Formal Elimination Theory. Applications to Control Theory

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Abstract

Following Douglas' ideas on the inverse problem of the calculus of variations, the purpose of this article is to show that we can use the formal integrability theory to develop a theory of elimination for systems of partial differential equations. In particular, we consider linear systems of partial differential equations with variable coefficients and we show that we can organize the integrability conditions on the coefficients to build an "intrinsic tree". Trees of integrability conditions naturally appear when we test the controllability, observability, invertibility, ... of linear control systems with some variable or unknown coefficients, or for linearized nonlinear systems. Many explicit examples will illustrate the main results.

Keywords: Elimination, resultant, formal integrability, D-module, control theory, controllability, observability, invertibility.

1 Introduction

Expansion into power series of analytic or formal solutions of a system of partial differential equations (PDE) has early been a powerful tool in mathematics, physics and engineering sciences. In particular, the wish to have a theory which computes the dimension of the space of the analytic solutions of a system of PDE, without integrating it explicitly, is not new, as Einstein explained it in 1952 in [8]: "*... we need a method which gives a measure of the strength of an equation system. We do as follows: we expand the field variables in the neighborhood of a point P, into Taylor series (which presupposes the analytic character of the field), the coefficients of these series, which are simply the derivatives of the field variables at the point P, fall into sets according to the degree of differentiation. In every such degree there appears, for the first time, a set of coefficients which would be free for arbitary choice if it were not that the field must satisfy a system of differential equations. Through this system of differential equations (and its derivatives with respect to the coordinates) the number of coefficients is restricted, so that in each degree a smaller number of coefficients is left free for arbitrary choice. The set of numbers of "free" coefficients for all degrees of differentiation is then directly a measure of the "weakness" of the system of equations, and through this, indirectly, also of its "strength"."*

This notion of "strength" or of "degree of generality" of a system of PDE was introduced by Ch. Riquier [27] and M. Janet [14] in the years 1910 - 1930. They developped effective algorithms in order to compute it without integrating the system explicitly. Their works have inspired J.F. Ritt while he was creating *differential algebra* (see the last two chapters of [28] for a exposition of Riquier and Janet works). More recently and indepently of these precursors, the *formal integrability theory* has been developped in a intrinsic way by D.C. Spencer, using fibered manifolds, jet, δ-cohomology, diagram chasing, ... [21, 26]. In the beginning of the eighties, the formal integrability theory of PDE with Lie pseudogroups methods has given new insight on mathematical physics (see [21]).

In this paper, we are interested in elimination problems. We consider a system of PDE with two sets of variables and our interest is to know under what conditions on the first set of variables the system admits a solution. The solution of this problem was found by Seidenberg in 1956 (see [30]) using differential algebra approach. The purpose of this article is to show that we can use the formal integrability theory to compute the resultants of a system of PDE. This approach seems to be more intrinsic and permits to have a new point of view on the resultants. In particular, following the *Spencer-Goldschmidt criterion*, only three kinds of inequalities may appear in the resultants: the first ones appear for rank conditions (fibered manifold), the second for the surjectivity of the restricted prolongations, and the third ones when testing a more technical property, that is, the 2-acyclicity of the symbol of the system. In case of linear system of PDE with variable coefficients, these inequalities on the coefficients can be arranged in order to build an "intrinsic tree". Each final leaf represents a formal solution of the system with its degree of generality. Hence, we can study the variation of the degree of generality of the formal solutions of a linear system of PDE with variable coefficients. Surprisingly, such a point of view has firstly been adopted by J. Douglas (1941) in his study of the inverse problem of the calculus of variations [7], using the ideas of M. Janet [14].

Recently, the theory of differential module (D-module) has given a new insight for studying the structural properties of control systems. See for example [1, 3, 9, 10, 17, 21, 22, 23, 24, 25]. Most of the intrinsic properties as controllability, observability and invertibility of control systems have been reformulated in terms of an algebraic nature of a differential module (as torsion, torsion-free, projective, free D-module, ...). Formal tests have been developped in [21, 22, 23, 25] to know whether a finitely generated differential module is respectively a torsion, torsion-free, projective and free D-module. Thus, if we consider control systems with variable coefficients or linearized nonlinear ones, certain trees of integrability conditions will naturally appear when testing these properties. Elimination problems in control theory have been introduced by S. Diop in [4, 5, 6], using effective methods of differential algebra. These methods are in general more effective than those of the formal integrability theory, but less intrinsic (dependence of coordinate system through the ranking, choice of differential polynomials in characteristic set) (see [20]). Roughly speaking, we can say that the effective character always competes with intrinsic one.

2 Formal Integrability Theory

2.1 Introduction

We now introduce the main ideas of the formal integrability theory, before exposing them, using more technical tools.

Roughly speaking, if we want to compute the "degree of generality" of a formal solution of a system of PDE, we have to know the number of "arbitrary" ("free", "parametric", ... depending on the author) derivatives at each order.

Let us consider a system of PDE $\Phi^\tau(x, \partial_\mu y^k(x)) = 0$ of order q, where $\tau = 1, ..., l$, $k = 1, ..., m$ and $\mu = (\mu_1, ..., \mu_n)$ is a multi-index with length $|\mu| = \mu_1 + ... + \mu_n$ (we shall frequently use the notation $\partial_{i_1}...\partial_{i_m} y^k = y^k_{i_1...i_m}$). We substitute the derivative of the unknowns y^k by jet coordinates with the same indice ($\partial_\mu y^k(x) \to y^k_\mu$)), that is to say, we regard any derivative of the y^k as new unknowns. We shall say that a jet coordinate with an indice of length lower or equal to q is at the order q, and at the order strictly q if the length of its indice is equal to q. Thus, $\Phi^\tau(x, \partial_\mu y^k(x)) = 0$ is transformed into a pure equation relating the jet coordinates: $\Phi^\tau(x, y^k_\mu) = 0$. Let us suppose that these equations define a fibered manifold $\mathcal{R}_q$ (no relation among the x only) in the space of jet coordinates of order q. Using implicit function theorem, we can locally determine some jet coordinates in function of $\dim \mathcal{R}_q$ (the fibre dimension) other jet coordinates (we try to write the greatest number of jet coordinates of order strictly equal to q in function of jet coordinates of lower order). We call the first ones "*principal*" jet coordinates and the second "*parametric*" jet coordinates. Thus, we have made a partition on the jet of order q into two sets, principal and parametric, where the first one can be expressed in terms of the second.

Now, we remark that if we differentiate once the equations of $\Phi^\tau(x, \partial_\mu y^k(x)) = 0$, with respect to each x^i (prolongation ρ_1), and substitute again the derivatives by the jet coordinates, we obtain:

$$d_i \Phi^\tau = \frac{\partial \Phi}{\partial x^i} + \sum_{|\mu|=q} \frac{\partial \Phi^\tau}{\partial y^k_\mu} y^k_{\mu+1_i} = 0, \tag{1}$$

where $i = 1, ..., n$. Thus, the terms of order $q+1$ appear linearly with coefficients defined on $\mathcal{R}_q$, that is with jets satisfying $\Phi^\tau(x, y^k_\mu) = 0$ (something well known in differential algebra). This simple remark will allow us to use linear algebra. Let us define $\mathcal{R}_{q+1} = \rho_1(\mathcal{R}_q)$ by

$$\left\{ \begin{array}{lclcl} d_i\Phi^\tau & & & = & 0, \\ 0 & + & \Phi^\tau & = & 0, \end{array} \right. \tag{2}$$

$i = 1, ..., n,\ \tau = 1, ..., l$. Now, let us call M_{q+1} the vector space defined by

$$\sum_{|\mu|=q} \frac{\partial \Phi^\tau}{\partial y^k_\mu} v^k_{\mu+1_i} = 0,\ i = 1, ..., n, \tau = 1, ..., l, \tag{3}$$

in the jet coordinates of order strictly equal to $q+1$. There are $\frac{m(q+n)!}{(q+1)!(n-1)!}$ jet coordinates of order strictly equal to $q+1$ and if we denote by $\sigma_1(\Phi)$ the left member of (3), then we have $\dim M_{q+1} = \frac{(q+n)!}{(q+1)!(n-1)!} - \mathrm{rk}\, \sigma_1(\Phi)$ parametric jet coordinates at the order stricly $q+1$. Indeed, we can find in (3), $\mathrm{rk}\, \sigma_1(\Phi)$ lineary independent equations and by linear algebra in the upper part of (2) and substitution of the principal jet coordinates of order q by the parametric ones, we obtain $\mathrm{rk}\, \sigma_1(\Phi)$ principal jet coordinates of order stricly $q+1$ which can be expressed with $\dim M_{q+1}$ parametric jet coordinates of order stricly $q+1$ and with $\dim \mathcal{R}_q$ ones of order q.

Now, the trouble begins if $\mathrm{rk}\ \sigma_1(\Phi) < ln$: we have certain equations of (3) which are linear combinations of $\mathrm{rk}\, \sigma_1(\Phi)$ others. Eliminating the jets of order $q+1$ in the corresponding equations of (2), we obtain equations of order q. Only two different cases may happen:

- Substituting the principal jet of order q in these new equations, we are led to 0, then we have no new equations relating the parametric jet coordinates up to the order q. Thus, we have determined for the moment the number of parametric jet coordinates of strict order $q+1$. If we put a second member z^τ in the equations $\Phi^\tau = 0$ and begin again the same operations, we obtain: $A^{ip}_\tau(x, y^k_\mu) d_i z^\tau + B^p_\tau(x, y^k_\mu) z^\tau = 0, |\mu| \le q$. We notice that it leads to compatibility conditions in the linear case.

