Variation of product function and numerical solution of some partial differential equations by low-discrepancy sequences

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August 1996

Abstract

An inequality on the variation of the product of two functions of bounded variation and some applications in numerical solution of certain class of parabolic partial differential equations by low-discrepancy sequences are given.

Key words: Variation, low-discrepancy sequences.

A.M.S. Classification: 11K05, 11K30, 65C05, 65C20

1 Introduction

In this paper, we are interested in two problems related by quasi-Monte Carlo method (cf. [3] and [2]): the variation of the product of two functions of bounded variation and the numerical solution of a certain class of parabolic partial differential equations by low-discrepancy sequences.

In numerical integration by quasi-Monte Carlo method, the key formula for error bounds is the Koksma-Hlawka inequality.

Theorem 1.1 Koksma-Hlawka inequality*[2]*

If a function f has bounded variation $V(f)$ on $[0,1]^s$ in the sense of Hardy and Krause and $\xi = (\mathbf{x}_n)_{n\geq 1}$ is a $[0,1]^s$-valued sequence in $[0,1]^s$, then for any $N > 0$,

$$\left|\frac{1}{N}\sum_{n=1}^{N} f(\mathbf{x}_n) - \int_{[0,1]^s} f(\mathbf{t})d\mathbf{t}\right| \leq V(f)D^*(\xi, N),$$

where $D^*(\xi, N)$ is the star-discrepancy of the first N terms of ξ.

Therefore, in order to accelerate the speed of the approximation, a sequence with low discrepancy (so called low-discrepancy sequence) must be used. We refer the reader to [3] and [2] for the special low-discrepancy sequences.

The concept of bounded variation in the sense of Hardy and Krause is complex to deal with when $s > 2$. However, in [1] and [4], a more convenient notion to work with (at least theoretically) is proposed for which the Koksma-Hlawka inequality still holds. To recall it, we use the following notations: for $\mathbf{x} = (x_1, \ldots, x_s)$ and $\mathbf{y} = (y_1, \ldots, y_s)$ in $\mathbf{R}^s$ with $x_i \leq y_i$ for $i = 1, \ldots, s$, we denote $\mathbf{x} \leq \mathbf{y}$ and $[\![\mathbf{x}, \mathbf{y}]\!] = \prod_{i=1}^{s}[x_i, y_i]$.

Definition 1.2 *A function $f : [0,1]^s \to \mathbf{R}$ is said to have* bounded variation *(in the measure sense) if there exists a bounded signed measure μ on $\mathcal{B}([0,1]^s)$ with support in $[0,1]^s \setminus \{\mathbf{0}\}$, such that*

$$f(\mathbf{x}) = f(\mathbf{1}) + \mu\{[\![\mathbf{0}, \mathbf{1} - \mathbf{x}]\!]\} \quad \textit{for all } \mathbf{x} \textit{ in } [0,1]^s,$$

where $\mathbf{1} = (1, \ldots, 1)$ and $\mathbf{0} = (0, \ldots, 0)$. This measure is unique and its mass $||\mu||$ is called the variation *of f and denoted by $V(f)$.*

Recall the connection with the class of functions of bounded variation in the sense of Hardy and Krause is given by the following proposition [4]:

Proposition 1.3 *(a) If f has bounded variation (in the measure sense), it also has in the sense of Hardy and Krause. (b) If f has bounded variation in the sense of Hardy and Krause then: $f_+(\mathbf{x}) = \lim_{\mathbf{y} \to \mathbf{x}, \mathbf{y} \in [\![\mathbf{x}, \mathbf{1}]\!] \setminus \{\mathbf{x}\}} f(\mathbf{y})$ exists for every $\mathbf{x} \in [\![\mathbf{0}, \mathbf{1}]\!] \setminus \{\mathbf{1}\}$ ($f_+(\mathbf{1}) = f(\mathbf{1})$) and satisfies*

1. *$f_+ = f$ $d\mathbf{x}$-a.s.*
2. *f_+ has bounded variation (in the measure sense) and $V(f_+) \leq V_{H\&K}(f)$.*

In this paper, the variation of the product of two functions of bounded variation (in the measure sense) is studied, and an inequality for its estimation is given in §2.

Another problem in which we are interested is the numerical solution of partial differential equations using some low-discrepancy sequences.

