Generalized Bezout Identity

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Abstract

We describe a new approach of the generalized Bezout identity for linear time-varying ordinary differential control systems. We also explain when and how it can be extended to linear partial differential control systems. We show that it only depends on the algebraic nature of the differential module determined by the equations of the system. This formulation shows that the generalized Bezout identity is equivalent to the splitting of the exact differential sequence made with the control system and its parametrization. This point of view gives an algebraic and geometric interpretation of the entries of the generalized Bezout identity. This method brings the computations of the generalized Bezout identity closer to basic concepts of differential geometry and algebra.

Keywords: Generalized Bezout Identity, controllability, parametrization, Janet sequence, formal integrability, D-module, commutative algebra.

1 Introduction

Let us denote $s = \frac{d}{dt}$, $\mathbb{R}[s]$ the polynomial algebra in s and M_{mp} the set of $m \times p$ matrices with entries in $\mathbb{R}[s]$. It is well known that if

$$P(s)y + Q(s)u = 0, \qquad (1)$$

is a left coprime polynomial system, i.e. controllable, where $P \in M_{mm}$, $\det P(s) \neq 0$ and $Q \in M_{mp}$ then we can find four polynomial matrices $X \in M_{mm}, \overline{X}, Y \in M_{pm}, \overline{P} \in M_{mp}, \overline{Y}, \overline{Q} \in M_{pp}$ such that:

$$\left[\begin{array}{cc} P(s) & Q(s) \\ \overline{X}(s) & \overline{Y}(s) \end{array} \right] \left[\begin{array}{cc} X(s) & \overline{P}(s) \\ Y(s) & \overline{Q}(s) \end{array} \right] = I, \qquad (2)$$

where I is the $(m+p) \times (m+p)$ identity matrix. This identity, generally called *generalized Bezout identity*, is useful in control theory [8, 21]. Recently, it has been shown in [5, 6, 7, 13, 15] that controllability of control system was a "built-in" property of the system and thus did not depend on a separation of the system variables between inputs and outputs. So, we are led to revisit the generalized Bezout identity with a more intrinsic point of view. For controllable *surjective* linear time-varying control system, the generalized Bezout identity is reformulating in terms of *the splitting of the short exact differential sequence* formed by the system and its parametrization. Moreover, it has been suggested in [13, 15, 16] to extend most of the algebraic and geometric concepts of ordinary differential control theory (OD control theory) within the framework of partial differential control theory (PD control theory), that is, linear or nonlinear input/output relations defined by systems of partial differential equations. Then, we can wonder if such a generalized Bezout identity exists for PD control systems. However, the existence of the generalized Bezout identity for (1) is deeply based on Bezout theorem which is not true in general for multivariable polynomial algebra. So (2) does not seem to have a generalization for PD control systems. We will show that its existence only depends on the algebraic nature of the differential module determined by the equations of the system. Such a generalized Bezout identity exists for *surjective* linear PD control system generating *a free differential module*. In this case, the generalized Bezout identity can be reformulated in terms of a splitting of the short exact differential sequence made by the system and its parametrization. In case the differential module is no longer free but *projective*, then only the upper part of (2) is satisfied, or in other words, the system admits a *parametrization* and a *right-inverse*. Finally, if the system is controllable, i.e. if it generates *a torsion-free differential module*, we only have the right upper part of (2), that is, the system admits a parametrization. Some tests are known for checking whether a finitely generated differential module is torsion-free, projective or free [15, 16, 19, 20]. Thus for linear PD control systems, we are able to know which parts of the generalized Bezout identity exist and to compute them.

Moreover, the extension of the generalized Bezout identity in the case of none surjective linear OD and PD control system is obtained. In this case, we have to build and split a long exact differential sequence. Many explicit examples will illustrate the main results.

2 Controllability

The use of the module language for control system was initiated by Kalman twenty years ago [9] and took a new insight with Blomberg and Ylinen [1]. Recently, its use seemed to have given some new results on structural properties of the system like controllability, observability, poles and zeros, motion planing...[3, 5, 6, 7, 11, 13, 15]. We recall a few results.

A *differential field* K with n commuting derivatives $\partial_1, \ldots, \partial_n$ is a field which satisfies: $\forall a, b \in K$, $\forall i = 1, \ldots, n$:

- $\partial_i(a+b) = \partial_i a + \partial_i b$,
- $\partial_i(ab) = (\partial_i a)b + a\partial_i b$,
- $\partial_i\partial_j = \partial_j\partial_i$.

For example, the field of rational functions $\mathbb{R}(t)$ is a differential field with derivative $\frac{d}{dt}$ (see [13] for more details). We form the ring of linear differential operators with coefficients in K and we denote it by $D = K[d_1, \ldots, d_n]$. For example, every element $p \in D = \mathbb{R}(t)[\frac{d}{dt}]$ has the form: $p = \sum_{\text{finite}} a_i(t)(\frac{d}{dt})^i$, with $a_i \in \mathbb{R}(t)$. D is a noncommutative ring which verifies

$$\forall\, a, b \in K : a d_i\,(b d_k) = ab\, d_i d_k + a(\partial_i b)\, d_k,$$

and possesses the Ore property: $\forall\,(p, q) \in D^2$, $\exists\,(u, v) \in D^2$ such that $u\,p = v\,q$. We introduce the differential indeterminates $z = \{z^k \mid k = 1, \ldots, m\}$ and denote by $Dz = Dz^1 + \ldots + Dz^m$ the left D-module spanned by the set z. Every element of Dz has the form $\sum_{\text{finite}} a_k^\mu d_\mu z^k$ where $\mu = (\mu_1, \ldots, \mu_n)$ is a multi-index with length $\mid \mu \mid = \mu_1 + \ldots + \mu_n$ and $a_k^\mu \in K$. We shall frequently use the notation $d_{i_1} \ldots d_{i_m} z^k = z^k_{i_1 \ldots i_m}$. If we have a finite set $\mathcal{R}$ of linear OD or PD equations, we form the finitely generated left D-module $[\mathcal{R}]$ of linear differential consequences of the system generators and the differential residual D-module $\mathcal{M} = Dz/[\mathcal{R}] = D\eta$ where η^k is the canonical image of z^k in $\mathcal{M}$.

We call *observable* any element of $\mathcal{M}$, or in other words, any linear combination of the system variables (inputs and outputs together) and their derivatives. Only two possibilities may happen for an observable: it may or may not verify a OD or a PD equation by itself. An observable which does not satisfy any OD or PD equation is called *free*. We find in [13] the following definition of controllability:

Definition 1 *A system is controllable if every observable is free.*

A characterization of the controllability in terms of *differential closure* is shown in [13]. In [5, 6, 11], the equivalent notion of *torsion-free D-module* has been used for linear time-varying OD and delay control systems. A *torsion element* m of a $D-$module is an element which satisfies $\exists\; a \in D, a \neq 0,$ such that $a\,m = 0$ [18] and we denote $\tau(M)$ the submodule of $\mathcal{M}$ made by all the torsion elements. We recall that a module is *torsion-free* if $\tau(M) = 0$. From Definition 1, a linear OD or PD control system is controllable if and only if the module determined by its equations is a torsion-free D-module [11, 16]. In any case, $\mathcal{M}/\tau(\mathcal{M})$ is a torsion-free module, a result leading to the concept of minimal realization [13].

