Propagation of chaos and fluctuations for a moderate model with smooth initial data

B.Jourdain*
S.Méléard†

October 2, 1997

Abstract

In this paper, we are interested in a stochastic differential equation which is nonlinear in the following sense : both the diffusion and the drift coefficients depend locally on the density of the time marginal of the solution. When the law of the initial data has a smooth density with respect to Lebesgue measure, we prove existence and uniqueness for this equation. Under more restrictive assumptions on the density, we approximate the solution by a system of moderately interacting diffusion processes and obtain a trajectorial propagation of chaos result. Finally, we study the fluctuations associated with the convergence of the empirical measure of the system to the law of the solution of the nonlinear equation. In this situation, the convergence rate is different from $\sqrt{n}$.

The first part of this paper is dedicated to the nonlinear stochastic differential equation

$$\begin{cases} \bar{X}_t = \zeta + \int_0^t \sigma(p(s,\bar{X}_s)).dB_s + \int_0^t b(p(s,\bar{X}_s))ds \\ p \in C_b^{1,2}([0,T]\times\mathbb{R}^d) \ \text{ is such that the law of } \bar{X}_t \text{ is } p(t,x)dx \end{cases} \tag{0.1}$$

where $\bar{X}_t \in \mathbb{R}^d$, B_t is a d-dimensional Brownian motion, σ and b are smooth and the density f_0 of the law of ζ belongs to the space $H^{2+\alpha}$ of C_b^2 functions on $\mathbb{R}^d$ with second order derivatives Hölder continuous with exponent α $(0<\alpha<1)$. To prove existence and uniqueness for this problem, we first study the linear stochastic differential equation similar to (0.1) where p is replaced by a given smooth function q. Our study is based on results given by Ladyzhenskaya Solonnikov and Ural'ceva in [6] for linear parabolic partial differential equations. Then we conclude thanks to results also given in [6] for the quasilinear partial differential equation satisfied by p.

Considering the propagation of chaos proved by Oelschläger [13] and generalized by Méléard and Roelly [9] in the case of the identity diffusion matrix, it is sensible to try to approximate the

*ENPC-CERMICS, 6-8 av Blaise Pascal, Cité Descartes, Champs sur Marne, 77455 Marne la Vallée Cedex 2, France - e-mail : jourdain@cermics.enpc.fr

†Laboratoire de Probabilités, Université Paris 6, 4 place Jussieu, 75231 Paris France (URA CNRS 224) et Université Paris 10, UFR SEGMI, 200 avenue de la République, 92000 Nanterre France

solution of (0.1) by the following sequence of moderately interacting particle systems :

$$X_t^{i,n} = \zeta^i + \int_0^t \sigma(V^n * \mu_s^n(X_s^{i,n})).dB_s^i + \int_0^t b(V^n * \mu_s^n(X_s^{i,n}))ds,\ 1 \leq i \leq n \tag{0.2}$$

where $B^i,\ i \in \mathbb{N}^*$ is a sequence of independent $\mathbb{R}^d$-valued Brownian motions, $\zeta^i,\ i \in \mathbb{N}^*$ is a sequence of random variables IID with law $f_0(x)dx$ independent of the Brownian motions, $\mu^n = \frac{1}{n}\sum_{i=1}^n \delta_{X^{i,n}}$ denotes the empirical measure and $V^n(x) = \frac{1}{\epsilon_n^d}V(\frac{x}{\epsilon_n})$ for V a Lipschitz continuous and bounded probability density on $\mathbb{R}^d$ and $(\epsilon_n)_n$ a sequence of positive numbers converging to 0. In the case of the identity diffusion matrix, Oelschläger [13] manages to control $V^n * \mu_n$ by direct computations concerning the particle system. But as our diffusion matrix depends on $V^n * \mu_n$, we need other techniques to prove the propagation of chaos.
Delocalizing the interaction to enter in the classical McKean-Vlasov framework (see McKean [8], Sznitman [14] or Léonard [7] for instance), we obtain existence and uniqueness for the following mollified versions of (0.1):

$$\begin{cases} \bar{Y}_t^{i,n} = \zeta^i + \int_0^t \sigma(V^n * P_s^n(\bar{Y}_s^{i,n})).dB_s^i + \int_0^t b(V^n * P_s^n(\bar{Y}_s^{i,n}))ds \\ P^n \text{ is the law of } \bar{Y}^{i,n} \end{cases}$$

