{p}}^{p} \int_{\mathbb{R}} g_k(t) f_l(x) v_t(x) dx dt - \int_{\frac{1}{p}}^{p} \int_{\mathbb{R}} g_k(t) f_l(x) m_t(dx) dt$$

As $G_{k,l}$ is continuous, $\mathbb{E}^{\tilde{\pi}_\infty}(G_{k,l}^2) \leq \liminf_{n\to+\infty} \mathbb{E}^{\tilde{\pi}_n}(G_{k,l}^2)$. Let $\bar{V}^n(x) = V^n(-x)$.

$$\begin{aligned}
\mathbb{E}^{\tilde{\pi}_n}(G_{k,l}^2) &= \mathbb{E}\Big(\Big(\int_{\frac{1}{p}}^{p} g_k(t) \int_{\mathbb{R}} (\bar{V}^n * f_l(x) - f_l(x)) \mu_t^n(dx) dt \Big)^2 \Big) \\
&\leq p \|g_k\|_{L^2}^2 \sup_{x\in\mathbb{R}} (\bar{V}^n * f_l(x) - f_l(x))^2
\end{aligned}$$

$$|\bar{V}^n * f_l(x) - f_l(x)| \leq \int_{\mathbb{R}} |f_l(x + \frac{y}{n^\beta}) - f_l(x)| V^1(y) dy \leq \frac{1}{n^\beta} \|\frac{df_l}{dx}\|_{L^\infty} \int_{\mathbb{R}} |y| V^1(y) dy$$

Hence $\lim_{n\to+\infty} \mathbb{E}^{\tilde{\pi}_n}(G_{k,l}^2) = 0$ and $\mathbb{E}^{\tilde{\pi}_\infty}(G_{k,l}^2) = 0$. We set $\mathcal{N}_p = \bigcup_{k,l\in\mathbb{N}} G_{k,l}^{-1}(\mathbb{R}^*)$. We have $\tilde{\pi}_\infty(\mathcal{N}_p) = 0$ and since $(g_k)_k$ is dense in $L^2([\frac{1}{p}, p])$,

$$\forall (m,v) \in \mathcal{N}_p^c, \text{ for a.e. } t \in [\frac{1}{p}, p], \forall l \in \mathbb{N}, \int_{\mathbb{R}} f_l(x) m_t(dx) = \int_{\mathbb{R}} f_l(x) v_t(x) dx$$

Let ϕ be a C^1 function on $\mathbb{R}$ with values in $[0,1]$ such that for $|x| \leq 1$, $\phi(x) = 1$ and for $|x| \geq 2$, $\phi(x) = 0$. We set $\phi_j(x) = \phi(\frac{x}{j})$ for $j \in \mathbb{N}^*$ and we impose that (f_l) includes all the functions $x \to \phi_j(x)P(x)$ where $j \in \mathbb{N}^*$ and P is a polynomial with rational coefficients. Then this sequence is dense in $C_K(\mathbb{R})$ (the space of continuous functions with compact support) for the sum of the L^2 norm and the sup norm. Hence if $\forall l \in \mathbb{N}$, $\int_{\mathbb{R}} f_l(x) m_t(dx) = \int_{\mathbb{R}} f_l(x) v_t(x) dx$,

$$\forall f \in C_K(\mathbb{R}), \int_{\mathbb{R}} f(x) m_t(dx) = \int_{\mathbb{R}} f(x) v_t(x) dx \tag{3.3}$$

Approximating $-v_t 1_{\{v_t \leq 0\}}$ in $L^2(\mathbb{R})$ by positive functions belonging to $C_K(\mathbb{R})$, we obtain that $v_t \geq 0$. Thus $v_t(x)dx$ is a Radon measure. By (3.3), the Radon measures m_t and $v_t(x)dx$ are equal and m_t has a density equal to v_t.
To conclude, we set $\mathcal{N} = \bigcup_{p\in\mathbb{N}^*} \mathcal{N}_p$. ∎

Let $p \in \mathbb{N}^*$, $\phi \in C_b^2(\mathbb{R})$, $g \in C_b(\mathbb{R}^p)$, $0 < s_1 \leq \ldots \leq s_p \leq s \leq t$. For $\mathcal{N}$ given by Lemma 3.5, we define $G : \mathcal{H} \to \mathbb{R}$ by

$$G = 1_{\mathcal{N}^c} < m, \left(\phi(X_t) - \phi(X_s) - \int_s^t \frac{1}{2} \frac{d^2\phi}{dx^2}(X_r) + F_q(r, v(r, X_r)) \frac{d\phi}{dx}(X_r) dr \right) g(X_{s_1}, \ldots, X_{s_p}) >$$

where $v(r,x)$ is a measurable representative of v. We are going to prove that $\mathbb{E}^{\tilde{\pi}_\infty}(G^2) = 0$. We introduce $(\psi_k)_k$ a sequence of C^∞ probability densities with compact support on $\mathbb{R}$ which converges to δ_0 and we set

$$G_k =< m, \left(\phi(X_t) - \phi(X_s) - \int_s^t \frac{1}{2} \frac{d^2\phi}{dx^2}(X_r) + F_q(r, \psi_k * v_r(X_r)) \frac{d\phi}{dx}(X_r) dr \right) g(X_{s_1}, \ldots, X_{s_p}) >$$