- Substituting the principal jet of order q in these new equations, we are led to some non identically zero equations $\Psi^\alpha(x, y^k_\mu) = 0,\ |\mu| \le q$, relating the parametric jet coordinates up to order q. This contradicts the fact that they are parametric jet coordinates. Then, we have to add these new equations to the system $\Phi^\tau = 0$ and start anew with the following system:

$$\mathcal{R}_q^{(1)} \left\{ \begin{array}{l} \Phi^\tau(x, y_\mu^k) = 0, \\ \Psi^\alpha(x, y_\mu^k) = 0. \end{array} \right. \tag{4}$$

We have just shown how to compute the number of parametric jet coordinates of order $q+1$. Similary, it can be done for each order. We have seen that the feedback of informations on the lower order derivatives (new equations $\Psi^\alpha(x, y_\mu^k) = 0$) modifies the calculus of the number of parametric jet coordinates and thus the calculus of the dimension of the space of solutions (the parametric jet coordinates determine the initial conditions that we have to give to compute the power series of the solutions). Hence, certain systems of PDE seem to be "nicer" than some others, that is, those in which no feedback of informations on the lower order derivatives appears when differentiating the equations of the system and projecting them on lower order jets space. Hence, we shall call a system of PDE *formally integrable* whenever the formal power series of its solutions can be determined step by step by successive derivations without obtaining backward new informations on lower-order derivatives. We may wonder how to recognize when a system of PDE is formally integrable, as we have to verify that no new lower order informations appear at each order, that is, for a infinity of orders. So, we can ask: does it exist a finite algorithm testing whether a system of PDE is formally integrable or not? In the case where the system is not formally integrable, we have seen that we have to add new equations. So, does it exist a procedure which adds enough equations to the system, in order to transform it into a formal integrable system, with the same solutions? D.C. spencer and coworkers have given positive answers [26] in the years 1960-1975. Their algorithms turn around the two following crucial points: first of all, we have seen that in (1), the jet coordinates of order strict $q+1$ appeared linearly, a fact permitting to use linear algebra. Thus, we fell that everything that we have done precedingly can be reformulated into a more intrinsic way, using homological algebra(ker-coker exact sequences) in place of Cramer rules. D.C. Spencer has introduced the δ-sequence and its cohomology to deal with this problem. Secondly, backward informations on derivatives of order q have appeared after we had projected $\mathcal{R}_{q+1}$ on the space of jet coordinates of order q: we have found more equations in (4) than in $\Phi^\tau = 0$ and thus the projection $\mathcal{R}_q^{(1)}$ of $\mathcal{R}_{q+1}$ on the space of jet coordinates of order q, is a stricly subset of $\mathcal{R}_q$ (not always a submanifold). These remarks will lead (at best) to a prolongation-projection procedure that we sketch.

2.2 Main Results of the Formal Integrability Theory

We now expose the main results of the formal integrability theory (see [26, 20] for more details). These results will be illustrated in the examples of the next sections.

Let us denote by X a manifold of dimension n with local coordinates $(x^1, ..., x^n)$, by $T(X)$ and $T^\star(X)$, its tangent and cotangent bundles. Let $\mathcal{E}$ be a fibered manifold over X with fiber dimension m and local coordinates (x^i, y^k). We define the q-jet bundle $J_q(\mathcal{E})$ as a fibered manifold with local coordinates (x, y_μ^k), $\mu = (\mu_1, ..., \mu_n)$, $0 \leq |\mu| \leq q$ and a nonlinear system of PDE of order q as a fibered submanifold $\mathcal{R}_q$ of $J_q(\mathcal{E})$, determined locally by $\Phi^\tau(x, y_\mu^k) = 0$. The r prolongation of $\mathcal{R}_q$ is $\mathcal{R}_{q+r} = \rho_r(\mathcal{R}_q) = J_r(\mathcal{R}_q) \cap J_{r+q}(\mathcal{E})$, and is obtained by substituting the jet coordinates by the derivatives, differentiating r times and substituting again the derivatives by jet coordinates. The projection π_{q+r}^{q+r+s} : $J_{q+r+s}(\mathcal{E}) \to J_{q+r}(\mathcal{E})$ induces a projection of $\mathcal{R}_{q+r+s}$ on $\mathcal{R}_{q+r}$. We denote the image of this projection by $\mathcal{R}_{q+r}^{(s)}$. Notice that $\mathcal{R}_{q+r}$ and $\mathcal{R}_{q+r}^{(s)}$ are not in general fibered manifolds for any $r, s \geq 0$. The *linearized system* R_q of $\mathcal{R}_q$ is defined locally as: $\dfrac{\partial \Phi^\tau}{\partial y_\mu^k} Y_\mu^k = 0$. Its is a linear system in Y_μ^k, with variable coefficients satisfying $\Phi^\tau = 0$. We define the symbol

M_q of $\mathcal{R}_q$, as the family of vector space over $\mathcal{R}_q$, by

$$\sum_{|\mu|=q} \frac{\partial \Phi^\tau}{\partial y^k_\mu} v^k_\mu = 0, \ \tau = 1, ..., l, \tag{5}$$

and denote by $\sigma(\Phi)$ the corresponding matrix. Then the symbol M_{q+r} of $\mathcal{R}_{q+r}$ defined by

$$\sum_{|\mu|=q, |\nu|=r} \frac{\partial \Phi^\tau}{\partial y^k_\mu} v^k_{\mu+\nu} = 0, \tag{6}$$

only depends on M_q. We call $\sigma_r(\Phi) = \sigma(\rho_r(\Phi))$ the left member of (6). Let us define the *δ-sequence* by

$$\Lambda^s T^\star \otimes M_{q+r+1} \xrightarrow{\delta} \Lambda^{s+1} T^\star \otimes M_{q+r},$$

with $(\delta(\omega))^k_\mu = dx^i \wedge \omega^k_{\mu+1_i}$ where $\omega = v^k_{\mu,I}\, dx^I \in \Lambda^s T^\star \otimes M_{q+r+1}$, $dx^I = dx^{i_1} \wedge ...$ $\wedge\, dx^{i_s}, i_1 < ... < i_s$ and $|\mu| = q + r$. We easily verified that $\delta \circ \delta = 0$. The cohomology at $\Lambda^s T^\star \otimes M_{q+r}$ of the sequence

$$\Lambda^{s-1} T^\star \otimes M_{q+r+1} \xrightarrow{\delta} \Lambda^s T^\star \otimes M_{q+r} \xrightarrow{\delta} \Lambda^{s+1} T^\star \otimes M_{q+r-1},$$

is denoted by $H^s_{q+r}(M_q)$.

Definition 1 The symbol M_q of $\mathcal{R}_q$ is said to be *s-acyclic* if $\forall\, r \geq 0 : H^1_{q+r} = ... = H^s_{q+r} = 0$. M_q is *involutive* if it is n-acyclic. In particular, every system $\mathcal{R}_q$ of ordinary differential equations (ODE) has an involutive symbol. A symbol M_q is *of finite type* if $\exists\, r \geq 0$ such that $M_{q+r} = 0$.

Theorem 1 *Let M_q be the symbol of the system $\mathcal{R}_q$ then there exists an integer r large enough such that M_{q+r} is involutive.*

A test checking the 2-acyclicity of the symbol is still lacking. Indeed, we have to verify $H^2_{q+r} = 0$ for any $r \geq 0$ and thus for an infinity of orders. Only the case of finite type symbol can be checked as we only have to verify $H^2_q = ... = H^2_{q+r-1} = 0$ where $M_{q+r} = 0$ and $M_{q+r-1} \neq 0$. But, we can test whether a symbol is involutive or not. However, it must be done only on "sufficently generic coordinates": the *δ-regular coordinates*. Roughly speaking, the δ-regular coordinates are not the most generic coordinates but "generic enough" to give the right dimension of M_{q+r}.

Let $x = (x^1, ..., x^n)$ be a system of local coordinates of X and let us order the multi-index μ of length q: $\mu < \mu'$ if there exists l such that $\mu_i = \mu'_i$ for $i = n, ..., l+1$ and $\mu_l < \mu'_l$. The order on the multi-index μ implies a preorder on the v^k_μ of M_q: $\mu < \mu' \Rightarrow v^k_\mu < v^{k'}_{\mu'}$. We say that v^k_μ is of class $i > 1$ if $\mu_1 = ... = \mu_{i-1} = 0$ and $\mu_i > 0$ and of class 1 if $\mu_1 > 0$. Now, using the equations defining M_q, we try to express the maximum number of v^k_μ of class n, in function of the others v^l_ν. Next, we substitute these v^k_μ in the other equations to make disappear the v^k_μ of class n. We respectively do the same for the v^k_μ of class $n-1$, ..., 1. We usually say that M_q is in *the solved form*. We associate a system of "dots" to these equations, as follows:

equations of class n	1	...	...	...	n
equations of class $n-1$	1	...	...	$n-1$	$\bullet$
....					
equations of class i	1	...	i	$\bullet$	$\bullet$
...					
equations of class 1	1	$\bullet$	...	...	$\bullet$

Though this classification looks like the original one of M. Janet, it is in fact quite different. For a detailed study, we refer the reader to the reference [12, 13]. Moreover, let M^i_q be the vector space defined locally by $\sigma(\Phi)$ where we have equal to zero the v^k_μ of class strictly lower than i. We call $\sigma(\Phi)^i$ the left member of the defining equations of M^i_q. We have $\dim M^i_q = \frac{m(q+n-i-1)!}{(q-1)!\,(n-i)!} - \mathrm{rk}\,\sigma(\Phi)^i$. Let us call $\alpha^i_q = \dim M^{i-1}_q - \dim M^i_q$ for $i = 1, ..., n$.

Theorem 2 *The symbol M_q is involutive if there exists a system of coordinates, called δ-regular coordiates, in which one of the following properties is satisfied:*

1. *$dim\, M_{q+1} = \alpha^1_q + 2\alpha^2_q + ... + n\alpha^n_q$.*
2. *Prolongation with respect to the dots does not bring new equations.*

Then $\forall\, r \geq 0 : dim\, M_{q+r} = \sum_{i=1}^{n} \frac{(r+i-1)!}{r!\,(i-1)!}\, \alpha^i_q$.

We have seen that a "good system" $\mathcal{R}_q$ of PDE was a system in which no lower order informations appeared when projecting its prolongations $\mathcal{R}_{q+r+s} = \rho_{r+s}(\mathcal{R}_q)$ on lower order jet space $J_{q+r}(\mathcal{E})$. Using the previous notation, it leads to the following definition.

Definition 2 A system $\mathcal{R}_q$ is said to be *formally integrable* if $\forall\, r, s \geq 0$, $\mathcal{R}_{q+r}$ is a fibered manifold and the projection $\pi^{q+r+s}_{q+r} : \mathcal{R}_{q+r+s} \to \mathcal{R}_{q+r}$ is surjective (or equivalently $\mathcal{R}^{(s)}_{q+r} = \mathcal{R}_{q+r}$).

A system $\mathcal{R}_q$ is said to be *involutive* if $\mathcal{R}_q$ is formally integrable with an involutive symbol. We now give two key theorems. See [21] for the non trivial demonstrations.

Theorem 3 *If M_q is 2-acyclic and M_{q+1} is a vector bundle over $\mathcal{R}_q$ then $\forall r \geq 1 :\ M_{q+r}$ is a vector bundle over $\mathcal{R}_q$.*

Theorem 4 *If $\mathcal{R}^{(1)}_q$ is a fibered manifold and M_q is 2-acyclic then $\forall r \geq 0 :\ \rho_r(\mathcal{R}^{(1)}_q) = \mathcal{R}^{(1)}_{q+r}$.*

These theorems lead to the following criterion.

