Some geometric and combinatorial properties of $(0, m, 2)$-nets in base $b \geq 2$

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Abstract

In this paper, we study the geometric generation of the $(0, m, 2)$-nets. Based on some combinatorial methods, a generation property on $(0, m, 2)$-nets with an applications to a Sobol' sequence and the number of the $(0, m, 2)$-nets in equivalent sense are given. We also present some symmetry properties of the Sobol' sequence and the Roth-Zaremba sequence.

Key words: Discrepancy, $(0, m, 2)$-net, Sobol' sequence, Roth-Zaremba sequence.

1 Introduction

Sequences of points in $I^s = [0, 1)^s$ with even distribution or low discrepancy play a fundamental role in Quasi-Monte Carlo methods. The discrepancy is a well-known measure for the irregularity of distribution of a sequence and the (t, s)-sequences are among the best of low discrepancy sequences (see [8],[9], [1], [5] and [6]).

Recall some definitions concerning with the discrepancy and the (t, s)-sequences. Denote $I^s = [0, 1)^s$ and let $b \geq 2$ be an integer fixed.

Definition 1.1 *The* discrepancy of N points $\mathbf{x}_1, \ldots, \mathbf{x}_N$ *in* I^s *is defined by*

$$D_N = \sup_{J \in \mathcal{J}} \left| \frac{A_N(J)}{N} - \lambda_s(J) \right|$$

where $\mathcal{J}$ denotes the family of all subintervals of I^s of the form $J = \Pi_{i=1}^{s}[0, u_i)$, $A_N(J)$ represents the number of n, $0 \leq n < N$, for which $\mathbf{x}_n \in J$ and λ_s stands for Lebesgue measure on I^s.

Definition 1.2 *An* elementary interval in base b *of* I^s *is an interval of the form*

$$E = \prod_{i=1}^{s}[\frac{a_i}{b^{d_i}}, \frac{a_i+1}{b^{d_i}}),$$

with integers $d_i \geq 0$ *and* $0 \leq a_i < b^{d_i}$ *for* $1 \leq i \leq s$.

Definition 1.3 *Let* $0 \leq t \leq m$ *be integers. A* (t, m, s)-net in base b *is a point set of* b^m *points in* I^s *such that every elementary interval in base* b *of volume* b^{t-m} *contains* b^t *points of this net.*

A sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots,$ *of points in* I^s *is called a* (t, s)-sequence in base b *if for all integers* $k \geq 0$ *and* $m > t$ *the point set consisting of the* $\mathbf{x}_n$ *with* $kb^m < n \leq (k+1)b^m$ *is a* (t, m, s)*-net in base* b.

(t, m, s)-nets in base b have very good geometrical properties illustrated in the following proposition (cf. [5]).

Proposition 1.4 *Let* P *be a* (t, m, s)*-net in base* b*, let* E *be an elementary interval in base* b *with volume* b^{-u}*, where* $0 \leq u \leq m - t$*, and let* T *be an affine transformation from* E *onto* I^s*. Then the points of* P *that belong to* E *are transformed by* T *into a* $(t, m-u, s)$*-net in base* b.

A (t, s)-sequence has very low discrepancy, and according to [6], its discrepancy satisties an effective bound $D_N \leq C_s (\log N)^s / N + O((\log N)^{s-1}/N)$ with $\lim_{s\to\infty} C_s = 0$.

However, the constructions of the (t, s)-sequences are often algebric, such as Sobol' [8] with linear recurring relations over $\mathbf{F}_2$, Faure [1] with Pascal matrix triangle and Niederreiter [6] with polynomial over finite fields which generalize the Sobol' and Faure methods, thus we have not a clear idea of how the (t, s)-sequences are constructed geometrically and why they are more uniformely distributed except for their discrepancy properties?

The purpose of this paper is to study the geometric generation of the $(0, m, 2)$-nets which makes us to have a better idea on its structure.

We only consider the $(0, m, s)$-nets of I^s in equivalent sense as follows.