The functions G_k are continuous and bounded on $\mathcal{H}$. Hence

$$\mathbb{E}^{\tilde{\pi}_\infty}(G^2) \leq 2 \liminf_{k \to +\infty} \mathbb{E}^{\tilde{\pi}_\infty}((G - G_k)^2) + 2 \liminf_{k \to +\infty} \lim_{n \to +\infty} \mathbb{E}(G_k^2(\mu^n, V^n * \mu^n)) \tag{3.4}$$

Let us show that both terms of the right hand side of (3.4) are equal to 0.

By the boundedness of G_k (uniform in k), the Lipschitz properties of F_q (see (0.9)), Lemma 3.5 and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\mathbb{E}^{\tilde{\pi}_\infty}((G - G_k)^2) &\leq C\mathbb{E}^{\tilde{\pi}_\infty}(|G - G_k|) \\
&\leq C\mathbb{E}^{\tilde{\pi}_\infty}\left(1_{\mathcal{N}^c} < m, \int_s^t |\psi_k * v_r(X_r) - v(r, X_r)| dr >\right) \\
&\leq C\mathbb{E}^{\tilde{\pi}_\infty}\left(1_{\mathcal{N}^c} \int_s^t \int_{\mathbb{R}} |\psi_k * v_r(x) - v(r,x)| v(r,x) dx dr\right) \\
&\leq C \left(\mathbb{E}^{\tilde{\pi}_\infty}\left(\int_s^t \|v_r\|_{L^2}^2 dr\right)\right)^{\frac{1}{2}} \left(\mathbb{E}^{\tilde{\pi}_\infty}\left(\int_s^t \|v_r - \psi_k * v_r\|_{L^2}^2 dr\right)\right)^{\frac{1}{2}}
\end{aligned} \tag{3.5}$$

By the Fourier isomorphism, $\mathbb{E}^{\tilde{\pi}_n}\left(\int_s^t \|v_r\|_{L^2}^2 dr\right) = \mathbb{E}\left(\int_s^t \|\mathcal{F}(V^n * \mu_r^n)\|_{L^2}^2 dr\right)$. Applying Lemma 3.3 with $U^1 = V^1$, $c = 0$ and using the continuity of $(m, v) \in \mathcal{H} \to \int_s^t \|v_r\|_{L^2}^2 dr$, we conclude that $\mathbb{E}^{\tilde{\pi}_\infty}\left(\int_s^t \|v_r\|_{L^2(\mathbb{R})}^2 dr\right) < +\infty$.
As for any $f \in L^2(\mathbb{R})$, $\lim_{k \to +\infty} \|\psi_k * f - f\|_{L^2} = 0$ and $\|v_r - \psi_k * v_r\|_{L^2} \leq 2\|v_r\|_{L^2}$, taking the limit $k \to +\infty$ in (3.5), we obtain

$$\lim_{k \to +\infty} \mathbb{E}^{\tilde{\pi}_\infty}((G - G_k)^2) = 0 \tag{3.6}$$

To prove that the second term of the right hand side of (3.4) is equal to 0, we upper-bound $G_k^2(\mu^n, V^n * \mu^n)$ by

$$\begin{aligned}
&2 < \mu^n, \left(\phi(X_t) - \phi(X_s) - \int_s^t \frac{1}{2} \frac{d^2\phi}{dx^2}(X_r) + F_q(r, V^n * \mu_r^n(X_r)) \frac{d\phi}{dx}(X_r) dr \right) g(X_{s_1}, \ldots, X_{s_p}) >^2 \\
&\quad +2 < \mu^n, g(X_{s_1}, \ldots, X_{s_p}) \int_s^t (F_q(r, \psi_k * V^n * \mu_r^n(X_r)) - F_q(r, V^n * \mu_r^n(X_r))) \frac{d\phi}{dx}(X_r) dr >^2
\end{aligned} \tag{3.7}$$

Let $\bar{W}^n(x) = W^n(-x)$ and $A_{k,n}$ denote the expectation of the second term of (3.7). By a computation similar to (3.5), we obtain

$$
\begin{aligned}
A_{k,n} &\leq C\mathbb{E}\left(\int_s^t < \mu_r^n, |W^n * (W^n * \psi_k * \mu_r^n - W^n * \mu_r^n)| > dr\right) \\
&\leq C\mathbb{E}\left(\int_s^t \bar{W}^n * \mu_r^n(y)|W^n * \psi_k * \mu_r^n(y) - W^n * \mu_r^n(y)|dydr\right) \\
&\leq C\left(\mathbb{E}\left(\int_s^t \|\bar{W}^n * \mu_r^n\|_{L^2}^2 dr\right)\right)^{\frac{1}{2}}\left(\mathbb{E}\left(\int_s^t \|W^n * \psi_k * \mu_r^n - W^n * \mu_r^n\|_{L^2}^2 dr\right)\right)^{\frac{1}{2}}
\end{aligned}
$$