Spencer-Goldschmidt criterion If M_q is 2-acyclic and $\mathcal{R}_{q+1}$ is a fibered manifold such that $\mathcal{R}^{(1)}_q = \mathcal{R}_q$ then $\mathcal{R}_q$ is formally integrable.

The reader have to keep in mind that the previous criterion gives only sufficient conditions in order to have a formally integrable system.

Example 1 The symbol of the system $\partial_i \xi_j + \partial_j \xi_i = 0$ is neither 2-acyclic nor involutive but the first prolongation gives $\partial_{ij}\xi = 0$ and the system is formally integrable. More generally, any homogeneous system is formally integrable even if the criterion is not satisfied.

We have the following corollary.

Corollary 1 *Let $\mathcal{R}_q$ be an involutive system and let us denote by $\mathcal{R}_{q-1}$ the projection of $\mathcal{R}_q$ on $J_{q-1}(\mathcal{E})$ then*

$$dim\, \mathcal{R}_{q+r} = dim\, \mathcal{R}_{q-1} + \sum_{i=1}^{n} \frac{(r+i)!}{r!\, i!} \alpha^i_q.$$

See [21] for the proof. In particular, if we want to determine the analytic solutions of the system $\mathcal{R}_q$, we have to fix α^1_q functions in x^1, α^2_q functions in (x^1, x^2), ..., and α^n_q functions in $(x^1, ..., x^n)$.

Definition 3 A system $\mathcal{R}_q$ is called *sufficiently regular* if:

1. $\forall\, r, s \geq 0$, $\mathcal{R}^{(s)}_{q+r}$ is a fibered manifold.
2. $\forall\, r, s \geq 0$ the symbol $M^{(s)}_{q+r}$ is induced from a vector bundle over X.

In the case where the system $\mathcal{R}_q$ is not formally integrable, the following theorem shows that there is a finite procedure which adds enough equations to the system, in order to obtain a formaly integrable system, with the same solutions.

Theorem 5 *If $\mathcal{R}_q$ is sufficiently regular system, we can find two integers, r, $s \geq 0$, such that $\mathcal{R}^{(s)}_{q+r}$ is formally integrable (involutive) with the same solutions as $\mathcal{R}_q$.*

Thus, we are led to the following algorithm [13].

Algorithm We start with $\mathcal{R}_q$. Find $r \geq 0$ such that $\mathcal{R}_{q+r}$ is 2-acyclic (involutive). Test whether $\mathcal{R}^{(1)}_{q+r} = \mathcal{R}_{q+r}$. If it is the case, then the algorithm stops, else, starts anew with $\mathcal{R}^{(1)}_{q+r}$. Hence, we finally find two integers r, s such that $\mathcal{R}^{(s)}_{q+r}$ is a formally integrable system (involutive) with the same solutions as $\mathcal{R}_q$.

Now, we illustrate the spirit of these results by showing how the ideas of the previous introduction are transformed in a more "intrinsic way". For the simplicity, we only use a linear system of PDE, which will be denoted by R_q and determined locally by $\Phi^\tau(x, y^k_\mu) = 0$. R_q is a subvector bundle of $J_q(E)$ and let us denote by F_0 the vector bundle $J_q(E)/R_q$. We have the following short exact sequence:

$$0 \to R_q \to J_q(E) \xrightarrow{\Phi} F_0 \to 0.$$

Prolongating R_q once with respect to each x^i, we obtain the following exact sequence

$$0 \to R_{q+1} \to J_{q+1}(E) \xrightarrow{\rho_1(\Phi)} J_1(F_0),$$

where $\rho_1(\Phi)$ is the left member of (2).

We can consider $S_{q+1}T^\star \otimes E$ as a subset of $J_q(E)$, where $S_{q+1}T^\star$ denote the $q+1$ covariant symetric tensor. $S_{q+1}T^\star \otimes E$ is nothing else than the space of jet coordinates of order stricly equal to $q+1$ and we easily verify that $\dim S_{q+1}T^\star \otimes E = \frac{m\,(q+n)!}{(q+1)!\,(n-1)!}$. Hence, we have

$$0 \to M_{q+1} \to S_{q+1}T^\star \otimes E \xrightarrow{\sigma_1(\Phi)} T^\star \otimes F_0,$$

and we denote by F_1 the cokernel of $\sigma_1(\Phi)$. Thus, we have the following exact sequences

$$0 \to M_{q+1} \to S_{q+1}T^\star \otimes E \to \operatorname{im} \sigma_1(\Phi) \to 0,$$

and

$$0 \to \operatorname{im} \sigma_1(\Phi) \to T^\star \otimes F_0 \to F_1 \to 0$$

which give:

$$\begin{cases} \dim M_{q+1} = \dim S_{q+1}T^\star \otimes E - \operatorname{rk} \sigma_1(\Phi), \\ \dim F_1 = \dim T^\star \otimes F_0 - \operatorname{rk} \sigma_1(\Phi), \end{cases}$$

where $\operatorname{rk} \sigma_1(\Phi)$ denotes the rank of $\sigma_1(\Phi)$. This is nothing else than the Cramer rules. We recognize that we have $\dim S_{q+1}T^\star \otimes E - \operatorname{rk} \sigma_1(\Phi)$ new parametric jet coordinates of order strictly equal to $q+1$ and $\dim T^\star \otimes F_0 - \operatorname{rk} \sigma_1(\Phi)$ equations which are linear combination of $\operatorname{rk} \sigma_1(\Phi)$ ones in the symbol M_{q+1}. Hence by linear elimination of the jet coordinates of order strictly equal to $q+1$, we can find $\dim F_1$ new equations of order q. Substituting in those equations the principal by parametric jet coordinates of order q, it leads to $R^{(1)}_q$, that is, to the image of the projection π^{q+1}_q. We have:

$$0 \to M_{q+1} \to R_{q+1} \xrightarrow{\pi^{q+1}_q} R_q \to \operatorname{coker} \pi^{q+1}_q \to 0.$$

and

$$0 \to M_{q+1} \to R_{q+1} \to \operatorname{im} \pi^{q+1}_q = R^{(1)}_q \to 0,$$

which leads to $\dim M_{q+1} = \dim R_{q+1} - \dim R^{(1)}_q$. Hence, we have $R^{(1)}_q = R_q$ if we does no have new equations of order q and we have dim M_{q+1}=dim R_{q+1}-dim R_q.

2.3 Formal Elimination Theory

Let us take a system of PDE defined by the equations

$$\Phi^\tau(x, y^k_\mu, z^l_\nu) = 0, \qquad \tau = 1, ..., k, |\mu| \leq q, |\nu| \leq p, \tag{7}$$

where $y = (y^1, ..., y^m)$ and $z = (z^1, ..., z^s)$ are two sets of unknowns. We would like to know what conditions z has to satisfy in order to have solutions of the system (7). Regarding the system (7) as a system in the set of unknowns y only, with coefficients in z,

$$\Psi^\tau(x, y^k_\mu) = 0, \qquad \tau = 1, ..., k, |\mu| \leq q, \tag{8}$$

we can study the formal integrability of (8). Roughly speaking, suppose that z is given, we can try to find locally the formal solutions of (8) in bringing this system to formal integrability. However, in bringing it to formal integrability, we have to compute certain determinants (testing fibered manifold conditions, computing the dimension of the symbols, projections, ...) which may depend on z. So, we are led to define family of resultants that z has to satisfy in order to have formal solutions of the system (7). We have to notice that there are three kind of inequalities which can appear when we bring a system of PDE to formal integrability:

1. inequalities which appear when testing fibered manifold conditions,
2. inequalities appearing when projecting prolongations of the system on lower order space jet,
3. inequalities appearing when testing the 2-acyclicity (or involutivity) of the symbol.

However, the third kind of inequalities is a "technical one". Indeed, the definition of formal integrability does no use 2-acyclicity but only fibered manifolds and projections. However, most of the time, we have to use the Spencer-Goldschmidt criterion in which the 2-acyclicity (or involutivity) has to be tested.

We now give an example, in which the first and the second kinds of inequalities appear when computing the resultants. This example is taken from [5] where the resultants were computed using differential algebra techniques.

Example 2 Let us consider the system defined by:

$$R_1 \left\{ \begin{array}{l} \dot z^1 - uz^2 = 0, \\ \dot z^2 - z^1 - uz^2 = 0. \\ z^1 - y = 0. \end{array} \right.$$

In the control framework, u is the input, z the state, y the ouput and we look for input-output relations by eliminating z, called input-output behaviour. The system R_1 is not formally integrable in $z = (z^1, z^2)$. As this system is a system of ordinary differential equations, we know that the symbol $M_1 = 0$ is trivially involutive and we have only to saturate the system by lower order equations. We have:

$$R_1^{(1)} \left\{ \begin{array}{l} \dot z^1 - uz^2 = 0, \\ \dot z^2 - z^1 - uz^2 = 0, \\ z^1 - y = 0, \\ u\,z^2 - \dot y = 0, \end{array} \right.$$

and $R_1^{(1)}$ is a fibered manifold iff $u \neq 0$.

1. If $u = 0$ then $R_1^{(1)}$ is defined by

$$\left\{ \begin{array}{l} \dot z^1 - uz^2 = 0, \\ \dot z^2 - z^1 - uz^2 = 0, \\ z^1 - y = 0, \\ \dot y = 0, \end{array} \right.$$

and $R_1^{(1)}$ is a fibered manifold iff $\dot{y}=0$. In this case, we have $R_1^{(2)}=R_1^{(1)}$ and $R_1^{(1)}$ is an involutive system. Moreover, $\dim R_1^{(1)}=\dim M_1^{(1)}+\dim R_0^{(1)}=0+(2-1)=1$, where $R_0^{(1)}$ is the projection of $R_1^{(1)}$ on $J_0(E)$ (i.e., the zero order equations of the system $R_1^{(1)}$).