Definition 1.5 *Let R_1 and R_2 be two $(0,m,s)$-nets in base b. We say that $R_1 \simeq R_2$ if for all $x \in R_1$ such as $x \in E = \prod_{i=1}^{s}[\frac{a_i}{b^m}, \frac{a_i+1}{b^m})$, a elementary interval in base b with volume b^{-sm}, there is a $y \in R_2$ such as $y \in E$.*

It is easy to show that the above relation is an equivalent relation. Denote

$$\mathcal{F}_m^{s,b} = \{(\frac{i_1}{b^m}, \ldots, \frac{i_s}{b^m}) \mid 0 \le i_k \le b^m - 1, 1 \le k \le s\}.$$

A $(0,m,s)$-net in base b, R, is called a *normalized $(0,m,s)$-net in base b* if $R \subset \mathcal{F}_m^{s,b}$; for each $(0,m,s)$-net in base b we can choose a unique representant in $\mathcal{F}_m^{s,b}$.

Remark 1.6 *If R_1 and R_2 be two $(0,m,s)$-nets in base b such that $R_1 \simeq R_2$, then $|D(R_1) - D(R_2)| \le \frac{s}{b^m}$. This property is also hold for the other version of discrepancy, such as L^p-discrepancy with $1 \le p < \infty$.*

To study the $(0,m,s)$-nets in equivalent sense, we need some notations.

Notation : Let $s \ge 2$ and $b \ge \sup(s-1, 2)$ be integers. For $i_1, \ldots, i_s = 0, 1, \cdots, b-1$, we note $E_{i_1,\ldots,i_s} = \prod_{k=1}^{s}[\frac{i_k}{b}, \frac{i_k+1}{b})$ the elementary interval in base b of volume b^{-s}, and $T_{i_1,\ldots,i_s}$ the affine transformations from $E_{i_1,\ldots,i_s}$ onto I^s. For any point set Q of I^s, we note $T_{i_1,\ldots,i_s}(Q \cap E_{i_1,\ldots,i_s})$ by $T_{i_1,\ldots,i_s}(Q)$ to simplify the notation.

For $1 \le i \le s$, we note T_i^{proj} the projection from I^s to I^{s-1} defined by

$$T_i^{proj}(x_1, \cdots, x_i, \cdots, x_s) = (x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_s) \quad \text{for all } (x_1, \cdots, x_s) \in I^s.$$

We also need the following important characteristic lemma of a $(0,m,s)$-net. It can be proved easily by the definition of a $(0,m,s)$-net in base b and Proposition 1.3.

Lemma 1.7 *Let $m \ge s$ be an integer. Then a set R of b^m points in I^s is a $(0,m,s)$-net in base b if and only if the following conditions are satisfied:*

- *For all $i_1, \ldots, i_s \in \{0, 1, \cdots, b-1\}$, $T_{i_1,\ldots,i_s}(R)$ is a $(0, m-s, s)$-net in base b.*
- *For $1 \le i \le s$, $T_i^{proj}(R)$ is a $(0, m, s-1)$-net in base b.*

In this paper, we give the proof parts of the results annonced in the Note [11] with some generalizitions. First, we generalize the result on the generation of a $(0, m+1, 2)$-net from a known $(0, m, 2)$-net with an applications to a Sobol' sequence in Section 2. In Section 3 we compute the number of all $(0, m, 2)$-nets in base b in equivalent sense, which brings us to introduce the notion of $(0, m, 2)$-sequence and ask some questions concerning it. In the end, some symmetry properties of the Sobol' sequence and the Roth-Zaremba sequence are presented in Section 4.

2 Generation of $(0, m, 2)$-net with an application to a Sobol' sequence

The following theorem shows that a $(0, m, 2)$-net in base b can be self-generated from an original net and the number of such generation is large if $b \geq 3$.

Theorem 2.1 *Let R_m be a $(0, m, 2)$-net in base b with $m \geq 0$. Then there are $(b-1)!^{(m+1)b^m}$ $(0, m+1, 2)$-nets in base b, in equivalent sense, which include R_m.*

We call that a $(0, m+1, 2)$-net in base b, R_{m+1}, is generated by $(0, m, 2)$-net in base b, R_m, if $R_m \supset R_{m+1}$.

