**High temperature regime for a multidimensional Sherrington - Kirkpatrick model of spin glass*
(running title: multidimensional SK model)**

Alain TOUBOL†

Abstract

Comets and Neveu have initiated in [5] a method to prove convergence of the partition function of disordred systems to a log-normal random variable in the high temperature regime by means of stochastic calculus. We generalize their approach to a multidimensional Sherrington-Kirkpatrick model with an application to the Heisenberg model of uniform spins on a sphere of $\mathbb{R}^d$, see [9]. The main tool that we use is a truncation of the partition function outside a small neighbourhood of the typical energy path.

Introduction

The Sherrington and Kirkpatrick model was introduced in [13] in 1975 as a simplified mean-field model of spin glass. It has been intensively studied by physicists ever since, as one can see from the broad survey [10] of physical results by Mézard, Parisi and Virasoro. However, rigorous mathematical results about it are rather scarce. In 1987, Aizemann, Lebowitz and Ruelle proved in [1] the convergence in law of the partition function of the model towards a log-normal random variable, but only for Ising spins, zero magnetic field and high temperature. In 1987, Fröhlich and Zegarlinski gave in [7] complementary results for n-dimensional spins on the sphere, obtaining bounds on the annealed free energy in arbitrary magnetic field but high temperature. Among the mathematically rigorous results, let us mention the papers by Ben Arous and Guionnet [3] who describe the thermodynamic limit of the Gibbs measure for arbitrary one dimensional spins by means of a

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†ENPC-CERMICS, La courtine F-93167 Noisy-le-Grand Cedex and Université Paris 7, URA 1321 “Statistique et modèles aléatoires” e-mail: toubol@cermics.enpc.fr

stochastic dynamical system, and the recent paper of Talagrand [15] where one can find very accurate results about the zero and even non-zero magnetic field case and the techniques of which could be probably extended to more general situations. However, Talagrand is not interested in the convergence in distribution of the partition function of the model.

In 1995, Comets and Neveu gave in [5] an entirely different proof of the result of Aizenman, Lebowitz and Ruelle, by using stochastic calculus and martingales. Studying real continuous spins is motivated in particular by [3] and studying vector spins is important for physicists as explained by Gabay and Toulouse in [8] and [9] , because "real" spins always are multidimensional. Thus our goal in this work is to prove some results of convergence in law for the partition function of arbitrary multidimensional symmetric spins, in the high temperature regime with zero magnetic field. We will apply our results to the Heisenberg model of uniformly distributed spins on the sphere of $\mathbb{R}^d$, see [9]. It seems that the original method of Aizenmann, Lebowitz and Ruelle wouldn't work in such general a situation. This shows the power of the martingale method. However, the counterpart of the method is the difficulty to reach what may be expected as the critical temperature.

Let us be now describe more precisely the problem that we are trying to work out. We consider N independent and identically distributed $\mathbb{R}^d$-valued random variables $\sigma(i)$, we denote by ρ their common distribution. Let $(J_{i,j})_{0\leq i\leq j}$ a family of independent $\mathcal{N}(0,1)$ random variable. We suppose that the energy of a configuration $\sigma = (\sigma(i))_{1\leq i\leq N}$ can be written as follow:

$$H_N^{(1)}(\sigma) = \frac{1}{\sqrt{N}} \sum_{1\leq i<j\leq N} J_{i,j}\langle\sigma(i)|\sigma(j)\rangle$$

In order to simplify the expressions, we add to this energy the following small term:

$$H_N^{(2)}(\sigma) = \frac{1}{\sqrt{2N}} \sum_{i=1}^{N} J_{i,i}\|\sigma(i)\|^2$$

Denote by $\mathbf{P}_\sigma$ the measure $\otimes_{i=1}^N \rho(d\sigma(i))$, and by $\mathbf{E}_\sigma$ the expectation w.r.t. $\mathbf{P}_\sigma$. The partition function of the system at the inverse temperature β is then given by:

$$\mathcal{Z}_N(\beta) = \mathbf{E}_\sigma \exp \beta[H_N^{(1)}(\sigma) + H_N^{(2)}(\sigma)] \tag{1}$$

Following [5], we replace $\beta J_{i,j}$ by $B_{i,j}(\beta^2)$, where $(B_{i,j})_{0\leq i\leq j}$ is a family of independent standard Brownian motions, and we set

$$H_N(\sigma;t)=\frac{1}{\sqrt{N}}\sum_{1\le i<j\le N}B_{i,j}(t)\langle\sigma(i)|\sigma(j)\rangle+\frac{1}{\sqrt{2N}}\sum_{i=1}^{N}B_{i,i}(t)\|\sigma(i)\|^2$$

The partition function that we are working with is $Z_N(t)=\mathbf{E}_\sigma\exp H_N(\sigma;t)$, which is related to $\mathcal{Z}_N(\beta)$ by the formula $\mathcal{Z}_N(\beta)=Z_N(\beta^2)$ in distribution. In the case of Ising spins, the behaviour of $Z_N(t)$ is studied by Comets and Neveu by means of the martingale $(\mathbf{E}_\sigma\exp(H_N(\sigma;t)-Nt/4))_{t\le1}$. In the situation of more general spins, the behaviour of $(\mathbf{E}_\sigma\exp(H_N(\sigma;t)-\langle H_N(\sigma)\rangle_t/2)$ and of $(\mathbf{E}_\sigma\exp H_N(\sigma;t))$ are extremely different because $\langle H_N(\sigma)\rangle_t$ depends on σ.

Hence we have had to introduce another sequence of martingales. Fix $t>0$. Denote by $\mathcal{F}_s^{(N)}$ the natural filtration of the Brownian motions up to s, for $s\le t$. The right martingale to work with here is the most natural one, that is $(\mathbb{E}(Z_N(t)|\mathcal{F}_s^{(N)}))_{s\le t}$.

In the first part, we shall state our assumptions and our main results. We then evaluate the scaling factor for Z_N, which is closely related to the thermodynamical limit of the free energy. We use this result to introduce two different types of constraints. On the one hand we keep $\sum_{i=1}^N\sigma(i)\otimes\sigma(i)/N$ is a neighbourhood of its typical value (see [16]) and on the other hand we try to keep a control on the whole energy path $H_N(\sigma;\,\cdot\,)$. Let us denote by $\mathcal{C}(\sigma,B)$ this set of constraints that depends on the disorder B and on the configuration $\sigma=(\sigma(1),\dots,\sigma(N)$. The sequence of martingales that we study is for $s\le t$:

$$Y_N^{\mathcal{C}}(s)=\mathbb{E}\left(\mathbf{E}_\sigma\exp(H_N(\sigma;t))\mathbf{1}_{\mathcal{C}(\sigma,B)}\middle|\mathcal{F}_s^{(N)}\right)/\mathbb{E}\left(\mathbf{E}_\sigma\exp(H_N(\sigma;t))\mathbf{1}_{\mathcal{C}(\sigma,B)}\right)$$

In order to prove the convergence of this sequence of auxiliary martingales on $[0,t]$ towards a log normal process, we need a strong control on its braket which we get by means of Malliavin calculus. Then using the terminal value of the martingale,that is approximately $Z_N(t)$, we shall have proved the convergence in law of $Z_N(t)$ toward a log-normal random variable. Note that we have not been able to consider the process $(Z_N(t))_t$, but the random variable $Z_N(t)$ for fixed t, which is differs from [5].

In the last section, we prove some convergence results for the quenched law of the spins showing that the Gibbs measure depends in general on t, which is not the case when dealing with Ising spins.

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1 Preliminary part and statement of the results

1.1 Notations

Let us first introduce a few notations before stating our main assumptions. If $x, y \in \mathbb{R}^d$, we write $\langle x|y\rangle = \sum_{i=1}^d x_i y_i$ for their canonical Euclidean product, and $\|x\|$ for the associated norm. The vector space of real $d \times d$ matrices is denoted by $\mathcal{M}_d$, and the subspace of symmetric matrices by $\mathcal{S}_d$. If $x, y \in \mathbb{R}^d$, we set $x \otimes y = (x_i y_j)_{1 \le i,j \le d} \in \mathcal{M}_d$. Scalar product and norm on $\mathcal{M}_d$ are defined accordingly, identifying $\mathcal{M}_d$ with $\mathbb{R}^{d^2}$.

We assume that we are given a symmetric probability measure ρ on $\mathbb{R}^d$ that is not supported by a strict affine subspace of $\mathbb{R}^d$, and such that for every $\alpha > 0$:

$$\int_{\mathbb{R}^d} \exp \alpha \|\sigma\|^4 \ \rho(d\sigma) < \infty \tag{2}$$

If μ is a probability measure on $\mathbb{R}^d$, we define its Cramer transform Λ_μ^* by:

$$\Lambda_\mu^* \left| \begin{array}{lll} \mathbb{R}^d & \to & \mathbb{R} \\ x & \mapsto & \sup\{\langle \lambda | x\rangle - \ln \int_{\mathbb{R}^d} \exp\langle \lambda|\sigma\rangle \ \mu(d\sigma) : \ \lambda \in \mathbb{R}^d\} \end{array} \right.$$

1.2 Assumptions and results

We are now ready to state our assumptions. Let Λ_1^* be the Cramer transform of the law of $\sigma \otimes \sigma$ under $\rho(d\sigma)$ on $\mathcal{S}_d$, Λ_2^* of the law of $(\sigma \otimes \sigma, \tau \otimes \tau, \sigma \otimes \tau)$ under $\rho(d\sigma) \otimes \rho(d\tau)$ on $\mathcal{S}_d \times \mathcal{S}_d \times \mathcal{M}_d$.

$(H1)$ The variational problem

$$\sup\{\frac{t}{4}\|x\|^2 - \Lambda_1^*(x) : x \in \mathcal{S}_d\}$$

admits a unique solution which we denote by $v = v(t)$. Let $\hat{\mu}_t$ be the measure on $\mathbb{R}^d$ with density:

$$\frac{d\hat{\mu}_t}{d\rho} \propto \exp \frac{t}{2}\langle v|\sigma \otimes \sigma\rangle$$

Let $\Gamma(t)$ stand for the covariance matrix of $\sigma \otimes \sigma$ under $\hat{\mu}_t$. Assume that the matrix $I - \frac{t}{2}\Gamma(t)$ is positive definite.

Let us comment on this assumption. First of all, it is clear that both parts of it are fulfilled when t is small enough. Now, thanks to (2) and lemma 4 in [4], one can easily see that the supremum is reached. Thus we assume here uniqueness and non-degeneracy.

$(H2)$ Let $v_1, \dots, v_d$ be the eigenvalues of $v \in \mathcal{S}_d$. Assume that:

1. $tv_iv_j < 1$, for every $i, j = 1 \dots, d$
2. For every $s \leq t$, $\quad z \in \mathcal{M}_d \mapsto \Lambda_2^*(v, v, z) + \frac{s\|v\|^4}{2(\|v\|^2+\|z\|^2)}$ achieves its minimum uniquely at $z = 0$.

Again, the assumption is fulfilled when t is small enough.

Assume that $t > 0$ fulfills assumptions $H1$ and $H2$. We shall not repeat this every time. Our main result is the following:

Theorem 1.1 *Set* $\gamma(t) = \frac{t}{4}\|v(t)\|^2 - \Lambda_1^*(v(t))$ *and*

$$\phi(t) = -\frac{1}{2} \sum_{i,j=1}^{d} \ln(1 - tv_i(t)v_j(t))$$

Let ξ *be a* $\mathcal{N}(0, \phi(t))$ *random variable. Then* $e^{-N\gamma(t)}\sqrt{\det(I - \frac{t}{2}\Gamma(t))}Z_N(t)$ *converges in law to the log-normal random variable* $\exp(\xi - \phi(t)/2)$.

Let G_N^t be the Gibbs measure, that is the measure on $(\mathbb{R}^d)^N$ given by:

$$G_N^t(d\sigma(1), \dots, d\sigma(N)) = \frac{\exp H_N(\sigma; t)}{Z_N(t)} \otimes_{i=1}^N \rho(d\sigma(i))$$

Let $L_N = \frac{1}{N}\sum_{i=1}^N \delta_{\sigma(i)}$ be the empirical measure of the configuration. Then we have:

Theorem 1.2 *Under the Gibbs measure* G_N^t*, the empirical measure* L_N *converges* $\mathbb{P}-$*almost surely weakly in probability to* $\hat{\mu}_t$*, in the sense that* $\mathbb{P}-$ *almost surely, we have for any continuous and bounded function* g *on* $\mathbb{R}^d$ *and for any* $\delta > 0$*:*

$$\lim_{N\to\infty} G_N^t\{|\langle L_N, g\rangle - \langle \hat{\mu}_t, g\rangle| \geq \delta\} = 0$$

Our last result is about some sort of quenched 'propagation of chaos'.Namely, we have:

Theorem 1.3 *Let* k *be an arbitrary integer. For any continuous and bounded function* g *on* $(\mathbb{R}^d)^k$*, the following convergence in probability holds:*

$$\mathbb{P} - \lim_{N\to\infty} G_N^t\left[g(\sigma_1, \dots, \sigma_k)\right] = \langle \hat{\mu}_t^{\otimes k}, g\rangle$$

At this point we should emphasize that in general, the Gibbs measure G_N^t explicitly depends on t which is not the case for Ising spins (see [5]).

2 Asymptotical evaluation of $\mathbb{E}Z_N(t)$ and the trajectorial localization

In this section, we assume that $t > 0$ fulfills assumption $H1$.