Example 1 We take $D = \mathbb{R}[\frac{d}{dt}]$ and we form the D-modules $[\mathcal{R}] = [\ddot{y}^1 + y^1 - y^2 + \;\alpha\, u, \ddot{y}^2 + y^2 - y^1 - u]$ and $\mathcal{M} = (Dy^1 + Dy^2 + Du)/[\mathcal{R}] = D\eta^1 + D\eta^2 + D\eta^3$ where $\alpha \in \mathbb{R}$ and η^1, η^2 and η^3 are the canonical image of y^1, y^2 and u. We have the following identities in $\mathcal{M}$:

$$\left\{ \begin{array}{l} \ddot{\eta}^1 + \eta^1 - \eta^2 + \alpha\eta^3 = 0, \\ \ddot{\eta}^2 + \eta^2 - \eta^1 - \eta^3 = 0, \end{array} \right. \tag{3}$$

and all the combination of their derivatives.

- For $\alpha = -1$, if we substract the first equation from the second, we find $\tau^1 = \eta^1 - \eta^2$ satisfying $(\frac{d^2}{dt^2} + 2)\tau^1 = 0$. The element τ^1 is a torsion element of $\mathcal{M}$.
- For $\alpha = 1$, if we add the first equation to the second, we find a torsion element $\tau^2 = y^1 + y^2$ satisfying $(\frac{d^2}{dt^2})\tau^2 = 0$.

If D is a principal ring (for example $K[\frac{d}{dt}]$) the module $\mathcal{M}$ is torsion-free if and only if $\mathcal{M}$ is *free*, that is to say, if there exists a basis of the D-module $\mathcal{M}$ (it is not always true for a general module) [18]. We recall that a basis of a D-module $\mathcal{M}$ is a set of elements which are independent on D and generate $\mathcal{M}$. In [7, 11], this basis is called flat outputs or linearizing outputs. We recall that a D-module $\mathcal{M}$ is projective if there exists a D-module $\mathcal{M}'$ such that the direct sum $\mathcal{M} \oplus \mathcal{M}'$ is free [18]. For non principal rings (for example $K[d_1, \ldots, d_n], n \geq 2$) a free module is a projective module and a projective module is a torsion-free module, which can be summed up by the following module inclusions:

$$\text{free} \subseteq \text{projective} \subseteq \text{torsion-free}.$$

Thus for non principal rings, a torsion-free module is no more in general a free module. Quillen and Suslin have independently demonstrated in 1976 the Serre conjecture of 1950 claiming that, over a polynomial ring $k[\chi_1, \ldots, \chi_n]$ where k is a field, any projective module is also a free module [18]. We can find in [11, 19, 20] some tests permitting to know if a finitely generated $K[d_1, ..., d_n]-$module $\mathcal{M}$ with K a field of constants (i.e. $\forall\, a \in K : \forall\, i = 1, ..., n,\ \partial_i a = 0$) is respectively torsion-free, projective and free. Remark that in this case, we can use the Quillen-Suslin theorem and any projective module is a free module. Recently, some formal tests have been found in [16] permitting to treat the situation where $D = K[d_1, ..., d_n]$ with K a general differential field (see [12] for more deeper results). We now recall these tests.

From a geometric point of view, a linear PD control system may be defined by as a linear PD operator $\mathcal{D}_1 : F_0 \to F_1$ where F_0, F_1 are vector bundles over a manifold X of dimension n. In other words, $\mathcal{D}_1$ is a PD linear operator acting on the system variables which are sections of F_0. We define its sheaf of solutions by $\mathcal{D}_1\eta = 0$. An operator $\mathcal{D}_1$ is *injective* if $\mathcal{D}_1\eta = 0 \Rightarrow \eta = 0$ and it is *surjective* if the equations $\mathcal{D}_1\eta = 0$ are differentially independent [13] or equivalently if $\mathcal{D}_1\eta = \zeta$ have no compatibility conditions, that is, if there does not exist an operator $\mathcal{D}_2$ such that $\mathcal{D}_1\eta = \zeta \Rightarrow \mathcal{D}_2\zeta = 0$. A control system defined by $\mathcal{D}_1$ will be called *surjective* if $\mathcal{D}_1$ is a surjective operator.

Example 2

- The operator $\mathcal{D}_1 : \eta \to \zeta$ defined by (we recall that we use the notation: $d_i\eta^j = \eta^j_i$):

$$\left\{\begin{array}{l} -\eta_2 = \zeta^1, \\ x^2\eta_1 + \eta = \zeta^2, \end{array}\right. \tag{4}$$

  where (x^1, x^2) are local coordinates on X, is an injective operator as we may easily verified that $\eta = \zeta^2 - x^2\zeta^1 - (x^2)^2\zeta^1_2 - x^2\zeta^2_2$. Thus, $(\zeta^1, \zeta^2) = (0,0) \Rightarrow \eta = 0$.

- We take the *Spencer operator* (see [13] for more details) $\mathcal{D}_1 : \eta \to \zeta$ defined by:

$$\left\{\begin{array}{l} \eta^1_1 - \eta^2 = \zeta^1, \\ \eta^1_2 - \eta^3 = \zeta^2, \\ \eta^2_2 - \eta^3_1 = \zeta^3, \end{array}\right. \tag{5}$$

  It is not a surjective operator. Indeed, if differentiating ζ^1 with respect to d_2 and ζ^2 to d_1 and substracting them, we find $\zeta^2_1 - \zeta^1_2 - \zeta^3 = 0$. The operator $\mathcal{D}_2 : \zeta \to \chi$, defined by the compatibility condition $\zeta^2_1 - \zeta^1_2 - \zeta^3 = \chi$ of $\mathcal{D}_1$, is surjective because it has only one equation.

A fundamental idea is to associate to each operator $\mathcal{D}_1 : \eta \to \zeta$ the D-module $\mathcal{M} = D\eta/[\mathcal{D}_1\eta]$ and we will say that that the operator $\mathcal{D}_1$ determines the D-module $\mathcal{M}$.

We recall the duality of differential operators [13, 15]. We denote $E^\star$ the dual bundle of E and $\tilde{E} = \bigwedge^n T^\star \otimes E^\star$ its adjoint bundle. If $\mathcal{D}_1 : F_0 \to F_1$ is a linear differential operator, its formal adjoint $\tilde{\mathcal{D}}_1 : \tilde{F}_1 \to \tilde{F}_0$ is defined by the following rules:

- the adjoint of a matrix (zero order operator) is the transposed matrix,
- the adjoint of d_i is $-d_i$,
- for two linear PD operators P, Q that can be composed: $\widetilde{P \circ Q} = \tilde{Q} \circ \tilde{P}$.

We have the relation

$$\mu^t \mathcal{D}_1 \xi = (\tilde{\mathcal{D}}_1 \mu)^t \xi + d(\cdot),$$

with d the exterior derivative. We can directly compute the adjoint of an operator by multiplying by test functions on the left and integrating by part.

Example 3 We compute the adjoint operator of the Spencer operator (5). We multiply $\mathcal{D}_1\eta$ by a row vector $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ and integrate the result by part, we obtain the operator $\tilde{\mathcal{D}}_1 : \lambda \to \mu$ defined by:

$$\left\{\begin{array}{l} -d_1\lambda_1 - d_2\lambda_2 = \mu_1, \\ -d_2\lambda_3 - \lambda_1 = \mu_2, \\ d_1\lambda_3 - \lambda_2 = \mu_3. \end{array}\right. \tag{6}$$

We call an operator $\mathcal{D}_1$ *parametrizable* if there exists a set of arbitrary functions $\xi = (\xi^1, \ldots, \xi^r)$ or "potentials" and a linear operator $\mathcal{D}_0$ such that all the compatibility conditions of the inhomogenous system $\mathcal{D}_0\xi = \eta$ are exactly generated by $\mathcal{D}_1\eta = 0$. We find in [13, 15] the following theorem:

Theorem 1 *A linear PD control system is controllable if and only if it is parametrizable.*

By a abuse of language, we will say that an operator is controllable, projective or free if the D-module $\mathcal{M}$ associated to the operator is respectively torsion-free, projective or free. We describe a formal test for checking if the operator $\mathcal{D}_1$ is controllable or not (compare with [10]):

1. Start with $\mathcal{D}_1$.
2. Construct its adjoint $\tilde{\mathcal{D}}_1$.
3. Find the compatibility conditions of $\tilde{\mathcal{D}}_1\lambda = \mu$ and denote this operator by $\tilde{\mathcal{D}}_0$.
4. Construct its adjoint $\mathcal{D}_0$.
5. Find the compatibility conditions of $\mathcal{D}_0\xi = \eta$ and call this operator by $\mathcal{D}'_1$.