2. If $u\neq 0$ then $R_1^{(2)}\subsetneq R_1^{(1)}$ where $R_1^{(2)}$ is defined by:

$$R_1^{(2)}\left\{\begin{array}{l} \dot{z}^1-uz^2=0,\\ \dot{z}^2-z^1-uz^2=0,\\ z^1-y=0,\\ u\,z^2-\dot{y}=0,\\ (\dot{u}+u^2)z^2-\ddot{y}+uy=0. \end{array}\right.$$

Now, as $u\neq 0$, the last two equations lead to:

$$R_1^{(2)}\left\{\begin{array}{l} \dot{z}^1-uz^2=0,\\ \dot{z}^2-z^1-uz^2=0,\\ z^1-y=0,\\ u\,z^2-\dot{y}=0,\\ u\ddot{y}-(\dot{u}+u^2)\dot{y}-u^2y=0. \end{array}\right.$$

$R_1^{(2)}$ is a fibered manifold iff $u\neq 0$ and $u\ddot{y}-(\dot{u}+u^2)\dot{y}-u^2y=0$ and, in this case, $R_1^{(2)}$ is an involutive system. Moreover, $\dim R_1^{(2)}=\dim M_1^{(2)}+\dim R_0^{(2)}=0+(2-2)=0$.

We can notice that the dimension of the fibre is generically equal to 0 and the dimension jumps to 1 in the differentialy algebraic set $\{u=0,\dot{y}=0\}$. Finally, the input-output behaviour is the disjunction of the two following systems:

$$\left\{\begin{array}{l} u=0,\\ \dot{y}=0, \end{array}\right. \qquad \left\{\begin{array}{l} u\neq 0,\\ u\ddot{y}-(\dot{u}+u^2)\dot{y}-u^2y=0. \end{array}\right.$$

2.3.1 Trees of Integrability Conditions

It is well known that the “degree of generality” of the formal solutions of a system of linear PDE with certain variable coefficients, highly depends on certain relations that these coefficients may verify or not. These relations are nothing else than the resultants on the coefficients that the system has to verified in order to have a solution. These resultants naturally appear as formal integrability conditions when we study the formal integrability of the system. We can organize those integrability conditions in order to build a tree and each final leaf represents a formal solution of the system with its degree of generality.

Example 3 Let us consider the following system (we recall that $\partial_{i_1}...\partial_{i_m}\,y=y_{i_1...i_m}$):

$$R_2\left\{\begin{array}{l} y_{22}-a(x)\,y_1=0,\\ y_{12}=0. \end{array}\right.$$

First of all, the symbol of the system, defined by

$$M_2\left\{\begin{array}{l} v_{22}=0,\\ v_{12}=0, \end{array}\right. \qquad \boxed{\begin{array}{cc} 1 & 2\\ 1 & \bullet \end{array}}$$

is involutive (differentiating with respect to the dot does not bring new equation) and so, we only have to test if we have $R_2^{(1)}=R_2$. We have:

$$R_2^{(1)}\left\{\begin{array}{l} y_{22}-a(x)\,y_1=0,\\ y_{12}=0,\\ a(x)\,y_{11}+\partial_1 a(x)\,y_1=0. \end{array}\right.$$

1. If $a = 0$ then $R_2^{(1)} = R_2$ and R_2 is an involutive system. We easily see that $\alpha_2^1 = 1$ and $\alpha_2^2 = 0$ which implies that $\forall\, r \geq 0:\ \dim\ R_{2+r} = \dim\ \ M_{2+r} + \dim\ \ R_1 = 1+3 = 4$. The solution of the system depends on one function of x^1 and certain constants. Indeed, we easily integrate the system and we find $y = c\,x^2 + d(x^1)$.

2. If $a \neq 0$ then

$$R_2^{(1)} \begin{cases} y_{22} - a(x)\, y_1 = 0, \\ y_{12} = 0, \\ y_{11} + b(x)\, y_1 = 0, \end{cases}$$

where $b(x) = \partial_1 a(x)/a(x)$. $M_2^{(1)} = 0$ is trivially involutive and we have to compute $R_2^{(2)}$:

$$\begin{cases} y_{22} - a(x)\, y_1 = 0, \\ y_{12} = 0, \\ y_{11} + b(x)\, y_1 = 0, \\ \partial_2 b(x) y_1 = 0. \end{cases}$$

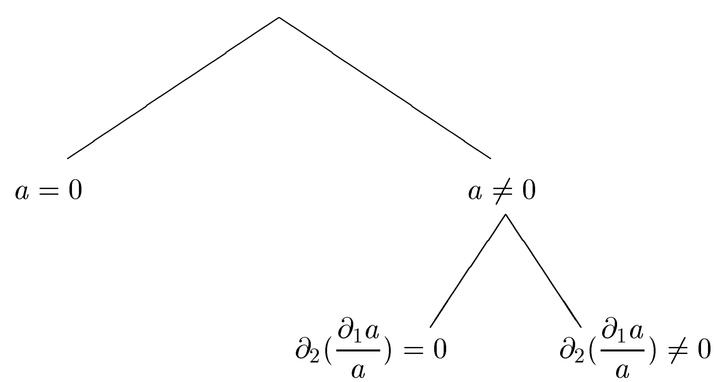
   (a) If $\partial_2 b(x) = 0$ then $R_2^{(2)} = R_2^{(1)}$ and $R_2^{(1)}$ is an involutive system. We have $\forall r \geq 0$ $\dim M_{2+r}^{(1)} = 0$ and the solution of the system depends only on constants.

   (b) If $\partial_2 b(x) \neq 0$ then $R_2^{(3)} = R_2^{(2)}$ and thus $R_2^{(2)}$ is an involutive system:

$$\begin{cases} y_{22} = 0, \\ y_{12} = 0, \\ y_{11} = 0, \\ y_1 = 0. \end{cases}$$

   The solution of the system depends only on $\dim\ R_1^{(2)} = 3 - 1 = 2$ constants. Indeed, we easily integrate it and find $y = c\,x^2 + d$.

We obtain the following intrinsic tree of integrability conditions.



We now give an example in which the third kind of inequalities appears when bringing the system to formal integrability. In particular, we are interested in knowing how the compatiblity conditions vary (number and order) with the variable coefficients of the system.

Example 4 Let us define the following system

$$R_2 \begin{cases} y_{33} - a\, y_{11} = 0, \\ y_{23} = 0, \\ y_{22} - b\, y_{11} = 0, \\ y_{13} = 0, \\ y_{12} = 0, \end{cases} \tag{9}$$

where a and $b \in \mathbb{R}$. We have the following multiplicative variables:

$$M_2 \quad \left\{ \begin{array}{l} v_{33} - a\,v_{11} = 0, \\ v_{23} = 0, \\ v_{22} - b\,v_{11} = 0, \\ v_{13} = 0, \\ v_{12} = 0. \end{array} \right. \qquad \boxed{\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \\ 1 & \bullet & \bullet \\ 1 & \bullet & \bullet \end{array}} \tag{10}$$

If we prolong with respect to the dots, we find two new equations: $a\,v_{111} = 0$ and $b\,v_{111} = 0$. Thus M_2 is involutive if $a = b = 0$. Else, if we prolong once the symbol M_2, we obtain $M_3 = 0$, i.e., M_2 is finite type and M_3 is a trivial involutive symbol. In that case, we can easily check whether the symbol M_2 is 2-acyclic or not: we have to compute the cohomology $H^2_2(M_2)$ of the following sequence

$$0 \longrightarrow \Lambda^2 T^\star \otimes M_2 \xrightarrow{\delta} \Lambda^3 T^\star \otimes T^\star.$$

Thus, we only have to check under what conditions on a and b, δ is injective:

$$\forall\, \omega = \; v_{\mu k, ij} dx^i \wedge dx^j \in \Lambda^2 T^\star \otimes M_2 : \delta(\omega)_\mu = (v_{\mu 3,12} + v_{\mu 1,23} + v_{\mu 2,31}) dx^1 \wedge dx^2 \wedge dx^3.$$

Thus $\delta(\omega) = 0$ with $v_{\mu k} \in M_2 \Rightarrow v_{11,23} = v_{22,31} = v_{33,12} = 0 \Rightarrow a\,v_{11,12} = 0,\ b\,v_{11,31} = 0$ and δ is injective iff $a \neq 0$ and $b \neq 0$. In this case, M_2 is 2-acyclic but not involutive otherwise we would have the exact sequence $\ldots \longrightarrow \Lambda^2 T^\star \otimes M_3 \xrightarrow{\delta} \Lambda^3 T^\star \otimes M_2 \longrightarrow 0$ and thus $M_3 = 0 \Rightarrow M_2 = 0$, which is obviously not true.

We obtain the following tree of integrability conditions:

$\left\{ \begin{array}{l} a = 0, \\ b = 0. \end{array} \right.$	$\left\{ \begin{array}{l} \max(\lvert a\rvert, \lvert b\rvert) \neq 0, \\ \min(\lvert a\rvert, \lvert b\rvert) = 0. \end{array} \right.$	$\left\{ \begin{array}{l} a \neq 0, \\ b \neq 0. \end{array} \right.$
M_2 involutive $M_3 \neq 0$	M_2 not 2-acyclic $M_3 = 0$ involutive	M_2 2-acyclic M_2 not involutive $M_3 = 0$ involutive

1. In case $a = 0$, $b = 0$, M_2 is involutive and we easily see that $R^{(1)}_2 = R_2$. Thus, R_2 is an involutive system. Moreover, $\dim M^0_2{=}1$, $\dim M^1_2{=}0$, $\dim M^2_2{=}0 \Rightarrow \alpha^1_2 = 1$, $\alpha^2_2 = 0$, $\alpha^3_2 = 0$. Thus $\dim M_{2+r}{=}\dim R_{2+r}{=}1$, $\forall r \geq 0$. We find the compatibility conditions of

$$\left\{ \begin{array}{l} y_{33} = z^1, \\ y_{23} = z^2, \\ y_{22} = z^3, \\ y_{13} = z^4, \\ y_{12} = z^5, \end{array} \right. \qquad \boxed{\begin{array}{ccc} 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \\ 1 & \bullet & \bullet \\ 1 & \bullet & \bullet \end{array}} \tag{11}$$

by derivating the equations with respect to the dots and projecting on the system R_2. We find 6 homogeneous *first order* compatibility conditions:

$$\left\{ \begin{array}{l} z^2_3 - z^1_2 = 0, \\ z^3_3 - z^2_2 = 0, \\ z^5_3 - z^2_1 = 0, \\ z^4_3 - z^1_1 = 0, \\ z^4_2 - z^2_1 = 0, \\ z^5_2 - z^3_1 = 0. \end{array} \right.$$

We let the reader check that this system is involutive (it is a general property of involutive systems [20]). Now, if we want to know the compatibility conditions of

$$\left\{\begin{array}{l} z^2_3 - z^1_2 = t^1, \\ z^3_3 - z^2_2 = t^2, \\ z^5_3 - z^2_1 = t^3, \\ z^4_3 - z^1_1 = t^4, \\ z^4_2 - z^2_1 = t^5, \\ z^5_2 - z^3_1 = t^6, \end{array}\right. \qquad \begin{array}{|ccc|} \hline 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \\ 1 & 2 & \bullet \\ 1 & 2 & \bullet \\ \hline \end{array} \tag{12}$$

we still differentiate the equations with respect to the dots and project the results on the system, we obtain 2 compatibility conditions:

$$\left\{\begin{array}{l} t^5_3 - t^4_2 + t^1_1 = 0, \\ t^6_3 - t^2_2 + t^2_1 = 0. \end{array}\right. \qquad \begin{array}{|ccc|} \hline 1 & 2 & 3 \\ 1 & 2 & 3 \\ \hline \end{array} \tag{13}$$

This system does not have compatibility conditions. We have just build the *Janet sequence* of the operator $\mathcal{D} : y \to z$ defined by (11). We have:

$$0 \to \Theta \to E \xrightarrow{\mathcal{D}} F_1 \xrightarrow{\mathcal{D}_1} F_2 \xrightarrow{\mathcal{D}_2} F_3 \longrightarrow 0,$$

where Θ is the kernel of $\mathcal{D}$ and the operators $\mathcal{D}_1 : z \to t$ and $\mathcal{D}_2 : t \to s$ are defined by (12) and (13) in which s is the second member.

