Volume-Discrepancy of Sequences and Numerical Tests

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Abstract

In this paper, we introduce the notion of volume-discrepancy (and of isotropic volume-discrepancy) of a sequences of points and we establish some of their basic properties. This notion is illustrated by an application to a reliability problem.

Key words: Lower volume of points, Discrepancy, Volume-Discrepancy.

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Introduction

An important problem in reliability analysis is the computation of the failure probability :

$$p = P\left(f(u_1, \ldots, u_s) \leq \lambda\right),$$

where f is a known function and λ a given level. Often the function f can be assumed to be increasing (or at least monotonous) in each of its coordinates, i.e. for each $1 \leq i \leq s$, if $u_i \leq u'_i$ then

$$f(u_1, \ldots, u_i, \ldots, u_s) \leq f(u_1, \ldots, u'_i, \ldots, u_s).$$

Clearly this monotonicity assumption implies that the function f is increasing for the partial order of $\mathbf{R}^s$, that is to say, if $\mathbf{x}$ and $\mathbf{y}$ are two points of $\mathbf{R}^s$ such that, for all i, $1 \leq i \leq s$, $\mathbf{x}_i \leq \mathbf{y}_i$ (we will denote this fact by $\mathbf{x} \leq \mathbf{y}$), then $f(\mathbf{x}) \leq f(\mathbf{y})$.

This monotonicity assumption can be used to improve the standard Monte-Carlo algorithm. The basic idea is to keep track of the values of f at the points $\mathbf{x}$ that have already been drawn, and to use this information to avoid the computation of $f(\mathbf{y})$, if $\mathbf{y}$ is chosen such that $\mathbf{y} \leq \mathbf{x}$ for a $\mathbf{x}$ with $f(\mathbf{x}) \leq \lambda$. If the points are chosen uniformly in $[0,1]^d$, this happens with probability equal to the volume of the set of points which are less than one point already drawn (we will call this volume the lower volume delimited by these points). One can easily prove that the lower volume converges to $P\left(f(u_1, \ldots, u_s) \leq \lambda\right)$. This probability, in reliability problems, is close to 1, say of order 0.95. So we can hope to compute f only for 5% of the points, at least asymptotically. This saving may be large, especially if f is difficult to compute. In order to evaluate the expected gain we will introduce the notion of volume discrepancy and of isotropic volume discrepancy. Using

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this notion, we can estimate the speed of convergence of the lower volume to the probability and the savings of the algorithm.

These notions will enable us to understand why savings are not as large as expected especially when the dimension is large. This fact is linked with the speed of convergence of the volume-discrepancy to 0.

Our work is organized as follows. In section 1, we recall some notations and definitions, then we establish results on the volume-discrepancy of dense sequences, and we give an estimate of the isotropic volume-discrepancy using volume-discrepancy. In section 2, we recall results concerning volume-discrepancy for sequences with low-discrepancy and we give an estimate on volume differences of random sequences. Numerical results using random and Faure sequences are presented in section 3. They show that these two families of sequence have almost the same efficiency for the previously described algorithm and that it is clearly better than a standard Monte Carlo algorithm when the dimension is not too large.

1 Definition and first properties of the volume discrepancy

1.1 Notations and definitions

Here we recall some notations and basic results. Let $\mathbf{x} = (x_1, \ldots, x_s)$ be a point of the s-dimensional closed unit cube $\bar{I}^s = [0,1]^s$ and let :

$$B(\mathbf{x}) = \{(y_1, \ldots, y_s) \in I^s = [0,1)^s | \; y_i < x_i \text{ for } 1 \leq i \leq s\}.$$

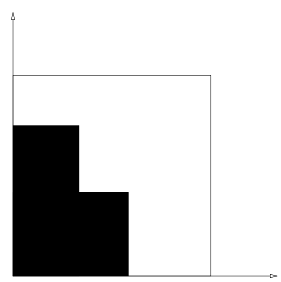

Figure 1: The lower volume of a set of points

If S is a point set (a point set is a finite sequence of points as in the terminology of [6]) of $\bar{I}^s$, we will extend the definition of B by setting

$$B_S = \bigcup_{\mathbf{x} \in S} B(\mathbf{x}).$$

Now, we define the lower volume of a set D.

Definition 1.1 *Let D is a Jordan-measurable subset of I^s. The* lower volume $V_D(S)$ *is :*

$$V_D(S) = \int_D 1_{B_{S \cap D}}(\mathbf{x}) d\mathbf{x}.$$

If $\sigma = (\mathbf{x}_n)_{n\geq 1}$ is an I^s-valued sequence, σ_N will denote the point set $\{\mathbf{x}_n \mid 1 \leq n \leq N\}$. We will call volume difference of σ_N associated with D *:*

$$EV_N(\sigma, D) = (\lambda(D) - V_D(\sigma_N)),$$

where λ denotes Lebesgue measure.

Remark 1.2 If D is monotonous (that is to say if $\mathbf{x}$ is in D then every $\mathbf{y} \leq \mathbf{x}$ is also in D) then this quantity is positive and represents the speed of convergence of the volume delimited by the points of the sequence which are in D to the volume of the set D.

Definition 1.3 *The* Volume-Discrepancy of the first N terms of σ, *$DV_N(\sigma)$ is :*

$$DV_N(\sigma) = \sup_{D\in\mathcal{P}_s} EV_N(\sigma, D),$$

where $\mathcal{P}_s$ is the family of all subintervals of I^s.

The star volume-discrepancy of the first N terms of σ *DV_N^* is :*

$$DV_N^*(\sigma) = \sup_{D\in\mathcal{P}_s^*} EV_N(\sigma, D)$$

where $\mathcal{P}_s^*$ is the family of all subintervals of I^s of kind $\prod_{k=1}^s [0, b_k[, 0 \leq b_k \leq 1$.