Proof . Denote $N(b, m) = (b-1)!^{(m+1)b^m}$. We will use induction on m.

For $m = 0$, the theorem is trivial.

For $m = 1$, decompose the I^2 into b^2 elementary intervals in base b, E_{i_1,i_2}, with $i_1, i_2 \in \{0, 1, \ldots, b-1\}$ and add one point in E_{i_1,i_2} if there is no point of R_1. Then, there are $b-1$ new points in each elementary inteval of the form

$$[\frac{u}{b}, \frac{u+1}{b}) \times [0,1) \quad \text{or} \quad [0,1) \times [\frac{v}{b}, \frac{v+1}{b})$$

with $u, v = 0, \ldots, b-1$. We have $(b-1)!$ possibilities to place these $b-1$ new points in order that there is one and only one point in each elementary inteval of the form

$$[\frac{ub+s}{b^2}, \frac{ub+s+1}{b^2}) \times [0,1) \quad \text{or} \quad [0,1) \times [\frac{vb+t}{b^2}, \frac{vb+t+1}{b^2})$$

where $s, t = 0, \ldots, b-1$. Thus, by Lemma 1.7 there are

$$(b-1)!^b \times (b-1)!^b = (b-1)!^{2b}$$

possibilities to construct a $(0, 2, 2)$-net in base b containing R_1.

Suppose that the theorem is true until $m-1$.

In the case of m, for each pair $i_1, i_2 \in \{0, 1, \ldots, b-1\}$,

$$T_{i_1,i_2}(R_m) = Q_{i_1,i_2}(m-2)$$

is a $(0, m-2, 2)$-net in base b. By the hypothesis of the induction, $Q_{i_1,i_2}(m-2)$ can generate $N_{b,m-2}$ $(0, m-1, 2)$-nets in base b. Let $Q_{i_1,i_2}(m-1)$ be one of such $(0, m-1, 2)$-nets in base b and write

$$R = \cup_{i_1,i_2=0,1,\ldots,b-1} T^1_{i_1,i_2}(Q_{i_1,i_2}(m-1)).$$

Then, $R_m \subset R$ and there are $b-1$ points of $R \backslash R_m$ in each elementary inteval of the form

$$[\frac{u}{b^m}, \frac{u+1}{b^m}) \times [0,1) \quad \text{or} \quad [0,1) \times [\frac{v}{b^m}, \frac{v+1}{b^m})$$

with $u, v = 0, \ldots, b^m - 1$. As in the case of $m = 1$, there are $(b-1)!$ possibilities to place these $b-1$ points in order that there is one and only one point in each elementary inteval of the form

$$[\frac{ub+s}{b^{m+1}}, \frac{ub+s+1}{b^{m+1}}) \times [0,1) \quad \text{or} \quad [0,1) \times [\frac{vb+t}{b^{m+1}}, \frac{vb+t+1}{b^{m+1}})$$

with $s, t = 0, \ldots, b-1$, and the image of the new net by T_{i_1,i_2} is $Q_{i_1,i_2}(m-1)$ in equivalent sense.

Finally, together with Lemma 1.7 and the hypothesis of the induction,

$$N(b,m) = (N(b, m-2))^{b^2}((b-1)!)^{2b^m}$$

and the result follows. □

Corollary 2.2 *Let R_m be a $(0, m, 2)$-net in base 2 with $m \geq 0$. Then the $(0, m+1, 2)$-net R_{m+1} in base 2 generated by R_m is unique in equivalent sense. Moreover, $R_{m+1} \backslash R_m$ is a $(0, m, 2)$-net.*

Corollary 2.3 *If $R_m \supset \mathcal{F}^{2,2}_{m+1}$ is a $(0, m, 2)$-net in base 2, then the unique normalized $(0, m+1, 2)$-net R_{m+1} generated by R_m satisfies $R_m \subset R_{m+1}$.*