2. In case $a \neq 0$, $b \neq 0$ (for example $a = b = 1$) then M_2 is 2-acyclic and $R^{(1)}_2 = R_2$. Hence, the system is formally integrable. In this case, we can compute the compatibility conditions of

$$\left\{\begin{array}{l} y_{33} - a\, y_{11} = z^1, \\ y_{23} = z^2, \\ y_{22} - b\, y_{11} = z^3 \\ y_{13} = z^4, \\ y_{12} = z^5, \end{array}\right. \tag{14}$$

by computing $R^{(1)}_2 = R_2$. We find only 5 homogeneous *first order* compatibility conditions, defined by:

$$\left\{\begin{array}{l} z^2_3 - z^1_2 - a\, z^5_1 = 0, \\ z^3_3 - z^2_2 + b\, z^4_3 = 0, \\ z^5_3 - z^2_1 = 0, \\ b\, z^4_3 - a z^5_2 - b\, z^1_1 + a\, z^3_1 = 0, \\ z^4_2 - z^2_1 = 0. \end{array}\right.$$

3. Finally, in case max($|a|$, $|b|$)$\neq 0$ and min($|a$ $|b|$)$= 0$, M_2 is not 2 acyclic and we have to prolong the system and see whether or not $R^{(1)}_3 = R_3$, as we already know that $M_3 = 0$ is a trivial involutive symbol. We let the reader check that it is the case and R_3 is an involutive system. Let us suppose that $a \neq 0$ and $b = 0$. Computing the compatibility conditions by differentiating with respect to the dots of M_3 and projecting on R_3, we find 6 homogeneous *first and second order* compatibility conditions:

$$\left\{\begin{array}{l} z^4_{33} - z^1_{13} - a\, z^4_{11} = 0, \\ z^5_3 - z^2_1 = 0, \\ z^5_2 - z^3_1 = 0, \\ z^3_3 - z^2_2 = 0, \\ z^2_3 - z^1_2 - a\, z^5_1 = 0, \\ z^2_4 - z^2_1 = 0. \end{array}\right.$$

We have to remark that R_2 is formally integrable ($\forall r, s \geq 0 : \mathcal{R}_{2+r+s}$ is a fibered manifold and $\mathcal{R}_{2+r+s} \to \mathcal{R}_{2+r}$ is surjective) even if the Spencer-Goldschmidt criterion is not satisfied (see example 1).

Notice that a simple change of the parameters a and b has totally changed the compatibility conditions of the system R_2 (the number and the orders). Moreover, in that example, the 2-acyclicity of M_2 is a generic property. Obviously, we can find examples combining the three kinds of inequations.

3 Applications to Control Systems Theory

Recently, the structural properties of control systems have received a new insight with the use of differential algebra, formal integrability theory and differential module (D-module) theory. See for example [3, 9, 10, 11, 17, 21, 22, 23, 25]. Certain intrinsic properties of the control systems have been reformulated in terms of the algebraic nature of its underlying differential module (as torsion, torsion-free, projective or free module). Formal tests have been found in [21, 22, 23, 25] to test whether a finitly generated $D = A\,[d_1, ..., d_n]$-module (A a differential ring containing $\mathbb{R}$) satisfies one of the above properties. These tests only use formal integrability theory and thus, most of the structural properties of control systems can be tested by bringing a system of ODE or PDE to formal integrability (testing the surjectivity or the injectivity of an operator, computing the compatibility conditions, ...). We are able to use the preceding results for control systems with variable coefficients or for linearized nonlinear ones. It will lead to trees of integrability conditions.

We first recall a few statments and results on D-module and linear operators. For more details, see [21, 23, 25]. In particular, the idea is to study how the algebraic nature of a differential module, determined by a system of PDE, changes with the variable coefficients of the system.

3.1 D-module and Linear Operator

Let $\mathcal{D}_0 : E \to F_0$ be a linear operator, where E and F_0 are vector bundles over X and $\dim E = m$. The operator $\mathcal{D}_0$ is injective if $\mathcal{D}_0\,\eta = 0 \Rightarrow \eta = 0$ and it is surjective if the equations $\mathcal{D}_0\eta = 0$ are linearly differential independent or if $\mathcal{D}_0\,\eta = \zeta$ has no compatibility conditions, i.e., if it does not exist an operator $\mathcal{D}_1$ such that $\mathcal{D}_0\,\eta = \zeta \Rightarrow \mathcal{D}_1\,\zeta = 0$ [21].

Let A be a differential ring with n commuting derivatives $\partial_1, ..., \partial_n$, containing $\mathbb{R}$. We denote by $D = A\,[d_1, ..., d_n]$ the ring of differential operators with coefficients in A where the d_i satisfies:

$$\forall\, a, b \in A : a\, d_i(b\, d_k) = ab\, d_i\, d_k + a(\partial_i\, b) d_k.$$

D is an integral domain which is commutative ring only when A is a ring of constants (with respect to the derivatives ∂_i, $i = 1, ..., n$). However, it possesses the left and right Ore properties: $\forall\,(p, q) \in D^2, \exists\,(r, s), (u, v) \in D^2 : r\,p = s\,q$ and $p\,u = q\,v$. Let $\eta = \{\eta^1, ..., \eta^m\}$ be some differential indeterminates and let us form the free left D-module generated by η and denote it by $[\eta]$. Every element of $[\eta]$ has the following form: $\sum_{\text{finite}} a_k^\mu\, d_\mu\, \eta^k$, where $\mu = (\mu_1, ..., \mu_n)$ is a multi-index. For all the algebraic concepts, see [29].

A fundamental idea is to associate with any operator $\mathcal{D}$ the left D-module $\mathcal{M} = [\eta]/[\mathcal{D}\,\eta]$. We will say, in the rest of the text, that the operator $\mathcal{D}$ determines the D-module $\mathcal{M}$.

Definition 4
- An element τ of $\mathcal{M}$ is called a torsion element if there exists a non zero element of D which kills τ, i.e., $\exists\, a \in D$, $a \neq 0$, $a\,\tau = 0$. We note by $t(\mathcal{M})$ the submodule formed by the torsion elements of $\mathcal{M}$.
- A D-module $\mathcal{M}$ is *torsion-free* if $t(\mathcal{M}) = 0$. The D-module $\mathcal{M}/t(\mathcal{M})$ is a torsion-free D-module.

Example 5 Let us consider the system $\mathcal{D}\eta = 0$ defined by

$$\left\{ \begin{array}{l} \ddot{\eta}^1 + \eta^1 - \eta^2 + \alpha\eta^3 = 0, \\ \ddot{\eta}^2 + \eta^2 - \eta^1 - \eta^3 = 0, \end{array} \right.$$

where $\alpha \in \mathbb{R}$ and the D-module $\mathcal{M} = [\eta]/[\mathcal{D}\eta]$ determined by the operator $\mathcal{D}$.

- For $\alpha = -1$, if we substract the first equation from the second, we find $\tau^1 = \eta^1 - \eta^2$ satisfying $(\frac{d^2}{dt^2} + 2)\tau^1 = 0$. The element τ^1 is a torsion element of $\mathcal{M}$.
- For $\alpha = 1$, if we add the first equation to the second, we find a torsion element $\tau^2 = y^1 + y^2$ satisfying $(\frac{d^2}{dt^2})\tau^2 = 0$.

It is quite difficult to see that, except these two values of the parameter α, the D-module $\mathcal{M}$ is torsion-free.

We now recall the duality of differential operators to give a formal test checking whether a finitely generated D-module $\mathcal{M}$ is torsion-free or not. If $\mathcal{M}$ is not a torsion-free D-module, the test gives the generators of $t(M)$ and the operators killing them. We denote $E^\star$ the dual bundle of E and $\tilde{E} = \bigwedge^n T^\star \otimes E^\star$ its adjoint bundle. If $\mathcal{D} : E \to F$ is a linear differential operator, its formal adjoint $\tilde{\mathcal{D}} : \tilde{F} \to \tilde{E}$ is defined by the following rules:

- the adjoint of a matrix (zero order operator) is the transposed matrix,
- the adjoint of d_i is $-d_i$,
- for two linear PD operators P, Q that can be composed: $\widetilde{P \circ Q} = \tilde{Q} \circ \tilde{P}$.

We have the relation

$$\mu^t \mathcal{D}_0\, \xi = (\tilde{\mathcal{D}}_0\, \mu)^t \xi + d(\cdot),$$

with d the exterior derivative. We can directly compute the adjoint of an operator by multiplying it by test functions on the left and integrating the result by part.

Example 6 Let us compute the adjoint of the operator $\mathcal{D} :\, \eta \to \zeta$ defined by:

$$\begin{cases} \eta^1_{22} - a(x)\eta^2_1 + \eta^1 = \zeta^1, \\ \eta^2_{12} - \eta^1_{11} = \zeta^2. \end{cases}$$

Multiplying the system by (λ_1, λ_2) on the left and integrating the result by part, we find the operator $\tilde{\mathcal{D}} : \lambda \to \mu$:

$$\begin{cases} d_{22}\lambda_1 - d_{11}\lambda_2 + \lambda_1 = \mu_1, \\ d_{12}\lambda_2 + a(x)d_1\lambda_1 + d_1 a(x)\lambda_1 = \mu_2. \end{cases}$$

Definition 5 We call an operator $\mathcal{D}_1$ *parametrizable* if there exists a set of arbitrary functions $\xi = (\xi^1, \ldots, \xi^r)$ or "potentials" and a linear operator $\mathcal{D}_0$ such that all the compatibility conditions of the inhomogenous system $\mathcal{D}_0\, \xi = \eta$ are exactly generated by $\mathcal{D}_1\eta = 0$. In that case, we will say that the sequence $E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1$ is formally exact.