In the work of Hua and Wang [3], the method of good lattice points were used to give an approximate solution of Cauchy's initial value problem for a class of parabolic partial differential equation. Let $E_s^\lambda(K)$, $\lambda > 2$, $K > 0$ denote the set of all functions defined on $\mathbf{R}^s$

$$f(\mathbf{x}) = \sum_{\mathbf{m} \in \mathbf{Z}^s} c(\mathbf{m}) e^{2\pi i \mathbf{m} \cdot \mathbf{x}} \quad \mathbf{x} = (x_1, \ldots, x_s)$$

with Fourier coefficients satisfing the decay conditions

$$|c(\mathbf{m})| \leq K(r(\mathbf{m}))^{-\lambda},$$

where $\mathbf{m} = (m_1, \ldots, m_s)$, $\mathbf{m} \cdot \mathbf{x} = \sum_{i=1}^s m_i x_i$ and

$$r(\mathbf{m}) = \Pi_{i=1}^s \max\{1, |m_i|\}.$$

Then a numerical solution is provided for the following equation:

$$\begin{cases} \frac{\partial}{\partial t} u(t, \mathbf{x}) = \Delta u(t, \mathbf{x}); & \mathbf{x} = (x_1, \cdots, x_s), \\ u(0, \mathbf{x}) = f(\mathbf{x}) & \text{with } f \in E_s^\lambda(K) \end{cases} \tag{1}$$

where $\Delta = \sum_{i=1}^s \frac{\partial^2}{\partial x_i^2}$ is Laplacian. Since then, their method has been extended to a more general class of partial differential equations in [5], [6] and [7].

In §3, the result of Hua and Wang is generalized to any low-discrepancy sequences using an inequality that will be proved in the next section.

2 An inequality on the variation of the product of two functions with bounded variation

In this section, we give an inequality on the variation of the product of two functions of bounded variation. Firstly, we need a lemma.

Lemma 2.1 *Let μ_1 and μ_2 be two signed measure on $\mathcal{B}([0,1]^s)$, then there exists a signed measure on $\mathcal{B}([0,1]^s)$, noted by $\mu_1 \star \mu_2$, such that*

$$\mu_1 \star \mu_2\{[\![\mathbf{0},\mathbf{x}]\!]\} = \mu_1\{[\![\mathbf{0},\mathbf{x}]\!]\}\mu_2\{[\![\mathbf{0},\mathbf{x}]\!]\} \quad \textit{if } \mathbf{x} \in [0,1]^s$$

and

$$||\mu_1 \star \mu_2|| \leq ||\mu_1||||\mu_2||.$$

Proof First, consider the case of μ_1 and μ_2 be positive measure. Define a positive measure $\mu_1 \star \mu_2$ on $\mathcal{B}([0,1]^s)$ by

$$\mu_1 \star \mu_2 = (\mu_1 \otimes \mu_2) \circ T^{-1},$$

where T is the function from $[0,1]^s \times [0,1]^s$ to $[0,1]^s$ defined by

$$T(\mathbf{y},\mathbf{z}) = (\max\{y_1,z_1\},\ldots,\max\{y_s,z_s\}).$$

Then we have for all $\mathbf{x} \in [0,1]^s$

$$\begin{aligned} \mu_1 \star \mu_2\{[\![\mathbf{0},\mathbf{x}]\!]\} &= \mu_1 \otimes \mu_2\{(\mathbf{y},\mathbf{z}) \,|\, \mathbf{y} \leq \mathbf{x} \text{ and } \mathbf{z} \leq \mathbf{x}\} \\ &= \mu_1 \otimes \mu_2\{[\![\mathbf{0},\mathbf{x}]\!] \times [\![\mathbf{0},\mathbf{x}]\!]\} \\ &= \mu_1\{[\![\mathbf{0},\mathbf{x}]\!]\}\mu_2\{[\![\mathbf{0},\mathbf{x}]\!]\} \end{aligned}$$

and

$$||\mu_1 \star \mu_2|| = \mu_1 \star \mu_2\{[\![\mathbf{0},\mathbf{1}]\!]\} = \mu_1\{[\![\mathbf{0},\mathbf{1}]\!]\}\mu_2\{[\![\mathbf{0},\mathbf{1}]\!]\} = ||\mu_1||||\mu_2||.$$