Moreover the associated propagation of chaos results imply that if ϵ_n converges to zero slowly enough, $\lim_{n\to+\infty} \mathbb{E}(\sup_{t\leq T} |X_t^{i,n} - \bar{Y}_t^{i,n}|^2) = 0$.
That is why we study the convergence for $n \to +\infty$ of $\bar{Y}^{i,n}$ to $\bar{X}^i$ where $\bar{X}^i$ denotes the solution of (0.1) for the Brownian motion B^i and the initial condition ζ^i. If the norm of f_0 in the space $H^{2+\alpha}$ is small enough, according to results concerning linear parabolic partial differential equations given in [6], for any $t \in [0,T]$, P_t^n is absolutely continuous with density $p_n(t,.)$. Moreover the sequence p_n is bounded in a Hölder space included in $C_b^{1,2}([0,T] \times \mathbb{R}^d)$. This boundedness property allows to prove that $\lim_{n\to+\infty} \mathbb{E}(\sup_{t\leq T} |\bar{X}_t^i - \bar{Y}_t^{i,n}|^2) = 0$. We conclude that, for ϵ_n converging to zero slowly enough,

$$\lim_{n\to+\infty} \mathbb{E}\biggl(\sup_{t\leq T} |\bar{X}_t^i - X_t^{i,n}|^2\biggr) = 0$$

which implies propagation of chaos for the moderately interacting particle system (0.2) and proves that the empirical measure μ_n provides a stochastic approximation of the solution of the Cauchy problem

$$\frac{\partial p}{\partial t} = \frac{1}{2}\sum_{i,j=1}^d \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij}(p)p) - \sum_{i=1}^d \frac{\partial}{\partial x_i}(b_i(p)p) \quad \text{and} \quad p(0,x) = f_0(x)$$

where a denotes the square of σ.

Finally, we study the fluctuations associated with this convergence. For the sake of simplicity, we limit ourselves to the case $d = 1$. The rate of convergence is $1/\epsilon_n^2$ where ϵ_n is chosen to minimize the upper-bound obtained for $\mathbb{E}(\sup_{t\leq T} |\bar{X}_s^i - X_s^{i,n}|^4)$. It is much smaller than $\sqrt{n}$, the rate obtained in the case of weak interaction. Let P denote the law of the solution of (0.1). We study the behaviour of $\eta^n = \frac{1}{\epsilon_n^2}(\mu^n - P)$ when n goes to infinity. The leading term is due to the convergence of V^n to δ_0 whereas the martingale part of the decomposition of η^n and the fluctuations related to the initial conditions, which would have non-trivial limits at rate $\sqrt{n}$, converge to zero. We follow the approach developped by Fernandez and Méléard in [2]. We prove that if σ, b and f_0 are smooth enough, the laws of the processes η^n are tight in $C([0,T], W_0^{-4,1})$

(the weighted Sobolev space $W_0^{-4,1}$ is defined further on) and that these processes converge in L^1 to a deterministic process characterized by a deterministic evolution equation.

Our results are obtained under restrictive assumptions on f_0. But, to our knowledge, the propagation of chaos result is the first one in the case of moderate interaction in the diffusion coefficient. The fluctuation result is the first one for moderately interacting systems and provides an example of a non-gaussian limit (since deterministic) with a rate different from $\sqrt{n}$.

Notations

We set $T > 0$, $d \in \mathbb{N}^*$. Let $C_b^{1,2}$ be the space of functions on $[0,T] \times \mathbb{R}^d$ continuous and bounded together with their first derivative with respect to the time variable (the first one) and their first and second derivatives with respect to the space variables. We introduce a few other functional spaces.