We are led to two different cases. If $\mathcal{D}'_1 = \mathcal{D}_1$ then the system $\mathcal{D}_1$ determines a torsion-free D-module $\mathcal{M}$, i.e. controllable, and $\mathcal{D}_0$ is a parametrization of $\mathcal{D}_1$. Otherwise, the operator $\mathcal{D}_1$ is among, but not exactly, the compatibility conditions of $\mathcal{D}_0$. The torsion elements of $\mathcal{M}$ are all the new compatibility conditions modulo the equations $\mathcal{D}_1\eta = 0$.

We recall that an exact differential sequence is a sequence of differential operators $\{D_i, i = 0, \ldots, l\}$, which verified Ker D_{i+1}=Im D_i. An injective operator $\mathcal{D}$ will be denoted by the following exact differential sequence $0 \longrightarrow E \xrightarrow{\mathcal{D}} F$ whereas the exact differential sequence $E \xrightarrow{\mathcal{D}} F \longrightarrow 0$ will mean that $\mathcal{D}$ is a surjective operator. An exact differential sequence is called *formally exact* if the all the sequences at any order, existing on the jet level, are exact [13]. In practise that means that each operator generates all the compatibility conditions of its preceding one. The exact sequence $0 \longrightarrow E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1 \longrightarrow 0$ is said to be a *splitting exact sequence* if we have one of the following equivalent properties [18]:

1. There exists an operator $\mathcal{P}_1 : F_1 \longrightarrow F_0$ such that $\mathcal{D}_1 \circ \mathcal{P}_1 = Id_{F_1}$.
2. There exists an operator $\mathcal{P}_0 : E \longrightarrow F_0$ such that $\mathcal{P}_0 \circ \mathcal{D}_0 = Id_E$.
3. $F_0 \simeq E \oplus F_1$ (on the level of sections).

We can represent the test by the following differential sequences where the number indicates the different stages:

$$
\begin{array}{ccccc}
 & & & 5 & \\
 & & & \xrightarrow{\mathcal{D}'_1} & F'_1 \\
E & \xrightarrow{\mathcal{D}_0} & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 \\
 & 4 & & 1 & \\
\tilde{E} & \xleftarrow{\tilde{\mathcal{D}}_0} & \tilde{F}_0 & \xleftarrow{\tilde{\mathcal{D}}_1} & \tilde{F}_1 \\
 & 3 & & 2 &
\end{array}
$$

In the preceding sequences, only the dual sequence and the sequence made with $\mathcal{D}_0$ and $\mathcal{D}'_1$ are formally exact. Thus, the defect of controllability of the operator $\mathcal{D}_1$ may be seen as a defect of the formally exactness of the upper sequence formed by $\mathcal{D}_0$ and $\mathcal{D}_1$.

Example 4 We wonder if the Spencer operator (5) is controllable. The adjoint operator of the Spencer operator is (6). Differentiating the second equation of $\tilde{\mathcal{D}}_1$ with respect to d_1, the third with respect to d_2 and adding them, we obtain the operator $\tilde{\mathcal{D}}_0 : \mu \to \nu$ defined by $-d_1\mu_2 - d_2\mu_3 + \mu_1 = \nu$. We multiply $\tilde{\mathcal{D}}_0$ by ξ and after one integration by part, we obtain the operator $\mathcal{D}_0 : \xi \to \eta$ defined by:

$$
\left\{
\begin{array}{l}
\xi = \eta^1, \\
\xi_1 = \eta^2, \\
\xi_2 = \eta^3.
\end{array}
\right. \tag{7}
$$

We find the compatibility conditions of $\mathcal{D}_0$ by differentiating the second equation by d_2, the third by d_1 and substracting them, we obtain the third equation of $\mathcal{D}_1$. Differentiating the first equation of $\mathcal{D}_0$ by respectively d_1 and d_2 and susbtracting it by respectively the second and the third equation, we obtain the first and the second equation of $\mathcal{D}_1$. Thus, all the compatibility conditions of $\mathcal{D}_0$ are exactly generated by $\mathcal{D}_1$ and the Spencer operator is controllable.

In the previous example, it was easy to compute the compatibility conditions but in the general case, it might be much more difficult and we have to use formal integrability theory [13] or differential algebra [4]. A system of partial differential equations is said to be *formally integrable* whenever the formal power series of the solutions can be determined step by step by successive derivations without obtaining backwards new informations on lower-order derivatives. For a sufficiently regular operator $\mathcal{D}$, we are always able to add to its equations new equations, made by differential consequences of the first one, in order to have a formally integrable and *involutive* operator [13]. Such a new operator is called involutive. If $\mathcal{D}$ is an involutive operator then the sequence starting with $\mathcal{D}$ and, in which, each operator exactly describe the compatibility conditions of the preceding one, is finite and stops after atmost n operators where n is the dimension of X or equivalently the number of independent variables. The sequence is formally exact and it is usually called the *Janet sequence* [13]. In the course of the text, we will always suppose that these regular conditions are satisfied.

We now give a theorical but non-trivial example of a computation of a torsion element.

Example 5 We consider the system $\ddot{\eta}^2+\alpha(t)\dot{\eta}^2+\dot{\alpha}(t)\eta^2+\ddot{\eta}^1-\eta^1=0$ where $\alpha(t)$ is a non zero function satisfying $\dot{\alpha}(t)+\alpha(t)^2-1=0$. See [17] for the general situation. We let the reader check that the operator $\mathcal{D}'_1 : \eta \to \zeta'$ is $\dot{\eta}^2+\dot{\eta}^1-\alpha(t)\eta^1-\frac{\dot{\alpha}(t)}{\alpha(t)}(\eta^2+\eta^1)=\zeta'$ (be careful, the adjoint of $\alpha(t)\dot{y}$ is $-\alpha(t)\dot{\lambda}-\dot{\alpha}(t)\lambda$). The compatibility condition of $\mathcal{D}_0$ is not the operator $\mathcal{D}_1$ and thus the system is not controllable. If we want to find the torsion element of the associated D-module, we only have to compute the compatibility conditions of the system:

$$\left\{\begin{array}{l} \dot{\eta}^2+\dot{\eta}^1-\alpha(t)\eta^1-\frac{\dot{\alpha}(t)}{\alpha(t)}(\eta^2+\eta^1)=\zeta', \\ \ddot{\eta}^2+\alpha(t)\dot{\eta}^2+\dot{\alpha}\eta^2+\ddot{\eta}^1-\eta^1=0. \end{array}\right.$$

After straightforward but tedious computations, we find that the torsion element ζ' satisfied $\alpha(t)\dot{\zeta}'+\zeta'=0$.