Now let μ_1 and μ_2 be signed measure with

$$\mu_1 = \mu_1^+ - \mu_1^- \quad \text{and} \quad \mu_2 = \mu_2^+ - \mu_2^-.$$

their Jordan-Hahn decompositions. Define a signed measure $\mu_1 \star \mu_2$ on $\mathcal{B}([0,1]^s)$ by

$$\mu_1 \star \mu_2 = \mu_1^+ \star \mu_2^+ + \mu_1^- \star \mu_2^- - \mu_1^+ \star \mu_2^- - \mu_1^- \star \mu_2^+.$$

Then, for any $\mathbf{x} \in [0,1]^s$, we have

$$\begin{aligned} \mu_1 \star \mu_2\{[\![\mathbf{0},\mathbf{x}]\!]\} &= (\mu_1^+ - \mu_1^-)\{[\![\mathbf{0},\mathbf{x}]\!]\}(\mu_2^+ - \mu_2^-)\{[\![\mathbf{0},\mathbf{x}]\!]\} \\ &= \mu_1\{[\![\mathbf{0},\mathbf{x}]\!]\}\mu_2\{[\![\mathbf{0},\mathbf{x}]\!]\} \end{aligned}$$

and

$$\begin{aligned} ||\mu_1 \star \mu_2|| &\leq ||\mu_1^+ \star \mu_2^+|| + ||\mu_1^- \star \mu_2^-|| + ||\mu_1^+ \star \mu_2^-|| + ||\mu_1^- \star \mu_2^+|| \\ &= ||\mu_1^+||||\mu_2^+|| + ||\mu_1^-||||\mu_2^-|| + ||\mu_1^+||||\mu_2^-|| + ||\mu_1^-||||\mu_2^+|| \\ &= ||\mu_1||||\mu_2||, \end{aligned}$$

the lemma follows. ◇

Theorem 2.2 *If two functions f and g on $[0,1]^s$ have respectively bounded variation (in the measure sense) $V(f)$ and $V(g)$. Then fg also has bounded variation on $[0,1]^s$ and*

$$V(fg) \leq V(f)V(g) + |g(\mathbf{1})|V(f) + |f(\mathbf{1})|V(g).$$

Proof Let μ and ν be two bounded signed measures on $\mathcal{B}([0,1]^s)$ with support in $[0,1]^s \setminus \{\mathbf{0}\}$, such that for all $\mathbf{x}$ in $[0,1]^s$

$$f(\mathbf{x}) = f(\mathbf{1}) + \mu\{[\![\mathbf{0}, \mathbf{1}-\mathbf{x}]\!]\} \quad \text{and} \quad g(\mathbf{x}) = g(\mathbf{1}) + \nu\{[\![\mathbf{0}, \mathbf{1}-\mathbf{x}]\!]\}.$$

Then, using Lemma 2.1, we have

$$\begin{aligned}
f(\mathbf{x})g(\mathbf{x}) &= \mu\{[\![\mathbf{0}, \mathbf{1}-\mathbf{x}]\!]\}\nu\{[\![\mathbf{0}, \mathbf{1}-\mathbf{x}]\!]\} + (g(\mathbf{1})\mu + f(\mathbf{1})\nu)\{[\![\mathbf{0}, \mathbf{1}-\mathbf{x}]\!]\} + f(\mathbf{1})g(\mathbf{1}) \\
&= (\mu \star \nu + g(\mathbf{1})\mu + f(\mathbf{1})\nu)\{[\![\mathbf{0}, \mathbf{1}-\mathbf{x}]\!]\} + f(\mathbf{1})g(\mathbf{1}) \\
&= \mu^*\{[\![\mathbf{0}, \mathbf{1}-\mathbf{x}]\!]\} + f(\mathbf{1})g(\mathbf{1})
\end{aligned}$$

where μ^* is a signed measure on $\mathcal{B}([0,1]^s)$ with support in $[0,1]^s \setminus \{\mathbf{0}\}$ defined by

$$\mu^* = \mu \star \nu + g(\mathbf{1})\mu + f(\mathbf{1})\nu$$

In addition,

$$\begin{aligned}
V(fg) &= ||\mu^*|| \\
&\leq ||\mu \star \nu|| + |g(\mathbf{1})|||\mu|| + |f(\mathbf{1})|||\nu|| \\
&\leq ||\mu||||\nu|| + |g(\mathbf{1})|||\mu|| + |f(\mathbf{1})|||\nu|| \\
&= V(f)V(g) + |g(\mathbf{1})|V(f) + |f(\mathbf{1})|V(g).
\end{aligned}$$