Hölder spaces

Let $\alpha \in (0,1)$. For any integer j, $H^{j+\alpha}$ is the space of real functions f on $\mathbb{R}^d$ which are continuous together with their partial derivatives up to order j and admit a finite norm

$$\|f\|_{j+\alpha} = \sum_{\overline{k} \leq j} \sup_{\mathbb{R}^d} |D^k f| + \sum_{\overline{k}=j} \sup_{\substack{x,x' \in \mathbb{R}^d \\ |x-x'| \leq 1}} \frac{|D^k f(x) - D^k f(x')|}{|x-x'|^\alpha}$$

(where for $k = (k_1, \ldots, k_d) \in \mathbb{N}^d$, $\overline{k} = \sum_{i=1}^d k_i$ and $D^k f = \frac{\partial^{\overline{k}} f}{\partial x_1^{k_1} \ldots \partial x_d^{k_d}}$)

For any integer j, $H^{\frac{j+\alpha}{2}, j+\alpha}$ is the space of real functions f on $[0,T] \times \mathbb{R}^d$ which are continuous together with their derivatives $D_t^r D_x^k f = \frac{\partial^{r+\overline{k}} f}{\partial_t^r \partial_{x_1}^{k_1} \ldots \partial_{x_d}^{k_d}}$ for $2r + \overline{k} \leq j$ and admit a finite norm

$$\begin{aligned} \|f\|_{\frac{j+\alpha}{2}, j+\alpha} = & \sum_{2r+\overline{k} \leq j} \sup_{[0,T] \times \mathbb{R}^d} |D_t^r D_x^k f| + \sum_{j-1 \leq 2r+\overline{k} \leq j} \sup_{\substack{x \in \mathbb{R}^d \\ t,t' \in [0,T] \\ |t-t'| \leq 1}} \frac{|D_t^r D_x^k f(t,x) - D_t^r D_x^k f(t',x)|}{|t-t'|^{\frac{j-2r-\overline{k}+\alpha}{2}}} \\ & + \sum_{2r+\overline{k}=j} \sup_{\substack{t \in [0,T] \\ x,x' \in \mathbb{R}^d \\ |x-x'| \leq 1}} \frac{|D_t^r D_x^k f(t,x) - D_t^r D_x^k f(t,x')|}{|x-x'|^\alpha} \end{aligned}$$

Weighted Sobolev spaces

For every integer j, $\beta \in \mathbb{R}_+$, let us consider the space of all real functions g defined on $\mathbb{R}$ with derivatives up to order j such that

$$\|g\|_{j,\beta} = \left(\sum_{k \leq j} \int_{\mathbb{R}} \frac{|g^{(k)}(x)|^2}{1+|x|^{2\beta}} dx \right)^{1/2} < +\infty$$

where $g^{(k)}$ denotes the kth derivative of g. Let $W_0^{j,\beta}$ be the closure of the set of functions of class C^∞ with compact support for this norm. $W_0^{j,\beta}$ is a separable Hilbert space with norm $\|\cdot\|_{j,\beta}$. We will denote by $W_0^{-j,\beta}$ its dual space.

Let $C^{j,\beta}$ be the space of functions g with continuous derivatives up to order j and such that $\lim_{|x| \to \infty} \frac{|g^{(k)}(x)|}{1+|x|^\beta} = 0, \ \forall k \leq j$. This space is normed with

$$\|g\|_{C^{j,\beta}} = \sum_{k \leq j} \sup_{x \in \mathbb{R}} \frac{|g^{(k)}(x)|}{1+|x|^\beta}$$

and $C^{j,0}$ is denoted by C_b^j. Let $C^{-j,\beta}$ be the dual space of $C^{j,\beta}$ and for $\beta=0$, C^{-j} is the dual space of C_b^j.
We have the following embeddings (See Adams [1], in particular the proofs of Theorem 5-4 and Theorem 6-53 can be adapted without difficulty for weighted Sobolev spaces):

$$\begin{aligned} W_0^{m+j,\beta} &\hookrightarrow C^{j,\beta} \text{ for } \quad m\geq 1, j\geq 0 \text{ and } \ \beta\geq 0, \text{ and } \|g\|_{C^{j,\beta}}\leq K\|g\|_{m+j,\beta} \\ C_b^j &\hookrightarrow W_0^{j,\beta}, \ \text{ for } \beta>1/2, j\geq 0, \text{ and } \|g\|_{j,\beta}\leq K\|g\|_{C_b^j}. \end{aligned} \tag{0.3}$$