Let us recall that we are given a family $(B_{i,j})_{i\leq j}$ of independent standard Brownian motions. The symbol $\mathbb{E}$ denotes the expectation w.r.t the Brownian motions, that is w.r.t. the disorder. We are interested in the partition function, that is:

$$Z_N(t) = \mathbf{E}_\sigma \exp\left[\frac{1}{\sqrt{N}} \sum_{1\leq i<j\leq N} B_{i,j}(t)\langle\sigma(i)|\sigma(j)\rangle + \frac{1}{\sqrt{2N}}\sum_{i=1}^{N} B_{i,i}(t)\|\sigma(i)\|^2\right]$$

2.1 An equivalent of $\mathbb{E}Z_N(t)$

The first step of the method is an evaluation of $\mathbb{E}Z_N(t)$ up to a factor $(1 + o(1))$, which provides the scaling factor for $Z_N(t)$. We first compute the following expression of $\mathbb{E}Z_N(t)$:

$$\begin{aligned}
\mathbb{E}Z_N(t) &= \mathbf{E}_\sigma \exp \frac{t}{4N}\sum_{i,j=1}^{N}\langle\sigma(i)\otimes\sigma(i)|\sigma(j)\otimes\sigma(j)\rangle \\
&= \mathbf{E}_\sigma \exp \frac{Nt}{4}\left\|\frac{1}{N}\sum_{i=1}^{N}\sigma(i)\otimes\sigma(i)\right\|^2
\end{aligned}$$

We set

$$\sigma.\tau = \sum_{i=1}^{N}\sigma(i)\otimes\tau(i)$$

We now give a logarithmic approximation of $\mathbb{E}Z_N(t)$ that will enable us to localize the problem. In order to do so, we use Varadhan's theorem, in the way stated in [6] theorem 2.1.10. Thanks to the convexity of $x \mapsto \|x\|^2$, we have for any $N \geq 1$:

$$\frac{Nt}{4}\left\|\frac{1}{N}\sum_{i=1}^{N}\sigma(i)\otimes\sigma(i)\right\|^2 \leq \frac{t}{4}\sum_{i=1}^{N}\|\sigma(i)\otimes\sigma(i)\|^2 = \frac{t}{4}\sum_{i=1}^{N}\|\sigma(i)\|^4$$

As the $\sigma(i)$ are independent, and thanks to (2), the assumption of theorem 2.1.10 in [6] is easily checked. Thus,

$$\lim_{N\to\infty}\frac{1}{N}\ln\mathbb{E}Z_N(t) = \sup\{\frac{t}{4}\|x\|^2 - \Lambda_1^*(x) : x \in \mathcal{S}_d\} = \gamma(t)$$

Let us notice that even if the law of $\sigma \otimes \sigma$ is supported by some strict affine subspace of $\mathcal{S}_d$, the matrix v is strictly positive. Indeed, Bolthausen shows in [4], eq.(1.8), that $v = \hat{\mu}_t(\sigma \otimes \sigma)$. Hence, for any $a \in \mathbb{R}^d \setminus \{0\}$, $a^T va = \hat{\mu}_t(\langle a|\sigma\rangle^2) > 0$ because of the hypothesis on the support of ρ. We also deduce from the result $v = \hat{\mu}_t(\sigma \otimes \sigma)$ and from Azencott [2], prop.I.9.7, that , if we only pay attention to the vector space spanned by the convex hull of the support of the law of $\sigma \otimes \sigma$ under ρ on $\mathcal{S}_d$, then $v \in \mathcal{D}om(\Lambda_1^*)^\circ$, and thus that Λ_1^* is of class $\mathcal{C}^\infty$ at v. Hence we may restrict ourselves to the case of a non-degenerate law for $\sigma \otimes \sigma$. Then, according to Bolthausen [4], we have:

$$\mathbb{E}Z_N(t) = \frac{\exp N\gamma(t)}{\sqrt{\det(I - \frac{t}{2}\Gamma(t))}}(1 + o(1)) \tag{3}$$

2.2 A trajectorial approach

In order to stay as near the problem as possible and to formulate weak assumptions on ρ, we reformulate the approximation result (3) using the whole trajectory of $H_N(\sigma)$.

More precisely, let W be a fresh independent (that is independent of everything introduced up to now) one-dimensional Brownian motion. The following identity in law clearly holds on $\mathcal{C}_0([0,t],\mathbb{R})$ under $\mathbb{P} \otimes \mathbf{P}_\sigma$:

$$\frac{H_N(\sigma;\cdot)}{N} \stackrel{\mathcal{L}}{=} \frac{1}{\sqrt{2}} \left\| \frac{\sigma.\sigma}{N} \right\| \frac{W.}{\sqrt{N}}$$

Hence we can write:

$$\mathbb{E}Z_N(t) = \mathbb{E} \otimes \mathbf{E}_\sigma \exp \frac{N}{\sqrt{2}} \left\| \frac{\sigma.\sigma}{N} \right\| \frac{W_t}{\sqrt{N}}$$

We denote by I_d^t the rate function for Schilder's theorem on $[0,t]$ in $\mathbb{R}^d$. We may again use Varadhan's theorem, and get:

$$\lim_{N\to\infty} \frac{1}{N} \ln \mathbb{E}Z_N(t) = \sup\{\frac{1}{\sqrt{2}}\|x\|w(t) - \Lambda_1^*(x) - I_1^t(w) : x \in \mathcal{S}_d, w \in \mathcal{C}_0([0,t],\mathbb{R})\}$$

Maximizing in w with x being fixed, we get $w^{(x)}(s) = \frac{\|x\|}{\sqrt{2}}s$, $0 \le s \le t$, and replacing this value into the variational problem, we are lead to solve the problem of $H1$. Hence we deduce the uniqueness of the solution, namely $x = v$ and $w(s) = f_0(s) = \frac{\|v\|}{\sqrt{2}}s, \quad 0 \le s \le t$. Denote by f the function

$$f(s) = \frac{\|v\|^2}{2}s, \quad 0 \le s \le t \tag{4}$$

Thanks to Varadhan's theorem, the dominating part of $\mathbb{E}Z_N(t)$ is $\frac{\sigma.\sigma}{N} \approx v$ and $\frac{W}{\sqrt{N}} \approx f_0$, which may also be written $\frac{\sigma.\sigma}{N} \approx v$ and $\frac{H_N(\sigma,\cdot)}{N} \approx f$. Thus f appears as the typical value of the energy. This means that if we set for every $\varepsilon, \delta > 0$

$$Z_N^{\varepsilon,\delta}(t) = \mathbf{E}_\sigma \exp H_N(\sigma,t) \mathbf{1}_{\left\|\frac{\sigma.\sigma}{N}-v\right\|\leq\delta} \mathbf{1}_{\sup_{s\leq t}\left|\frac{H_N(\sigma;s)}{N}-f(s)\right|\leq\varepsilon}$$

then :

$$\lim_{N\to\infty} \mathbb{E}\left|\frac{Z_N(t) - Z_N^{\varepsilon,\delta}(t)}{\mathbb{E}Z_N(t)}\right| = 0 \tag{5}$$

3 An interpolating martingale

From now on, the parameter t is assumed to fulfill $H1$ and $H2$.

3.1 Definition of $Y_N^{\varepsilon,\delta}$

Following Comets and Neveu, we are to define a sequence of interpolating martingales converging in distribution to a log-normal process. For any $s \geq 0$ and any function $g \in \mathcal{C}_0([0,s],\mathbb{R})$, we set $\|g\|_{[0,s]} = \sup\{|g(u)| : 0 \leq u \leq s\}$. Let $\varepsilon, \delta > 0$ be fixed real numbers. For $s \leq t$ we denote by $\mathcal{B}_s(g,\varepsilon)$ the $\varepsilon-$ball of center g in $\mathcal{C}_0([0,s])$.Let $(\mathcal{F}_s^{(N)})_{s\leq t}$ be the natural filtration of the Brownian motions, that is

$$\mathcal{F}_s^{(N)} = \sigma(B_{i,j}(u), 1 \leq i \leq j \leq N, u \leq s)$$

We introduce a few other notations:

$$e_N(\sigma;s) = \exp\left[H_N(\sigma;s) - \frac{Ns}{4}\left\|\frac{\sigma.\sigma}{N}\right\|^2\right]$$

We intend to truncate $e_N(\sigma;s)$ outside a neighbourhood of the typical value of $H_N(\sigma)$, that is outside a ball of center f – see eq. (4)–. Therefore, we set:

$$\overline{e}_N^\varepsilon(\sigma;s) = e_N(\sigma;s)\mathbf{1}_{\mathcal{B}_s(f,\varepsilon)}\left(\frac{H_N(\sigma)}{N}\right) \tag{6}$$

Clearly $e_N(\sigma;\cdot)$ is an exponential $(\mathcal{F}_s^{(N)})_{s\leq t}$-martingale, whereas $\overline{e}_N^\varepsilon(\sigma;\cdot)$ is a supermartingale. We are also going to truncate the probability measure by using instead of $\mathbf{P}_\sigma$ a modified probability measure, namely $\mathbf{P}_{\sigma,\tau}$:

$$\frac{d\mathbf{P}_\sigma^{t,\delta}}{d\mathbf{P}_\sigma} \propto \exp\left[\frac{Nt}{4}\left\|\frac{\sigma.\sigma}{N}\right\|^2\right]\mathbf{1}_{\|\frac{\sigma.\sigma}{N}-v\|\le\delta}$$

Obviously we have:

$$Z_N(t) = \mathbf{E}_\sigma^{t,\infty} e_N(\sigma;t)$$

We now define for $s \le t$ the interpolating martingale:

$$Y_N^{\varepsilon,\delta}(s) = \mathbb{E}\left[\mathbf{E}_\sigma^{t,\delta} e_N(\sigma;t)\mathbf{1}_{\mathcal{B}_t(f,\varepsilon)}\left(\frac{H_N(\sigma)}{N}\right)\Big|\mathcal{F}_s^{(N)}\right] = \mathbb{E}\left[\mathbf{E}_\sigma^{t,\delta}\overline{e}_N^\varepsilon(\sigma;t)\Big|\mathcal{F}_s^{(N)}\right]$$

We also set $\mathbb{E}Y_N^{\varepsilon,\delta} = \mathbb{E}[Y_N^{\varepsilon,\delta}(s)]$. It clearly follows from (5) that $Y_N^{\varepsilon,\delta} = 1+o(1)$. The result that we are going to prove in this section is the following:

Proposition 3.1 *Under H1, H2, for every sufficiently small ε, δ and for every $s \le t$,*

$$\lim_{N\to\infty} \mathbb{E}[Y_N^{\varepsilon,\delta}(s)^2] = \frac{1}{\sqrt{\prod_{1\le i,j\le N}(1 - sv_i(t)v_j(t))}} =: \exp\phi_t(s)$$

Proof: Let $\tilde{W}$ be a fresh independent one dimensional Brownian motion. We set $\tilde{e}_N(\sigma;s) = \exp\left(\frac{\|\sigma.\sigma\|}{N\sqrt{2}}\tilde{W}_s - \frac{Ns}{4}\left\|\frac{\sigma.\sigma\|}{N}\right\|^2\right)$. Thanks to the Markov property, we have $\mathbb{P}$-almost surely:

$$\begin{aligned}
Y_N^{\varepsilon,\delta}(s) &= \mathbb{E}\left[\mathbf{E}_\sigma^{t,\delta} e_N(\sigma;t)\mathbf{1}_{\mathcal{B}_t(f,\varepsilon)}\left(\frac{H_N(\sigma)}{N}\right)\Big|\mathcal{F}_s^{(N)}\right] \\
&= \mathbf{E}_\sigma^{t,\delta}\left[e_N(\sigma;s)\mathbf{1}_{\mathcal{B}_s(f,\varepsilon)}\left(\frac{H_N(\sigma)}{N}\right)\right. && (7) \\
& {}_quad\tilde{\mathbb{E}}\left.\left(\tilde{e}_N(\sigma;t-s);\left\|\frac{\|\sigma.\sigma\|}{N\sqrt{2N}}\tilde{W} + \frac{H_N(\sigma;s)}{N} - f(s+\cdot)\right\|_{[0,t-s]} \le \varepsilon\right)\right] \\
&\le \mathbf{E}_\sigma^{t,\delta}\left[\overline{e}_N^\varepsilon(\sigma;s)\right] =: A_N(\varepsilon,\delta;s) && (8)
\end{aligned}$$

Similarly, on $\{H_N(\sigma)/N \in \mathcal{B}_s(f,\varepsilon/2)\} \subset \{H_N(\sigma)/N \in \mathcal{B}_s(f,\varepsilon)\}$, since f is linear, we have:

$$\left\{\left\|\frac{\|\sigma.\sigma\|}{N\sqrt{2N}}\tilde{W} - f\right\|_{[0,t-s]} \le \varepsilon/2\right\} \subset \left\{\left\|\frac{\|\sigma.\sigma\|}{N\sqrt{2N}}\tilde{W} + \frac{H_N(\sigma;s)}{N} - f(s+\cdot)\right\|_{[0,t-s]} \le \varepsilon\right\}$$

Whence we deduce that:

$$
\begin{aligned}
&Y_N^{\varepsilon,\delta}(s)\\
&\geq \mathbf{E}_\sigma^{t,\delta}\left[\overline{e}_N^{\varepsilon/2}(\sigma;s)\tilde{\mathbb{E}}\left(\tilde{e}_N(\sigma;t-s);\left\|\frac{\|\sigma.\sigma\|}{N\sqrt{2N}}\tilde{W}+\frac{H_N(\sigma;s)}{N}-f(s+\cdot)\right\|_{[0,t-s]}\leq\varepsilon\right)\right]\\
&\geq \mathbf{E}_\sigma^{t,\delta}\left[\overline{e}_N^{\varepsilon/2}(\sigma;s)\tilde{\mathbb{E}}\left(\tilde{e}_N(\sigma;t-s);\left\|\frac{\|\sigma.\sigma\|}{N\sqrt{2N}}\tilde{W}-f(\cdot)\right\|_{[0,t-s]}\leq\frac{\varepsilon}{2}\right)\right]\\
&=: B_N(\varepsilon/2,\delta;s)
\end{aligned}
$$