Remark 2.4 *In dimension 3, the above generation theorem of the net does not hold in base 2. For example, there is no $(0,2,3)$-net in base 2 which contains the $(0,1,3)$-net in base 2 defined by $\{(0,0,0),(0.5,0.5,0.5)\}$. This can be proved by using the projection nets on two dimensions, which are the same thanks to Theorem 2.1.*

Now, we give an application to two-dimensional Sobol' sequences which is a $(0,2)$-sequence in base 2. Recall its definition. Let $n \geq 0$ be an integer with $n = \sum_{j=0}^{\infty} n_j 2^j$, then the Sobol' sequence is defined by

$$(\phi(n),\psi(n))_{n\geq 0} = (\sum_{j=0}^{\infty} \frac{n_j}{2^{j+1}}, \sum_{j=0}^{\infty} \frac{h_j}{2^{j+1}}),$$

where $h_j = \sum_{i=j}^{\infty} \begin{pmatrix} i \\ j \end{pmatrix} n_i (\mathrm{mod}\ 2)$. Using the fact that the 2^m first terms of Sobol' sequence is a normalized $(0,m,2)$ net in base 2, and Theorem 2.1, we can prove the following proposition.

Proposition 2.5 *Let $S_0 = \{(0,0)\}$ be a $(0,0,2)$-net in base 2. For $m \geq 1$, let S_m denote the normalized $(0,m,2)$-net in base 2 generated by the normalized $(0,m-1,2)$-net S_{m-1} in base 2. Then S_m is the set of 2^m first terms of the Sobol' sequence.*

This proposition showes that the Sobol's sequence is a nature result of the characteristic of the $(0,2$-sequence in base 2 and the initial point $(0,0)$.

3 Number of $(0,m,2)$-net in base b and some questions

In the following, we compute the number of all different $(0,m,2)$-nets in base b in equivalent sense.

Theorem 3.1 *Let $m \geq 0$ be an integer, then the number of the $(0,m,2)$-nets in base b is $b!^{mb^{m-1}}$.*

Proof . We proceed by induction on m. Denote $N(b,m) = b!^{mb^{m-1}}$.

The result is trivial for $m = 0,1$.

Suppose that the theorem is true until $m-1$.

For m, all $(0,m,2)$-nets in base b can be constructed as follows. Given b^2 normalized $(0,m-2,2)$-net in base b, Q_{i_1,i_2} with $i_1,i_2=0,\dots,b-1$, denote

$$R_{i_1,i_2} = T^{-1}_{i_1,i_2} Q_{i_1,i_2}.$$

Then in every elementary intervals of volume $\frac{1}{b^m}$ of the form

$$[\frac{u}{b^{m-1}}, \frac{u+1}{b^{m-1}}) \times [\frac{v}{b}, \frac{v+1}{b}) \quad \text{or} \quad [\frac{v}{b}, \frac{v+1}{b}) \times [\frac{u}{b^{m-1}}, \frac{u+1}{b^{m-1}})$$

with $u=0,\dots,b^m-1$ and $v=0,\dots,b-1$, there is one and only one point of the set $R=\cup_{i_1,i_2=0}^{b-1} R_{i_1,i_2}$. By Lemma 1.7, in order that R is a $(0,m,2)$-net in base b, it suffices that 1) $T_{i_1,i_2}(R) \simeq Q_{i_1,i_2}$, and 2) only one point of R belongs to the elementary intervals of volume $\frac{1}{b^m}$ of the form

$$[\frac{u}{b^m}, \frac{u+1}{b^m}) \times [0,1) \quad \text{or} \quad [0,1) \times [\frac{v}{b^m}, \frac{v+1}{b^m})$$

with $u,v=0,1,\dots,b^m-1$. Under the condition of 1), there are $b!$ possibilities to satisfy the condition 2) in each elementary interval of the form

$$[\frac{u}{b^{m-1}}, \frac{u+1}{b^{m-1}}) \times [0,1) \quad \text{or} \quad [\frac{v}{b^{m-1}}, \frac{v+1}{b^{m-1}})$$

with $u,v=0,1,\dots,b^{m-1}-1$. In addition that for each $i_1,i_2=0,\dots,b-1$ there are $N_{b,m-2}$ possibilities to choose Q_{i_1,i_2}, so the number of all different R is

$$b!^{2b^{m-1}} N_{b,m-2}^{b^2} = N_{b,m},$$

and the result follows. □

Remark 3.2 *1. In the proof, we give indeed a formal method to construct all of $(0,m,2)$-nets in base b.*