Applying Lemma 3.3 with $U^1 = \bar{W}^1$ and $c = 0$, we deduce

$$
A_{k,n} \leq C\left(\mathbb{E}\left(\int_s^t \|W^n * \psi_k * \mu_r^n - W^n * \mu_r^n\|_{L^2}^2 dr\right)\right)^{\frac{1}{2}}
$$

Using the Fourier isomorphism then Lemma 3.3 with $U^1 = W^1$ and $c = r \wedge \frac{1-\beta}{2}$, we obtain

$$
\begin{aligned}
A_{k,n}^2 &\leq C\mathbb{E}\left(\int_s^t \int_{|\lambda|\leq M} |\sqrt{2\pi}\mathcal{F}(\psi_k)(\lambda) - 1|^2 |\mathcal{F}(W^n * \mu_r^n)(\lambda)|^2 d\lambda dr\right) \\
&\quad + C\mathbb{E}\left(\int_s^t \int_{|\lambda|>M} (|\sqrt{2\pi}\mathcal{F}(\psi_k)(\lambda)| + 1)^2 |\mathcal{F}(W^n * \mu_r^n)(\lambda)|^2 \frac{1+|\lambda|^{2c}}{1+M^{2c}} d\lambda dr\right) \\
&\leq C\left(M \sup_{|\lambda|\leq M} |\sqrt{2\pi}\mathcal{F}(\psi_k)(\lambda) - 1|^2 + \frac{1}{1+M^{2c}}\right)
\end{aligned}
$$

where the constant C depends neither on n nor on k. Since the probability measures $\psi_k(x)dx$ converge weakly to δ_0, applying Lévy's theorem we conclude $\lim_{k\to+\infty} \sup_n A_{k,n} = 0$.
As, by Itô's formula, the first term of (3.7) is equal to $(\frac{1}{n}\sum_{i=1}^n g(X_{s_1}^{i,n}, \ldots, X_{s_p}^{i,n}) \int_s^t \frac{d\phi}{dx}(X_r^{i,n})dB_r^i)^2$, its expectation goes to 0 when $n \to +\infty$. Hence $\liminf_{k\to+\infty} \lim_{n\to+\infty} \mathbb{E}(G_k^2(\mu^n, V^n * \mu^n)) = 0$. With (3.4) and (3.6), this result implies $\mathbb{E}^{\tilde{\pi}_\infty}(G^2) = 0$.
Restricting $\phi, g, s_1, \ldots, s_p, s, t$ to countable subsets then taking limits by Lebesgue theorem, we get that $\tilde{\pi}_\infty$ a.s., $\forall p \in \mathbb{N}^*$, $\forall \phi \in C_b^2(\mathbb{R})$, $\forall g \in C_b(\mathbb{R}^p)$, $\forall 0 \leq s_1 \leq \ldots \leq s_p \leq s \leq t$,

$$
1_{\mathcal{N}^c} < m, \left(\phi(X_t) - \phi(X_s) - \int_s^t \frac{1}{2}\frac{d^2\phi}{dx^2}(X_r) + F_q(r, v(r, X_r))\frac{d\phi}{dx}(X_r)dr\right) g(X_{s_1}, \ldots, X_{s_p}) >= 0
$$

As $\forall n \in \mathbb{N}^*$, $\tilde{\pi}_n \circ m_0^{-1} = \delta_{\delta_0}$ and the map $(m, v) \in \mathcal{H} \to m_0$ is continuous, $\tilde{\pi}_\infty \circ m_0^{-1} = \delta_{\delta_0}$. Hence there is a Borel set $\tilde{\mathcal{N}}$ with $\mathcal{N} \subset \tilde{\mathcal{N}}$ and $\tilde{\pi}_\infty(\tilde{\mathcal{N}}) = 0$ such that $\forall (m, v) \in \tilde{\mathcal{N}}^c$, $\forall \phi \in C_b^2(\mathbb{R})$, $\phi(X_t) - \phi(0) - \int_0^t \frac{1}{2}\frac{d^2\phi}{dx^2}(X_r) + F_q(r, v(r, X_r))\frac{d\phi}{dx}(X_r)dr$ is a m-martingale.
Let $(m, v) \in \tilde{\mathcal{N}}^c$. The process $X_t - \int_0^t F_q(r, v(r, X_r))dr$ is a m-Brownian motion. By Girsanov's theorem, we obtain that $m \in \tilde{\mathcal{P}}(\Omega)$. If p is a measurable version of the densities for m, since $(m, v) \in \mathcal{N}^c$, by Lemma 3.5, m a.s., $\forall t > 0$, $\int_0^t F_q(r, v(r, X_r))dr = \int_0^t F_q(r, p(r, X_r))dr$. Therefore m solves the nonlinear martingale problem (M_q), which puts an end to the proof.

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