The proof of proposition 3.1 will the a consequence of the following results:

$$\mathbb{E}[A_N(\varepsilon,\delta;s)]^2=\exp\phi_t(s)(1+o(1)) \tag{9}$$

$$\mathbb{E}[A_N(\varepsilon,\delta;s)-B_N(\varepsilon/2,\delta;s)]^2=o(1) \tag{10}$$

■

3.2 Proof of (9)

Let us rewrite $\mathbb{E}A_N(\varepsilon,\delta;s)^2$ using two independent "replicas" of the system:

$$\mathbb{E}A_N(\varepsilon,\delta;s)^2=\mathbb{E}\otimes\mathbf{E}_{\sigma,\tau}^{t,\delta}\left[\overline{e}_N^{\varepsilon}(\sigma;s)\overline{e}_N^{\varepsilon}(\tau;s)\right] \tag{11}$$

where $\mathbf{P}_{\sigma,\tau}^{t,\delta}=\mathbf{P}_\sigma^{t,\delta}\otimes\mathbf{P}_\tau^{t,\delta}$. We notice that $(H_N(\sigma;\cdot)/N,H_N(\tau;\cdot)/N))$ is a two-dimensional Brownian motion with variance $K(\frac{\sigma.\sigma}{N},\frac{\tau.\tau}{N},\frac{\sigma.\tau}{N})/N$ where K is defined by:

$$K\left|\begin{array}{ccc}\mathcal{S}_d\times\mathcal{S}_d\times\mathcal{M}_d & \to & \mathcal{S}_2\\ x,y,z & \mapsto & \frac{1}{2}\begin{pmatrix}\|x\|^2 & \|z\|^2\\ \|z\|^2 & \|y\|^2\end{pmatrix}\end{array}\right.$$

Hence, by introducing a fresh independent two dimensional Brownian motion W, we get the following identity in law on $\mathcal{C}_0([0,t],\mathbb{R}^2)$ under $\mathbb{P}\otimes\mathbf{P}_{\sigma,\tau}^{t,\delta}$:

$$(H_N(\sigma;\cdot)/N,H_N(\tau;\cdot)/N))\stackrel{\mathcal{L}}{=}K(\frac{\sigma.\sigma}{N},\frac{\tau.\tau}{N},\frac{\sigma.\tau}{N})^{1/2}\frac{W}{\sqrt{N}}$$

The equivalent required in (11) is obviously related to the following variational problem ($^{\mathbf{T}}$ means transpose):

$$
\begin{aligned}
\sup\Big\{&\frac{t-s}{4}(\|x\|^2+\|y\|^2)+\langle K(x,y,z)^{1/2}(1,1)^{\mathbf{T}}|\varphi(s)\rangle\\
&\qquad\qquad-\Lambda_2^*(x,y,z)-I_2^s(\varphi)-2\gamma(t):\\
&x,y\in\mathcal{S}_d,z\in\mathcal{M}_d,\varphi\in\mathcal{C}_0([0,s],\mathbb{R}^2),\\
&\|x-v\|\leq\varepsilon,\|y-v\|\leq\varepsilon,\|K(x,y,z)^{1/2}\varphi-(f,f)^{\mathbf{T}}\|_{[0,s]}\leq\delta\Big\}
\end{aligned}
\tag{12}
$$

Lemma 3.2 *Under H2, for every sufficiently small ε, δ, the variational problem (12) admits $(v, v, 0, f_0, f_0)$ as unique solution. Furthermore, the maximum is then non-degenerate in the sense of [4].*

We prove this lemma later on, after remark 3.2. We now complete the proof of (9) by means of Laplace method. Let us write the following Taylor expansion in a neighbourhood of $(v, v, 0, f_0, f_0)$:

$$\begin{aligned} &\frac{t-s}{4}(\|x+v\|^2 + \|y+v\|^2) + \langle K(x+v, y+v, z)^{1/2}(1,1)^{\mathbf{T}} | \varphi(s) + (f_0, f_0)^{\mathbf{T}} \rangle \\ &\qquad = \frac{t}{2}(\langle v|x\rangle + \langle v|y\rangle) + \frac{t\|v\|^2}{2} + \frac{t}{4}(\|x\|^2 + \|y\|^2) + \frac{s}{2}\|z\|^2 \\ &\qquad + \frac{1}{\|v\|\sqrt{2}}(\langle v|x\rangle\varphi_1(s) + \langle v|y\rangle\varphi_2(s)) - \frac{s}{4\|v\|^2}(\langle v|x\rangle^2 + \langle v|y\rangle^2) \end{aligned}$$

Let Σ be the covariance matrix of $(\sigma\otimes\sigma, \tau\otimes\tau, \sigma\otimes\tau)$ under $\hat{\mu}_t(d\sigma)\otimes\hat{\mu}_t(d\tau)$ on $\mathcal{S}_d\times\mathcal{S}_d\times\mathcal{M}_d$, and (ξ, η, ζ) be a $\mathcal{N}(0, \Sigma)$ random vector. Then , according to Bolthausen's results, the following convergence holds:

$$\lim_{N\to\infty} \mathbb{E}[A_N(\varepsilon, \delta; s)]^2 = \frac{\mathbb{E}\exp\frac{t}{4}(\|\xi\|^2 + \|\eta\|^2) + \frac{s}{2}\|\zeta\|^2)}{[\mathbb{E}\exp\frac{t}{4}(\|\xi\|^2)]^2}$$

One easily checks that the matrix Σ may be written as follows:

$$\Sigma = \begin{pmatrix} \Gamma(t) & 0 & 0 \\ 0 & \Gamma(t) & 0 \\ 0 & 0 & V \end{pmatrix}$$

The operator V may be expressed on $\mathcal{M}_d$ by $VM = vMv$. A simple calculation shows that it may be diagonalized, with $v_i(t)v_j(t)$, $1 \le i, j \le d$ as eigenvalues.

Remark 3.1 By making use of Varadhan's theorem, one can again localize $\mathbb{E}[A_N(\varepsilon, \delta; s)]^2$. This means that if we set:

$$\tilde{A}_N(\varepsilon, \delta; s) = \mathbf{E}^{t,\delta}_{\sigma,\tau}\left[\overline{e}^{\varepsilon}_N(\sigma; s)\overline{e}^{\varepsilon}_N(\tau; s)\mathbf{1}_{\|\frac{\sigma.\tau}{N}\|\le\delta}\right]$$

we can prove as in (5) that:

$$\mathbb{E}\left|[A_N(\varepsilon, \delta; s)]^2 - \tilde{A}_N(\varepsilon, \delta; s)\right| = o\left(\mathbb{E}[A_N(\varepsilon, \delta; s)]^2\right) \qquad (13)$$

Remark 3.2 In a similar way, we have:

$$\mathbb{E}[A_N(\varepsilon,\delta;s) - A_N(\varepsilon/2,\delta;s)]^2 = o(\mathbb{E}[A_N(\varepsilon,\delta;s))^2)$$

Proof of Proposition 3.2: We define a function Ψ by:

$$\Psi \left| \begin{array}{l} \mathcal{S}_d \times \mathcal{S}_d \times \mathcal{M}_d \times \mathcal{C}_0([0,s],\mathbb{R}^2) \to \mathbb{R} \cup \{-\infty\} \\ \quad x,y,z,\varphi \mapsto \dfrac{t-s}{4}(\|x\|^2 + \|y\|^2) + \langle K(x,y,z)^{1/2}(1,1)^{\mathbf{T}} | \varphi(s)\rangle \\ \qquad\qquad\qquad\qquad -\Lambda_2^*(x,y,z) - I_2^s(\varphi) - 2\gamma(t) \end{array} \right.$$

Note that $\Psi(v,v,0,f_0,f_0) = 0$. We first prove that $(v,v,0,f_0,f_0)$ is a non degenerate local maximum for the function Ψ, and then we show that $H2$ is a necessary and sufficient condition for the lemma to hold. We of course restrict ourselves to $\varphi \in H^1$. The Cramer transform Λ_2^* is of class $\mathcal{C}^\infty$ in a neighbourhood of $(v,v,0)$. We may write its Taylor expansion up to the second order as follows:

$$\begin{aligned} \Lambda_2^*(v+x,v+y,z) &= \Lambda_2^*(v,v,0) + \frac{t}{2}(\langle v|x\rangle + \langle v|y\rangle) + \frac{1}{2}D^2\Lambda_1^*(v)[x^2] \\ &\quad + \frac{1}{2}D^2\Lambda_1^*(v)[y^2] + \frac{1}{2}V^{-1}[z^2] + o(\|x\|^2 + \|y\|^2 + \|z\|^2) \end{aligned} \tag{14}$$

with V the covariance of $\sigma\otimes\tau$ on $\mathcal{M}_d$ under $\hat{\mu}_t(d\sigma)\otimes\hat{\mu}_t(d\tau)$. In a neighbourhood of $(v,v,0)$, the function $K^{1/2}$ is also of class $\mathcal{C}^\infty$ as it is easily checked by means of the implicit functions theorem. One checks that:

$$\begin{aligned} K^{1/2}(v+x,v+y,z) &= \frac{1}{\sqrt{2}}\begin{pmatrix} \|v\| & 0 \\ 0 & \|v\| \end{pmatrix} + \frac{1}{\|v\|\sqrt{2}}\begin{pmatrix} \langle v|x\rangle & 0 \\ 0 & \langle v|y\rangle \end{pmatrix} \\ &\quad + \frac{1}{\|v\|\sqrt{2}}\begin{pmatrix} \|x\|^2 - \frac{\langle v|x\rangle^2}{\|v\|^2} & \|z\|^2 \\ \|z\|^2 & \|y\|^2 - \frac{\langle v|y\rangle^2}{\|v\|^2} \end{pmatrix} + o(\|x\|^2 + \|y\|^2 + \|z\|^2) \end{aligned}$$

Summing up both expansions, we have:

$$\begin{aligned} &\Psi(v+x,v+y,z,f_0+\varphi_1,f_0+\varphi_2) - \Psi(v,v,0,f_0,f_0) \\ &= -\frac{1}{2}(D^2\Lambda_1^*(v)[x]^2 - \frac{t}{2}\|x\|^2) - \frac{1}{2}(D^2\Lambda_1^*(v)[y]^2 - \frac{t}{2}\|y\|^2) - \frac{1}{2}(V^{-1}[z]^2 - s\|z\|^2) \\ &\quad - \frac{1}{2}\int_0^s \left\| \dot{\varphi}_1(u)\frac{v}{\|v\|} - \frac{x}{\sqrt{2}} \right\|^2 du - \frac{1}{2}\int_0^s \left\| \dot{\varphi}_2(u)\frac{v}{\|v\|} - \frac{y}{\sqrt{2}} \right\|^2 du \\ &\qquad + o(\|x\|^2 + \|y\|^2 + \|z\|^2) \end{aligned}$$

This clearly proves that $(v,v,0,f_0,f_0)$ is a non degenerate local maximum for Ψ.

Let us now take $\varepsilon=\delta=0$ in (12) and evaluate the following quantity depending on z:

$$\sup\Big\{\frac{t-s}{2}\|v\|^2+\langle(1,1)^{\mathbf{T}}|(f(s),f(s))^{\mathbf{T}}\rangle-\Lambda_2^*(v,v,z)-I_2^s(\varphi)-2\gamma(t):\\ \varphi\in\mathcal{C}_0([0,s],\mathbb{R}^2),\|K(v,v,z)^{1/2}\varphi-(f,f)^{\mathbf{T}}\|_{[0,s]}=0\Big\} \quad (15)$$

Let us study the constraint ion $\varphi=(\varphi_1,\varphi_2)$: If $\|z\|\neq\|v\|$, the matrix $K(v,v,z)^{1/2}$ is invertible and the only possibility is $\varphi_1=\varphi_2=\sqrt{2}f/\sqrt{\|v\|^2+\|z\|^2}$. Hence we have:

$$I_2^s(\varphi)=\frac{s\|v\|^4}{2(\|v\|^2+\|z\|^2)}$$

If now $\|z\|=\|v\|$, then for every φ satisfying the constraint there exists a continuous function w such that $\varphi_1=\sqrt{2}f/\sqrt{\|v\|^2+\|z\|^2}+w$, $\varphi_2=\sqrt{2}f/\sqrt{\|v\|^2+\|z\|^2}-w$. Denote this φ by φ^w. One easily checks that $I_2(\varphi^w)$ is minimum for $w=0$. Hence in any case,

$$I_2^s(\varphi)=\frac{s\|v\|^4}{2(\|v\|^2+\|z\|^2)}$$

It is now clear that $H2$ implies that when $\varepsilon=\delta=0$, $(v,v,0,f_0,f_0)$ is the maximum of Ψ. Since this point is a strict local maximum for Ψ, and Ψ is an upper semi-continuous function with compact level sets, for sufficiently small $\varepsilon,\delta>0$, the point $(v,v,0,f_0,f_0)$ remains the only maximum. The proof has also shown the necessity of $H2$ for the lemma to hold. This completes the proof of lemma 3.2.