2. Note that $b!^{mb^{m-1}}$ is a very large number. For example, in the case $b=2$ and $m=4$, we have 2^{32} $(0,4,2)$-nets in base 2. The usual discrepancy estimation is valid for all $(0,m,2)$-net in base b apart from the Roth sequence and the Zaremba sequence (see below). Thus it is interesting to find the $(0,m,2)$-net in base b with the lowest discrepancy.

Since there are so many $(0, m, 2)$-nets, to find the best one, we want first limit our research by introducing so called $(0, m, 2)$-sequence.

The following is given in dimension s.

Definition 3.3 *A finite sequence* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{b^m}$ *of* b^m *points in* I^s *is called a* (t, m, s)-sequence in base b *if for all integers* $m \geq m' \geq t$ *and* $b^{m-m'} > k \geq 0$ *the point set consisting of the* $\mathbf{x}_n$ *with* $kb^{m'} < n \leq (k+1)b^{m'}$ *is a* (t, m', s)*-net in base* b*. A* (t, m, s)*-net in base* b *is called* (t, m, s)*-sequence in base* b *if it coincide with a point set consisting of a* (t, m, s)*-sequence in base* b*.*

We have immediately the following proposition.

Proposition 3.4 *Let* $m \geq t + 2$*. A finite sequence* $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_{b^m}$ *of* b^m *points in* I^s *is a* (t, m, s)*-sequence in base* b *if and only if*

1. *The point set* $\{\mathbf{x}_n \mid 1 \leq n \leq b^m\}$ *is a* (t, m, s)*-net in base* b*.*
2. *For all* $0 \leq k \leq b - 1$*, the finie sequences,* $\mathbf{y}_n^k = \mathbf{x}_{n+kb^{m-1}}$*,* $1 \leq n \leq b^{m-1}$ *are* $(t, m-1, s)$*-sequences in base* b*.*

For $s = 2$, we have a result on the affine transformations of $(0, m, 2)$-sequence in base b.

Proposition 3.5 *Let* $m \geq 2$ *be an integer and* R_m *be a* $(0, m, 2)$*- sequence in base* b*. Then for all* $i_1, i_2 \in \{0, 1, \cdots, b-1\}$*,* $T_{i_1,i_2}(R_m)$ *is a* $(0, m-2, 2)$*-sequence in base* b*.*

Proof . We show it by induction on m.

If $m = 2$ and $m = 3$, it is obvious that the proposition is true.

Suppose that the proposition is hold until $m - 1$.

In the case of m, by Proposition 3.4, there exist two $(0, m-1, 2)$-sequence in base b, R^1_{m-1} and R^2_{m-1} such that $R_m = R^1_{m-1} \cap R^2_{m-1}$. For all $i_1, i_2 \in \{0, 1, \cdots, b-1\}$, by the hypothesis of the induction. $T_{i_1,i_2}(R^1_{m-1})$ and $T_{i_1,i_2}(R^2_{m-1})$ are two $(0, m-3, 2)$-sequences in base b. However

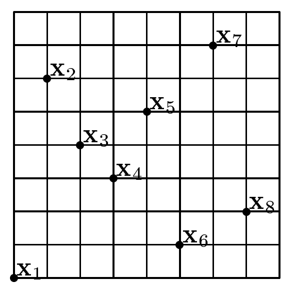
$$T_{i_1,i_2}(R_m) = T_{i_1,i_2}(R^1_{m-1}) \cup T_{i_1,i_2}(R^1_{m-1})$$

is a $(0, m-2, 2)$-net in base b. Applying once Proposition 3.4, we have that $T_{i_1,i_2}(R_m)$ is a $(0, m-2, 2)$-sequence in base b. $\square$