Remark 1.4 Clearly :

$$DV_N^*(\sigma) = \sup_{\mathbf{x}\in\bar{I}^s} (\lambda(B(\mathbf{x})) - V_{B(\mathbf{x})}(\sigma_N))$$

where $\bar{I}^s = [0,1]^s$ and for $\mathbf{x} = (x_1, \ldots, x_s) \in \bar{I}^s$, $\lambda(B(\mathbf{x})) = \prod_{i=1}^s x_i$.

Remark 1.5 When $s = 1$, the volume-discrepancy of a set of points $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$ can be computed as follows. Let $(\mathbf{y}_1, \ldots, \mathbf{y}_n)$ be the increasing ordering of $(\mathbf{x}_1, \ldots, \mathbf{x}_n)$. If we set $\mathbf{y}_0 = 0$ and $\mathbf{y}_{n+1} = 1$ then :

$$DV_N^* = \max_{i=0,\ldots,n} \mathbf{y}_{i+1} - \mathbf{y}_i.$$

The following proposition can be proved using a slight variation of the proof of proposition 2.4 in [6].

Proposition 1.6 *For all $s \geq 1$, $DV_N^*(\sigma) \leq DV_N(\sigma) \leq 2^s DV_N^*(\sigma)$.*

1.2 Volume discrepancy as a density test

In this section, we will prove that the volume discrepancy can lead to a density criterium for a sequence of points.

Theorem 1.7 *Let $\sigma = (\mathbf{x}_n)_{n\geq 1}$ be a sequence of points of I^s, then $\lim_{N\to\infty} DV_N^*(\sigma) = 0$, if and only if σ is a dense sequence in I^s.*

Remark 1.8 Note that, if D_N^* is the usual star-discrepancy, we have :

$$\lim_{N \to \infty} D_N^*(\sigma) = 0$$

if and only if σ is uniformly distributed in I^s (see [6]). So the notion of the volume-discrepancy is weaker than that of the discrepancy. Sequences having small discrepancies, (which are efficient sequences for the quasi Monte Carlo methods, see [6] and [3]) might be expected to have small volume-discrepancy.

Proof : of theorem 1.7. Let σ be a sequence which is not dense. Clearly there exists a subinterval of I^s such that no point of the sequence falls in this subinterval. One can easily deduce from this fact that :

$$\lim_{N \to \infty} DV_N^*(\sigma) > 0.$$

Now, it remains to prove that the density of σ implies that the volume discrepancy goes to 0. Le us first prove that for a given point $\mathbf{x}$, if $\sigma = (\mathbf{x}_n)_{n \geq 1}$ is a dense sequence in I^s, we have :

$$\lim_{N \to \infty} EV_N(\sigma, B(\mathbf{x})) = 0.$$

Clearly $\lim_{N \to \infty} EV_N(\sigma, B(\mathbf{x}))$ exists because $EV_N(\sigma, B(\mathbf{x}))$ decreases with N. Assume that there exists a point $z = (z_1, \ldots, z_s) \in \bar{I}^s$ such that :

$$\lim_{N \to \infty} EV_N(\sigma, B(z)) = a > 0.$$

We can suppose that each $z_i > 0$ for $1 \leq i \leq s$, because if $z_i = 0$ then $EV_N(\sigma, B(z)) = 0$ for all $N \geq 1$. Now, let us denote by g the following function :

$$g(x) = \prod_{i=1}^{s} z_i - \prod_{i=1}^{s} (z_i - x).$$

This function is clearly continuous and increasing, $g(0) = 0$, moreover if $m = \min_{1 \leq i \leq s} z_i$ then $g(m) = \prod_{i=1}^{s} z_i > 0$. Thus there exist a number y, $y \in (0, m)$ such that :

$$g(y) = b < a.$$

Now we will see that there is no point of the sequence σ in the interval $\prod_{i=1}^{s} [z_i - y, z_i)$ of measure $y^s > 0$. To prove this, let us suppose that there exists such a point. Thus

$$\begin{aligned} a &\leq EV_N(\sigma, B(z)) = \prod_{i=1}^{s} z_i - \int_{B_{B(\mathbf{z}) \cap \sigma_N}} dt \\ &\leq \prod_{i=1}^{s} z_i - \int_{B(\mathbf{z} - y)} dt = g(y) = b < a, \end{aligned}$$

where $\mathbf{z} - y$ is the point $(z_1 - y, \ldots, z_s - y)$. This leads to a contradiction. ∎

Now, we will use a generalized version of Dini lemma.

Lemma 1.9 (Dini lemma) *If $(f_n(\mathbf{x}))_{n\geq 1}$ is a decreasing sequence of upper semi-continuous functions on $\bar{I}^s$ such that, for all $\mathbf{x} \in \bar{I}^s$:*

$$\lim_{n\to\infty} f_n(\mathbf{x}) = 0,$$

then $\lim_{n\to\infty} \sup_{\mathbf{x}\in\bar{I}^s} f_n(\mathbf{x}) = 0$.

In order to use the previous lemma in our context we need to prove that $\mathbf{x} \mapsto V_{B(\mathbf{x})}(E)$ is lower semi-continuous.