We have also

$$W_0^{m+j,\beta}\hookrightarrow_{H.S.} W_0^{j,\beta+\gamma} \quad m\geq 1,\ j\geq 0,\ \beta\geq 0,\ \gamma>\frac{1}{2},$$

where $H.S.$ means that the embedding is of Hilbert-Schmidt type, and

$$\|g\|_{j,\beta+\gamma}\leq K\|g\|_{m+j,\beta}. \tag{0.4}$$

We deduce the following dual embeddings:

$$\begin{aligned} C^{-j,\beta} &\hookrightarrow W_0^{-(m+j),\beta}, \quad m\geq 1,\ j\geq 0,\ \beta\geq 0, \\ W_0^{-j,\beta} &\hookrightarrow C^{-j}, \quad \beta>1/2,\ j\geq 0, \\ W_0^{-j,\beta+\gamma} &\hookrightarrow_{H.S.} W_0^{-(m+j),\beta}, \quad m\geq 1,\ j\geq 0,\ \beta\geq 0,\ \gamma>\frac{1}{2}. \end{aligned}$$

The following lemma, proved in [2], gives estimates of the norm of some elementary linear operators in a well-chosen weighted Sobolev space.

Lemma 0.1 *For every fixed $x,y\in\mathbb{R}^d$ the linear mappings $D_{xy},\ D_x,\ H_x : W_0^{2,2}\to\mathbb{R}$ defined by $D_{xy}(\varphi)=\varphi(x)-\varphi(y)$; $D_x(\varphi)=\varphi(x)$; $H_x(\varphi)=\varphi'(x)$ are continuous and*

$$\|D_{xy}\|_{-2,2} \leq K_1|x-y|(1+|x|^2+|y|^2) \tag{0.5}$$
$$\|D_x\|_{-2,2} \leq K_2(1+|x|^2) \tag{0.6}$$
$$\|H_x\|_{-2,2} \leq K_3(1+|x|^2) \tag{0.7}$$

Hypotheses

If E is a Borel set, let $\mathcal{P}(E)$ denote the set of probability measures on E.
Let $\Omega=C([0,T],\mathbb{R}^d)$ endowed with the topology of uniform convergence, X be the canonical process on Ω. If $P\in\mathcal{P}(\Omega)$, $(P_t)_{t\in[0,T]}$ is the set of time marginals of P.

$\tilde{\mathcal{P}}(\Omega)=\{P\in\mathcal{P}(\Omega);\ \forall t\in[0,T],\ P_t$ is absolutely continuous with respect to Lebesgue measure$\}$

If $P\in\tilde{\mathcal{P}}(\Omega)$, there is a measurable function $p(s,x)$ on $[0,T]\times\mathbb{R}^d$ such that for any $s\in[0,T]$, $p(s,.)$ is a density of P_s with respect to Lebesgue measure. See for example Meyer [10] pages 193-194. Such a function is called a measurable version of the densities.

In all the following, we assume that σ is a Lipschitz continuous mapping on $\mathbb{R}$ with values in the space of symmetric non-negative $d\times d$ matrices such that :

$$\exists m_\sigma>0,\ \forall x\in\mathbb{R}^d, \forall y\in\mathbb{R},\ x^*\sigma(y)x\geq m_\sigma|x|^2 \tag{0.8}$$

and that b is a Lipschitz continuous $\mathbb{R}^d$-valued mapping on $\mathbb{R}$. The matrix $\sigma\sigma^*$ is denoted by a. Let V be a Lipschitz continuous (constant K_v) and bounded (constant M_v) probability density on $\mathbb{R}^d$ such that $\int_{\mathbb{R}^d} |x|^3 V(x)dx < +\infty$ and $\int_{\mathbb{R}^d} xV(x)dx = 0$.