3.3 Proof of (10)

Thanks to remark 3.2 , we are lead to prove that for sufficiently small $\varepsilon,\delta>0$ we have :

$$\mathbb{E}[A_N(\varepsilon,\delta;s)-B_N(\varepsilon,\delta;s)]^2=o(\mathbb{E}[A_N(\varepsilon,\delta;s))^2)$$

Write this expectation as follows, using two fresh independent one dimensional Brownian motions $\tilde{W}$ and $\hat{W}$:

$$\mathbb{E}[A_N(\varepsilon,\delta;s)-B_N(\varepsilon,\delta;s)]^2=\mathbb{E}\otimes\mathbf{E}^{t,\delta}_{\sigma,\tau}\Bigg[\overline{e}^{\varepsilon}_N(\sigma;s)\overline{e}^{\varepsilon}_N(\tau;s)$$

$$\tilde{\mathbb{E}}\left(\tilde{e}_N(\sigma;t-s)\mathbf{1}_{\left\|\frac{\|\sigma.\sigma\|}{N\sqrt{2N}}\tilde{W}-f\right\|_{[0,t-s]}>\varepsilon}\right)\hat{\mathbb{E}}\left(\hat{e}_N(\tau;t-s)\mathbf{1}_{\left\|\frac{\|\tau.\tau\|}{N\sqrt{2N}}\hat{W}-f\right\|_{[0,t-s]}>\varepsilon}\right)\Bigg]$$

Using Varadhan's theorem again, we get:

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \ln \mathbb{E}[A_N(\varepsilon,\delta;s) - B_N(\varepsilon,\delta;s)]^2 \leq$$

$$\sup \Big\{ \langle K(x,y,z)^{1/2}(1,1)^{\mathbf{T}} | \varphi(s)\rangle + \frac{\|x\|}{\sqrt{2}}\tilde{\varphi}(t-s) + \frac{\|y\|}{\sqrt{2}}\hat{\varphi}(t-s)$$
$$- \Lambda_2^*(x,y,z) - I_2^s(\varphi) - I_1^{t-s}(\tilde{\varphi}) - I_1^{t-s}(\hat{\varphi}) - 2\gamma(t)$$
$$x,y \in \mathcal{S}_d, z \in \mathcal{M}_d, \varphi \in \mathcal{C}_0([0,s],\mathbb{R}^2), \tilde{\varphi} \in \mathcal{C}_0([0,t-s],\mathbb{R}), \hat{\varphi} \in \mathcal{C}_0([0,t-s],\mathbb{R}),$$
$$\|x-v\| \leq \varepsilon, \|y-v\| \leq \varepsilon, \|K(x,y,z)^{1/2}\varphi - (f,f)^{\mathbf{T}}\|_{[0,s]} \leq \delta,$$
$$\left\| \frac{\|x\|}{\sqrt{2}}\tilde{\varphi} - f \right\|_{[0,t-s]} \geq \varepsilon, \left\| \frac{\|y\|}{\sqrt{2}}\hat{\varphi} - f \right\|_{[0,t-s]} \geq \varepsilon \Big\}$$

We first relax the constraints on $\tilde{\varphi}$ and $\hat{\varphi}$. We then maximize in $\tilde{\varphi}$ and $\hat{\varphi}$, with x, y, z, φ being fixed and we get exactly problem (16). Hence the maximum is uniquely achieved at $(v, v, 0, f_{0|[0,s]}, f_{0|[0,s]}, f_{0|[0,t-s]}, f_{0|[0,t-s]})$ and the value is obviously $2\gamma(t)$. Since we are trying to maximize an upper semi-continuous with compact level sets on the closed set of the constraints, the maximum is achieved at least at a point. As this one cannot be $(v, v, 0, f_{0|[0,s]}, f_{0|[0,s]}, f_{0|[0,t-s]}, f_{0|[0,t-s]})$, the maximum is strictly smaller than $2\gamma(t)$, whence the result follows.

4 Convergence in law to a log-normal process

As $(Y_N^{\varepsilon,\delta}(s))_{s\leq t}$ is a positive continuous martingale, it is possible to define its martingale logarithm, namely:

$$M_N^{\varepsilon,\delta}(s) = \int_0^s \frac{dY_N^{\varepsilon,\delta}(u)}{Y_N^{\varepsilon,\delta}(u)}, \qquad s \leq t$$

Furthermore, we know that $Y_N^{\varepsilon,\delta}(s) = \mathbb{E}[Y_N^{\varepsilon,\delta}]\exp[M_N^{\varepsilon,\delta}(s) - \langle M_N^{\varepsilon,\delta}\rangle(s)/2]$. The method in [5] is to prove the convergence of $\langle M_N^{\varepsilon,\delta}\rangle(s)$ towards ϕ_t uniformly in probability. This requires a strong control on the derivative of the bracket.

In order to get such a control, we are going to write the predictable representation for $Y_N^{\varepsilon,\delta}$, which will enable us to give an explicit formula for the bracket by means of the Clark-Haussmann-Ocone formula [12]. This however induces some difficulties because of the truncation function $\mathbf{1}_{\mathcal{B}_s(f,\varepsilon)}\left(\frac{H_N(\sigma)}{N}\right)$. We will first replace it by a smooth function $F_N^{\varepsilon,\delta}$ and then use our machinery.

4.1 A smoothly truncated partition function

Concerning the Malliavin calculus, we will use the notations of Nualart[11]. In particular $\mathbb{D}^{+\infty}$ stands for the space of smooth functions on the Wiener space, and D is the derivation operator.

Let us recall a result of Sugita [14]. A careful reading of the proofs of Sugita's lemmas 2.2 and 2.5 shows that the following result holds:

Lemma 4.1 *Let $0 < R_1 < R_2$ two real numbers. There exists a sequence $F_N : \mathcal{C}_0([0,t],\mathbb{R}) \to \mathbb{R}$, such that:*

1. *F_N is continuous, $F_N \in \mathbb{D}^{+\infty}$,*
2. *$0 \leq F_N \leq 1$,*
3. *$F_N(w) = 1$ if $\|w\| \leq R_1\sqrt{N}$, and $F_N(w) = 0$ if $\|w\| \geq R_2\sqrt{N}$,*
4. *$\sup_N \int_0^t |D_u F_N|^2\, du \in L^\infty$.*

The only modification is the uniformity result *4* which is straightforward following the lines of [14].

Let us now take $R_1 = \varepsilon/2$ and $R_2 = \varepsilon$. Let F_N^ε be a sequence of smooth truncation functions such as defined in the previous lemma. Obviously we have:

$$\mathbf{1}_{\{w:\|w\|_{[0,t]}\leq\varepsilon\sqrt{N}/2\}} \leq F_N^\varepsilon \leq \mathbf{1}_{\{w:\|w\|_{[0,t]}\leq\varepsilon\sqrt{N}\}} \tag{17}$$

We are now considering the following partition function:

$$Z_N^{F_N^\varepsilon,\delta}(t) = \mathbf{E}_\sigma^{t,\delta}\left[e_N(\sigma;t)F_N^\varepsilon\left(\frac{H_N(\sigma;\cdot)}{\frac{\|\sigma.\sigma\|}{\sqrt{2N}}} - \frac{Nf(\cdot)}{\frac{\|\sigma.\sigma\|}{\sqrt{2N}}}\right)\right]$$

We may do so because the law of the argument in F_N is equivalent to the Wiener measure. We then define as previously:

$$Y_N^{F_N^\varepsilon,\delta}(s) = \mathbb{E}[Z_N^{F_N^\varepsilon,\delta}(t)|\mathcal{F}_s^{(N)}]$$

In order to simplify the notations let us write:

$$F_N(\sigma) = F_N^\varepsilon\left(\frac{H_N(\sigma;\cdot)}{\frac{\|\sigma.\sigma\|}{\sqrt{2N}}} - \frac{Nf(\cdot)}{\frac{\|\sigma.\sigma\|}{\sqrt{2N}}}\right)$$

According to (17) the following inequalities hold under $\mathbf{P}_\sigma^{t,\delta}$:

$$\mathbf{1}_{\mathcal{B}_t(f,\frac{\|v\|-\delta}{\sqrt{2}}\varepsilon)}(\frac{H_N(\sigma)}{N}) \leq F_N(\sigma) \leq \mathbf{1}_{\mathcal{B}_t(f,\frac{\|v\|+\delta}{\sqrt{2}}\varepsilon)}(\frac{H_N(\sigma)}{N})$$

Hence clearly:

$$Y_N^{\varepsilon\frac{\|v\|-\delta}{2\sqrt{2}},\delta}(s) \leq Y_N^{F_N^\varepsilon,\delta}(s) \leq Y_N^{\varepsilon\frac{\|v\|+\delta}{\sqrt{2}},\delta}(s) \tag{18}$$

Consequently, $\lim_{N\to\infty}\mathbb{E}[Y_N^{F_N^\varepsilon,\delta}(s)^2] = \exp\phi_t(s)$.

Moreover, for every sufficiently small ε, δ , as consequence of Doob's inequality and of Varadhan's theorem, we have:

$$\mathbb{E}\left[\sup_{s\leq t}[Y_N^{\varepsilon\frac{\|v\|+\delta}{\sqrt{2}},\delta}(s) - Y_N^{\varepsilon\frac{\|v\|+\delta}{2\sqrt{2}},\delta}(s)]^2\right] \leq 4\mathbb{E}\left[Y_N^{\varepsilon\frac{\|v\|+\delta}{\sqrt{2}},\delta}(t) - Y_N^{\varepsilon\frac{\|v\|+\delta}{2\sqrt{2}},\delta}(t)\right]^2 = o(1) \tag{19}$$

The next lemma, which is more general than lemma 3.1 in [5], is the key of the proof as shown by Comets and Neveu.

Lemma 4.2 *Under H1, H2, for every sufficiently small $\varepsilon, \delta > 0$, the following convergence result holds:*

$$\lim_{N\to\infty}\int_0^t \mathbb{E}\left|\frac{d}{ds}\langle Y_N^{F_N^\varepsilon,\delta}\rangle_s - [Y_N^{F_N^\varepsilon,\delta}(s)]^2\phi_t'(s)\right| \, ds = 0$$

The end of the section is devoted to the proof of this lemma. Define $c_{i,j} = 1$ if $i \neq j$ and $1/\sqrt{2}$ else. With this notation we can write:

$$H_N(\sigma;t) = \frac{1}{\sqrt{N}}\sum_{1\leq i\leq j\leq N} c_{i,j}B_{i,j}(t)\langle\sigma(i)|\sigma(j)\rangle$$

Hence the Clark-Haussmann-Ocone formula for $Y_N^{F_N^\varepsilon,\delta}$ reads:

$$\begin{aligned}
&Y_N^{F_N^\varepsilon,\delta}(s) - \mathbb{E}Y_N^{F_N^\varepsilon,\delta} = \\
&+\sum_{1\leq i\leq j\leq N}\int_0^s \mathbb{E}\left[\mathbf{E}_\sigma^{t,\delta}c_{i,j}\frac{\langle\sigma_i|\sigma_j\rangle}{\sqrt{N}}e_N(\sigma;t)F_N(\sigma)\,\Big|\mathcal{F}_u^{(N)}\right]\, dB_{i,j}(u) \qquad (C) \\
&+\sum_{1\leq i\leq j\leq N}\int_0^s \mathbb{E}\left[\mathbf{E}_\sigma^{t,\delta}c_{i,j}\frac{\frac{\langle\sigma_i|\sigma_j\rangle}{\sqrt{N}}}{\frac{\|\sigma.\sigma\|}{\sqrt{2N}}}e_N(\sigma;t)D_uF_N(\sigma)\,\Big|\mathcal{F}_u^{(N)}\right]\, dB_{i,j}(u) \qquad (D)
\end{aligned}$$

Let us introduce again some notations:

$$C_N^{i,j}(u) = \mathbb{E}\left[\mathbf{E}_\sigma^{t,\delta} c_{i,j}\frac{\langle \sigma_i|\sigma_j\rangle}{\sqrt{N}} e_N(\sigma;t)F_N(\sigma)\Big|\mathcal{F}_u^{(N)}\right]$$

$$D_N^{i,j}(u) = \mathbb{E}\left[\mathbf{E}_\sigma^{t,\delta} c_{i,j}\frac{\sqrt{2}\langle \sigma(i)|\sigma(j)\rangle}{\|\sigma.\sigma\|} e_N(\sigma;t)D_uF_N(\sigma)\,\Big|\mathcal{F}_u^{(N)}\right]$$

Proof of lemma 4.2 requires two preliminary lemmas which we now state but that will be proved later on. We shall first show that the term (D) converges to zero in a sufficiently good sense:

Lemma 4.3 *Under H1,H2, for every sufficiently small ε, δ we have:*

$$\int_0^t \sum_{1\le i\le j\le N} \mathbb{E}[D_N^{i,j}(u)^2]\,du = o(1)$$

Then we prove the following convergence for the term (C):

Lemma 4.4 *Under H1, H2, we have for every sufficiently small ε, δ and every $s \le t$:*

$$\mathbb{E}\left|\sum_{1\le i\le j\le N}[C_N^{i,j}(s)]^2 - [Y_N^{\varepsilon,\delta}(s)]^2\phi_t'(s)\right| = o(1)$$

Moreover,

$$\sup_{N\ge 1, s\le t}\mathbb{E}\left|\sum_{i\le j}[C_N^{i,j}(s)]^2 - [Y_N^{\varepsilon,\delta}(s)]^2\phi_t'(s)\right| < \infty \qquad (20)$$

It is now obviously possible by making use of (18) and (19) to replace $Y_N^{\varepsilon,\delta}$ by $Y_N^{F_N^\varepsilon,\delta}$ in lemma 4.4. Now recall that:

$$\frac{d}{ds}\langle Y_N^{F_N^\varepsilon,\delta}\rangle - [Y_N^{F_N^\varepsilon,\delta}(s)]^2\phi_t'(s) = \sum_{1\le i\le j\le N}[C_N^{i,j}(s)]^2 - [Y_N^{\varepsilon,\delta}(s)]^2\phi_t'(s)$$
$$-([Y_N^{F_N^\varepsilon,\delta}(s)]^2 - [Y_N^{\varepsilon,\delta}(s)]^2)\phi_t'(s) + \sum_{1\le i\le j\le N}[D_N^{i,j}(s)]^2 + 2\sum_{1\le i\le j\le N} C_N^{i,j}(s)D_N^{i,j}(s)$$

Hence ,integrating, we get:

$$\int_0^t \mathbb{E}\left|\frac{d}{ds}\langle Y_N^{F_N^\varepsilon,\delta}\rangle - [Y_N^{F_N^\varepsilon,\delta}]^2\phi_t'(s)\right| \, ds \le \int_0^t \sum_{i\le j} \mathbb{E}[D_N^{i,j}(s)]^2 \, ds$$

$$+ 2\left[\int_0^t \sum_{i\le j} \mathbb{E}[C_N^{i,j}(s)]^2 \, ds\right]^{1/2} \left[\int_0^t \sum_{i\le j} \mathbb{E}[D_N^{i,j}(s)]^2 \, ds\right]^{1/2}$$

$$+ \int_0^t \mathbb{E}|\sum_{i\le j}[C_N^{i,j}(s)]^2 - [Y_N^{\varepsilon,\delta}(s)]^2\phi_t'(s)| \, ds$$

$$+ t\sup_{s\le t}\phi_t'(s)\left[\mathbb{E}\sup_{s\le t}[Y_N^{\varepsilon,\delta}(s) - Y_N^{F_N^\varepsilon,\delta}(s)]^2\right]^{1/2}\left[2\mathbb{E}[Y_N^{\varepsilon,\delta}(t)]^2 + 2\mathbb{E}[Y_N^{F_N^\varepsilon,\delta}(t)]^2\right]^{1/2}$$

The first term goes to 0 according to lemma 4.3, the second one by means of lemma 4.3 and (21), the third one by means of lemma 4.4 and (20),the last one by (19). The completes the proof of lemma 4.2.