The examples of (t,m,s)-sequences in base b can be obtained by taking the first b^m terms in a (t,s)-sequence in base b. In one dimension, every $(0,m,1)$-net in base b is a copy (in equivalent sense) of the first b^m terms of the Van der Corput sequence in base b, so it is a $(0,m,1)$-sequence in base b. But if the dimension $s \geq 2$, whether all (t,m,s)-nets in base b are (t,m,s)-sequences in base b ?

It is easy to see that this is true for $(0,m,2)$-nets in base 2 in the case $m \leq 2$. However, in the case of $m = 3$, we give the following negative example, which implies that for $m \geq 3$ there exists a $(0,m,2)$-net but not a $(0,m,2)$-sequence in base 2.

Example 3.6 *Let R_3 be the set of* $\mathbf{x}_1 = (0,0)$, $\mathbf{x}_2 = (0.125, 0.75)$, $\mathbf{x}_3 = (0.25, 0.5)$, $\mathbf{x}_4 = (0.375, 0.375)$, $\mathbf{x}_5 = (0.5, 0.625)$, $\mathbf{x}_6 = (0.625, 0.125)$, $\mathbf{x}_7 = (0.75, 0.875)$ *and* $\mathbf{x}_8 = (0.875, 0.25)$ *as in the figure bellow:*



It is easy to see that R_3 is a $(0,3,2)$-net in base 2. If R_3 is a $(0,3,2)$-sequence, then there are two disjoint $(0,2,2)$-nets noted by R_2^1 and R_2^2 such that $R_2^1 \cup R_2^2 = R_3$. We can suppose that $\mathbf{x}_1$ belongs to R_2^1, then $\mathbf{x}_3$ belongs to R_2^1. But in this case neither $\mathbf{x}_5$ nor $\mathbf{x}_6$ can be in R_2^1, it is impossible because there is not point of R_2^1 in the elementary interval $[0.5, 0.75) \times [0,1)$ of volume $\frac{1}{4}$.

Finally, for the Roth sequence and the Zaremba sequences, we do not know whether they are $(0,m,2)$-sequences.

Thus, we want ask the following questions: 1) Whether (t,m,s)-sequences are more uniformly distributed than (t,m,s)-nets in base b which are not the sequences? 2) Can we generate a $(t,m+1,s)$-sequence in base b, as well as a (t,s)-sequence in base b from any (t,m,s)-sequence in base b? 3) In the equivalent sense of $(0,m,s)$-nets in base b, how many $(0,m,s)$-sequences are there in base b?

4 Some symmetry properties of the Sobol' sequence and the Roth-Zaremba sequence

We first present a symmetric property of the 2^m first terms of the Sobol' sequence S_m.

Proposition 4.1 *For $m \geq 1$, we have*

$$T_{0,0}(S_m) \simeq T_{1,1}(S_m) \quad \textit{and} \quad T_{0,1}(S_m) \simeq T_{1,0}(S_m).$$

This proposition can be shown using the following lemma on the succession symmetric property of a $(0, m, 2)$-net in base 2.

Lemma 4.2 *Let R_m be a normalized $(0, m, 2)$-net in base 2 and let R_{m+1} be the unique normalized $(0, m+1, 2)$-net in base 2 generated by R_m. If*

$$T_{0,0}(R_m) \simeq T_{1,1}(R_m) \quad \textit{and} \quad T_{0,1}(R_m) \simeq T_{1,0}(R_m),$$

then

$$T_{0,0}(R_{m+1}) \simeq T_{1,1}(R_{m+1}) \quad \textit{and} \quad T_{0,1}(R_{m+1}) \simeq T_{1,0}(R_{m+1}).$$