◇

To apply the above inequality in the following section, we will use the following essential example of function with bounded variation (cf. [4]). Let us introduce some additional notations : let $I \subset \{1, \ldots, d\}$, we sets :

$$m_I^i = dx_i \text{ if } i \in I,\ m_I^i = \delta_1 \text{ if } i \notin I \text{ and } d\mathbf{x}^I = \bigotimes_{1 \leq i \leq d} m_I^i,$$
$$x_I^i = x_i \text{ if } i \in I, x_I^i = 1 \text{ if } i \notin I \text{ and } \mathbf{x}_I = (x_I^i)_{1 \leq i \leq d},\ \mathbf{x}^I = (x^i)_{i \in I}$$

Example : Let f be a function from $[0,1]^s$ to $\mathbf{R}$. If, for every $I \subset \{1, \ldots, d\}$, $\frac{\partial f}{\partial \mathbf{x}^I}$ (in the distribution sense) lies in L^1, then f has bounded variation with

$$\mu_f(d\mathbf{x}) = \sum_{I \subset \{1, \ldots, d\}} (-1)^{card(I)-1} \frac{\partial f}{\partial \mathbf{x}^I}(\mathbf{x}_I) d\mathbf{x}^I$$

and

$$V(f) = \sum_{I \subset \{1, \ldots, d\}} \int_{[0,1]^s} |\frac{\partial f}{\partial \mathbf{x}^I}(\mathbf{x}_I)| d\mathbf{x}^I.$$

3 Numerical solution of certain class of parabolic partial differential equations by low-discrepancy sequences

In this section, we study the numerical solution of equation (1) using low-discrepancy sequences. We have the following result.

Theorem 3.1 *Let $u(t, \mathbf{x})$ be the solution of (1) and let $\xi = (\mathbf{x}_n)_{n \geq 1}$ be a $[0,1]^s$-valued sequence in $[0,1]^s$. For any $N \geq 1$, denote*

$$u_N(t, \mathbf{x}) = \sum_{r(\mathbf{m}) \leq N} (\frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x}_n) e^{-2\pi i \mathbf{m} \cdot \mathbf{x}_n}) e^{-4\pi^2 t \mathbf{m} \cdot \mathbf{m} + 2\pi i \mathbf{m} \cdot \mathbf{x}}. \tag{2}$$

Then

$$|u(t, \mathbf{x}) - u_N(t, \mathbf{x})| \leq C(\lambda, f, t, s) D^*(\xi, N) \tag{3}$$

with $C(\lambda, f, t, s) > 0$ a constant depending on λ, f, t and s.

Proof Denoting the Fourier coefficients of $f(\mathbf{x})$ by $c(\mathbf{m})$. As in [3], we have

$$u(t, \mathbf{x}) = \sum_{\mathbf{m} \in \mathbf{Z}^s} c(\mathbf{m}) e^{-4\pi^2 t \mathbf{m} \cdot \mathbf{m} + 2\pi i \mathbf{m} \cdot \mathbf{x}}$$

Note that because of $\lambda > 2$ all calculations with this infinite series are justified, and by the Example in §2 f is a bounded variation function with variation $V(f)$.

Now we estime the error of the approximate solution. We have

$$|u(t, \mathbf{x}) - u_N(t, \mathbf{x})| \leq \Sigma_1 + \Sigma_2 \tag{4}$$

with

$$\Sigma_1 = \sum_{r(\mathbf{m}) < N} |c(\mathbf{m}) - \frac{1}{N} \sum_{n=1}^{N} f(\mathbf{x}_n) e^{-2\pi i \mathbf{m} \cdot \mathbf{x}_n}| e^{-4\pi^2 t \mathbf{m} \cdot \mathbf{m}}$$

and

$$\Sigma_2 = \sum_{r(\mathbf{m}) \geq N} |c(\mathbf{m})| e^{-4\pi^2 t \mathbf{m} \cdot \mathbf{m}}$$