Let f_0 be a probability density on $\mathbb{R}^d$, B_t and ζ be a d-dimensional Brownian motion and a random variable on $\mathbb{R}^d$ independent of the Brownian motion with law $f_0(x)dx$.

For any integer $j \geq 2$, $[\mathbf{Hyp_j}]$ denotes the following hypothesis : σ is C^{j+1} (continuously differentiable up to order $j+1$), b is C^j and f_0 belongs to $H^{j+\alpha}$.

1 The nonlinear stochastic differential equation (0.1)

1.1 A linear stochastic differential equation

Let $q \in H^{1+\frac{\alpha}{2},2+\alpha}$. With q, we associate the second order operator

$$L_q = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(q(s,y))\frac{\partial^2 .}{\partial y_i \partial y_j} + \sum_{i=1}^{d} b_i(q(s,y))\frac{\partial .}{\partial y_i} \tag{1.1}$$

The adjoint of this operator is

$$L_q^* = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(q(t,x))\frac{\partial^2 .}{\partial x_i \partial x_j} + \sum_{i=1}^{d} B_i(t,x)\frac{\partial .}{\partial x_i}(t,x) + C(t,x).$$

where

$$\begin{cases} B_i(t,x) = \sum_{j=1}^d a'_{ij}(q(t,x))\frac{\partial q}{\partial x_j}(t,x) - b_i(q(t,x)) \\ C(t,x) = \frac{1}{2}\sum_{i,j=1}^d \left(a''_{ij}(q(t,x))\frac{\partial q}{\partial x_i}\frac{\partial q}{\partial x_j} + a'_{ij}(q(t,x))\frac{\partial^2 q}{\partial x_i \partial x_j}(t,x)\right) - \sum_{i=1}^d b'_i(q(t,x))\frac{\partial q}{\partial x_i}(t,x) \end{cases}$$

Proposition 1.1 *If* $[\mathbf{Hyp_2}]$ *holds, the law of the unique strong solution of the stochastic differential equation*

$$X_t = \zeta + \int_0^t \sigma(q(s,X_s)).dB_s + \int_0^t b(q(s,X_s))ds \tag{1.2}$$

belongs to $\tilde{\mathcal{P}}(\Omega)$ *and admits a measurable version of the densities* $p \in H^{1+\frac{\alpha}{2},2+\alpha}$ *which is the unique solution of the partial differential equation*

$$\frac{\partial p}{\partial t} = L_q^* p \;\; on \;\; [0,T]\times\mathbb{R}^d \quad and \quad p(0,x) = f_0(x) \tag{1.3}$$

in $C_b^{1,2}$. *Moreover,*

$$\|p\|_{1+\frac{\alpha}{2},2+\alpha} \leq F_2(T,\sigma,b,\|q\|_{1+\frac{\alpha}{2},2+\alpha})\|f_0\|_{2+\alpha} \tag{1.4}$$

with F_2 nondecreasing in its last variable.
If $[\mathbf{hyp_j}]$ *holds for some* $j>2$ *and* $q \in H^{\frac{j+\alpha}{2},j+\alpha}$, *then* $p \in H^{\frac{j+\alpha}{2},j+\alpha}$ *and*

$$\|p\|_{\frac{j+\alpha}{2},j+\alpha} \leq F_j(T,\sigma,b,\|q\|_{\frac{j+\alpha}{2},j+\alpha})\|f_0\|_{j+\alpha} \tag{1.5}$$

with F_j nondecreasing in its last variable.