Proof . For any $i_1, i_2 \in \{0, 1\}$, let $Q_{m-1}(i_1, i_2)$ be the normalized $(0, m-1, 2)$-net in base 2 generated by the $(0, m-2, 2)$-net in base 2, $T_{i_1,i_2}(R_m)$. Since $T_{i_1,i_2}(R_m) \in \mathcal{F}^{2,2}_{m-1}$, using the hypothesis of the lemma and Corollary 2.3, we have

$$Q_{m-1}(0,0) = Q_{m-1}(1,1) \text{ and } Q_{m-1}(0,1) = Q_{m-1}(1,0) \tag{1}$$

Let $P_{m-1}(i_1, i_2) = T^{-1}_{i_1,i_2}(Q_{m-1}(i_1, i_2) \backslash T_{i_1,i_2}(R_m))$. Since $Q_{m-1}(i_1, i_2) \in \mathcal{F}^{2,2}_{m-1}$, we have $P_{m-1}(i_1, i_2) \in \mathcal{F}^{2,2}_{m}$. Because R_m is a normalized $(0, m, 2)$-net, for each point $(x, y) \in P_{m-1}(i_1, i_2)$, there exist a point of R_m with abscissa x and an anthor point of R_m with ordinate y. To construct a $(0, m+1, 2)$-net in base 2, we transform (x, y) into $(x + \frac{1}{2^{m+1}}, y + \frac{1}{2^{m+1}})$. Denote the new points set by

$$P_m = \{(x + \frac{1}{2^{m+1}}, y + \frac{1}{2^{m+1}}) \mid (x, y) \in \cup_{i_1,i_2=0,1} P_{m-1}(i_1, i_2)\},$$

we have

$$T_{0,0}(P_m) \simeq T_{1,1}(P_m) \text{ and } T_{0,1}(P_m) \simeq T_{1,0}(P_m),$$

which imply

$$T_{0,0}(P_m \cup R_m) \simeq T_{1,1}(P_m \cup R_m) \text{ and } T_{0,1}(P_m \cup R_m) \simeq T_{1,0}(P_m \cup R_m).$$

It is clear that $P_m \cup R_m \in \mathcal{F}^{2,2}_{m+1}$ satisfies the conditions of the Lemma 1.7, so it is a $(0, m+1, 2)$-net in base 2. Moreover, by Corollary 2.2 and Corollary 2.3,

$$P_m \cup R_m = R_{m+1}$$

$\square$

Now, we give the definition of the Roth-Zaremba sequence (see [3], [4], [10] and [2]). For $n = \sum_{i=0}^{\infty} a_i(n) b^i \in \mathbf{N}$, let $\phi_b^{\Sigma}(n) = \sum_{i=0}^{\infty} \sigma_i(a_i(n)) b^{-i-1}$ be the generalized van der Corput sequences in base b where $\Sigma = (\sigma_i)_{i \geq 0}$ being an infinite sequence of permutations of the set $\{0, 1, \ldots, b-1\}$. Then, for each positive integer m, the Roth-Zaremba sequence is defined by

$$Z_{b,m} = (\frac{n}{b^m}, \phi_b^{\Sigma}(n))_{0 \leq n \leq b^m - 1}.$$

In the case of $\sigma_i = I$ with I the identity permutations of the set $\{0, 1, \ldots, b-1\}$ for all $i \geq 0$, we have the original Roth sequence $R_{b,m} = (\frac{n}{b^m}, \phi_b(n))_{0 \leq n \leq b^m - 1}$. In the case of $b = 2$, and $\sigma_i = (01)$ for i even, $\sigma_i = I$ for i odd, we get the original Zaremba sequence.