Lemma 1.10 *If $E = \{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is a finite set of points in I^s, then the function $\mathbf{x} \mapsto f(\mathbf{x}) = V_{B(\mathbf{x})}(E)$ is lower semi-continuous on $\bar{I}^s$.*

Proof : Clearly $f(\mathbf{x})$ is increasing for the partial order on $\bar{I}^s$. For each point $\mathbf{x}_k = (x_{k,1}, \ldots, x_{k,s}) \in E, 1 \leq k \leq n$, we consider the s hyperplanes defined by

$$\{\mathbf{y} = (y_1, \ldots, y_s) \mid y_i = x_{k,i}\}.$$

At most ns hyperplanes cut $\bar{I}^s$ in, at most, $(n+1)^s$ subintervals of form:

$$\prod_{i=1}^{s} (u_i, v_i] \text{ if } 0 < u_i < v_i \leq 1$$

(if $u_i = 0$, we take $[u_i, v_i]$ rather than $(u_i, v_i]$). Now, if $P = \prod_{i=1}^{s}(u_i, v_i]$ is such an interval, and if $\mathbf{x} = (x_i)_{1\leq i\leq s} \in P$:

$$f(\mathbf{x}) = f((u_1, \ldots, u_s)). \tag{1}$$

To prove that f is lower semi-continuous we will verify that for all $\mathbf{x} = (x_i)_{1\leq i\leq s} \in \bar{I}^s$, there exists a neighborhood $V_{\mathbf{x}} \subset \bar{I}^s$ of $\mathbf{x}$ such that for all $\mathbf{y} \in V_{\mathbf{x}}$:

$$f(\mathbf{x}) \leq f(\mathbf{y}) \tag{2}$$

Clearly, it suffices to prove (2) for $\mathbf{x} \in \partial P = P \backslash \prod_{i=1}^{s}(u_i, v_i)$. Moreover, we will suppose that $u_i > 0$ for all $1 \leq i \leq s$, because if $\mathbf{x} = (x_i)_{1\leq i\leq s}$ and $x_{i_0} = 0$, for one i_0 then $f(\mathbf{x}) = 0$.

Now, let $\mathbf{x} = (a_i)_{1\leq i\leq s} \in \partial P$. Obviously there exists $\epsilon > 0$ such that :

$$V_{\mathbf{x}}^{-}(\epsilon) = \{\mathbf{y} = (y_1, \ldots, y_s) \mid a_i - \epsilon < y_i \leq a_i\} \subset P.$$

For $\mathbf{y} \in V_{\mathbf{x}}^{-}(\epsilon)$, $f(\mathbf{x}) = f(\mathbf{y})$, let us set

$$V_{\mathbf{x}}(\epsilon) = \{\mathbf{y} = (y_1, \ldots, y_s) \mid |y_i - a_i| < \epsilon\} \cap \bar{I}^s.$$

If $\mathbf{y} \in V_{\mathbf{x}}(\epsilon) \backslash V_{\mathbf{x}}^{-}(\epsilon)$, we define $\mathbf{y}' = (y_1', \ldots, y_s')$ with $y_i' = a_i$ if $y_i > a_i$ and $y_i' = y_i$ otherwise. This point $\mathbf{y}'$ is such that :

- $\mathbf{y}' \in V_{\mathbf{x}}^{-}(\epsilon)$, so $f(\mathbf{y}') = f(\mathbf{x})$,
- $\mathbf{y}' \leq \mathbf{y}$, so $f(\mathbf{y}') \leq f(\mathbf{y})$.

Hence, we have $f(\mathbf{x}) \leq f(\mathbf{y})$. ∎

1.3 Isotropic Volume-Discrepancy

We will now introduce the notion of isotropic volume-discrepancy of sequences which looks like the usual isotropic discrepancy (cf. [4] et [5]).

Definition 1.11 *The* Isotropic Volume-Discrepancy *of the first N points of a sequence $\sigma = (\mathbf{x}_n)_{n\geq 1}$ is defined by :*

$$DJ_N(\sigma) = \sup_{C \in \mathcal{C}_s} EV_N(\sigma, C),$$

where $\mathcal{C}_s$ is the family of all convex sets included in I^s.

Theorem 1.12 *For all sequence $\sigma = (\mathbf{x}_n)_{n\geq 1}$ of elements of I^s, we have*

$$DV_N(\sigma) \leq DJ_N(\sigma) \leq (4s\sqrt{s}+1)(DV_N(\sigma))^{\frac{1}{s}}.$$

For the proof, we will use the following two lemmas.

Lemma 1.13 *Let C be a convex open set and E be a finite subset of C. For $\epsilon > 0$, there exists a closed convex polytope K containing E and included in C satisfying*

$$\lambda(C \backslash K) < \epsilon.$$

Proof : First, note that for $\epsilon > 0$ there exists a compact set K_1 such that $K_1 \subset C$, $E \subset K_1$ and

$$\lambda(C \backslash K_1) < \epsilon.$$

Since C is open, for all $\mathbf{x} \in K_1$ there exists $\epsilon_{\mathbf{x}} > 0$ such that $\overline{V}_{\mathbf{x}} \subset C$, with $V_{\mathbf{x}} = \prod_{i=1}^{s}(x_i - \epsilon_{\mathbf{x}}, x_i + \epsilon_{\mathbf{x}})$. The family of open sets $\{V_{\mathbf{x}} \mid \mathbf{x} \in K_1\}$ is an open covering of the compact set K_1, so there exists a finite subset of points $\mathbf{x}_j \in K_1, 1 \leq j \leq m$ such that $\cup_{j=1}^{m} V_{\mathbf{x}_j} \supset K_1$.

For $1 \leq j \leq m$, let E_j be the set of vertices of $V_{\mathbf{x}_j}$, and then define $K = \mathrm{conv}(\cup_{j=1}^{m} V_{\mathbf{x}_j})$, where conv denotes convex hull. We have :

$$K = \mathrm{conv}(\cup_{j=1}^{m} E_j).$$

Hence K is a closed convex polytope included in C, containing $K_1 \supset E$ and such that $\lambda(C \backslash K) < \epsilon$. ∎

Lemma 1.14 *If $\mathcal{F}_s$ is the family of all closed convex polytopes of I^s then*

$$DJ_N(\sigma) = \sup_{P \in \mathcal{F}_s} EV_N(\sigma, P).$$

Proof : For all $C \in \mathcal{C}_s$, we have

$$\lambda(\mathrm{int}(C)) = \lambda(C) \ \text{ and } \ V_{\mathrm{int}(C)}(\sigma_N) \leq V_C(\sigma_N),$$

where $\text{int}(C)$ denotes the interior of C. Hence

$$\begin{aligned} EV_C(\sigma_N) &= \lambda(C) - V_C(\sigma_N) \\ &\leq \lambda(\text{int}(C)) - V_{\text{int}(C)}(\sigma_N) = EV_{\text{int}(C)}(\sigma_N). \end{aligned}$$

So :

$$DJ_N(\sigma) = \sup_{C \in \mathcal{L}_s} EV_N(\sigma, C)$$

where $\mathcal{L}_s$ denotes the family of the open convex sets in I^s.