By Lemma 7.7 of [3],

$$\Sigma_2 \leq \sum_{r(\mathbf{m}) \geq N} K r(\mathbf{m})^{-\lambda} \leq \frac{C_2}{N^{\lambda - 1}} \tag{5}$$

with C_2 a positive constant. For Σ_1, applying Koksma-Hlawka inequality, we have

$$\Sigma_1 \leq \sum_{r(\mathbf{m}) < N,\ \mathbf{m} \neq \mathbf{0}} V_{\mathbf{m}} D^*(\xi, N) e^{-4\pi^2 t \mathbf{m} \cdot \mathbf{m}} + V(f) D^*(\xi, N)$$

where

$$V_{\mathbf{m}} = V(f(\mathbf{x})\cos(2\pi\mathbf{m}\cdot\mathbf{x})) + V(f(\mathbf{x})\sin(2\pi\mathbf{m}\cdot\mathbf{x}))$$

with

$$\begin{aligned} V(\cos(2\pi\mathbf{m}\cdot\mathbf{x})) &= V(\sin(2\pi\mathbf{m}\cdot\mathbf{x})) \\ &= \sum_{k=1}^{s}\sum_{1\le i_1<i_2<\ldots<i_k}\int_0^1\cdots\int_0^1 \left|\frac{\partial^k\cos(2\pi\mathbf{m}\cdot\mathbf{x})}{\partial t_{i_1}\ldots\partial t_{i_k}}\right| dt_{i_1}\cdots dt_{i_k} \\ &\le \sum_{k=1}^{s}\sum_{1\le i_1<i_2<\ldots<i_k}(2\pi)^k|m_{i_1}|\cdots|m_{i_k}| \\ &\le (4\pi)^s|m_1|\cdots|m_s| \end{aligned}$$

Thanks to Thereom 2.3 and for all $t>0$

$$\sum_{m=1}^{\infty}\frac{m}{e^{4\pi^2tm^2}} < +\infty,$$

we obtain

$$V_{\mathbf{m}} \le 2(2V(f)+f(\mathbf{1}))V(\cos(2\pi\mathbf{m}\cdot\mathbf{x}))$$

and therefore

$$\begin{aligned} \Sigma_1 &\le \sum_{r(\mathbf{m})<N,\ \mathbf{m}\neq\mathbf{0}} V_{\mathbf{m}}D^*(\xi,N)e^{-4\pi^2 t\mathbf{m}\cdot\mathbf{m}} + V(f)D^*(\xi,N) \\ &\le 2(2V(f)+f(\mathbf{1}))D^*(\xi,N)(\sum_{r(\mathbf{m})<N,\ \mathbf{m}\neq\mathbf{0}}|m_1|\cdots|m_s|e^{-4\pi^2 t\mathbf{m}\cdot\mathbf{m}}+1) \\ &\le 2(2V(f)+f(\mathbf{1}))D^*(\xi,N)(\sum_{m_1=1}^{N}\cdots\sum_{m_s=1}^{N}|m_1|\cdots|m_s|e^{-4\pi^2 t\mathbf{m}\cdot\mathbf{m}}+1) \\ &= 2(2V(f)+f(\mathbf{1}))D^*(\xi,N)[(2\sum_{m=1}^{N}\frac{m}{e^{4\pi^2tm^2}})^s+1)] \\ &\le C_1D^*(\xi,N) \end{aligned} \tag{6}$$

with C_1 a positive constant depending only f, t and s. Together with (4) and (5), the result of (3) follows easily. $\diamond$

Remark : The above result can be extended to more general class of parabolic partial differential equation as follows:

$$\begin{cases} \frac{\partial}{\partial t}u(t,\mathbf{x}) = (\sum_{i=1}^{s}\sum_{j=1}^{s}a_{ij}\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{i=1}^{s}b_i\frac{\partial}{\partial x_i}+c)u(t,\mathbf{x}) \quad , \\ u(0,\mathbf{x}) = f(\mathbf{x}) \ \in E_s^{\lambda}(C), \end{cases} \tag{7}$$

where $\mathbf{x}\in\mathbf{R}^s$, $A=(a_{ij})$ is positive definite matrix, and b_i with $1\le i\le s$ and c are real constants. But our method does not naturally extend to the equations as there studied in the work of [5], [6] and [7].

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