Proposition 4.3 *For the Roth-Zaremba sequence $Z_{b,m}$, we have, for all $i, j \in \{0, 1, \ldots, b-1\}$,*

$$T_{i,j}(Z_{b,m}) \simeq T_{0,0}(Z_{b,m}).$$

For the original Roth sequence $R_{b,m}$,

$$T_{i,j}(R_{b,m}) \simeq R_{b,m-2}.$$

Proof . For $0 \leq n \leq b^m - 1$, write $n = n_0 + n_1 b + \ldots + n_{m-1} b^{m-1}$ with $n_0, n_1, \ldots, n_{m-1} = 0, 1, \ldots, b-1$. We have

$$\frac{n}{b^m} = \frac{n_0}{b^m} + \ldots + \frac{n_{m-1}}{b} \text{ and } \phi_b^{\Sigma}(n) = \frac{\sigma_0(n_0)}{b} + \ldots + \frac{\sigma_{m-1}(n_{m-1})}{b^m}.$$

Thus $(\frac{n}{b^m}, \phi_b^\Sigma(n)) \in P_{i,j} = [\frac{i}{b}, \frac{i+1}{b}) \times [\frac{j}{b}, \frac{j+1}{b})$ if and only if $n_{m-1} = i$ and $\sigma_0(n_0) = j$, and

$$\begin{aligned} T_{i,j}(Z_{b,m}) &= \{(b(\frac{n}{b^m} - \frac{i}{b}), b(\phi_b^\Sigma(n) - \frac{j}{b}))\}_{0\le n\le b^m-1, n_{m-1}=i, \sigma_0(n_0)=j} \\ &= \{(\frac{n_0}{b^{m-1}} + \ldots + \frac{n_{m-2}}{b}, \frac{\sigma_1(n_1)}{b} + \ldots + \frac{\sigma_{m-1}(n_{m-1})}{b^{m-1}})\}^{0\le n\le b^m-1}_{n_{m-1}=i, \sigma_0(n_0)=j} \\ &\simeq \{(\frac{n_1}{b^{m-2}} + \ldots + \frac{n_{m-2}}{b}, \frac{\sigma_1(n_1)}{b} + \ldots + \frac{\sigma_{m-2}(n_{m-2})}{b^{m-2}})\}_{n_1,\ldots,n_{m-2}=0,1,\ldots,b-1}. \end{aligned}$$

In the case of the Roth sequence, $\sigma_i = I$ and

$$R_{b,m-2} = \{(\frac{n_1}{b^{m-2}} + \ldots + \frac{n_{m-2}}{b}, \frac{n_1}{b} + \ldots + \frac{n_{m-2}}{b^{m-2}})\}_{n_1,\ldots,n_{m-2}=0,1,\ldots,b-1},$$

therefore, the result follows. □

Applying Lemma 1.7, we can construct directly some "symmetrie" $(0, m, 2)$-nets in base 2.

Proposition 4.4 *Let R_m be a normalized $(0, m, 2)$-net in base 2, then we can construct a "symmetrie" $(0, m+2, 2)$-net R_{m+2} in base 2 in the following meaning:*

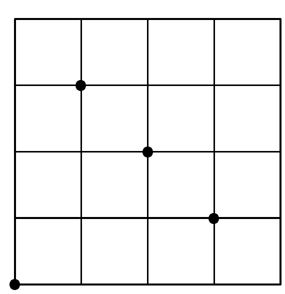
$$T_{0,0}(R_{m+2}) \simeq T_{1,1}(R_{m+2}) \simeq R_m$$

and

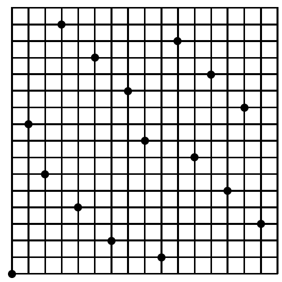
$$T_{0,1}(R_{m+2}) \simeq T_{1,0}(R_{m+2}) \simeq T^{\frac{1}{2^{m+1}}}(R_m).$$

where $T^{\frac{1}{2^{m+1}}}(x, y) = (\{x + \frac{1}{2^{m+1}}\}, \{y + \frac{1}{2^{m+1}}\})$ with $\{\cdot\}$ the fraction part of a real.

Example 4.5 *Let $R_2 = F_2$ as in the following figure.*



Then the figure of the R_4 symmetric $(0,4,2)$-net in base 2 is as follows.



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