Proof : The proof consists in bringing together results of Friedman [3] and Ladyzenskaya Solonnikov and Ural'ceva [6]. It would be possible to obtain that the law of X belongs to $\tilde{\mathcal{P}}(\Omega)$ by the Malliavin calculus (see for instance Nualart [12] Theorem 2.3.1 p.110). But for the sake of consistency, we do not insist on this approach.
We first suppose the $[\mathbf{Hyp_2}]$ holds. The operator L_q^* is uniformly parabolic and its coefficients belong to $H^{\frac{\alpha}{2},\alpha}$. By Friedman [3] Chap.6, there exists a fundamental solution $\Gamma_q^*(x,t,y,s)$, $0 \leq s < t \leq T$ of $L_q^* - \frac{\partial}{\partial t}$ and for any $t \in [0,T]$, the law of X_t has a density with respect to Lebesgue measure given by $p(t,x) = \int_{\mathbb{R}^d} \Gamma_q^*(x,t,y,0) f_0(y)dy$.
In [6] Chap.IV, Ladyzenskaya, Solonnikov and Ural'ceva deal with uniformly parabolic operators of the second order with coefficients in $H^{\frac{\alpha}{2},\alpha}$. We apply their results to L_q^*. As f_0 belongs to $H^{2+\alpha}$, by equations (14.3) p.389 and (14.5) p.390 we conclude that p belongs to $H^{1+\frac{\alpha}{2},2+\alpha}$ and solves (1.3). Inequality (5.9) p.320 then implies that $\|p\|_{1+\frac{\alpha}{2},2+\alpha} \leq C\|f_0\|_{2+\alpha}$. The proof of (5.9) shows that the constant C depends only on T, on m_σ and on the norm of the coefficients of L_q^* in $H^{\frac{\alpha}{2},\alpha}$ and increases with this norm. Hence (1.4) holds.
Uniqueness for equation (1.3) in $C_b^{1,2}$ is an easy consequence of the maximum principle.

If, for $j>2$, $[\mathbf{hyp_j}]$ holds and $q \in H^{\frac{j+\alpha}{2},j+\alpha}$, then the coefficients of L_q^* belong to $H^{\frac{j-2+\alpha}{2},j-2+\alpha}$ and $f_0 \in H^{j+\alpha}$. By Theorem 5.1 p.320 [6], (1.3) admits a solution in $H^{\frac{j+\alpha}{2},j+\alpha} \subset C_b^{1,2}$. As uniqueness holds for (1.3) in $C_b^{1,2}$, we deduce that this solution is equal to p. Hence $p \in H^{\frac{j+\alpha}{2},j+\alpha}$. Inequality (1.5) is like (1.4) a consequence of equation (5.9) p.320. ∎

1.2 Existence and Uniqueness for the nonlinear stochastic differential equation (0.1)

This section is dedicated to the nonlinear stochastic differential equation (0.1) :

$$\begin{cases} \bar{X}_t = \zeta + \int_0^t \sigma(p(s,\bar{X}_s)).dB_s + \int_0^t b(p(s,\bar{X}_s))ds \\ p \in C_b^{1,2}([0,T]\times\mathbb{R}^d) \ \text{ is a measurable version of the densities for the law of } \bar{X} \end{cases}$$

Let us assume that $[\mathbf{Hyp_2}]$ holds. We are going to prove existence of a unique strong solution $(\bar{X},p)$ for this equation under a new hypothesis on σ.
If $(\bar{X},p)$ is a solution of (0.1), applying Itô's formula and taking expectations, we obtain that p is a weak solution of the quasilinear partial differential equation :

$$\frac{\partial p}{\partial t} = L_p^* p \ \text{ on } \ [0,T]\times\mathbb{R}^d \quad \text{and} \quad p(0,x) = f_0(x) \tag{1.6}$$

As $p \in C_b^{1,2}([0,T] \times \mathbb{R}^d)$, it is in fact a classical solution. Our existence and uniqueness result for (0.1) is based on results concerning (1.6) given by Ladyzenskaya, Solonnikov and Ural'ceva in [6]. As these authors deal with equations in divergence form, we put (1.6) in divergence form and obtain :

$$\frac{\partial p}{\partial t} = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \left(\frac{1}{2} \sum_{j=1}^{d} (a'_{ij}(p)p + a_{ij}(p)) \frac{\partial p}{\partial x_j} - b_i(p)p \right) \text{ on } [0,T] \times \mathbb{R}^d \quad \text{and} \quad p(0,x) = f_0(x) \tag{1.7}$$