Let $\mathcal{D}_1$ be a surjective operator with a injective adjoint $\tilde{\mathcal{D}}_1$. As $\tilde{\mathcal{D}}_1$ is an injective operator, among the consequences of the equations $\tilde{\mathcal{D}}_1\lambda=\mu$, we must find $\lambda=\tilde{\mathcal{P}}_1\mu$. A natural way to compute $\tilde{\mathcal{P}}_1$ is to bring $\tilde{\mathcal{D}}_1$ to become formally integrable [13]. Thus, bringing $\tilde{\mathcal{D}}_1$ to formal integrability, we form an operator $\tilde{\mathcal{P}}_1$ satisfying $\tilde{\mathcal{P}}_1\circ\tilde{\mathcal{D}}_1=Id_{\tilde{F}_1}$ where $Id_{\tilde{F}_1}$ is the identity operator of $\tilde{F}_1$. The operator $\tilde{\mathcal{P}}_1$ is then a left-inverse of $\tilde{\mathcal{D}}_1$. Dualizing $\tilde{\mathcal{P}}_1\circ\tilde{\mathcal{D}}_1=Id_{\tilde{F}_1}$, we obtain $\mathcal{D}_1\circ\mathcal{P}_1=Id_{F_1}$ or in other words, $\mathcal{D}_1$ admits a right-inverse. We also say that $\mathcal{P}_1$ is a *differential lift* of the sequence [18]:

$$E \xrightarrow{\mathcal{D}_0} F_0 \underset{\xleftarrow{\mathcal{P}_1}}{\xrightarrow{\mathcal{D}_1}} F_1 \longrightarrow 0.$$

It is equivalent to say that the D-module $\mathcal{M}$ determined by the surjective operator is a projective module [11, 16, 20].

Theorem 2 *A surjective differential operator determines a projective D-module if and only if its adjoint is injective.*

Example 6 To illustrate what has been said, we show that the system:

$$\eta^2_2-x^2\eta^1_1+\eta^1=0, \qquad (8)$$

where (x^1,x^2) are local coordinates on X, determines a projective module and we find a right-inverse. Its adjoint is just (4). We have seen that $\tilde{\mathcal{D}}_1$ is an injective operator and thus it determines a projective module. If we denote the operator $\tilde{\mathcal{P}}_1\mu=\mu^2-x^2\mu_1-(x^2)^2d_2\mu_1-x^2d_2\mu^2=\lambda$ then its adjoint $\mathcal{P}_1:\zeta\to\eta$, given by

$$\left\{\begin{array}{l} x^2\zeta_2+2\zeta=\eta^1, \\ (x^2)^2\zeta_1-x^2\zeta=\eta^2, \end{array}\right.$$

is a right-inverse of (8). Indeed, we easily verified that $\mathcal{D}_1\circ\mathcal{P}_1=Id_{F_1}$.

In the general case where $\mathcal{D}_1$ is no longer a surjective operator, a characterization of projective module can be found in [2, 16]. We recall it. As $\mathcal{D}_1$ is not a surjective operator, there exists a compatibility conditions operator $\mathcal{D}_2$. $\mathcal{D}_1$ defines a projective D-module $\mathcal{M}$ if and only if there exists an operator $\mathcal{P}_1: F_1\longrightarrow F_0$ such that $\mathcal{D}_1\circ\mathcal{P}_1=Id_{\mathrm{Im}\,\mathcal{D}_1}=Id_{F_1}$ modulo $\mathcal{D}_2$. However, $\mathcal{D}_1\circ\mathcal{P}_1=Id_{\mathrm{Im}\,\mathcal{D}_1}$ is equivalent to

$$\mathcal{D}_1\circ\mathcal{P}_1\circ\mathcal{D}_1=\mathcal{D}_1, \qquad (9)$$

Indeed, the direct way is trivial whereas the reciprocity can be demonstrated as follows. From (9), we have $(Id_{F_1}-\mathcal{D}_1\circ\mathcal{P}_1)\circ\mathcal{D}_1=0$ and thus $Id_{F_1}-\mathcal{D}_1\circ\mathcal{P}_1$ must factorized by $\mathcal{D}_2$ (see [14]), that is to say, there exists an operator $\mathcal{P}_2$ such that:

$$\mathcal{D}_1\circ\mathcal{P}_1+\mathcal{P}_2\circ\mathcal{D}_2=Id_{F_1}, \qquad (10)$$

which proves the inverse way. Moreover, the identity (10) implies $\mathcal{D}_2\circ\mathcal{P}_2\circ\mathcal{D}_2=\mathcal{D}_2$ and $\mathcal{D}_2$ defines, at its turn, a projective D-module. In a similar way, all the successive operators of compatibility conditions define a projective D-module. Now, if we dualize (9), we obtain $\tilde{\mathcal{D}}_1\circ\tilde{\mathcal{P}}_1\circ\tilde{\mathcal{D}}_1=\tilde{\mathcal{D}}_1$ and thus $\tilde{\mathcal{D}}_1$ defines a projective D-module. The adjoints of (10) and $\mathcal{D}_2\circ\mathcal{D}_1=0$ are respectively $\tilde{\mathcal{P}}_1\circ\tilde{\mathcal{D}}_1+\tilde{\mathcal{D}}_2\circ\tilde{\mathcal{P}}_2=Id_{\tilde{F}_1}$ and $\tilde{\mathcal{D}}_1\circ\tilde{\mathcal{D}}_2=0$. The last identity shows that $\operatorname{Im}\tilde{\mathcal{D}}_2 \subseteq \operatorname{Ker}\tilde{\mathcal{D}}_1$ whereas if we take $\lambda \in \operatorname{Ker}\tilde{\mathcal{D}}_1$, the second shows that $\tilde{\mathcal{D}}_2(\tilde{\mathcal{P}}_2\lambda) = \lambda$ and thus $\lambda \in \operatorname{Im}\tilde{\mathcal{D}}_2$. We have the following exact sequence:

$$\tilde{F}_2 \xrightarrow{\tilde{\mathcal{D}}_2} \tilde{F}_1 \xrightarrow{\tilde{\mathcal{D}}_1} \tilde{F}_0.$$

For a none surjective operator $\mathcal{D}_1$, a test for checking if the D-module $\mathcal{M}$ determined by the operator $\mathcal{D}_1$ is a projective module can be found in [16]. We recall this test:

1. Construct the Janet sequence starting with $\mathcal{D}_1$.
2. Ckeck if the adjoint of the last operator of the sequence is injective.
3. Check if the backward sequence made with the adjoint of the Janet sequence is an exact sequence.

Example 7 The Spencer operator $\mathcal{D}_1$ is not a surjective operator as we have seen in the example 2. The operator $\mathcal{D}_2 : \zeta \to \chi$ defining the compatibility conditions of $\mathcal{D}_1$ is

$$\zeta_1^2 - \zeta_2^1 - \zeta^3 = \chi, \tag{11}$$

and it is surjective. Dualizing the operator $\mathcal{D}_2$ by multiplying it by β and integrating the result by part, we obtain the injective operator $\tilde{\mathcal{D}}_2 : \beta \to \lambda$ defined by:

$$\begin{cases} \beta_2 = \lambda_1, \\ -\beta_1 = \lambda_2, \\ -\beta = \lambda_3. \end{cases}$$

Thus, we have only to verify that all the compatibility conditions of the operator $\tilde{\mathcal{D}}_2$ are exactly defined by the operator $\tilde{\mathcal{D}}_1$. Up to a change of sign, it is the same as to verify that all the compatibility conditions of $\mathcal{D}_0$ are defined by $\mathcal{D}_1$ (see the example 4). We conclude that the Spencer operator determines a projective module $\mathcal{M}$. We easily find that $\mathcal{P}_2 : \chi \to \zeta$ defined by:

$$\begin{cases} 0 = \zeta^1, \\ 0 = \zeta^2, \\ -\chi = \zeta^3, \end{cases} \tag{12}$$

is a right-inverse of $\mathcal{D}_2$. As the Spencer operator is a PD system with constant coefficients, then according to the theorem of Quillen-Suslin, it determines a free D-module. Indeed, the D-module $\mathcal{M}$ determined by the Spencer operator is equal to the module $D\xi = D\eta^1$ which is a free D-module (see the parametrization (7) of $\mathcal{D}_1$).