As an immediate consequence of this result as in [5] we get the convergence result for $((Y_N^{F_N^\varepsilon,\delta}(s))_{s\le t})_{N\ge1}$ and thus for $((Y_N^{\varepsilon,\delta}(s))_{s\le t})_{N\ge1}$:

Theorem 4.5 *Assume H1 and H2. For sufficiently small $\varepsilon,\delta>0$ the sequence of martingales $((Y_N^{\varepsilon,\delta}(s))_{s\le t})_{N\ge1}$ converges in distribution on $[0,t]$ to a log-normal process $(\exp(M_\infty^t(s)-\phi_t(s)))_{s\le t}$ where $(M_\infty^t(s))_{s\le t}$ is a centered gaussian process with independent increments with covariance*

$$\mathbb{E}[M_\infty^t(s') - M_\infty^t(s)]^2 = \phi_t(s') - \phi_t(s), \quad s\le s'\le t$$

Taking $s=t$ in the previous theorem we obtain the following corollary:

Corollary 4.6 *Let ξ be a $\mathcal{N}(0,\phi(t))$ random variable. Then*

$$Z_N^{\varepsilon,\delta}(t)/\mathbf{E}_\sigma\left[\exp\left[\frac{Nt}{4}\left\|\frac{\sigma.\sigma}{N}\right\|^2\right]\mathbf{1}_{\|\frac{\sigma.\sigma}{N}-v\|\le\delta}\right] \xrightarrow{\mathcal{L}} \exp(\xi - \phi(t)/2)$$

Using now (5) the proof of theorem 1.1 is complete.

4.2 Proof of lemma 4.3

Let us recall that the derivation operator on the Wiener space is a local operator (see [11]). In particular, for almost every w we have for $u\le t$:

$D_u F_N^\varepsilon(w)\mathbf{1}_{\|w\|\geq\varepsilon\sqrt{N}} = 0$. Set $\tilde\varepsilon = \frac{\|v\|+\delta}{\sqrt{2}}\varepsilon$:

$$\mathbb{E}\left[D_N^{i,j}(u)^2\right] = \mathbb{E}\left(\mathbb{E}\left[\mathbf{E}_\sigma^{t,\delta} c_{i,j}\frac{\sqrt{2}\langle\sigma(i)|\sigma(j)\rangle}{\|\sigma.\sigma\|}e_N(\sigma;t)D_uF_N(\sigma)\,\Big|\mathcal{F}_u^{(N)}\right]\right)^2$$

$$= \mathbb{E}\left(\mathbb{E}\left[\mathbf{E}_\sigma^{t,\delta} c_{i,j}\frac{\sqrt{2}\langle\sigma(i)|\sigma(j)\rangle}{\|\sigma.\sigma\|}e_N(\sigma;t)D_uF_N(\sigma)\mathbf{1}_{\mathcal{B}_t(f,\tilde\varepsilon)}\left(\frac{H_N(\sigma)}{N}\right)|\mathcal{F}_u^{(N)}\right]\right)^2$$

$$\leq \mathbb{E}\left[\mathbf{E}_\sigma^{t,\delta} c_{i,j}\frac{\sqrt{2}\langle\sigma(i)|\sigma(j)\rangle}{\|\sigma.\sigma\|}\overline{e}_N^{\tilde\varepsilon}(\sigma;t)D_uF_N(\sigma)\right]^2$$

$$\leq \mathbb{E}\left[\left(\mathbf{E}_\sigma^{t,\delta} c_{i,j}^2\frac{2\langle\sigma(i)|\sigma(j)\rangle^2}{\|\sigma.\sigma\|^2}\overline{e}_N^{\tilde\varepsilon}(\sigma;t)\right)\left(\mathbf{E}_\sigma^{t,\delta}\overline{e}_N^{\tilde\varepsilon}(\sigma;t)\left(D_uF_N(\sigma)\right)^2\right)\right]$$

Let us sum over (i,j):

$$\sum_{1\leq i\leq j\leq N}\mathbb{E}[D_N^{i,j}(u)]^2 \leq \mathbb{E}\left[\left(\mathbf{E}_\sigma^{t,\delta}\overline{e}_N^{\tilde\varepsilon}(\sigma;t)\right)\left(\mathbf{E}_\sigma^{t,\delta}\overline{e}_N^{\tilde\varepsilon}(\sigma;t)\left(D_uF_N(\sigma)\right)^2\right)\right]$$

Thus, after integrating, we get:

$$\int_0^t\sum_{1\leq i\leq j\leq N}\mathbb{E}[D_N^{i,j}(u)]^2\,du \leq \mathbb{E}\left[\left(\mathbf{E}_\sigma^{t,\delta}\overline{e}_N^{\tilde\varepsilon}(\sigma;t)\right)\left(\mathbf{E}_\sigma^{t,\delta}\overline{e}_N^{\tilde\varepsilon}(\sigma;t)\int_0^t D_uF_N(\sigma)^2\,du\right)\right]$$

$$\leq C\mathbb{E}\left[\left(\mathbf{E}_\sigma^{t,\delta}\overline{e}_N^{\tilde\varepsilon}(\sigma;t)\right)\left(\mathbf{E}_\sigma^{t,\delta}e_N(\sigma;t)\mathbf{1}_{\frac{\tilde\varepsilon}{2}\leq\left\|\frac{H_N(\sigma;\cdot)}{N}-f(\cdot)\right\|\leq\tilde\varepsilon}\right)\right]$$

$$\leq C\left[\mathbb{E}\left(\mathbf{E}_\sigma^{t,\delta}\overline{e}_N^{\tilde\varepsilon}(\sigma;t)\right)^2\right]^{1/2}\left[\mathbb{E}\left(\mathbf{E}_\sigma^{t,\delta}e_N(\sigma;t)\mathbf{1}_{\frac{\tilde\varepsilon}{2}\leq\left\|\frac{H_N(\sigma;\cdot)}{N}-f(\cdot)\right\|\leq\tilde\varepsilon}\right)^2\right]^{1/2}$$

$$= C\left[\mathbb{E}[Y_N^{\tilde\varepsilon,\delta}]^2\right]^{1/2}\left[\mathbb{E}[Y_N^{\tilde\varepsilon,\delta}(t) - Y_N^{\frac{\tilde\varepsilon}{2},\delta}(t)]^2\right]^{1/2}$$

Now, thanks to (19) , the proof of lemma 4.3 is complete.

$((Y_N^{\varepsilon,\delta}(s))_{s\leq t})_{N\geq 1}$

4.3 Proof of lemma 4.4

We first prove the second part of the lemma, assuming that the first one holds. Using it for $s=t$ we get:

$$\sup_{N\geq 1}\mathbb{E}\sum_{i\leq j}[C_N^{i,j}(t)]^2 < \infty \tag{21}$$

As $([C_N^{i,j}(s)]^2)_{s\le t}$ and $([Y_N^{\varepsilon,\delta}(s)]^2)_{s\le t}$ are positive continuous submartingales,we also have:

$$\sup_{N\ge 1, s\le t} \mathbb{E}\left|\sum_{i\le j}[C_N^{i,j}(s)]^2 - [Y_N^{\varepsilon,\delta}(s)]^2\phi_t'(s)\right| \le \sup_{N\ge 1}\mathbb{E}\sum_{i\le j}[C_N^{i,j}(t)]^2 + \sup_{N\ge 1}\mathbb{E}[Y_N^{\varepsilon,\delta}(s)]^2\phi_t'(t) < \infty$$

Let us now make some simple remarks:

$$\begin{aligned}
&[C_N^{i,j}(s)]^2 \\
&\quad = \left(\mathbf{E}_\sigma^{t,\delta} c_{i,j}\frac{\langle\sigma(i)|\sigma(j)\rangle}{\sqrt{N}}\mathbb{E}\left[e_N(\sigma;t)F_N(\sigma)|\mathcal{F}_s^{(N)}\right]\right)^2 \\
&\quad = \mathbf{E}_{\sigma,\tau}^{t,\delta} c_{i,j}^2\frac{\langle\sigma(i)|\sigma(j)\rangle}{\sqrt{N}}\frac{\langle\tau(i)|\tau(j)\rangle}{\sqrt{N}}\mathbb{E}\left[e_N(\sigma;t)F_N(\sigma)|\mathcal{F}_s^{(N)}\right]\mathbb{E}\left[e_N(\tau;t)F_N(\tau)|\mathcal{F}_s^{(N)}\right]
\end{aligned}$$

Hence, summing over (i,j), we get:

$$\sum_{1\le i\le j\le N}[C_N^{i,j}(s)]^2 = \mathbf{E}_{\sigma,\tau}^{t,\delta}\frac{1}{2}\left\|\frac{\sigma.\tau}{\sqrt{N}}\right\|^2\mathbb{E}\left[e_N(\sigma;t)F_N(\sigma)|\mathcal{F}_s^{(N)}\right]\mathbb{E}\left[e_N(\tau;t)F_N(\tau)|\mathcal{F}_s^{(N)}\right]$$

Let us now introduce the truncation by setting:

$$X_N^{\varepsilon,\delta}(s) = \mathbf{E}_{\sigma,\tau}^{t,\delta}\frac{1}{2}\left\|\frac{\sigma.\tau}{\sqrt{N}}\right\|^2\mathbb{E}[\overline{e}_N^\varepsilon(\sigma;t)|\mathcal{F}_s]\,\mathbb{E}[\overline{e}_N^\varepsilon(\tau;t)|\mathcal{F}_s]$$

According to (17), we get:

$$X_N^{\varepsilon\frac{\|v\|-\delta}{2\sqrt{2}},\delta}(s) \le \sum_{1\le i\le j\le N}[C_N^{i,j}(s)]^2 \le X_N^{\varepsilon\frac{\|v\|+\delta}{\sqrt{2}},\delta}(s) \tag{22}$$

Even if $X_N^{\varepsilon,\delta}$ seems to be still rather complicated, it is in fact much more tractable than $C_N^{i,j}$.

It is very useful to make a careful localization in $X_N^{\varepsilon,\delta}$ in order to stay as near our problem as possible. Thanks to (13), we know that we may keep $\sigma.\tau/N$ as small as needed. Hence we set:

$$\tilde{X}_N^{\varepsilon,\delta}(s) = \mathbf{E}_{\sigma,\tau}^{t,\delta}\frac{1}{2}\left\|\frac{\sigma.\tau}{\sqrt{N}}\right\|^2\mathbf{1}_{\|\frac{\sigma.\tau}{N}\|\le\delta}\mathbb{E}\left[\overline{e}_N^\varepsilon(\sigma;t)|\mathcal{F}_s^{(N)}\right]\mathbb{E}\left[\overline{e}_N^\varepsilon(\tau;t)|\mathcal{F}_s^{(N)}\right]$$

Then, since $\|\sigma.\tau\|^2 \le \|\sigma.\sigma\|.\|\tau.\tau\| \le N^2(\|v\|+\delta)^2$:

$$\mathbb{E}\left|X_N^{\varepsilon,\delta}(s) - \tilde{X}_N^{\varepsilon,\delta}(s)\right| \le \frac{N(\|v\|+\delta)^2}{2}\mathbf{E}_{\sigma,\tau}^{t,\delta}\mathbf{1}_{\|\frac{\sigma.\tau}{N}\|>\delta}\mathbb{E}[\overline{e}_N^{\varepsilon}(\sigma;t)|\mathcal{F}_s]\,\mathbb{E}[\overline{e}_N^{\varepsilon}(\tau;t)|\mathcal{F}_s]$$

Hence, according to (13),we get for $s \le t$:

$$\mathbb{E}\left|X_N^{\varepsilon,\delta}(s) - \tilde{X}_N^{\varepsilon,\delta}(s)\right| = o(1) \tag{23}$$

We are now going to prove the following intermediate lemma and then lemma 4.4 will be an easy corollary of it according to (22).