Using the previous lemma, we can see that for all $O \in \mathcal{L}_s$ and for all $\epsilon > 0$, there exists a $P_\epsilon \in \mathcal{F}_s$ included in O such that

$$\lambda(O) \leq \lambda(P_\epsilon) + \epsilon$$

and

$$\text{Card}\{\mathbf{x}_n \in O \cap \sigma_N\} = \text{Card}\{\mathbf{x}_n \in P_\epsilon \cap \sigma_N\}.$$

Thus we have :

$$\begin{aligned} EV_O(\sigma_N) &= \lambda(O) - V_O(\sigma_N) \\ &\leq \lambda(P_\epsilon) - V_{P_\epsilon}(\sigma_N) + \epsilon \\ &= EV_{P_\epsilon}(\sigma_N) + \epsilon. \end{aligned}$$

Hence $DJ_N(\sigma) = \sup_{P \in \mathcal{F}_s} EV_N(\sigma, P)$. ∎

Proof : of Theorem 1.12. We use the same method as in [4] (p. 94-97) to estimate the classical isotropic discrepancy.

By the above lemma, it suffices to estimate $EV_N(\sigma, P)$ for a $P \in \mathcal{F}_s$. To simplify this estimation we will construct a subset of P, say P_1 easier to handle than P. Since $P_1 \subset P$:

$$\begin{aligned} EV_P(\sigma_N) &= \lambda(P) - V_P(\sigma_N) \\ &\leq \lambda(P_1) - V_{P_1}(\sigma_N) + \lambda(P) - \lambda(P_1) \\ &= EV_{P_1}(\sigma_N) + \lambda(P) - \lambda(P_1). \end{aligned} \tag{3}$$

The set P_1 is constructed as follows. For a positive integer r, for a lattice point $(h_1, h_2, ..., h_s)$ with $0 \leq h_i < r$ and for all $1 \leq i \leq s$ we denote by $J^{(r)}_{h_1 h_2 \ldots h_s}$ the interval $\{(x_1, \ldots, x_s) \in \mathbf{R}^s \mid \frac{h_i}{r} \leq x_i < \frac{h_i+1}{r}, 1 \leq i \leq s\}$. The set $\mathcal{L}^{(r)}$ of all these intervals is a partition of I^s. Let P_1 by the union of all the intervals of $\mathcal{L}^{(r)}$ included in P. If we fix $s-1$ integers $h_1, \ldots, h_{s-1}$ satisfying the above restriction, then the integers $h, 0 \leq h < r$, with $J^{(r)}_{h_1 \ldots h_{s-1} h} \subset P$ are consecutive because P is a convex set. Hence the union of these intervals $J^{(r)}_{h_1 \ldots h_{s-1} h}$ is again an interval. It follows that P_1 can be written as the union of at most r^{s-1} pairwise disjoint intervals. Hence :

$$EV_{P_1}(\sigma_N) \leq r^{s-1} DV_N(\sigma). \tag{4}$$

By [4] p. 96-97, we have

$$\lambda(P) - \lambda(P_1) \leq \frac{2s\sqrt{s}}{r},$$

now, using (4) and (3)

$$EV_P(\sigma_N) \leq r^{s-1} DV_N(\sigma) + \frac{2s\sqrt{s}}{r},$$

and since the upper bound is independent of P, we have :

$$DJ(\sigma_N) \leq r^{s-1} DV_N(\sigma) + \frac{2s\sqrt{s}}{r},$$

for all positive integers r. If we choose $r = [(DV_N(\sigma))^{-\frac{1}{s}}]$ we obtain :

$$DJ(\sigma_N) \leq (4s\sqrt{s}+1)(DV_N(\sigma))^{\frac{1}{s}}.$$

∎

2 Some estimations of volume discrepancies

In this section we give some properties of volume discrepancies. Most of then are already known, some are new. We begin with the case of deterministic sequences.

2.1 Deterministic sequences

The case $s = 1$ In this case (see [10]) the notion of discrepancy is closely related to the notion of dispersion and of maximal spacing. As quoted in remark 1.5, the volume-discrepancy of a sequence of points $\sigma = \{\mathbf{x}_n, n \geq 1\}$ is given by :

$$DV_N^*(\sigma) = \max\left\{\mathbf{x}_{\tau(1)}, \mathbf{x}_{\tau(2)} - \mathbf{x}_{\tau(1)}, \ldots, \mathbf{x}_{\tau(N)} - \mathbf{x}_{\tau(N-1)}, 1 - \mathbf{x}_{\tau(N)}\right\},$$

if τ is a permutation of $\{1, 2, \ldots, N\}$ such that :

$$\mathbf{x}_{\tau(1)} \leq \mathbf{x}_{\tau(2)} \leq \cdots \leq \mathbf{x}_{\tau(N)}.$$

With the same notation the *dispersion* $d_N(x)$ is given by :

$$d_N(\sigma) = \max\left\{\mathbf{x}_{\tau(1)}, \frac{1}{2}\left(\mathbf{x}_{\tau(2)} - \mathbf{x}_{\tau(1)}\right), \ldots, \frac{1}{2}\left(\mathbf{x}_{\tau(N)} - \mathbf{x}_{\tau(N-1)}\right), 1 - \mathbf{x}_{\tau(N)}\right\}.$$

So $d_N(\sigma) \leq DV_N^*(\sigma) \leq 2d_N(\sigma)$ and the asymptotic behavior of $d_N(\sigma)$ and $DV_N^*(\sigma)$, as N goes to infinity, are the same.