We describe a formal test checking if the operator $\mathcal{D}_1$ determines a torsion-free D-module $\mathcal{M}$ or not (see [21] and compare with [16]):

1. Start with $\mathcal{D}_1$.
2. Construct its adjoint $\tilde{\mathcal{D}}_1$.
3. Find the compatibility conditions of $\tilde{\mathcal{D}}_1\lambda = \mu$ and denote this operator by $\tilde{\mathcal{D}}_0$.
4. Construct its adjoint $\mathcal{D}_0$.
5. Find the compatibility conditions of $\mathcal{D}_0\, \xi = \eta$ and call this operator by $\mathcal{D}'_1$.

We are led to two different cases. If $\mathcal{D}'_1 = \mathcal{D}_1$ then the system $\mathcal{D}_1$ determines a torsion-free D-module $\mathcal{M}$ and $\mathcal{D}_0$ is a parametrization of $\mathcal{D}_1$. Otherwise, the operator $\mathcal{D}_1$ is among, but not exactly, the compatibility conditions of $\mathcal{D}_0$. The torsion elements of $\mathcal{M}$ are all the new compatibility conditions modulo the equations $\mathcal{D}_1\eta = 0$.

We can represent the test by the following differential sequences where the number indicates the different stages:

$$
\begin{array}{ccccc}
 & & & 5 & \\
 & & & \xrightarrow{\mathcal{D}_1'} & F_1' \\
E & \xrightarrow{\mathcal{D}_0} & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 \\
 & 4 & & 1 & \\
\tilde{E} & \xleftarrow{\tilde{\mathcal{D}}_0} & \tilde{F}_0 & \xleftarrow{\tilde{\mathcal{D}}_1} & \tilde{F}_1 \\
 & 3 & & 2 &
\end{array}
$$

Hence, summerizing the above results, we obtain the following useful theorem.

Theorem 6 *A system $\mathcal{D}_1$ determines a torsion-free D-module $\mathcal{M}$ iff the operator $\mathcal{D}_1$ is parametrizable. Hence, for systems of PDE with variable coefficients, we have to construct two trees of integrability conditions (one for $\tilde{\mathcal{D}}_1$ and the other for $\mathcal{D}_0$) to know whether $\mathcal{D}_1$ determines a torsion-free D-module or not.*

Such examples with two trees of integrability conditions are very rare. We take an example of [24].

Example 7 Let us consider the finite transformation $y = f(x)$ satisfying the Pfaffian system:

$$dy^3 - a(y^2)dy^1 = \rho(x)(dx^3 - a(x^2)dx^1).$$

Linearizing such a transformation around the identity by setting $y = x + t\xi(x) + \ldots$ and making $t \to 0$, after eliminating $\rho(x)$, we discover that infinitesimal transformations are defined, trough the use of a correct geometric object, by the kernel of the differential system $\mathcal{D}_0\xi = \eta$ as following (see p. 237 of [21]):

$$\left\{\begin{array}{lcl}
-a(x^2)\xi_1^1 + \xi_1^3 + \frac{1}{2}a(x^2)(\xi_1^1 + \xi_2^2 + \xi_3^3) - \xi^2\partial_2 a(x^2) & = & \eta^1, \\
-a(x^2)\xi_2^1 + \xi_2^3 & = & \eta^2, \\
-a(x^2)\xi_3^1 + \xi_3^3 - \frac{1}{2}(\xi_1^1 + \xi_2^2 + \xi_3^3) & = & \eta^3.
\end{array}\right.$$

From the theory of Lie pseudogroups [21], we can prove that the PD system $\mathcal{D}_0\xi = 0$ is formally integrable if and only if $\partial_2 a(x^2) = c = cst$, the "classical case" of *contact transformations* corresponding to $a(x^2) = x^2$ ($\Rightarrow c = 1$). It follows that the only compatibility condition $\mathcal{D}_1\eta = 0$ is

$$-a(x^2)(\eta_2^3 - \eta_3^2) + \eta_1^2 - \eta_2^1 + \partial_2 a(x^2)\eta^3 = 0,$$

and the operator $\mathcal{D}_1$ is surjective. The adjoint operator $\tilde{\mathcal{D}}_1$ is defined by:

$$\left\{\begin{array}{lcl}
\lambda_2 & = & \mu_1, \\
-a(x^2)\lambda_3 - \lambda_1 & = & \mu_2, \\
a(x^2)\lambda_2 + 2c\lambda & = & \mu_3.
\end{array}\right.$$

As $\mu_3 - a(x^2)\mu_1 = 2c\lambda$, the operator $\tilde{\mathcal{D}}_1$ is injective if and only if $c \neq 0$. In that case, the two independent compatibility conditions can be written:

$$\left\{\begin{array}{lcl}
d_2\mu_3 - a(x^2)d_2\mu_1 - 3c\mu_1 & = & 2\nu_2, \\
-a(x^2)d_3(\mu_3 - a(x^2)\mu_1) - d_1(\mu_3 - a(x^2)\mu_1) - 2c\mu_2 & = & -2(\nu_1 + a(x^2)\nu_3),
\end{array}\right.$$

after introducting the adjoint $\tilde{\mathcal{D}}_0$ of $\mathcal{D}_0$ as follows:

$$\left\{\begin{array}{lcl}
\frac{1}{2}a(x^2)d_1\mu_1 + \frac{1}{2}d_1\mu_3 + a(x^2)d_3\mu_3 + a(x^2)d_2\mu_2 + \partial_2 a(x^2)\mu_2 & = & \nu_1, \\
-\frac{1}{2}a(x^2)d_2\mu_1 + \frac{1}{2}d_2\mu_3 - \frac{3}{2}\,\partial_2 a(x^2)\mu_1 & = & \nu_2, \\
-d_1\mu_1 - \frac{1}{2}a(x^2)d_3\mu_1 - d_2\mu_2 - \frac{1}{2}d_3\mu_3 & = & \nu_3.
\end{array}\right.$$

Now, let us start with the operator $\tilde{\mathcal{D}}_0$ depending on the arbitrary function $a(x^2)$ and let us question about the algebraic nature of $\mathcal{M} = [\mu]/[\tilde{\mathcal{D}}_0\mu]$. According to the general test, we must construct the adjoint of $\tilde{\mathcal{D}}_0$ which is $\mathcal{D}_0$ and look for its compatibility conditions $\mathcal{D}_1$, a result bringing out the condition $\partial_2 a(x^2) = c$, where c is an arbitary constant. When $c = 0$, we should find the zero order compatibility condition $\mu_3 - a(x^2)\mu_1 = 0$ which is not a consequence of $\tilde{\mathcal{D}}_0$ and thus $\tilde{\mathcal{D}}_0$ determines a D-module $\mathcal{M}$ with torsion elements. Indeed, we call easy verify that the element non zero element $\tau = \mu_3 - a(x^2)\mu_1 \in \mathcal{M}$ satisfies $d_2\tau = 0$. When $c \neq 0$, the adjoint $\tilde{\mathcal{D}}_1$ admits the compatibility condition expressed by $\tilde{\mathcal{D}}_0$ because we have in that case:

$$a(x^2)d_3\nu_3 + d_3\nu_1 - d_2\nu_2 - a(x^2)d_1\nu_2 + \nu_3 = 0,$$

which gives $\nu_3 = (d_2 + a(x^2)d_1)\nu_2 - d_3(\nu_1 + a(x^2)\nu_3)$ and $\mathcal{M}$ is a torsion-free D-module.

We recall the definition of a free D-module.

Definition 6 A D-module $\mathcal{M}$ is a free D-module if there exists a basis of $\mathcal{M}$, i.e., elements of $\mathcal{M}$ which are independent on D and which generate $\mathcal{M}$.

Theorem 7 *An operator $\mathcal{D}_1$ determines a free D-module iff it admits an injective parametrization $\mathcal{D}_0$, i.e., if the sequence $0 \longrightarrow E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1$ is formally exact.*

We turn now to an important class of D-module: projective module. We only give some basic results. See [23, 25] for more general ones, and in particular, how to use of projective D-module and formal duality in order to split certain differential sequences of operators. See [18] for more deeper results.

Definition 7 A D-module $\mathcal{M}$ is a projective D-module if there exists a D-module $\mathcal{N}$ and a free D-module $\mathcal{F}$ such that $\mathcal{F} = \mathcal{M} \oplus \mathcal{N}$, i.e., $\mathcal{M}$ is a summand of a free D-module. Remark that $\mathcal{N}$ is a projective D-module too.

We have the following theorem, which allows us to test if $\mathcal{M}$ is a projective D-module.

Theorem 8 *A surjective operator $\mathcal{D}_1 : F_0 \to F_1$ determines a projective D-module if there exists an operator $P_1 : F_1 \to F_0$ such that $\mathcal{D}_1 \circ \mathcal{P}_1 = id_{F_1}$, where id_{F_1} is the identity operator of F_1, i.e., if its dual $\tilde{\mathcal{D}}_1$ is injective. In particular, if the system has variable coefficients, only one tree of integrability conditions has to be built in order to decide whether the operator determines a projective D-module or not.*

Let us describe a formal test for checking whether a D-module $\mathcal{M}$ is projective or not:

1. Start with $\mathcal{D}_1$ and check its surjectivity.
2. Construct its adjoint $\tilde{\mathcal{D}}_1$.
3. Check whether $\tilde{\mathcal{D}}_1$ is an injective operator or not.

We are led to two different cases. If $\tilde{\mathcal{D}}_1$ is an injective operator then $\mathcal{D}_1$ determines a projective D-module. If we want to compute its right-inverse, we have to bring the system $\tilde{\mathcal{D}}_1\lambda = \mu$ to formal integrability. We have to find $\lambda = \tilde{\mathcal{P}}_1\mu$. Dualize this operator, we find the operator $\mathcal{P}_1$ which satisfies $\mathcal{P}_1 \circ \mathcal{D}_1 = Id_{F_1}$. In the case when the adjoint of $\mathcal{D}_1$ is not injective, then $\mathcal{D}_1$ does not determine a projective D-module.