Lemma 4.7 *Under 1, H2, we have for every sufficiently small ε, δ:*

$$\mathbb{E}|\tilde{X}_N^{\varepsilon,\delta}(s) - [Y_N^{\varepsilon,\delta}(s)]^2\phi_t'(s)| = o(1)$$

Proof of lemma 4.7: Let us transform the quantity we intend to evaluate:

$$\begin{aligned}
&\tilde{X}_N^{\varepsilon,\delta}(s) - [Y_N^{\varepsilon,\delta}(s)]^2\phi_t'(s)\\
&= \mathbf{E}_{\sigma,\tau}^{t,\delta}\left[\left(\frac{1}{2}\left\|\frac{\sigma.\tau}{\sqrt{N}}\right\|^2 - \phi_t'(s)\right)\mathbf{1}_{\|\frac{\sigma.\tau}{N}\|\le\delta}\mathbb{E}\left(\overline{e}_N^{\varepsilon}(\sigma;t)|\mathcal{F}_s^{(N)}\right)\mathbb{E}\left(\overline{e}_N^{\varepsilon}(\tau;t)|\mathcal{F}_s^{(N)}\right)\right]\\
&= e^{\phi_t(s)}\mathbf{E}_{\sigma,\tau}^{t,\delta}\left[G\left(\frac{\sigma.\tau}{\sqrt{N}}\right)\mathbf{1}_{\|\frac{\sigma.\tau}{N}\|\le\delta}e^{-\frac{s}{2}\left\|\frac{\sigma.\tau}{\sqrt{N}}\right\|^2}\mathbb{E}\left(\overline{e}_N^{\varepsilon}(\sigma;t)|\mathcal{F}_s^{(N)}\right)\mathbb{E}\left(\overline{e}_N^{\varepsilon}(\tau;t)|\mathcal{F}_s^{(N)}\right)\right]
\end{aligned}$$

where we have set:

$$G(x) = \frac{d}{ds}\left(e^{\frac{s}{2}\|x\|^2 - \phi_t(s)}\right) \tag{24}$$

The result we wish to prove will be a staightforward corollary of the next lemma. For $A > 0$, denote by $\mathcal{C}_{(A)}$ the following space:

$$\mathcal{C}_{(A)} = \{g \in \mathcal{C}(\mathcal{M}_d, \mathbb{R}), g(x) = o(\exp A\|x\|^2/2) \text{ at } \infty\}$$

Endowed with the norm $\|g\|_{(A)} = \sup_x |g(x)|\exp(-A\|x\|^2/2)$, $\mathcal{C}_{(A)}$ is a Banach space. A careful reading of *H1* and *H2* shows that if t fulfills these assumptions then there exists $\varepsilon(t) > 0$ such that *H2* is true for $s \le t + \varepsilon(t)$.

Lemma 4.8 *Let $\zeta \rightsquigarrow \mathcal{N}(0,V)$ on $\mathcal{M}_d$. Let $\mathcal{C}^0_{(t+\varepsilon(t))} = \{G \in \mathcal{C}_{(t+\varepsilon(t))} : \mathbb{E}G(\zeta) = 0\}$, which is a Banach subspace of $\mathcal{C}_{(t+\varepsilon(t))}$. Define the following linear form on $\mathcal{C}^0_{(t+\varepsilon(t))}$:*

$$\Lambda_N(G) \stackrel{def}{=} \mathbf{E}_{\sigma,\tau}^{t,\delta}\left[G\left(\frac{\sigma.\tau}{\sqrt{N}}\right)\mathbf{1}_{\|\frac{\sigma.\tau}{N}\|\le\delta}e^{-\frac{s}{2}\left\|\frac{\sigma.\tau}{\sqrt{N}}\right\|^2}\mathbb{E}(\overline{e}_N^{\varepsilon}(\sigma;t)|\mathcal{F}_s)\,\mathbb{E}(\overline{e}_N^{\varepsilon}(\tau;t)|\mathcal{F}_s)\right]$$

Then Λ_N is continuous and

$$\forall G \in \mathcal{C}^0_{(t+\varepsilon(t))}, \quad \lim_{N\to\infty}\Lambda_N(G) = 0$$

As our function G in (24) belongs to $\mathcal{C}^0_{(t+\varepsilon(t))}$, we have:

$$\tilde{X}_N^{\varepsilon,\delta}(s) - [Y_N^{\varepsilon,\delta}(s)]^2\phi_t'(s) = e^{\phi_t(s)}\Lambda_N(G) = 1 + o(1)$$

This completes the proof of lemma 4.7

Proof of lemma 4.8: Indeed, the variational problem

$$\sup_{x,y,z}\{\frac{t}{4}(\|x\|^2+\|y\|^2)+\frac{t+\varepsilon(t)}{4}z^2-\Lambda_2^*(x,y,z) : \|x-v\| \leq \delta, \|y-v\| \leq \delta, \|z\| \leq \delta\}$$

admits $(v,v,0)$ as unique maximum for every sufficiently small ε, δ. A Taylor expansion then proves that it is non-degenerate. A standard application of Laplace method then enables to write:

$$\sup_{N\geq 1}\mathbf{E}^{t,\delta}_{\sigma,\tau}e^{\frac{N(t+\varepsilon(t))}{2}\left\|\frac{\sigma.\tau}{N}\right\|^2}\mathbf{1}_{\|\frac{\sigma.\eta}{N}\|\leq\delta} < \infty$$

And since

$$|\Lambda_N(G)| \leq \|G\|_{(t+\varepsilon(t))}\mathbf{E}^{t,\delta}_{\sigma,\tau}e^{\frac{N(t+\varepsilon(t))}{2}\left\|\frac{\sigma.\tau}{N}\right\|^2}\mathbf{1}_{\|\frac{\sigma.\eta}{N}\|\leq\delta}$$

the proof of the continuity of Λ_N is completed.

Let us prove the second part of the lemma. Continuous and bounded functions are dense in $\mathcal{C}_{(t+\varepsilon(t))}$ which enables us to replace a function $G \in \mathcal{C}^0_{(t+\varepsilon(t))}$ by such a function. So we now assume G to be continuous and bounded. As a first step, we are going to replace the conditional expectation $\mathbb{E}\left(\overline{e}^{\varepsilon}_N(\sigma;t)|\mathcal{F}_s^{(N)}\right)$ by $\overline{e}^{\varepsilon}_N(\sigma;s)$ and the same for τ. This will be possible because $\overline{e}^{\varepsilon}_N(\sigma;s)$ is a supermartingale (see (6)). Define:

$$\begin{aligned}\Delta_N(G) = \mathbb{E}\otimes\mathbf{E}^{t,\delta}_{\sigma,\tau}&\left[G\left(\frac{\sigma.\tau}{\sqrt{N}}\right)\mathbf{1}_{\|\frac{\sigma.\tau}{N}\|\leq\delta}e^{-\frac{s}{2}\left\|\frac{\sigma.\tau}{\sqrt{N}}\right\|^2}\mathbb{E}\left(\overline{e}^{\varepsilon}_N(\sigma;t)|\mathcal{F}_s\right)\mathbb{E}\left(\overline{e}^{\varepsilon}_N(\tau;t)|\mathcal{F}_s\right)\right]\\ &- \mathbb{E}\otimes\mathbf{E}^{t,\delta}_{\sigma,\tau}\left[G\left(\frac{\sigma.\tau}{\sqrt{N}}\right)\mathbf{1}_{\|\frac{\sigma.\tau}{N}\|\leq\delta}e^{-\frac{s}{2}\left\|\frac{\sigma.\tau}{\sqrt{N}}\right\|^2}\overline{e}^{\varepsilon}_N(\sigma;s)\overline{e}^{\varepsilon}_N(\tau;s)\right]\end{aligned}$$

Now using the supermartingale property and the Cauchy-Schwarz inequality we get:

$$
\begin{aligned}
|\Delta_N(G)| = \Bigg| \mathbb{E}\otimes \mathbf{E}^{t,\delta}_{\sigma,\tau}\Bigg[& G\left(\frac{\sigma.\tau}{\sqrt{N}}\right)\mathbf{1}_{\|\frac{\sigma.\tau}{N}\|\leq\delta} e^{-\frac{s}{2}\left\|\frac{\sigma.\tau}{\sqrt{N}}\right\|^2} \\
& \Bigg(\left(\mathbb{E}\left(\overline{e}^{\varepsilon}_N(\sigma;t)|\mathcal{F}_s\right) - \overline{e}^{\varepsilon}_N(\sigma;s)\right)\mathbb{E}\left(\overline{e}^{\varepsilon}_N(\tau;t)|\mathcal{F}_s\right) \\
& \quad + \overline{e}^{\varepsilon}_N(\sigma;s)\left(\mathbb{E}\left(\overline{e}^{\varepsilon}_N(\tau;t)|\mathcal{F}_s\right) - \overline{e}^{\varepsilon}_N(\tau;s)\right)\Bigg)\Bigg]\Bigg| \\
& \leq 2\|G\|_\infty \left[\mathbb{E}(Y^{\varepsilon,\delta}_N(s) - A_N(\varepsilon,\delta;s))^2\right]^{1/2}\left[\mathbb{E}A_N(\varepsilon,\delta;s)^2\right]^{1/2}
\end{aligned}
$$

As the first term goes to zero and the second one is bounded, the difference $\Delta_N(G)$ goes to zero.

The proof now follows the lines of [5]:

$$
\begin{aligned}
\left(\mathbb{E}\otimes \mathbf{E}^{t,\delta}_{\sigma,\tau}\left[G\left(\frac{\sigma.\tau}{\sqrt{N}}\right)\mathbf{1}_{\|\frac{\sigma.\tau}{N}\|\leq\delta} e^{-\frac{s}{2}\left\|\frac{\sigma.\tau}{\sqrt{N}}\right\|^2}\overline{e}^{\varepsilon}_N(\sigma;s)\overline{e}^{\varepsilon}_N(\tau;s)\right]\right)^2 & \\
= U_N(G)\times \mathbb{E}\otimes \mathbf{E}^{t,\delta}_{\sigma}\overline{e}^{\varepsilon}_N(\sigma;t) &
\end{aligned}
$$

with

$$
\begin{aligned}
U_N(G) = \mathbb{E}\otimes \mathbf{E}^{t,\delta}_{\sigma,\tau,\eta}\Bigg[& G\left(\frac{\sigma.\tau}{\sqrt{N}}\right)G\left(\frac{\sigma.\eta}{\sqrt{N}}\right)e^{-\frac{Ns}{2}\|\frac{\sigma.\tau}{N}\|^2-\frac{Ns}{2}\|\frac{\sigma.\eta}{N}\|^2} \\
& \overline{e}^{\varepsilon}_N(\sigma;s)\overline{e}^{\varepsilon}_N(\tau;s)\overline{e}^{\varepsilon}_N(\eta;s)\mathbf{1}_{\|\frac{\sigma.\tau}{N}\|\leq\delta}\mathbf{1}_{\|\frac{\sigma.\eta}{N}\|\leq\delta}\Bigg] \qquad (25)
\end{aligned}
$$

As in section 3.1 (8), we now prove the convergence of $U_N(G)$.

Limit of $U_N(G)$: We obviously wish to apply the central limit theorem, which will be a consequence of Laplace method. Let $\mathcal{K}$ be the following function:

$$
\mathcal{K}\left|\begin{array}{ccc}
\mathcal{S}_d\times\mathcal{S}_d\times\mathcal{S}_d\times\mathcal{M}_d\times\mathcal{M}_d\times\mathcal{M}_d & \to & \mathcal{S}_3 \\
x,y,z,x_1,x_2,x_3 & \mapsto & \frac{1}{2}\begin{pmatrix} \|x\|^2 & \|x_1\|^2 & \|x_2\|^2 \\ \|x_1\|^2 & \|y\|^2 & \|x_3\|^2 \\ \|x_2\|^2 & \|x_3\|^2 & \|z\|^2 \end{pmatrix}
\end{array}\right.
$$

The precise asymptotic in (25) is clearly related to the following variational problem:

$$\sup\{\langle \mathcal{K}(x,y,z,x_1,x_2,x_3)^{1/2}(1,1,1)^{\mathbf{T}}|\varphi(s)\rangle + \frac{t-s}{4}[\|x\|^2+\|y\|^2+\|z\|^2]$$
$$-\frac{s}{2}[\|x_1\|^2+\|x_2\|^2] - \Lambda_3^*(x,y,z,x_1,x_2,x_3) - I_3(\varphi) - 3\gamma(t) :$$
$$\|x-v\|\leq\delta, \|y-v\|\leq\delta, \|z-v\|\leq\delta, \|x_1\|\leq\delta, \|x_2\|\leq\delta,$$
$$\|\mathcal{K}(x,y,z,x_1,x_2,x_3)^{1/2}\varphi - (f,f,f)^*\|_{[0,s]}\leq\varepsilon\Big\} \tag{26}$$

As we are now used to doing, we first take $\varepsilon=\delta=0$. As we have:

$$\Lambda_3^*(v,v,v,0,0,x_3) = \Lambda_2^*(v,v,x_3) + \Lambda_1^*(v)$$

we are exactly lead to the problem (15). Hence, the maximum is achieved at $x_3=0$ and $\varphi=(f_0,f_0,f_0)$. A Taylor expansion enables to check that $(v,v,v,0,0,0,(f_0,f_0,f_0))$ is a non-degenerate local maximum. Hence, for sufficiently small ε,δ, it is the unique solution of problem (26). We may now complete the proof by using a standard Laplace method:

$$\lim_{N\to\infty}\frac{U_N(G)}{U_N(1)} = \mathbb{E}_{\mathcal{N}(0,V\otimes V)}G(\zeta_1)G(\zeta_2) = 0$$

The only point that remains to be noticed is the following:

$$U_N(1) = (\det(I_d - sV))^{-1}(1+o(1))$$

This completes the proof of lemma 4.8.

5 Example: Heisenberg spins

In this example, we consider 2−dimensional spins with uniform distribution on the circle of radius $R>0$. Clearly the assumption (2) holds because ρ is of compact support. Let us denote by $v=R^2I_2/2$ the expectation of $\sigma\otimes\sigma$. (I_2 is the identity matrix of dimension 2)

The first result is the following: for $t>0$ small enough, the variational problem in $(H1)$ admits v as unique and non degenerate solution.