Moreover it can be shown that :

$$DV_N^*(\sigma) \geq \frac{1}{N+1},$$

and that

$$\limsup_{N\to\infty} N DV_N^*(\sigma) \geq \frac{1}{\log(2)}.$$

It is easy to construct a sequence of points with "minimal" volume discrepancy (that is to say which goes to zero at speed at least K/N, with $K = 1/\log(2)$) . For instance, if :

$$\mathbf{x}_1 = 1 \text{ and } \mathbf{x}_n = \left\{ \frac{\log(2n-3)}{\log 2} \right\}, \text{ for } n \geq 2,$$

one can prove that :

$$\limsup_{N\to\infty} NDV_N^*(\sigma) = \frac{1}{\log 2}.$$

Note that this sequence is dense but not uniformly distributed in $[0, 1]$.

The volume discrepancy of a one dimensional Van Der Corput sequence can also be computed and is "small". The following result is proved in [10].

Theorem 2.1 *Let ϕ_b be the Van Der Corput sequences in base b, then*

$$\lim_{N\to\infty} NDV_N^*(\phi_b) = \begin{cases} \frac{(b+2)^2}{4b} & \text{if } b \text{ even,} \\ \frac{(b+1)(b+3)}{4b} & \text{if } b \text{ odd.} \end{cases}$$

The case $s > 1$ In dimension greater than one, the problem is harder and only very few results are known. In [10], the following result concerning two dimensional sequences is proved :

Theorem 2.2 *If R is a $(0, m, 2)$-net in base b with $m \geq 1$ be an integer, then*

$$DV^*(R) \leq \frac{(3b-2)m}{b^m}.$$

This implies that, if σ is a $(0, 2)$-sequence in base b, then for all $N \geq 1$,

$$DV_N^*(sigma) \leq \frac{b(3b-2)\log_b N}{N}.$$

Moreover, for all $m \geq 1$, there exists a $(0, m, 2)$-net Q in base b such that

$$DV^*(Q) \geq \frac{(b-1)(1/2 + 1/b)m + 1}{b^m}.$$

2.2 The case random sequences

Here, $\sigma = (\mathbf{x}_n)_{n\geq 1}$ will be a sequence of independent random variables uniformly distributed on I^s.

Note that, if $s = 1$, the result of Deheuvels [1] on maximal spacings for multivariate order statistics implies that $DV_N^*(\sigma) = O\left(\frac{\log N}{N}\right)$ almost surely.

When the dimension is greater than one, we conjecture that the asymptotic behavior of the volume discrepancy of a random sequence is almost surely less than $K \log^\alpha(N)/N$, where $\alpha > 0$ is some positive number. But, we can only prove the following result on volume differences.

Theorem 2.3 *For all $\mathbf{x} \in I^s$, for all $\alpha > s$, for almost all ω, there exists a constant $C(\omega)$ such that :*

$$EV_N(\sigma, B(\mathbf{x})) \leq C(\omega)\frac{\log^\alpha(N)}{N}$$

To prove this result we need the following lemma.

Lemma 2.4 *For all $N \geq 1$ and for every $\mathbf{x} = (x_1, \ldots, x_s) \in \bar{I}^s$, we have*

$$\mathbf{E}(EV_N(\sigma, B(\mathbf{x}))) = O\left(\frac{(\log N)^{s-1}}{N}\right).$$

Proof : If D is a set of I^s, let us set :

$$B_k(\omega) = \begin{cases} B(\mathbf{x}_k(\omega)) \cap D & \text{if } \mathbf{x}_k(\omega) \in D \\ \emptyset & \text{otherwise.} \end{cases}$$

Clearly, using independence :

$$\mathbf{E}(\lambda(D) - V_D(\sigma_N)) = \mathbf{E}\int_D 1_{(\cup_{k=1}^N B_k(\omega))^c}(\mathbf{x})d\mathbf{x} = \int_D \prod_{k=1}^N \mathbf{E}1_{B_k^c(\omega)}(\mathbf{x})d\mathbf{x}.$$

This leads to :

$$\mathbf{E}(EV_N(\sigma, D)) = \int_D [1 - \mathbf{P}(\omega \mid \mathbf{x} \leq \mathbf{x}_1(\omega) \in D)]^N d\mathbf{x}.$$

Now, choose $D = B(\mathbf{x})$ where $\mathbf{x} = (x_1, \ldots, x_s) \in \bar{I}^s$ with $x_i > 0$ for all $1 \leq i \leq s$. We have

$$\begin{aligned} \mathbf{E}(EV_N(\sigma, D)) &= \int_{B(\mathbf{x})} [1 - \mathbf{P}(\omega \mid y \leq \mathbf{x}_1(\omega) \in B(\mathbf{x}))]^N d\mathbf{y} \\ &= \int_{B(\mathbf{x})} (1 - \prod_{i=1}^s (x_i - y_i))^N dy_1 \cdots dy_s \\ &= \int_{B(\mathbf{x})} (1 - \prod_{i=1}^s z_i)^N dz_1 \cdots dz_s \qquad \text{where } z_i = x_i - y_i \\ &\leq \int_{I^s} (1 - \prod_{i=1}^s z_i)^N dz_1 \cdots dz_s, \end{aligned}$$

To conclude, it remains to prove that

$$\int_{I^s} (1 - \prod_{i=1}^s x_i)^N dx_1 \cdots dx_s \leq \frac{(\log N)^{s-1}}{N}. \tag{5}$$