Let $\mathcal{D}_1$ be an operator defining a projective D-module. Thus, we have the two following formally exact sequences:

$$\begin{array}{l} F_0 \xrightarrow{\mathcal{D}_1} F_1 \ldots F_n \xrightarrow{\mathcal{D}_{n+1}} F_{n+1} \longrightarrow 0 \\ \tilde{F}_0 \xleftarrow{\tilde{\mathcal{D}}_1} \tilde{F}_1 \ldots \tilde{F}_n \xleftarrow{\tilde{\mathcal{D}}_{n+1}} \tilde{F}_{n+1} \longleftarrow 0 \end{array}$$

As $\mathcal{D}_{n+1}$ is a surjective operator with an injective adjoint $\tilde{\mathcal{D}}_{n+1}$ there exists an operator $\mathcal{P}_{n+1} : F_{n+1} \longrightarrow F_n$ such that $\mathcal{D}_{n+1} \circ \mathcal{P}_{n+1} \circ \mathcal{D}_{n+1} = \mathcal{D}_{n+1}$. Let us denote $\mathcal{Q}_n = Id_{F_n} - \mathcal{P}_{n+1} \circ \mathcal{D}_{n+1}$. We have $\mathcal{D}_{n+1} \circ \mathcal{Q}_n = \mathcal{D}_{n+1} - \mathcal{D}_{n+1} \circ \mathcal{P}_{n+1} \circ \mathcal{D}_{n+1} = 0$ and thus $\tilde{\mathcal{Q}}_n \circ \tilde{\mathcal{D}}_{n+1} = 0$. However, we have $\tilde{\mathcal{D}}_n \circ \tilde{\mathcal{D}}_{n+1} = 0$ which implies that $\tilde{\mathcal{Q}}_n$ factorizes by $\tilde{\mathcal{D}}_n$: $\tilde{\mathcal{Q}}_n = \tilde{\mathcal{P}}_n \circ \tilde{\mathcal{D}}_n \Rightarrow \mathcal{Q}_n = \mathcal{D}_n \circ \mathcal{P}_n \Rightarrow \mathcal{D}_n \circ \mathcal{P}_n + \mathcal{P}_{n+1} \circ \mathcal{D}_{n+1} = Id_{F_n} \Rightarrow \mathcal{D}_n \circ \mathcal{P}_n \circ \mathcal{D}_n = \mathcal{D}_n$. In a similar way, we can find $\mathcal{P}_i$ for $i \in \{1, \ldots, n\}$ satisfying $\mathcal{D}_i \circ \mathcal{P}_i \circ \mathcal{D}_i = \mathcal{D}_i$. If we want to compute effectively the different $\mathcal{P}_i$, we first have to construct the Janet sequence starting with $\mathcal{D}_1$. The last operator $\mathcal{D}_{n+1}$ of the Janet sequence is a surjective operator with a injective adjoint and $\mathcal{P}_{n+1}$ can be computed as it has been explained precedingly. So let us suppose that we know $\mathcal{P}_i$ then $\mathcal{P}_{i-1}$ can be computed as it follows:

1. Compute $\mathcal{Q}_{i-1} = Id_{F_{i-1}} - \mathcal{P}_i \circ \mathcal{D}_i$ and $\tilde{\mathcal{Q}}_{i-1}$.
2. As before, $\tilde{\mathcal{Q}}_{i-1}$ must factorized through $\tilde{\mathcal{D}}_{i-1}$ and we find $\tilde{\mathcal{P}}_{i-1}$ such that $\tilde{\mathcal{Q}}_{i-1} = \tilde{\mathcal{P}}_{i-1} \circ \tilde{\mathcal{D}}_{i-1}$ and dualizing, we have $\mathcal{P}_{i-1}$.

Example 8 We have seen that the Spencer operator defined a projective D-module. We show how to compute $\mathcal{P}_1$. We start by defining $Q_1 = Id_{F_1} - \mathcal{P}_2 \circ \mathcal{D}_2$. The operator $Q_1 : \zeta \to \zeta'$ is thus:

$$\begin{cases} \zeta^1 = \zeta'^1, \\ \zeta^2 = \zeta'^2, \\ \zeta_1^2 - \zeta_2^1 = \zeta'^3. \end{cases}$$

Taking its adjoint, we obtain $\tilde{Q}_1 : \lambda \to \phi$:

$$\begin{cases} d_2\lambda_3 + \lambda_1 = \phi_1, \\ -d_1\lambda_3 + \lambda_2 = \phi_2, \\ 0 = \phi_3, \end{cases}$$

whereas $\tilde{\mathcal{D}}_1$ is given by (6). We easily find that $\tilde{\mathcal{P}}_1$ is defined by:

$$\begin{cases} -\mu_2 = \phi_1, \\ -\mu_3 = \phi_2, \\ 0 = \phi_3, \end{cases}$$

and we have $\mathcal{P}_1 : \zeta \to \eta$:

$$\begin{cases} 0 = \eta^1, \\ -\zeta^1 = \eta^2, \\ -\zeta^2 = \eta^3. \end{cases} \tag{13}$$

We let the reader check that $\mathcal{D}_1 \circ \mathcal{P}_1 \circ \mathcal{D}_1 = \mathcal{D}_1$.

We now state a very useful theorem [13, 15].

Theorem 3 *A surjective linear time-varying OD control system is controllable if and only if its adjoint is injective.*

Proof In a principal ring, the notion of torsion-free and projective module are equivalent. Thus, a linear OD control system is controllable if and only if the module $\mathcal{M}$ is projective. $\mathcal{D}_1$ is a surjective operator with its adjoint $\tilde{\mathcal{D}}_1$ which is injective, then $\mathcal{M}$ is projective and the system is controllable. Conversely, if $\tilde{\mathcal{D}}_1$ is not injective then we can find a test vector λ which satisfies $\tilde{\mathcal{D}}_1\lambda = 0$. Thus $\lambda^t\mathcal{D}_1\eta$ is a total derivative of an observable which is therefore a torsion element as its derivative is null as soon as η is a solution of the system and the system is not controllable.

Example 9 We take again the first example. Multiplying it by a row vector $\lambda = (\lambda_1, \lambda_2)$ and integrating the result by part, we obtain $\tilde{\mathcal{D}}_1 : \lambda \to \mu$ defined by:

$$\begin{cases} \ddot{\lambda}_1 + \lambda_1 - \lambda_2 = \mu_1, \\ \ddot{\lambda}_2 + \lambda_2 - \lambda_1 = \mu_2, \\ -\lambda_2 + \alpha\lambda_1 = \mu_3. \end{cases}$$

Differentiating twice the zero-order equation and substituting it, we obtain

$$(\alpha + 1)(\alpha - 1)\lambda_1 = 0,$$

and thus the operator $\tilde{\mathcal{D}}_1$ is injective and thus controllable if and only if $\alpha \neq -1$ and $\alpha \neq 1$.

Theorem 4 *An operator $\mathcal{D}_1$ determines a free D-module $\mathcal{M}$ if an only if there exists an injective parametrization of $\mathcal{D}_1$.*

Indeed, let $\mathcal{D}_0\xi = \eta$ be an injective parametrization of $\mathcal{D}_1\eta = \zeta$. As ξ does not satisfy any equation, the D-module $D\xi$ generated by ξ is a free module. If $\mathcal{D}_1$ is controllable then we have $\mathcal{M} \subseteq D\xi$. Remark that it is nothing else than a reformulation of the property of a torsion-free module: any submodule of a free module is torsion-free. Now, if $\mathcal{D}_0\xi = \eta$ is an injective parametrization of $\mathcal{D}_1$ then there exists a left-inverse $\mathcal{P}_0$ of $\mathcal{D}_0$ such that $\xi = \mathcal{P}_0 \circ \mathcal{D}_0\xi \Leftrightarrow \xi = \mathcal{P}_0\eta \Rightarrow D\xi \subseteq \mathcal{M}$. Thus, $\mathcal{M} = D\xi$.