Example 8 Let us show that the operator $\mathcal{D} : \eta \to \zeta$ defined by

$$\eta_2^2 - x^2\,\eta_1^1 + \eta^1 = \zeta$$

determines a projective D-module $\mathcal{M}$. We first dualize $\mathcal{D}$ and we find the $\tilde{\mathcal{D}} : \lambda \to \mu$:

$$\left\{ \begin{array}{l} x^2\lambda_1 + \lambda = \mu_1, \\ -\lambda_2 = \mu_2. \end{array} \right.$$

We now study the formal integrability of the preceding system in λ and we find a new equation $\lambda = -x^2(d_2\mu_1 + x^2 d_1\mu_2 + \mu_2) + \mu_1$. Thus $\tilde{\mathcal{D}}\lambda = 0 \Rightarrow \lambda = 0$ and thus $\mathcal{M}$ is a projective D-module with a right-inverse $\mathcal{P}$ given by the dual of the operator $-x^2(d_2\mu_1 + x^2 d_1\mu_2 + \mu_2) + \mu_1 = \nu$. We obtain $\mathcal{P} : \zeta \to \eta$ defined by

$$\left\{ \begin{array}{l} x^2\zeta_2 + 2\zeta = \eta^1, \\ (x^2)^2\zeta_1 - x^2\zeta = \eta^2, \end{array} \right.$$

and we let the reader verify that $\mathcal{D} \circ \mathcal{P} = Id$.

For non surjective operator, see [25]. It is quite easy to see that every free D-module is projective and every projective D-module is a torsion-free, which can be summed up by the following module inclusions:

$$\text{free} \subseteq \text{projective} \subseteq \text{torsion-free}.$$

For a principal ideal ring D (for example $D = k[\frac{d}{dt}]$ with k a field), every torsion-free D-module is a free D-module. Thus, we have the following useful corollary:

Corollary 2 *A surjective OD operator determines a free D-module iff its dual is injective. In particular, if the operator has some variable coefficients, we only need one tree of integrability conditions to know whether it determines a free D-module or not.*

Quillen and Suslin have demonstrated independently in 1976 the Serre conjecture of 1950 claiming that every projective module over a polynomial ring $k[\chi_1, ..., \chi_n]$, where k is a field, is free (see [29]). It is typically the case where $D = k[\partial_1, ..., \partial_n]$ and k is a constant field (i.e., $\forall\, i = 1, ..., n, \forall\, a \in k : \partial_i(a) = 0$). Thus, we have the following corollary:

Corollary 3 *A surjective operator with constant coefficients determines a free D-module iff its formal adjoint is injective.*

Now, we give an example showing that the algebraic nature of a D-module, determined by a linear PD system with variable or unknown coefficients, depends on some integrability conditions on the coefficients.

Example 9 Let $\mathcal{D}_1 : \eta \to \zeta$ be the operator defined by:

$$\eta^1_2 - \alpha\,\eta^1_1 - \eta^2_2 + a(x)\eta^2 = \zeta, \tag{15}$$

with $\alpha \in \mathbb{R}$ and let $\mathcal{M}$ be the D-module determined by $\mathcal{D}_1$. We would like to know how the algebraic nature of the D-module $\mathcal{M}$ (torsion, torsion-free, projective and free) depends on the coefficients. Dualizing $\mathcal{D}_1$, we obtain the operator $\tilde{\mathcal{D}_1} : \lambda \to \mu$ defined by:

$$\left\{ \begin{array}{l} -\lambda_2 + \alpha\,\lambda_1 = \mu_1, \\ \lambda_2 + a(x)\lambda = \mu_2, \end{array} \right.$$

which can be rearrange under the following form:

$$\left\{ \begin{array}{l} -\lambda_2 + \alpha\,\lambda_1 = \mu_1, \\ \alpha\,\lambda_1 + a(x)\lambda = \mu_1 + \mu_2. \end{array} \right.$$

We put $\mu_1 = \mu_2 = 0$ and we call R_1 the corresponding system:

$$R_1 \left\{ \begin{array}{l} -\lambda_2 + \alpha\lambda_1 = 0, \\ \alpha\,\lambda_1 + a(x)\lambda = 0. \end{array} \right. \tag{16}$$

We now study the formal integrability of the system R_1. First of all, M_1 is an involutive symbol $\forall\, \alpha$ but its dimension depends on whether α is equal to zero or not.

1. If $\alpha = 0$ then $\dim M_1$=2-1=1 and M_1 is involutive. Now, the dimension of R_1 depends whether $a = 0$ or not.

   (a) If $a = 0$ then $\dim R_0$=1-0=1 and R_1 is formally integrable and we easily find that $\mathcal{M}$ is a torsion D-module generated by $\tau = \eta^1 - \eta^2$ satisfying $\partial_2 \tau = 0$.

   (b) If $a \neq 0$ then $\tilde{\mathcal{D}}_1$ is an injective operator. Thus $\mathcal{D}_1$ determined a projective D-module and we have $\tilde{\mathcal{P}}_1 : \mu \rightarrow \lambda$ defined by $(\mu_1 + \mu_2)/a(x) = \lambda$. Dualizing, we obtain the right-inverse $\mathcal{P}_1 : \zeta \rightarrow \eta$ of $\mathcal{D}_1$ with

   $$\left\{ \begin{array}{l} \dfrac{\zeta}{a(x)} = \eta^1, \\ \dfrac{\zeta}{a(x)} = \eta^2. \end{array} \right.$$

   We let the reader check that a parametrization $\mathcal{D} : \xi \rightarrow \eta$ of $\mathcal{D}_1$ is defined by

   $$\left\{ \begin{array}{l} -a(x)\xi_2 + (a(x)^2 - 2\,\partial_2 a(x))\xi = \eta^1, \\ -a(x)\xi_2 - 2\,\partial_2 a(x)\,\xi = \eta^2. \end{array} \right.$$

   It is an injective parametrization as we easily see that $\xi = \dfrac{\eta^1 - \eta^2}{a(x)^2}$ and $\mathcal{M} = [\xi]$.

2. If $\alpha \neq 0$ then $\dim M_1$=2-2=0 and $M_1 = 0$ is a trivial involutive symbol. We only have to study the projection $\pi_1^2 : R_2 \rightarrow R_1$, i.e., $R_1^{(1)}$. We have:

   $$R_1^{(1)} \left\{ \begin{array}{l} -\lambda_2 + \alpha\,\lambda_1 = 0, \\ \alpha\,\lambda_1 + a(x)\lambda = 0, \\ (\partial_2 a(x) - \alpha\,\partial_1 a(x))\,\lambda = 0. \end{array} \right.$$

   The dimension of $R_0^{(1)}$ depends on whether $\partial_2 a(x) - \alpha\,\partial_1 a(x)$ is equal to zero or not.

   (a) If $\partial_2 a(x) - \alpha\,\partial_1 a(x) = 0$ then $\mathcal{D}_1$ does not determined a projective D-module $\mathcal{M}$. But we easily find a parametrisation $\mathcal{D} : \xi \rightarrow \eta$ defined by:

   $$\left\{ \begin{array}{l} \xi_2 - a(x)\,\xi = \eta^1, \\ \xi_2 - \alpha\,\xi = \eta^2. \end{array} \right.$$

   Thus $\mathcal{D}_1$ determines a torsion-free but not projective D-module $\mathcal{M}$.

   (b) If $\partial_2 a(x) - \alpha\,\partial_1 a(x) \neq 0$ then $\tilde{\mathcal{D}}_1$ is an injective operator and $\mathcal{D}_1$ determines a projective D-module $\mathcal{M}$. We let the reader check that

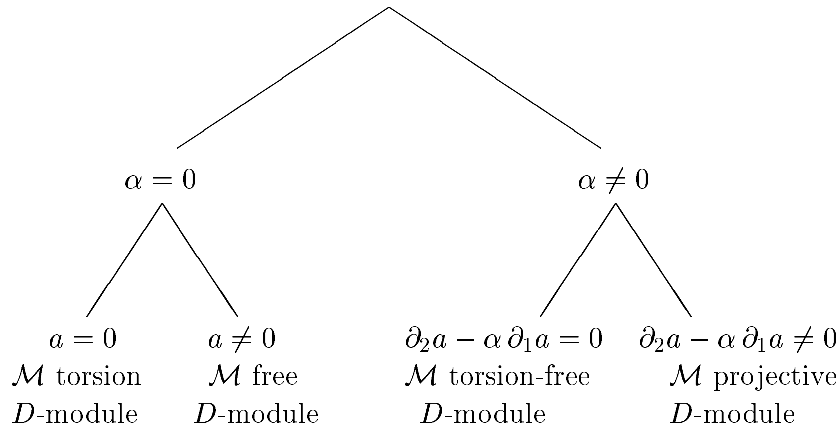
   $$\lambda = \frac{d_2\mu_1 + d_2\mu_2 - \alpha\,d_1\mu_1 + a(x)\mu_1}{\partial_2 a(x) - \alpha\,\partial_1 a(x)}.$$

   If we pose $\phi = \dfrac{\zeta}{\partial_2 a(x) - \alpha\,\partial_1 a(x)}$ then $\mathcal{P}_1 : \zeta \rightarrow \eta$ defined by:

   $$\left\{ \begin{array}{l} -\phi_2 + a(x)\phi = \eta^1, \\ -\phi_2 + \alpha\phi_1 = \eta^2, \end{array} \right.$$

   is a right-inverse of $\mathcal{D}_1$.

We can sum up the previous study by the following intrinsic tree of integrability conditions:



3.2 Structural Properties of Linear Control Systems

We now give a mathematical definition of a control system (see [15, 9, 21] for more details).

Definition 8 A *control system* is a system $\mathcal{R}_q \subset J_q\,(\mathcal{E} \times_X \mathcal{F} \times_X \mathcal{G})$, where $\mathcal{E} \times_X \mathcal{F} \times_X \mathcal{G}$ denotes the *fibered product*, that is the fibered manifold over X consiting of all elements (u, y, z) in $\mathcal{E} \times \mathcal{F} \times \mathcal{G}$ having the same projection into X. If $\mathcal{R}_q$ is defined by a differential prime ideal $\mathcal{I}$ of $k\{U, Y, Z\}$ (k a differential field) then a control system is the extension $k\langle u, y, z\rangle = Q(k\{U, Y, Z\}/\mathcal{I})$ over k. We call u the input, y the output and z the latent variable. In the linear case, a control system is the D-module $\mathcal{M}$ determined by an operator $\mathcal{D}_1 : F_0 \to F_1$, where a section of F_0 is $\eta = (u, y, z)$.