We prove it by induction on s. Clearly for $s = 1$, we have

$$\int_0^1 (1 - x_1)^N dx_1 = \frac{1}{N+1}.$$

Moreover if (5) is true for $s - 1$:

$$\int_{I^s} (1 - \prod_{i=1}^s x_i)^N dx_1 \cdots dx_s = \frac{1}{N+1} \int_0^1 dx_1 \cdots \int_0^1 dx_{s-1} \frac{1 - (1 - x_1 \cdots x_{s-1})^{N+1}}{x_1 \cdots x_{s-1}},$$

and, by induction :

$$\begin{aligned}
\int_{I^s}(1-\prod_{i=1}^{s} x_i)^N dx_1\cdots dx_s &= \frac{1}{N+1}\int_0^1 dx_1\cdots\int_0^1 dx_{s-1}\sum_{k=0}^{N}(1-x_1\cdots x_{s-1})^k \\
&\leq \frac{1}{N+1}(\sum_{k=1}^{N}\frac{(\log k)^{s-2}}{k}+1) \\
&\leq \tfrac{(\log N)^{s-1}}{N}.
\end{aligned}$$

This completes the proof of the lemma. ∎

Proof : Now we prove proposition 2.3 using Borel Cantelli lemma. Using Markov inequality and the previous lemma, we have for $\epsilon > 0$

$$\mathbf{P}\left(EV_N(\sigma, B(\mathbf{x})) \geq \frac{\log^{s+\epsilon}(N)}{N}\right) \leq \frac{1}{\log^{1+\epsilon}(N)}.$$

Now let us set $n_k = 2^k$. Clearly

$$\sum_{k\geq 1}\mathbf{P}\left(EV_{n_k}(\sigma, B(\mathbf{x})) \geq \frac{\log^{s+\epsilon}(n_k)}{n_k}\right) < +\infty.$$

So, for almost all ω there exists $k_0(\omega)$ such that, if $k \geq k_0(\omega)$:

$$EV_{n_k}(\sigma, B(\mathbf{x})) \leq \frac{\log^{s+\epsilon}(n_k)}{n_k}.$$

This implies, that for almost every ω there exists a constant $C(\omega)$ such that, for all $k \geq 1$:

$$EV_{n_k}(\sigma, B(\mathbf{x})) \leq C(\omega)\frac{\log^{s+\epsilon}(n_k)}{n_k}.$$

Now, for each N, let us denote by $k(N)$ the unique integer such that $n_{k(N)} \leq N < n_{k(N)+1}$. Clearly :

$$\frac{NEV_N(\sigma, B(\mathbf{x}))}{\log^{s+\epsilon}(N)} \leq \frac{n_{k(N)+1}EV_{k(N)(\sigma,B(\mathbf{x}))}}{\log^{s+\epsilon}(n_{k(N)})} = 2\frac{n_{k(N)}EV_{k(N)}(\sigma, B(\mathbf{x}))}{\log^{s+\epsilon}(n_{k(N)})} \leq 2C(\omega).$$

This completes the proof. ∎

3 Description of the algorithm and examples

3.1 The algorithm

In this section, we will describe more precisely the algorithm of the introduction, then give numerical results illustrating it and see why the isotropic volume discrepancy gives a useful a priori estimate for the speed of the algorithm.

We are interested in the computation of a failure probability in a reliability problem. A reliability problem can be described within the following scheme. Let f be a function from I^s to $\mathbf{R}$ and let $\mathbf{x} = (x^1, \ldots, x^s)$ be a random vector following a uniform law on I^s. Of course, practical problems are not strictly of this type (for instance, the law of the random vector $\mathbf{x}$ can be arbitrary), but standard result of simulation prove that they can be reduced to this form. One of the main problems of reliability analysis is to compute, given a confidence level λ, the probability :

$$p = \mathbf{P}\left(f(\mathbf{x}) \leq \lambda\right).$$

The probability is typically near to 1 (say 0.9, 0.95, 0.99).

In the standard Monte Carlo procedure we simulate a sequence of independent, uniformly distributed on I^s, vectors $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n, \ldots)$ and we approximate p by p_N with :

$$p_N = \frac{1}{N}\left(\mathbf{1}_{\{f(\mathbf{x}_1) \leq \lambda\}} + \cdots + \mathbf{1}_{\{f(\mathbf{x}_N) \leq \lambda\}}\right).$$

Note that, to compute p_N we need to compute f at N different points. Often the complexity of the computation of f is so large that the previous algorithm is almost unuseful : it is vital to reduce the number of computations of f. We will now see how one can achieve this task if we assume that the function f is increasing, or at least monotonous, for the partial order of $\mathbf{R}^N$. In what follows, for sake of simplicity, we will suppose that f is increasing.

Under this assumption, suppose that we have already drawn N points $S_N = \{\mathbf{x}_1, \ldots, \mathbf{x}_N\}$. Clearly, the set $D = \{\mathbf{x} \in I^s, f(\mathbf{x}) \leq \lambda\}$ contains, in the terminology of paragraph 1

$$B_{S_N \cap D} = \{\mathbf{x} \in I^s, \exists i \leq N, f(\mathbf{x}_i) \leq \lambda \text{ and } \mathbf{x} \leq \mathbf{x}_i\}.$$

The assumption on f shows that the set D is monotonous. So $EV_N(\mathbf{x}, D)$ goes to zero (that is to say the volume of $B_{S_N \cap D}$ goes to the volume of D). So, if the next point $\mathbf{x}_{N+1}$ fall in $B_{S_N \cap D}$ we *need not* compute f on this point. This seems to lead to very large savings because it happens with probability $\mathrm{Vol}\,(B_{S_N \cap D}) \approx \mathrm{Vol}\,(D)$, and this number is near to 1.