Example 10 We take the operator made with the compatibilty condition of the Spencer operator $\mathcal{D}_2 : \zeta \to \chi$ defined by $\zeta^2_1 - \zeta^1_2 - \zeta^3 = \chi$. Its adjoint $\tilde{\mathcal{D}}_2$ is injective and $\tilde{\mathcal{D}}_1$, defined by (6), is the operator made by the compatibility conditions of $\tilde{\mathcal{D}}_2$. But, $\mathcal{D}_1$ is not an injective parametrization of $\mathcal{D}_2$. However, we have $\mu_1 = d_1\mu_2 + d_2\mu_3$ and thus if we take only the first and the second equation of $\tilde{\mathcal{D}}_1$ as a new operator, we easily see that its adjoint $\mathcal{D}^\sharp_1 : \theta \to \zeta$, defined by

$$\begin{cases} -\theta^1 = \zeta^1, \\ -\theta^2 = \zeta^2, \\ -\theta^2_1 + \theta^1_2 = \zeta^3, \end{cases}$$

is an injective parametrization of $\mathcal{D}_2$ and $\mathcal{D}_2$ determines a free D-module.

3 Generalized Bezout Identity

Let $\mathcal{D}_1 : F_0 \longrightarrow F_1$ be an operator determining a projective D-module $\mathcal{M}$ then we have seen that we could construct its Janet sequence

$$F_0 \xrightarrow{\mathcal{D}_1} F_1 \ldots F_n \xrightarrow{\mathcal{D}_{n+1}} F_{n+1} \longrightarrow 0.$$

Let us suppose that we have found some operators $\mathcal{P}_i$ such that $\mathcal{D}_i \circ \mathcal{P}_i \circ \mathcal{D}_i = \mathcal{D}_i$ for $i = 1 \ldots n$ and $\mathcal{P}_{n+1} \circ \mathcal{D}_{n+1} = Id_{F_{n+1}}$. Let us foccus, only for the moment, on the exact differential sequence $F_{i-1} \xrightarrow{\mathcal{D}_i} F_i \xrightarrow{\mathcal{D}_{i+1}} F_{i+1}$ with $\mathcal{D}_{i+1} \circ \mathcal{P}_{i+1} \circ \mathcal{D}_{i+1} = \mathcal{D}_{i+1}$ and $\mathcal{D}_i \circ \mathcal{P}_i \circ \mathcal{D}_i = \mathcal{D}_i$. We have $\forall\, \eta \in F_i$: $\mathcal{D}_{i+1} \circ (Id_{F_i} - \mathcal{P}_{i+1} \circ \mathcal{D}_{i+1})\eta = 0 \Rightarrow \exists\, \xi \in F_{i-1}$: $(Id_{F_i} - \mathcal{P}_{i+1} \circ \mathcal{D}_{i+1})\eta = \mathcal{D}_i\xi$ as the sequence made by $\mathcal{D}_i$ and $\mathcal{D}_{i+1}$ is exact. However, we have $\forall\, \xi \in F_{i-1}$: $(\mathcal{D}_i \circ \mathcal{P}_i - Id_{F_i}) \circ \mathcal{D}_i\xi = 0 \Rightarrow (\mathcal{D}_i \circ \mathcal{P}_i - Id_{F_i}) \circ (Id_{F_i} - \mathcal{P}_{i+1} \circ \mathcal{D}_{i+1})\eta = 0, \forall \eta \in F_i$. Finally, we obtain the new identity $Id_{F_i} = \mathcal{D}_i \circ \mathcal{P}_i + \mathcal{P}_{i+1} \circ \mathcal{D}_{i+1} - \mathcal{D}_i \circ \mathcal{P}_i \circ \mathcal{P}_{i+1} \circ \mathcal{D}_{i+1}$. This identity can be rewritten under the two different following forms:

$$\begin{cases} \mathcal{P}'_i = \mathcal{P}_i \circ (Id_{F_i} - \mathcal{P}_{i+1} \circ \mathcal{D}_{i+1}), \\ Id_{F_i} = \mathcal{D}_i \circ \mathcal{P}'_i + \mathcal{P}_{i+1} \circ \mathcal{D}_{i+1}. \end{cases} \tag{14}$$

or

$$\begin{cases} \mathcal{P}''_{i+1} = (Id_{F_i} - \mathcal{D}_i \circ \mathcal{P}_i) \circ \mathcal{P}_{i+1}, \\ Id_{F_i} = \mathcal{D}_i \circ \mathcal{P}_i + \mathcal{P}''_{i+1} \circ \mathcal{D}_{i+1}, \end{cases} \tag{15}$$

Now, let us suppose that $\mathcal{P}_{i+1} \circ \mathcal{D}_{i+1} \circ \mathcal{P}_{i+1} = \mathcal{P}_{i+1}$, then we have $\mathcal{P}'_i \circ \mathcal{P}_{i+1} = 0 \Rightarrow \operatorname{Im} \mathcal{P}_{i+1} \subseteq \operatorname{Ker} \mathcal{P}'_i$. Let us take $\eta \in \operatorname{Ker} \mathcal{P}'_i$ then from the second equation of (14) we have, $\eta = \mathcal{P}_{i+1}(\mathcal{D}_{i+1}\eta) \Rightarrow \eta \in \operatorname{Im} \mathcal{P}_{i+1}$ showing that

$$F_{i+1} \xrightarrow{\mathcal{P}_{i+1}} F_i \xrightarrow{\mathcal{P}'_i} F_{i-1},$$

is an exact differential sequence. Moreover, from the second equation of (14), we have $\mathcal{P}'_i \circ \mathcal{D}_i \circ \mathcal{P}'_i = \mathcal{P}'_i$. For showing that $\mathcal{P}_{i+1} \circ \mathcal{D}_{i+1} \circ \mathcal{P}_{i+1} = \mathcal{P}_{i+1}$, we have only to prove it for $i = n$. However, $\mathcal{D}_{n+1} \circ \mathcal{P}_{n+1} = Id_{F_{n+1}} \Rightarrow \mathcal{P}_{n+1} \circ \mathcal{D}_{n+1} \circ \mathcal{P}_{n+1} = \mathcal{P}_{n+1}$. Finally, we obtain the following exact differential sequence:

$$0 \to F_{n+1} \xrightarrow{\mathcal{P}_{n+1}} F_n \xrightarrow{\mathcal{P}'_n} F_{n-1} \dots F_1 \xrightarrow{\mathcal{P}'_1} F_0,$$

with $\mathcal{D}_i \circ \mathcal{P}'_i \circ \mathcal{D}_i = \mathcal{D}_i$ and $\mathcal{P}'_i \circ \mathcal{D}_i \circ \mathcal{P}'_i = \mathcal{P}'_i$. Now, as $\mathcal{D}_1$ determines a projective D-module, there exists a parametrization $\mathcal{D}_0$ and we can prolong the above differential sequence in order to have the following exact differential sequence:

$$0 \to F_{n+1} \xrightarrow{\mathcal{P}_{n+1}} F_n \xrightarrow{\mathcal{P}'_n} F_{n-1} \dots F_1 \xrightarrow{\mathcal{P}'_1} F_0 \xrightarrow{\mathcal{P}'_0} E,$$

with $\mathcal{D}_0 \circ \mathcal{P}'_0 \circ \mathcal{D}_0 = \mathcal{D}_0$ and $\mathcal{P}'_0 \circ \mathcal{D}_0 \circ \mathcal{P}'_0 = \mathcal{P}'_0$. We now explain the link of the preceding results with the generalized Bezout identity.