3.2.1 Controllability

Controllability is one of the key concepts of the control systems theory. Its definition and test for time-varying linear systems go back to Kalman's pioneering work [15]. We recall some recent improvements. See [3, 9, 10, 11, 17, 21, 23, 25] for more details.

We call *observable* any element of $\mathcal{M}$, i.e., any linear combination of the system variables (input and outputs together) and their derivatives. Only two possibilities may happen for an observable: it may or may not verify a PD equation by itself. An observable which does not satisfy any PD equation is called *free*. In [21], we can find the following definition:

Definition 9 A control system is *controllable* iff every observable is free.

A characterization of the controllability in terms of differential closure is shown in [21]. In [9, 11], the definition has been reformulated in the differential module framework:

Definition 10 A linear control system is controllable iff it determines a torsion-free D-module.

Hence, using the results of the previous section, we obtain:

Corollary 4 *A linear control system $\mathcal{D}_1$ is controllable iff it is parametrizable by an operator $\mathcal{D}_0$. Hence, the controllability depends at most on two problems of formal integrability and thus at most on two trees of integrability conditions.*

We know that a nonlinear control system is controllable if its generic linearization is controllable [23]. However, the linearized control system has variable coefficients, satisfying the nonlinear control system. Hence, we have to build trees of integrability conditions to know whether a nonlinear control system is controllable or not.

Example 10 Let us consider the system defined by the operator $\tilde{\mathcal{D}}_0$ of the example 7 and let us question about its controllability. We have seen that the module determined by $\tilde{\mathcal{D}}_0$ was a torsion-free D-module, i.e., controllable, if $\partial_2 a(x^2) = c$ and $c \neq 0$. The first condition is a particular branch of a first tree of integrability conditions (formal integrability of $\mathcal{D}_0$) and the second is another branch of a second tree (formal integrability of $\tilde{\mathcal{D}}_1$) depending of the first tree. Hence, the study of the controllability of that system depends on two trees of formal obstructions (see [24]). Examples of a double tree of formal obstructions to the controllability are very rare.

It is quite often supposed that the inputs are linearly differentially independent, a fact that leads to the surjectivity of the corresponding operator. Then, we have the useful corollary:

Corollary 5 *A surjective OD control system $\mathcal{D}_1$ is controllable iff it adjoint $\tilde{\mathcal{D}}_1$ is injective. In this case, the controllability depends only on one tree of integrability conditions.*

Example 11 We study the controllability of the system

$$\left\{ \begin{array}{l} \ddot{y}^1 + y^1 - y^2 + \alpha\, u = 0, \\ \ddot{y}^2 + y^2 - y^1 \;- u = 0, \end{array} \right.$$

where $\alpha \in \mathbb{R}$. Dualizing the surjective operator $\mathcal{D}_1$, we obtain $\tilde{\mathcal{D}}_1$ defined by:

$$\left\{ \begin{array}{l} \ddot{\lambda}_1 + \lambda_1 - \lambda_2 = \mu_1, \\ \ddot{\lambda}_2 + \lambda_2 - \lambda_1 = \mu_2, \\ -\lambda_2 + \alpha\, \lambda_1 = \mu_3. \end{array} \right.$$

We put $\mu_1 = \mu_2 = \mu_3 = 0$ and bring this system to formal integrability, we obtain the new equation

$$(\alpha + 1)(\alpha - 1)\lambda_1 = 0$$

and $\tilde{\mathcal{D}}_1$ is injective and thus controllable iff $\alpha \neq -1$ and $\alpha \neq 1$.

We notice that these results show that the controllability is a "built-in" property of a control system that does not depend on the separation of the variables between inputs and outputs, a fact very far from engineering intuition.

Recently, the important class of *flat* nonlinear control systems has been found in [11]. This class of systems is particulary useful for the motion planning.

Definition 11 A linear control system $\mathcal{D}_1$ is a *flat* system if $\mathcal{D}_1$ determines a free D-module.

For linear control systems, we prefer to call this notion *differential freedom* than flatness. Indeed, we do not have to confuse *flat* control systems with the algebraic notion of flat D-module, which in that case, is equivalent to the notion of projective D-module ($\mathcal{M}$ is finitely presented). For more details, see the nice reference [29].

Theorem 9 *A control system $\mathcal{D}_1$ determines a free control system iff $\mathcal{D}_1$ is parametrized by an injective operator $\mathcal{D}_0$. If $\mathcal{D}_1$ is a surjective operator with constant coefficients, it determines a free control system iff its adjoint is injective.*

The basis of the D-module $\mathcal{M}$ of a free control system are called the *flat output* or *linearizing outputs*. If $\mathcal{D}_1$ is parametrized by an injective operator $\mathcal{D}_0$ then the basis ξ is obtained by bringing the operator $\mathcal{D}_0$ to formal integrability. Indeed, we find an operator $\mathcal{P}_0$ such that $\mathcal{P}_0 \circ \mathcal{D}_0 = Id_E$ and thus $\xi = \mathcal{P}_0 \eta$.

Example 12 In the example 9, only one tree of integrability conditions has to been built in order to conclude about the algebraic nature of the D-module determined by $\mathcal{D}_1$. In the example 7, the module $\mathcal{M}$ determined by $\tilde{\mathcal{D}}_0$ is a free D-module if $\partial_2 a(x^2) = c$ and $c \neq 0$. Indeed in that case, $\tilde{\mathcal{D}}_0$ is parametrized by the injective operator $\tilde{D}_1$, and we have $\mathcal{M} = [\mu_3 - a(x^2)\mu_1]$. In this example, the freedom of the control system $\tilde{\mathcal{D}}_0$ depends on two sucessive trees of integrability conditions.

3.2.2 Observability

Another key concept in control theory is the concept of *observability.* It has been reformulated recently in the D-module framework in [3, 9, 10, 21] as follows:

Definition 12 A system of control $\mathcal{M} = [\eta', \eta'']$ is said to be *observable* with respect to η' iff $[\eta'] = \mathcal{M}$, that is, iff every component of η'' can be expressed as a linear combination of the components of η' and their derivatives.

We can reformulate the above definition, saying that the system $\mathcal{M}$ is observable with respect to η' iff $\mathcal{M}/[\eta'] = 0$ or equivalently, in the language of operator theory:

Theorem 10 *Let $\mathcal{D}_1 : F_0 \to F_1$ be an operator determining the D-module $\mathcal{M}$ and F_0' be the subbundle of F_0 with sections $(0, \eta'')$ then $\mathcal{M}$ is observable with respect to η' if the operator induced $\mathcal{D}_1' : F_0' \to F_1$ is injective.*

Example 13 Let us study the observability of the following system

$$\left\{ \begin{array}{l} z^2_{22} + z^1_{12} + z^1 - u = 0, \\ z^2_{12} - z^1_{11} + z^2 - y = 0, \end{array} \right.$$

with respect to u and y. Bringing the above system to formal integrability in $z = (z^1, z^2)$, we find that the symbol M_2 is involutive (differentiating with respect to the dot does not bring new equation) and we let the reader check by himself that we obtain:

$$R_2^{(2)} \left\{ \begin{array}{l} z^2_{22} + z^1_{12} + z^1 = u, \\ z^2_{12} + z^1_{11} - z^2 = y, \\ z^2_2 + z^1_1 = u_1 - y_2, \\ z^1 = u_{22} - y_{12} + y, \\ z^2 = y_{11} - u_{12} - u, \end{array} \right.$$

and the system is observable.

From the above theorem, we have only to study the formal integrability of one system of ODE or PDE and thus we have the following corrolary.

Corollary 6 *The observability of a control system with variable or unknown coefficients only depends on one tree of integrability conditions.*

Example 14 Let us consider the following control system

$$\left\{ \begin{array}{l} \ddot{x}^1 + x^1 - x^2 - u^1 = 0, \\ \ddot{x}^2 + x^2 - x^1 - u^2 = 0, \\ -y - x^2 + \alpha\, x^1 = 0, \end{array} \right.$$

where $x = (x^1, x^2)$ is the state, $u = (u^1, u^2)$ the input, y the output and $\alpha \in \mathbb{R}$. This system is observable with respect to (y, u) iff $[x, y, u] = [y, u]$, i.e., if the operator $\mathcal{D}_1' : x \to \zeta$ defined by

$$\left\{ \begin{array}{l} \ddot{x}^1 + x^1 - x^2 = \zeta^1, \\ \ddot{x}^2 + x^2 - x^1 - u^2 = \zeta^2, \\ -x^2 + \alpha\, x^1 = \zeta^3, \end{array} \right.$$

is injective. We recognize that this operator is nothing else than the dual of the operator $\mathcal{D}_1$ in the preceding example and thus the system is observable iff $\alpha \neq -1$ and $\alpha \neq 1$. If $\alpha \neq -1$ and $\alpha \neq 1$, we have:

$$\left\{ \begin{array}{l} x^1 = \frac{1}{(\alpha+1)(\alpha-1)}(\ddot{y} + (1+\alpha)y + u^2 - \alpha u^1), \\ x^2 = \frac{\alpha}{(\alpha+1)(\alpha-1)}(\ddot{y} + (1+\alpha)y + u^2 - \alpha u^1) - y. \end{array} \right.$$

Many others properties of the control systems have been reformulated using formal integrability theory [21]. Look also at [6] to have a general view of certain properties of control systems theory that can be tested by effective differential algebraic methods. For example:

- computation of *differential transcendence degree*: we bring the system to be involutive and compute the last character $\alpha_q^n \Rightarrow$ computation of *output rank* $\Rightarrow$ invertibility [6].
- *state elimination*: we bring the system in (z, y, u) to formal integrability in the state z and obtain resultants in u and y which define the input-output behaviour (see example 2).
- *structure at infinity*: we bring the system in (y, u) to formal integrability in u [19], ...

Thus these properties depend on trees of integrability conditions if the system has variable coefficients.

4 Conclusion

In this article, we have begun to develop a theory of formal elimination for system of PDE, using the formal integrability theory. We hope that we have convince the reader that this approach seems to be natural in many problems and in particular for linear systems of PDE with variable coefficients. In his paper [30], Seidenberg has given a general solution for this problem based on differential algebraic approach. But, as we have already noticed, the effective character always competes with the intrinsic character. Thus, we think that the formal theory of elimination will give more intrinsic results than the purely differential algebra methods. We have mainly studied the linear case with variable coefficients for its simplicity and also because it gave some nice results in control theory. We shall develop the nonlinear case later on in future papers. We think that an interesting problem sould be to revisit the Douglas' classification of the inverse problem of the calculus of variations (see [7]) in this modern framework.

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