Nevertheless, as we will see on an example, the convergence of $EV_N(\mathbf{x}, D)$ to 0 is very slow, especially when the dimension s becomes large.

3.2 The example

In this example we compute

$$p = \mathbf{P}(g_1^2 + \cdots + g_s^2 \leq \lambda),$$

where $(g_1, \ldots, g_n)$ are s independent standard Gaussian random variables ($\mathbf{E}(g_1) = 0$,$\mathbf{E}(g_1^2) = 1$). Of course, the computation of p can be done directly using a χ^2-table.

In order to put this problem on the previous form, let F be the distribution function of a standard random variable $\mathbf{x} \to \mathbf{P}(g_1 \leq \mathbf{x})$, let F^{-1} be the inverse of F and $g(u) = F^{-1}\left(\frac{u+1}{2}\right)^2$. An easy computation shows that, if U is a uniformly distributed random variable on $[0,1]$, the distribution of $g(U)$ is the same as the distribution of g_1^2. Hence

$$p = \mathbf{P}(g(U_1) + \cdots + g(U_s) \leq \lambda),$$

if $(U_1, \ldots, U_s)$ are s independent random variables uniformly distributed on $[0,1]$. Of course, the function

$$f(u_1, \ldots, u_s) = g(u_1) + \cdots + g(u_s),$$

is increasing for the partial order of I^s.

In the example λ has been chosen in order to have $p = 0.95$. We simulate N points using the classical Monte Carlo method and we denote by N_c the number of time the algorithm actually compute the function f. We are particularly interested in the fraction $\frac{N_c}{N}$ which qualify the efficiency of the method. Asymptotically, $\frac{N_c}{N}$ goes to $1-p$ as N goes to infinity, so in good situation we can expect that this fraction would be near to $1/20$. We will see in numerical example that saving for realistic value of N (10000 to 1000000) are far lower than this value.

3.3 Numerical results

First we have tested the behavior of the algorithm with standard random sequences.

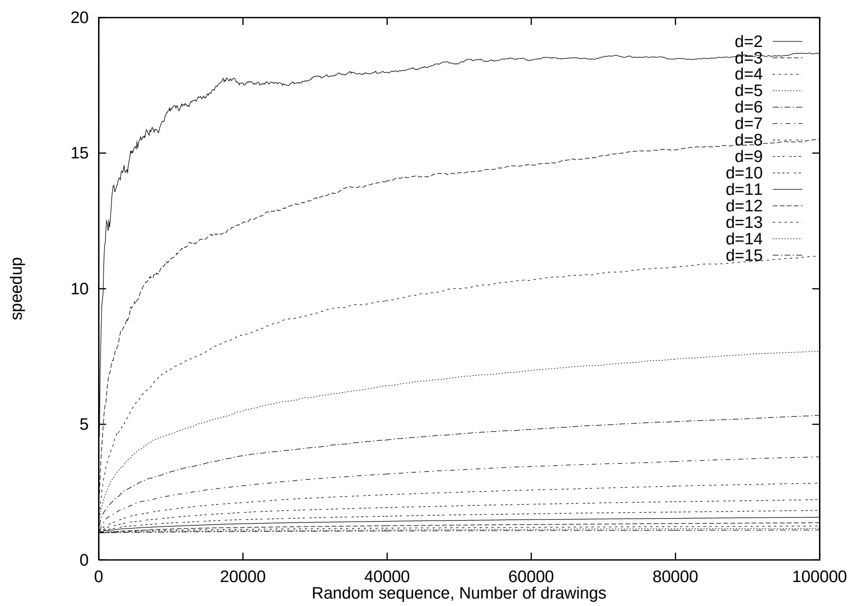

Figure 2: Random sequences

One can note on figure 2 that as the dimension s increase the gain of the algorithm decrease to zero. With $N = 10000$, our procedure is very useful if $s \leq 6$ but the improvement is very small if $s > 10$.

We have also replace random sequences by a low discrepancy sequence : the Faure sequence. In this case the algorithm exhibit the same kind of behavior (see figure 3).

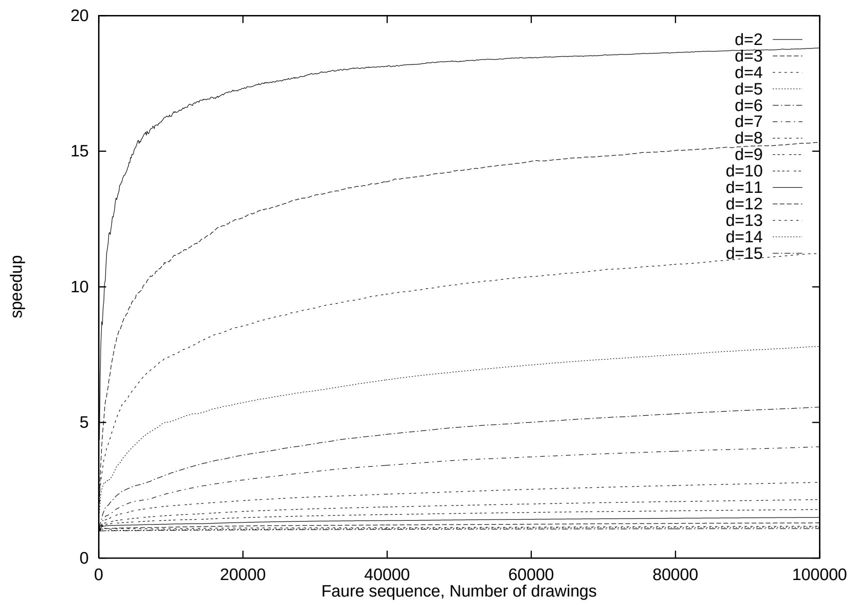

Figure 3: Faure sequences

We first thought that, at least in low dimension, the asymptotic behavior of $\frac{N_c}{N}$ would be better for low discrepancy sequences. This is not true, at least in the case of Faure sequences as shown on figure 4.