3.1 PD Control Systems with Variable Coefficients

We sum up the different results of the preceding sections in the following theorem. We insist on the fact that everything that follows can be computed. See the examples illustrating the main results.

Theorem 5 *Let $\mathcal{D}_1 : F_0 \to F_1$ be a PD control system with variable coefficients.*

1. *If $\mathcal{D}_1$ determines a free D-module $\mathcal{M}$ then there exists three operators $\mathcal{D}_0 : E \to F_0$, $\mathcal{P}_0 : F_0 \to E$ and $\mathcal{P}_1 : F_1 \to F_0$ such that:*

$$\begin{cases} \mathcal{D}_1 \circ \mathcal{D}_0 = 0, \\ \mathcal{P}_0 \circ \mathcal{D}_0 = Id_{F_0}, \\ \mathcal{D}_1 \circ \mathcal{P}_1 \circ \mathcal{D}_1 = \mathcal{D}_1, \\ \mathcal{P}_0 \circ \mathcal{P}_1 = 0. \end{cases}$$

   The sequences $0 \longrightarrow E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1$ and $F_1 \xrightarrow{\mathcal{P}_1} F_0 \xrightarrow{\mathcal{P}_0} E \longrightarrow 0$ are exact.

2. *If $\mathcal{D}_1$ determines a projective D-module $\mathcal{M}$ then there exists three operators $\mathcal{D}_0 : E \to F_0$, $\mathcal{P}_0 : F_0 \to E$ and $\mathcal{P}_1 : F_1 \to F_0$ such that:*

$$\begin{cases} \mathcal{D}_1 \circ \mathcal{D}_0 = 0, \\ \mathcal{D}_0 \circ \mathcal{P}_0 \circ \mathcal{D}_0 = \mathcal{D}_0, \\ \mathcal{D}_1 \circ \mathcal{P}_1 \circ \mathcal{D}_1 = \mathcal{D}_1, \\ \mathcal{P}_0 \circ \mathcal{P}_1 = 0. \end{cases}$$

   The sequence $E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1$ and $F_1 \xrightarrow{\mathcal{P}_1} F_0 \xrightarrow{\mathcal{P}_0} E$ are exact.

3. *If $\mathcal{D}_1$ determines a torsion-free D-module $\mathcal{M}$, i.e. $\mathcal{D}_1$ is controllable, then there exists one operator $\mathcal{D}_0 : E \to F_0$ such that:*

$$\mathcal{D}_1 \circ \mathcal{D}_0 = 0.$$

   The sequence $E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1$ is a formally exact sequence.

Example 11 Let us take again the Spencer operator $\mathcal{D}_1$ defined by (5). We have shown that $\mathcal{D}_1$ determines a free D-module and that the following sequence

$$0 \to E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} F_2 \to 0, \tag{16}$$

was a formally exact, where $\mathcal{D}_0$ and $\mathcal{D}_2$ are defined respectively by (7) and by (11). The operator $\mathcal{P}_2$ defined by (12) is a right-inverse of $\mathcal{D}_2$ and $\mathcal{P}_1$, defined by (13), satisfies $\mathcal{D}_1 \circ \mathcal{P}_1 \circ \mathcal{D}_1 = \mathcal{D}_1$. We let the reader check that $\mathcal{P}'_1 = \mathcal{P}_1$ and $\mathcal{P}'_0 = \mathcal{P}_0 : \eta \to \xi$ defined by $\eta^1 = \xi$ satisfies $\mathcal{P}_0 \circ \mathcal{D}_0 = Id_{F_0}$. Thus, we have:

$$\begin{cases} \mathcal{D}_1 \circ \mathcal{D}_0 = 0, \\ \mathcal{D}_2 \circ \mathcal{D}_1 = 0, \\ \mathcal{P}_0 \circ \mathcal{D}_0 = Id_{F_0}, \\ \mathcal{D}_1 \circ \mathcal{P}_1 \circ \mathcal{D}_1 = \mathcal{D}_1, \\ \mathcal{P}_1 \circ \mathcal{D}_1 \circ \mathcal{P}_1 = \mathcal{P}_1, \\ \mathcal{D}_2 \circ \mathcal{P}_2 = Id_{F_2}, \\ \mathcal{P}_1 \circ \mathcal{P}_2 = 0, \\ \mathcal{P}_0 \circ \mathcal{P}_1 = 0. \end{cases}$$

Moreover, the sequences

$$0 \longrightarrow F_2 \xrightarrow{\mathcal{P}_2} F_1 \xrightarrow{\mathcal{P}_1} F_0 \xrightarrow{\mathcal{P}_0} E \longrightarrow 0,$$

and (16) are exact.

In the case where $\mathcal{D}_1$ is an surjective operator, the previous theorem leads to the existence of the generalized Bezout identity.

Corollary 1 *Let $\mathcal{D}_1 : F_0 \longrightarrow F_1$ be a surjective PD control system with variable coefficients.*

1. *If the operator $\mathcal{D}_1$ determines a free D-module then we have:*

$$\begin{bmatrix} \mathcal{D}_1 \\ \mathcal{P}_0 \end{bmatrix} \circ \begin{bmatrix} \mathcal{P}_1 & \mathcal{D}_0 \end{bmatrix} = \begin{bmatrix} Id_{F1} & 0_{F1} \\ 0_E & Id_E \end{bmatrix},$$

   and the generalized Bezout identity is equivalent to the splitting of the following formally exact differential sequence:

$$0 \longrightarrow E \underset{\xleftarrow{\mathcal{P}_0}}{\xrightarrow{\mathcal{D}_0}} F_0 \underset{\xleftarrow{\mathcal{P}_1}}{\xrightarrow{\mathcal{D}_1}} F_1 \to 0.$$

2. *If $\mathcal{D}_1$ determines a projective D-module then we have:*

$$\left[\ \mathcal{D}_1\ \right] \circ \left[\ \mathcal{P}_1 \quad \mathcal{D}_0\ \right] = \left[\ Id_{F1} \quad 0_{F1}\ \right].$$

In the next example, we illustrate each situation for a surjective operator.

Example 12 1. The system $\eta_2^1 - x^2\eta_1^2 - \eta^3 = 0$ determines a free-module and we have:

$$\begin{bmatrix} d_2 & -x^2 d_1 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & -d_2 & -x^2 d_1 \end{bmatrix} = I.$$

2. We have seen that the system (8) defined by $\eta_2^2 - x^2\eta_1^1 + \eta^1 = 0$, was generating a projective module and we found a right-inverse. We let the reader check that:

$$\left[\ -x^2 d_1 + 1 \quad d_2\ \right] \circ \begin{bmatrix} x^2 d_2 + 2 & x^2 d_2^2 + 2d_2 \\ (x^2)^2 d_1 - x^2 & (x^2)^2 d_1 d_2 - x^2 d_2 - 1 \end{bmatrix} = [1, 0].$$

3. The system $\eta_2^2 - x^1\eta_1^1 + \eta^1 = 0$ determines only a torsion-free module and we have:

$$\left[\ -x^1 d_1 + 1 \quad d_2\ \right] \circ \begin{bmatrix} -d_2 \\ -x^1 d_1 + 1 \end{bmatrix} = 0.$$

Projective module and right-inverse are useful if we want to know whether a system of polynomial equations admits some solutions. We give an example.