Under $\rho(d\sigma)$ it holds that $trace(\sigma\otimes\sigma)=R^2$. Hence Λ_1^* is infinite outside the affine hyperplane $\mathcal{H}=\{trace(x)=R^2\}$ in $\mathcal{S}_2$. Thus

$$\Lambda_1^*(x) = \sup_{\substack{\lambda\in\mathcal{S}_2\\ trace(\lambda)=0}}\left\{\langle x-v|\lambda\rangle - \ln\mathbf{E}_\sigma\exp\langle\sigma\otimes\sigma|\lambda\rangle\right\}, \qquad x\in\mathcal{H}$$

We can also compute:

$$\mathbf{E}_\sigma \exp\langle \sigma\otimes\sigma|\lambda\rangle = \mathcal{I}_0\left(\frac{R^2\|\lambda\|}{\sqrt{2}}\right)$$

where $\mathcal{I}_0$ denotes the Bessel function $\mathcal{I}_0(r) = 1/2\pi\int_0^{2\pi}\exp(r\cos\theta)\,d\theta$, whence we deduce:

$$\Lambda_1^*(x) = \sup_{r\geq 0}\left\{r\|x-v\| - \ln\mathcal{I}_0(\frac{R^2 r}{\sqrt{2}})\right\}$$

Restricting us to $x\in\mathcal{H}$, we have to maximize:

$$\frac{t}{4}\|x\|^2 - \Lambda_1^*(x) = \frac{t}{4}\|v\|^2 + \frac{t}{4}\|x-v\|^2 - \Lambda_1^*(x)$$

One can check that $\ln\mathcal{I}_0(r)\leq r^2/4$ thus $\Lambda_1^*(x)\geq\frac{2\|x-v\|^2}{R^4}$. As a conclusion, as soon as $tR^4<8$, the maximum in $(H1)$ is uniquely achieved at v and is non-degenerate. If $tR^4=8$, point v is still the unique maximum but is degenerate and if $tR^4>8$, point v is not a maximum any longer.

The real problem is to check assumption $H2.2$. Let us denote by Λ_3^* the Cramer transform of the distribution of $\sigma\otimes\tau$ under $\rho^{\otimes 2}$. It is easy to see that:

$$\Lambda_2^*(v,v,z) - \Lambda_2^*(v,v,0)\geq\Lambda_3^*(z) \text{ and } \frac{s\|v\|^4}{2(\|v\|^2+\|z\|^2)}\geq\frac{s\|v\|^2}{2}-\frac{s\|z\|^2}{2}$$

Hence it is clear that assumption $(H2.2)$ is satisfied as soon as $\Lambda_3^*(z)-\frac{s\|z\|^2}{2}$ achieves its minimum uniquely at $z=0$ for every $s\leq t$ which is easily seen to be equivalent to $\Lambda_3^*(z)\geq\frac{t\|z\|^2}{2}$ with equality only when $z=0$.

Let $\lambda = \begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\mathcal{M}_2$. We compute the Laplace transform of the distribution of $\sigma\otimes\tau$:

$$\mathbf{E}_{\sigma,\tau}\exp\langle\sigma\otimes\tau|\lambda\rangle = \frac{1}{2\pi}\int_0^{2\pi}\mathcal{I}_0\left(R^2\sqrt{\frac{a^2+b^2+c^2+d^2}{2}+\alpha\cos\theta}\right)\,d\theta$$

for some positive $\alpha=\alpha(a,b,c,d)\leq\frac{a^2+b^2+c^2+d^2}{2}$. Function $\mathcal{I}_0$ is increasing on $\mathbb{R}^+$ thus

$$\begin{aligned}\ln\mathbf{E}_{\sigma,\tau}\exp\langle\sigma\otimes\tau|\lambda\rangle &\leq \ln\mathcal{I}_0\left(R^2\sqrt{a^2+b^2+c^2+d^2}\right)\\ &\leq R^4\frac{a^2+b^2+c^2+d^2}{4} = \frac{R^4\|\lambda\|^2}{4}\end{aligned}$$

As a immediate consequence, we get $\Lambda_3^*(z) \geq \|z\|^2/R^4$. Hence, if $t < 2/R^4$, assumption $(H2.2)$ is fulfilled.

Our results now lead to the following conclusions. Let $t < 2/R^4$. Then

$$\mathbb{E}Z_N(t) = \left(1 - \frac{tR^4}{8}\right)^{-1} \exp \frac{NtR^2}{8}(1 + o(1))$$

Furthermore,

$$Z_N(t)/\mathbb{E}Z_N(t) \xrightarrow{\mathcal{L}} \left(1 - \frac{tR^4}{4}\right) \exp \xi$$

where ξ is a $\mathcal{N}(0, -2\ln\left(1 - \frac{tR^4}{4}\right))$ random variable.

We can generalize the previous results to $d-$dimensional spins. Let ρ be the uniform distribution on the sphere of radius R in $\mathbb{R}^d$, then there exists a $t_d^* > 0$ such that for $t < t_d^*$ we have:

$$\mathbb{E}Z_N(t) = \left(1 - \frac{tR^4}{d(d+2)}\right)^{-\frac{(d-1)(d+2)}{4}} \exp \frac{NtR^2}{4d}(1 + o(1))$$

and

$$Z_N(t)/\mathbb{E}Z_N(t) \xrightarrow{\mathcal{L}} \left(1 - \frac{tR^4}{d^2}\right)^{d^2/4} \exp \xi$$

where ξ is a $\mathcal{N}(0, -\frac{d^2}{2}\ln\left(1 - \frac{tR^4}{d^2}\right))$ random variable.

We recognize Gabay and Toulouse temperatures $\beta_c = d/R^2$ and $\beta^* = \sqrt{d(d+2)}/R^2$, see [9].

6 Convergence of the Gibbs measure

6.1 Convergence of the empirical measure

We are going to prove some results about the quenched law of the spins. In order to do so, we are first working with the empirical measure $L_N = \frac{1}{N}\sum_{i=1}^N \delta_{\sigma_i}$ under the Gibbs measure G_N^t associated with $Z_N(t)$. Let ϱ_N^t be defined by:

$$d\varrho_N^t = \frac{e^{H_N(\sigma;t)}}{\mathbf{E}_\sigma e^{\frac{Nt}{4}\left\|\frac{\sigma.\sigma}{N}\right\|^2}} d\mathbf{P}_\sigma$$

Let $q(\sigma) = \sigma \otimes \sigma$. For any borel subset B of $\mathcal{P} = \mathcal{P}(\mathbb{R}^d)$, the space of probability measures on $\mathbb{R}^d$, we have:

$$\mathbb{E}\varrho_N^t(\sigma : L_N(\sigma) \in B) = \frac{\mathbf{E}_\sigma \exp\left[N\frac{t}{4}\left\|\frac{\sigma.\sigma}{N}\right\|^2\right] \mathbf{1}_{L_N \in B}}{\mathbb{E}Z_N(t)}$$

Hence according to Varadhan's theorem, we have:

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \ln \mathbb{E}\varrho_N^t(\sigma : L_N(\sigma) \in B) \leq \sup\left\{\frac{t}{4}\|\langle \nu, q\rangle\|^2 - H(\nu|\rho) : \nu \in \overline{B}\right\} - \gamma(t)$$

where $H(\cdot|\rho)$ denotes the relative entropy w.r.t. ρ. Denote by J the function of the right hand side, that is:

$$J(\nu) = H(\nu|\rho) - \frac{t}{4}\langle \nu, q\rangle^2 + \left(\frac{t}{4}\|v\|^2 - \Lambda_1^*(v)\right)$$

One can easily check easily that J is a good rate function on $\mathcal{P}$ that achieves its minimum uniquely at $\hat{\mu}_t$.

We are now proving theorem 1.2. Let us first consider continuous functions with compact support. Since the space of such functions is separable, we just have to consider one such function. Let then g be a continuous function with compact support, and let $\delta > 0$ be an arbitrary number. We have:

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \ln \mathbb{E}\varrho_N^t\{|\langle L_N, g\rangle - \langle \hat{\mu}_t, g\rangle| \geq \delta\} \leq -\inf\{J(\nu); |\langle \nu, g\rangle - \langle \hat{\mu}_t, g\rangle| \geq \delta\} < 0$$

The infimum is actually achieved because J is a good rate function and the set that we consider is a closed set. Hence there exists an integer $N(g,\delta)$ such that for $N \geq N(g,\delta)$, we have for a $\psi(g,\delta) > 0$:

$$\mathbb{E}\varrho_N^t\{|\langle L_N, g\rangle - \langle \hat{\mu}_t, g\rangle| \geq \delta\} \leq e^{-N\psi(g,\delta)}$$

As a consequence, for $N \geq N(g,\delta)$, we have:

$$\mathbb{P}\left\{\varrho_N^t\{|\langle L_N, g\rangle - \langle \hat{\mu}_t, g\rangle| \geq \delta\} \geq e^{-N\psi(g,\delta)/2}\right\} \leq e^{-N\psi(g,\delta)/2}$$

The measure ϱ_N^t is related to G_N^t by the relation:

$$G_N^t = \frac{\mathbb{E}Z_N(t)}{Z_N(t)}\varrho_N^t$$

Let us state a lemma that we will prove afterwards:

Lemma 6.1 *Under H1, H2, for any $u > 0$, there exists a constant $C(u,t) > 0$ such that:*

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \ln \mathbb{P}\left\{|\frac{1}{N}\ln Z_N(t) - \frac{1}{N}\ln \mathbb{E}Z_N(t)| > u\right\} \leq -C(u,t)$$

Choose now $\eta > 0$. We have for $N \geq N(g, \delta)$:

$$
\begin{aligned}
&\mathbb{P}\left\{G_N^t\{|\langle L_N, g\rangle - \langle \hat{\mu}_t, g\rangle| \geq \delta\} \geq \eta\right\} \\
&\qquad = \mathbb{P}\left\{\frac{\mathbb{E}Z_N(t)}{Z_N(t)}\varrho_N^t\{|\langle L_N, g\rangle - \langle \hat{\mu}_t, g\rangle| \geq \delta\} \geq \eta\right\} \\
&\qquad \leq \mathbb{P}\left\{\varrho_N^t\{|\langle L_N, g\rangle - \langle \hat{\mu}_t, g\rangle| \geq \delta\} \geq e^{-N\psi(g,\delta)/2}\right\} \\
&\qquad\quad + \mathbb{P}\left\{\frac{\mathbb{E}Z_N(t)}{Z_N(t)} \geq \eta e^{N\psi(g,\delta)/2}\right\} \\
&\qquad \leq e^{-N\psi(g,\delta)/2} + \mathbb{P}\left\{\frac{\mathbb{E}Z_N(t)}{Z_N(t)} \geq \eta e^{N\psi(g,\delta)/2}\right\}
\end{aligned}
$$

Moreover, for $N \geq M(g, \delta) \geq N(g, \delta)$, we have:

$$
\begin{aligned}
&\mathbb{P}\left\{\frac{\mathbb{E}Z_N(t)}{Z_N(t)} \geq \eta e^{N\psi(g,\delta)/2}\right\} \\
&\qquad = \mathbb{P}\left\{-\frac{1}{N}\ln Z_N(t) + \frac{1}{N}\ln \mathbb{E}Z_N(t) \geq \frac{1}{2}\psi(g,\delta) + \frac{1}{N}\ln\eta\right\} \\
&\qquad \leq \mathbb{P}\left\{-\frac{1}{N}\ln Z_N(t) + \frac{1}{N}\mathbb{E}\ln Z_N(t) \geq \frac{1}{4}\psi(g,\delta)\right\} \\
&\qquad \leq \exp\left[-\frac{NC(\psi^2(g,\delta)/4, t)}{2}\right]
\end{aligned}
$$

Hence:

$$
\sum_N \mathbb{P}\left\{G_N^t\{|\langle L_N, g\rangle - \langle \hat{\mu}_t, g\rangle| \geq \delta\} \geq \eta\right\} < \infty
$$

According to Borel-Cantelli's lemma, we obtain a set $\Omega_{g,\delta}$ of full probability such that:

$$
\forall \omega \in \Omega_{g,\delta}, \qquad \lim_{N\to\infty} G_N^t\{|\langle L_N, g\rangle - \langle \hat{\mu}_t, g\rangle| \geq \delta\} = 0
$$

We now conclude the proof for continuous functions with compact support by considering a dense sequence g_n and the following set of full probability:

$$
\tilde{\Omega} = \bigcap_{\delta \in \mathbb{Q}_+^*, n \geq 0} \Omega_{g_n,\delta}
$$

Let now g be such an arbitrary countinuous and bouded function, and $M = [\sup|g|] + 1$. Let $\delta > 0$. Chose a constant $A(M, \delta) > 0$ such that:

$$
\hat{\mu}_t\left\{x \in \mathbb{R}^d : \|x\| \geq A(M,\delta)\right\} \leq \frac{\delta}{6M}
$$

Let $t_{M,\delta}$ be a continuous truncation function such that $0 \leq t_{M,\delta} \leq 1$, $t_{M,\delta}(x) = 1$ if $\|x\| \leq A(M,\delta)$, $t_{M,\delta}(x) = 0$ if $x \geq A(M,\delta)+1$. Then we have:

$$\begin{aligned}\{|\langle L_N, g\rangle - \langle \hat{\mu}_t, g\rangle| \geq \delta\} \subset \{|\langle L_N, gt_{M,\delta}\rangle - \langle \hat{\mu}_t, gt_{M,\delta}\rangle| \geq \delta/2\} \\ \cup \{|\langle L_N, g(1-t_{M,\delta})\rangle - \langle \hat{\mu}_t, g(1-t_{M,\delta})\rangle| \geq \delta/2\}\end{aligned}$$

Moreover,

$$\begin{aligned}\{|\langle L_N, g(1-t_{M,\delta})\rangle - \langle \hat{\mu}_t, g(1-t_{M,\delta})\rangle| \geq \delta/2\} \subset \{|\langle L_N, g(1-t_{M,\delta})\rangle| \geq \delta/3\} \\ \subset \left\{L_N\left\{x \in \mathbb{R}^d : \|x\| \geq A(M,\delta)\right\} \geq \delta/3M\right\}\end{aligned}$$

Since $\{\nu : \nu\left\{x \in \mathbb{R}^d : \|x\| \geq A(M,\delta)\right\} \geq \delta/3M\}$ is a closed set that does not contain $\hat{\mu}_t$, we conclude as in the first part of the proof that there exists a $\Omega_{M,\delta}$ of full probability such that:

$$\forall \omega \in \Omega_{M,\delta}, \qquad \lim_{N\to\infty} G_N^t\left\{L_N\left\{x \in \mathbb{R}^d : \|x\| \geq A(M,\delta)\right\} \geq \delta/3M\right\} = 0$$