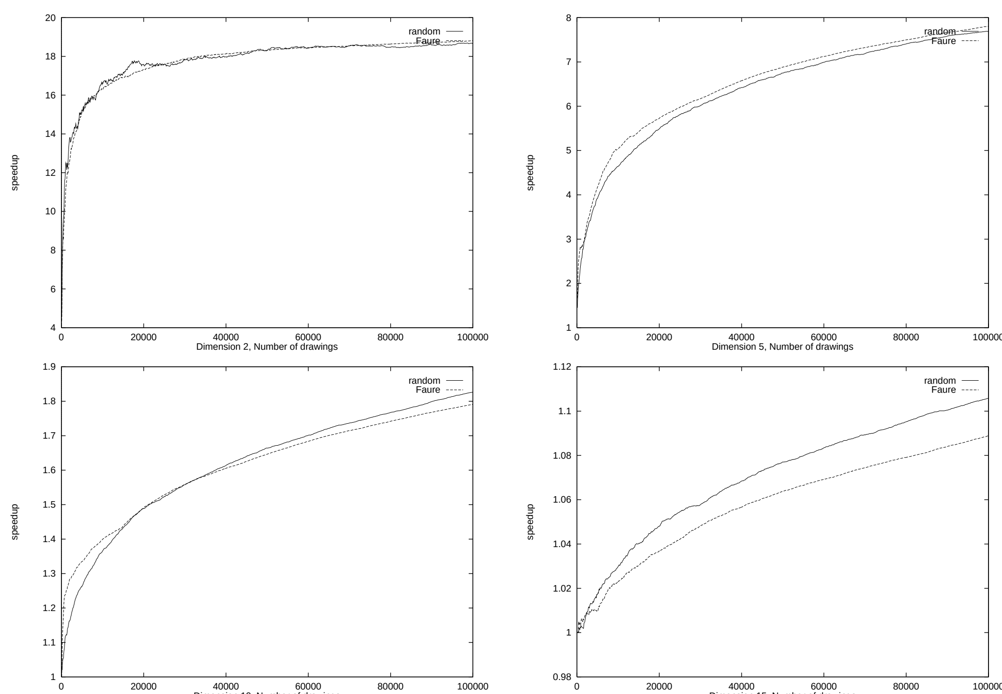

Figure 4: Random and Faure sequence, $s = 2, 5, 10$ and 15

3.4 Volume discrepancy and the efficiency of the algorithm : remarks et conjectures

Numerical experiments lead us to notice two important facts

- the improvement of the algorithm is affected by the dimension s of the problem.
- low discrepancy sequences do not seem to improve the efficiency of the algorithm.

One way to "explain" these points is to note that, when f is a convex function the set D is also convex, so $EV_N(\mathbf{x}, D)$ is less than the isotropic volume discrepancy $DJ_N(\mathbf{x})$. Moreover, using theorem 1.12 we know that :

$$DJ_N(\mathbf{x}) \leq KDV_N(\mathbf{x})^{\frac{1}{s}}.$$

Now suppose that , for a random sequence, the asymptotic behavior of $DV_N(\mathbf{x})$ is $\log^{\alpha}(N)/N$ (this fact can be conjectured but has not yet been proved). This leads to a possible upper bound for the asymptotical behavior of $DJ_N(\mathbf{x})$ of type $\log(N)^{\beta}/N^{\frac{1}{s}}$. We think that this upper bound gives the right behavior of $DJ_N(\mathbf{x})$ as N goes to infinity. Moreover, for "most" of the set D (but of course not all) this speed of convergence also seems to be the right one.

Moreover we conjecture that the asymptotical behavior of $DV_N(\mathbf{x})$ for low discrepancy sequences follows the same speed for random sequences that is to say $\log(N)^{\alpha}/N$. The fact that low discrepancy sequences can not help improving the algorithm seems to be linked with this conjecture.

References

[1] P. Deheuvels, Strong bounds for multidimensional spacings, *Z.Wahrch. Verw. Geb., 64 (1983), pp. 411-424.*

[2] H.Faure, Discrépance de suites associées à un systême de numération (en dimension s) *Acta. Arithmetica.XLI. (1982), 337-351.*

[3] L.K. Hua and Y. Wang, Application of Number Theory to Numerical Analysis, Springer-Verlag 1981.

[4] L.Kuipers and H.Niederreiter, Uniform distribution of sequences, Wiley, New York, 1974.

[5] H.Niederreiter, Random Number Generation and Quasi-Monte Carlo Methods, SIAM, Philadelphia Pennsylvania, 1992.

[6] H.Niederreiter, Quasi Monte Carlo methods and pseudo-random numbers. *Bull. AMS. 84, (1978), p.957-1041.*

[7] H.Niederreiter, Point sets and sequences with small discrepancy, *Monatsh. Math. 104, (1987), 273-337.*

[8] Y.J. Xiao, Estimations de volume pour les suites de type $(0, s)$, *C.R.Acad. Sci. Paris, t. 319, Série I, N^{o}2 p.101-104, 1994.*

[9] Y.J. Xiao, Estimates for the volume of points of $(0, s)$-sequences in base $b \geq s \geq 2$, In : Monte Carlo and Quasi-Monte Carlo Methods in Scientific Computing, (H.Niederreiter and P.J.S. Shine, Eds.), *Lecture Notes in Statist.*, 106, (1995), 362-372, Springer-Verlag, New York.

[10] Y.J. Xiao and H.Faure, Volume-Discrepancy estimates in one and two dimension, *to appear in Journal of the Mathematical and Computer Modeling.*, Vol. 23, No. 819 (1996), p. 25-36.