Example 13 The Hilbert theorem claims that the system $P_1, \ldots, P_m \in k[\chi_1, \ldots, \chi_n]$ has no solution if and only if there exists $Q_1, \ldots, Q_m \in k[\chi_1, \ldots, \chi_n]$ such that $Q_1 P_1 + Q_2 P_2 + \ldots + Q_m P_m = 1$. We can reformulate the Hilbert theorem saying that the system of polynomial equations $P_1, \ldots, P_m \in k[\chi_1, \ldots, \chi_n]$ has no solution if and only if the adjoint of the surjective operator $\mathcal{D}_1 : \eta \to \zeta$ defined by $P_1\eta^1 + \ldots + P_m\eta^m = \zeta$, where we have substituted χ_i by d_i in P_j, is injective. We give an example.

We search the common solutions of the following set of polynomial equations:

$$\left\{ \begin{array}{l} P_1 = \chi_3^3 + \chi_1\chi_3 + 1, \\ P_2 = \chi_3^2 + \chi_2\chi_3, \\ P_3 = \chi_2^2 + \chi_1. \end{array} \right. \tag{17}$$

We define $\mathcal{D}_1$ by $(d_3^3 + d_1 d_3 + 1)\eta^1 + (d_3^2 + d_2 d_3)\eta^2 + (d_2^2 + d_1)\eta^3 = \zeta$. It is quite easy to see that we obtain $\lambda = \mu_1 + (d_3 - d_2)\mu_2 + d_3\mu_3$ from $\tilde{\mathcal{D}}_1\lambda = \mu$. Thus, $\mathcal{P}_1 : \zeta \to \eta$, defined by

$$\left\{ \begin{array}{l} \zeta = \eta^1, \\ (d_2 - d_3)\zeta = \eta^2, \\ -d_3\zeta = \eta^3, \end{array} \right.$$

is a right-inverse of $\mathcal{D}_1$, we have $P_1 + (\chi_2 - \chi_3)P_2 - \chi_3 P_1 = 1$ and the system (17) has no solution.

3.2 Time-varying OD Control Systems

The following theorem leads to the existence of the generalized Bezout identity.

Theorem 6 *Let $\mathcal{D}_1 : F_0 \to F_1$ be a controllable time-varying OD control system then there exists three operators $\mathcal{D}_0 : E \to F_0$, $\mathcal{P}_0 : F_0 \to E$ and $\mathcal{P}_1 : F_1 \to F_0$ such that:*

$$\left\{ \begin{array}{l} \mathcal{D}_1 \circ \mathcal{D}_0 = 0, \\ \mathcal{P}_0 \circ \mathcal{D}_0 = Id_{F_0}, \\ \mathcal{D}_1 \circ \mathcal{P}_1 \circ \mathcal{D}_1 = \mathcal{D}_1, \\ \mathcal{P}_1 \circ \mathcal{P}_0 = 0. \end{array} \right.$$

Moreover, the sequences $0 \longrightarrow E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1$ and $F_1 \xrightarrow{\mathcal{P}_1} F_0 \xrightarrow{\mathcal{P}_0} E \longrightarrow 0$ are exact. If $\mathcal{D}_1$ is a surjective operator then we have:

$$\begin{bmatrix} \mathcal{D}_1 \\ \mathcal{P}_0 \end{bmatrix} \circ \left[\ \mathcal{P}_1 \quad \mathcal{D}_0\ \right] = \begin{bmatrix} Id_{F1} & 0_{F1} \\ 0_E & Id_E \end{bmatrix},$$

and the generalized Bezout identity is equivalent to the splitting of the following exact differential sequence:

$$0 \longrightarrow E \underset{\xleftarrow{\mathcal{P}_0}}{\xrightarrow{\mathcal{D}_0}} F_0 \underset{\xleftarrow{\mathcal{P}_1}}{\xrightarrow{\mathcal{D}_1}} F_1 \to 0.$$

If we start with the system (1), we can rewrite it under the form $\mathcal{D}_1\eta = \zeta$ where $\mathcal{D}_1 = [P(s), Q(s)]$, $s = \frac{d}{dt}$ and $\eta = (y, u)^t$. The asumption of $\det\, P(s) \neq 0$ amounts to the surjectivity of $\mathcal{D}_1$ and by the theorem 6, we have (2).

We now give an example of a computation of the generalized Bezout identity for a time-varying OD control system.

Example 14 We compute a generalized Bezout identity for the following time-varying OD control system:

$$\ddot{\eta}^1 + \alpha(t)\dot{\eta}^1 + \eta^1 - \dot{\eta}^2 - \alpha(t)\eta^2 = 0.$$

We take the surjective operator $\mathcal{D}_1$ associated with the previous system and dualizing it, we obtain the operator $\tilde{\mathcal{D}}_1 : \lambda \to \mu$:

$$\left\{ \begin{array}{l} \ddot{\lambda} - \alpha(t)\dot{\lambda} - \dot{\alpha}(t)\lambda + \lambda = \mu_1, \\ \dot{\lambda} - \alpha(t)\lambda = \mu_2. \end{array} \right.$$

It is easy to see that $\tilde{\mathcal{D}}_1$ is an injective operator as we have $\tilde{\mathcal{P}}_1 \circ \tilde{\mathcal{D}}_1 = Id_{\tilde{E}}$ where $\tilde{\mathcal{P}}_1 : \mu \to \lambda$ is given by: $-\dot{\mu}_2 + \mu_1 = \lambda$. Thus, the adjoint of $\tilde{\mathcal{P}}_1$ is a right-inverse of $\mathcal{D}_1$ and we find $\mathcal{P}_1 : \zeta \to \eta$ defined by:

$$\left\{ \begin{array}{l} \zeta = \eta^1, \\ \dot{\zeta} = \eta^2. \end{array} \right.$$

Substituting $\lambda = \tilde{\mathcal{P}}_1\mu$ in $\tilde{\mathcal{D}}_1$, we find the operator $\tilde{\mathcal{D}}_0\mu = -\ddot{\mu}^2 + \alpha(t)\dot{\mu}^2 - \mu^2 + \dot{\mu}^1 - \alpha(t)\mu^1 = \nu$. Dualizing $\tilde{\mathcal{D}}_0$, we obtain a parametrization $\mathcal{D} : \xi \to \eta$ defined by:

$$\left\{ \begin{array}{l} \dot{\xi} + \alpha(t)\xi = \eta^1, \\ \ddot{\xi} + \alpha(t)\dot{\xi} + (1 + \dot{\alpha}(t))\xi = \eta^2. \end{array} \right.$$

This parametrization is injective and we have the left-inverse of $\mathcal{P}\eta = -\dot{\eta}^1 + \eta^2 = \xi$. We easily check that $\mathcal{P}' = \mathcal{P}$ generates exactly the compatibility conditions of $\mathcal{P}_1$. We have $\mathcal{M} = D(-\dot{\eta}^1 + \eta^2)$ and the $\mathcal{M}$ is a free D-module. We have:

$$\left[\begin{array}{cc} s^2 + \alpha s + 1 & -s - \alpha \\ -s & 1 \end{array} \right] \circ \left[\begin{array}{cc} 1 & s + \alpha \\ s & s^2 + \alpha s + (1 + \dot{\alpha}) \end{array} \right] = I.$$

4 Conclusion

We have seen how the generalized Bezout identity could be extended to none surjective linear time-varying OD control system. In the case where the linear time-varying OD control system is surjective, we have shown that the generalized Bezout identity was, in fact, the well-kwown algebraic notion of splitting exact sequence: the generalized Bezout identity is a splitting of the formally exact differential sequence made with the system and its parametrization. We have seen when and how it could be extended for general linear PD control system with variable coefficients. We have shown that it was only depend on the algebraic nature of the differential module determined by the system. This new formulation has the advantage to bring the generalized Bezout identity and its computation closer to algebraic and geometric concepts. In particular, we have made clear that it did not depend at all on a separation of the system variables between inputs and outputs.

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