Since $gt_{M,\delta}$ is continuous with compact support, we also have:

$$\forall \omega \in \tilde{\Omega}, \qquad \lim_{N\to\infty} G_N^t\left\{|\langle L_N, gt_{M,\delta}\rangle - \langle \hat{\mu}_t, gt_{M,\delta}\rangle| \geq \delta/2\right\} = 0$$

We now complete the proof by considering now the set of full probability

$$\hat{\Omega} = \tilde{\Omega} \cap \bigcap_{\substack{M\in\mathbb{N}^* \\ \delta\in\mathbb{Q}_+^*}} \Omega_{M,\delta}$$

Proof of lemma 6.1

In order to prove this lemma, let us prove first the result for $Z_N^\delta(t) = \mathbf{E}_\sigma \mathbf{1}_{\|\frac{\sigma.\sigma}{N}-v\|\leq\delta} \exp H_N(\sigma;t)$. We are going to use an exponential inequality for the martingale associated with $\frac{1}{N}\ln Z_N^\delta(t)$ in its predictable representation. Let us recall that $c_{i,j}=1$ if $i \neq j$ and $1/\sqrt{2}$ otherwise:

$$\begin{aligned}&\frac{1}{N}\ln Z_N^\delta(t) = \mathbb{E}\frac{1}{N}\ln Z_N^\delta(t) \\ &+\frac{1}{N}\int_0^t \sum_{i\leq j} \mathbb{E}\left[\frac{1}{Z_N^\delta(t)}\mathbf{E}_\sigma \mathbf{1}_{\|\frac{\sigma.\sigma}{N}-v\|\leq\delta} c_{i,j}\frac{\langle\sigma(i)|\sigma(j)\rangle}{\sqrt{N}} e^{H_N(\sigma;t)}\middle|\mathcal{F}_s^{(N)}\right]\, dB_{i,j}(s)\end{aligned}$$

Let $G_N^{t,\delta}$ denote the Gibbs measure associated with $Z_N^\delta(t)$. We get the following upper bound for the bracket:

$$\sum_{i\leq j}\left[\mathbb{E}\frac{1}{Z_N^\delta(t)}\mathbf{E}_\sigma \mathbf{1}_{\|\frac{\sigma.\sigma}{N}-v\|\leq\delta} c_{i,j}\langle\sigma_i|\sigma_j\rangle e^{H_N(\sigma;t)}\middle|\mathcal{F}_s^{(N)}\right]^2$$
$$= \sum_{i\leq j}\left[\mathbb{E}c_{i,j}G_N^{t,\delta}(\langle\sigma_i|\sigma_j\rangle)\middle|\mathcal{F}_s^{(N)}\right]^2 \leq \sum_{i\leq j}\mathbb{E}^{\mathcal{F}_s}\left[c_{i,j}^2(G_N^{t,\delta}(\langle\sigma_i|\sigma_j\rangle))^2\right]$$
$$\leq \mathbb{E}\left[G_N^{t,\delta}\left(\sum_{i\leq j}c_{i,j}^2\langle\sigma_i|\sigma_j\rangle^2\right)\middle|\mathcal{F}_s^{(N)}\right] = \mathbb{E}\left[\langle\frac{\|\sigma.\sigma\|^2}{2}\rangle_\delta\middle|\mathcal{F}_s^{(N)}\right]$$
$$\leq \frac{N^2(\|v\|+\delta)^2}{2}$$

Hence:

$$\left\langle\frac{1}{N}\int_0^\cdot\sum_{i\leq j}\mathbb{E}^{\mathcal{F}_s}\left[\frac{1}{Z_N^\delta(t)}\mathbf{E}_\sigma\mathbf{1}_{\|\frac{\sigma.\sigma}{N}-v\|\leq\delta}c_{i,j}\frac{\langle\sigma_i|\sigma_j\rangle}{\sqrt{N}}e^{H_N(\sigma;t)}\middle|\mathcal{F}_s^{(N)}\right]\,dB_{i,j}(s)\right\rangle_s \leq \frac{t(\|v\|+\delta)^2}{2N}$$

An inequality for exponential martingales gives for any $u>0$:

$$\mathbb{P}(|\frac{1}{N}\ln Z_N^\delta(t)-\mathbb{E}\frac{1}{N}\ln Z_N^\delta(t)|>u)\leq 2\exp\left[-\frac{Nu^2}{(\|v\|+\delta)^2 t}\right]$$

Let us now prove the lemma for $Z_N(t)$.

1. We have previously proved that $\mathbb{P}-$almost surely,

$$\lim_{N\to\infty}\left\{\frac{1}{N}\ln Z_N^\delta(t)-\mathbb{E}\frac{1}{N}\ln Z_N^\delta(t)\right\}=0$$

2. Since $Z_N^\delta(t)/\mathbb{E}Z_N^\delta(t)$ converges in law to an almost surely non-zero random variable, we have in $\mathbb{P}$-probability:

$$\lim_{N\to\infty}\left\{\frac{1}{N}\ln Z_N^\delta(t)-\frac{1}{N}\ln\mathbb{E}Z_N^\delta(t)\right\}=0$$

Hence we get:

$$\lim_{N\to\infty}\left\{\frac{1}{N}\mathbb{E}\ln Z_N^\delta(t)-\frac{1}{N}\ln\mathbb{E}Z_N^\delta(t)\right\}=0$$

As a consequence, for any $u > 0$, as soon as $|\ln \mathbb{E}Z_N(t) - \ln \mathbb{E}Z_N^\delta(t)| + |\ln \mathbb{E}Z_N^\delta(t) - \mathbb{E}\ln Z_N^\delta(t)| \leq Nu/2$ we have :

$$\begin{aligned}
\mathbb{P}(|\frac{1}{N}\ln Z_N(t) - \frac{1}{N}\ln \mathbb{E}Z_N(t)| > u) &\leq \mathbb{P}(|\frac{1}{N}\ln Z_N(t) - \frac{1}{N}\mathbb{E}\ln Z_N^\delta(t)| > u/2) \\
&\leq \mathbb{P}(|\frac{1}{N}\ln Z_N(t) - \frac{1}{N}\ln Z_N^\delta(t)| > u/4) \\
&\quad + \mathbb{P}(|\frac{1}{N}\ln Z_N^\delta(t) - \mathbb{E}\frac{1}{N}\ln Z_N^\delta(t)| > u/4)
\end{aligned}$$

Let us study these expressions. Since $Z_N(t) \geq Z_N^\delta(t)$, we have:

$$\begin{aligned}
\mathbb{P}(|\frac{1}{N}\ln Z_N(t) - \frac{1}{N}\ln Z_N^\delta(t)| > u/4) &= \mathbb{P}(\frac{Z_N(t)}{Z_N^\delta(t)} \geq \exp(Nu/4)) \\
&= \mathbb{P}(\frac{Z_N(t) - Z_N^\delta(t)}{\mathbb{E}Z_N^\delta(t)} \geq (\exp(Nu/4) - 1)\frac{Z_N^\delta(t)}{\mathbb{E}Z_N^\delta(t)}) \\
&\leq \mathbb{P}(\frac{Z_N(t) - Z_N^\delta(t)}{\mathbb{E}Z_N^\delta(t)} \geq 1 - e^{-Nu/4}) + \mathbb{P}(\frac{Z_N^\delta(t)}{\mathbb{E}Z_N^\delta(t)} \leq e^{-Nu/4}) \\
&\leq \frac{1}{1 - e^{-Nu/4}}\frac{\mathbb{E}[Z_N(t) - Z_N^\delta(t)]}{\mathbb{E}Z_N^\delta(t)} \\
&\quad + \mathbb{P}(\frac{1}{N}\ln Z_N^\delta(t) - \mathbb{E}\frac{1}{N}\ln Z_N^\delta(t) \leq -u/4)
\end{aligned}$$

As a conclusion,

$$\begin{aligned}
\mathbb{P}(|\frac{1}{N}\ln Z_N(t) - \frac{1}{N}\ln \mathbb{E}Z_N(t)| > u) &\leq (1 - e^{-Nu/4})^{-1}\frac{\mathbb{E}[Z_N(t) - Z_N^\delta(t)]}{\mathbb{E}Z_N^\delta(t)} \\
&\quad + 2\mathbb{P}(|\frac{1}{N}\ln Z_N^\delta(t) - \mathbb{E}\frac{1}{N}\ln Z_N^\delta(t)| \geq u/4)
\end{aligned}$$

Since both expressions go exponentially quickly to zero, the result is proved.

■

Corollary 6.2 *Let $G_N^t L_N$ be the random probability measure on $\mathbb{R}^d$ defined by:*

$$G_N^t L_N(dx) = \int G_N^t(d\sigma) L_N(\sigma, dx)$$

Then $\mathbb{P}-$almost surely, $G_N^t L_N$ converges weakly to $\hat{\mu}_t$.

6.2 Finite dimensional marginals of the Gibbs measure

We are now proving theorem 1.3. Let k be a fixed integer. We study the convergence in law of $(\sigma_1, \dots, \sigma_k)$ under G_N^t.

The proof of the proposition relies on the use of two replicas. Let μ_N^t be defined by:

$$d\mu_N^t = \frac{e^{H_N(\sigma;t)} \mathbf{1}_{\left\| \frac{\sigma.\sigma}{N} - v \right\| \le \delta} \mathbf{1}_{\mathcal{B}_t(f,\varepsilon)} \left(\frac{H_N(\sigma)}{N} \right)}{\mathbb{E} \otimes \mathbf{E}_\sigma e^{H_N(\sigma;t)} \mathbf{1}_{\left\| \frac{\sigma.\sigma}{N} - v \right\| \le \delta} \mathbf{1}_{\mathcal{B}_t(f,\varepsilon)} \left(\frac{H_N(\sigma)}{N} \right)} \, d\mathbf{P}_\sigma$$

Set $\alpha_N^{\varepsilon,\delta} = [\mathbb{E} Z_N^{\varepsilon,\delta}(t)]^2 / \mathbb{E}[Z_N^{\varepsilon,\delta}(t)^2]$. Define $q_1(\sigma,\sigma) = \sigma \otimes \sigma$, $q_2(\sigma,\tau) = \tau \otimes \tau$ and $q_3(\sigma,\tau) = \sigma \otimes \tau$. Under $\alpha_N^{\varepsilon,\delta} \mathbb{E}(\mu_N^t \otimes \mu_N^t)$, the empirical measure $L_N^{(2)} = \frac{1}{N} \sum_{i=1}^N \delta_{\sigma(i),\tau(i)}$ statisfies for any Borel subset B of $\mathcal{P}(\mathbb{R}^d \times \mathbb{R}^d)$:

$$\begin{aligned}
&\overline{\lim_{N\to\infty}} \frac{1}{N} \ln \alpha_N^{\varepsilon,\delta} \mathbb{E} \mu_N^t \otimes \mu_N^t \left\{ (\sigma,\tau) : L_N^{(2)} \in B \right\} \le \\
&\sup \Big\{ \langle K(\langle \nu, q_1 \rangle, \langle \nu, q_2 \rangle, \langle \nu, q_3 \rangle)^{1/2} (1,1)^{\mathbf{T}} | \varphi \rangle - H(\nu | \rho^{\otimes 2}) - I_2(\varphi) : \nu \in \overline{B}, \\
&\| \langle \nu, q_1 \rangle - v \| \le \delta, \| \langle \nu, q_2 \rangle - v \| \le \delta, \| K(\langle \nu, q_1 \rangle, \langle \nu, q_2 \rangle, \langle \nu, q_3 \rangle^{1/2} \varphi - (f,f)^{\mathbf{T}} \| \le \varepsilon \Big\}
\end{aligned}$$

We can then check as previously that under $\alpha_N^{\varepsilon,\delta} \mathbb{E} \mu_N^t \otimes \mu_N^t$, $L_N^{(2)}$ weakly converges to $\hat{\mu}_t \otimes \hat{\mu}_t$. By exchangeability, we get the propagation of chaos, in the sense that for any integer k, the law of $((\sigma_1,\tau_1), \dots, (\sigma_k,\tau_k))$ under $\alpha_N^{\varepsilon,\delta} \mathbb{E} \mu_N^t \otimes \mu_N^t$ weakly converges to $(\hat{\mu}_t \otimes \hat{\mu}_t)^{\otimes k}$. Hence we get for any conitnuous and bounded function g on $\mathbb{R}^d \times \mathbb{R}^d$:

$$\lim_{N\to\infty} \alpha_N^{\varepsilon,\delta} \mathbb{E} \mu_N^t \otimes \mu_N^t g(\sigma_1, \dots, \sigma_k) g(\tau_1, \dots, \tau_k) = \langle \hat{\mu}_t^{\otimes k}, g \rangle^2$$

As a consequence, by polarizing the result and taking one of the functions equal to 1, we get:

$$\lim_{N\to\infty} \alpha_N^{\varepsilon,\delta} \mathbb{E} \mu_N^t \otimes \mu_N^t g(\sigma_1, \dots, \sigma_k) = \langle \hat{\mu}_t^{\otimes k}, g \rangle$$

As a result, we get:

$$\lim_{N\to\infty} \mathbb{E} \left[\mu_N^t \left(g(\sigma_1, \dots, \sigma_k) \right) - \frac{Z_N^{\varepsilon,\delta}(t)}{\mathbb{E} Z_N^{\varepsilon,\delta}(t)} \langle \hat{\mu}_t^{\otimes k}, g \rangle \right]^2 = 0$$

We now divide by $Z_N^{\varepsilon,\delta}(t) / \mathbb{E}[Z_N^{\varepsilon,\delta}(t)]$ which converge in law to a almost surely non zero random variable and according to (5) the proof is complete.

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