Like in [6] p.494, it is possible to express the difference of two classical solutions of (1.7) as the solution of a linear Cauchy problem (with coefficients depending on both the solutions). If we assume that the leading matrix $a'_{ij}(p)p + a_{ij}(p)$ is nonnegative i.e.

$$\forall x \in \mathbb{R}^d,\ \forall y \in \mathbb{R},\ x^*(a'(y)y + a(y))x \geq 0, \tag{1.8}$$

then the maximum principle (Theorem 2.5 p.18 [6]) implies that the difference is equal to zero and that uniqueness holds for (1.7). We deduce uniqueness for (0.1):

Proposition 1.2 *Under the assumptions* $[\mathbf{Hyp_2}]$ *and (1.8), the nonlinear stochastic differential equation (0.1) has no more than one solution.*

Proof : We suppose that $(\bar{X}^p, p)$ and $(\bar{X}^q, q)$ are two solutions of (0.1). Applying Itô's formula and taking expectations, we obtain that p and q solve the nonlinear equation (1.6) in the sense of distributions. As p and q belong to $C_b^{1,2}([0,T], \mathbb{R}^d)$, these functions are in fact classical solutions. Since the equations (1.6) and (1.7) are equivalent as far as they are considered in the classical sense, p and q solve (1.7). By the uniqueness result for this equation, we deduce that $p = q$. It follows immediately that $\bar{X}^p = \bar{X}^q$. ∎

Under a stronger assumption on the leading matrix

$$\exists \mu_a > 0,\ \forall x \in \mathbb{R}^d,\ \forall y \in \mathbb{R},\ x^*(a'(y)y + a(y))x \geq \mu_a |x|^2, \tag{1.9}$$

applying Theorem 8.1 p.495 [6] to our particular framework, we obtain existence in $H^{1+\frac{\alpha}{2}, 2+\alpha}$ for the Cauchy problem (1.7). We are now ready to state the main result of the section.

Proposition 1.3 *Under the assumptions* $[\mathbf{Hyp_2}]$ *and (1.9), the nonlinear stochastic differential equation (0.1) admits a unique strong solution* $(\bar{X}, p)$

Proof : Uniqueness is a consequence of the previous proposition. To prove existence, we remark that the solution q of (1.7) solves (1.6). According to Proposition 1.1, the law of the unique strong solution of the linear stochastic differential equation

$$X_t = \zeta + \int_0^t \sigma(q(s, X_s)).dB_s + \int_0^t b(q(s, X_s))ds$$

belongs to $\tilde{\mathcal{P}}(\Omega)$ and admits the unique solution of the partial differential equation

$$\frac{\partial p}{\partial t} = L_q^* p \ \text{ on } \ [0,T] \times \mathbb{R}^d \quad \text{and} \quad p(0,x) = f_0(x)$$

in $C_b^{1,2}([0,T] \times \mathbb{R}^d)$ as a measurable version for its densities. As q solves this equation, q is a measurable version of the densities for the law of X. Hence the couple (X,q) solves (0.1). ∎

2 The propagation of chaos result

For $j \geq 2$, let $[\mathbf{Hyp'_j}]$ mean that $[\mathbf{Hyp_j}]$ and $\|f_0\|_{j+\alpha} \leq 1/F_j(T,\sigma,b,1)$ hold. (F_2 is defined in (1.4) and for $j > 2$, F_j is defined in (1.5)).

Remark 2.1 *There exists probability densities on $\mathbb{R}^d$ belonging to $H^{j+\alpha}(\mathbb{R}^d)$ with an arbitrary small norm in this space. Indeed $\|\frac{1}{k^d} f_0(\frac{\cdot}{k})\|_{j+\alpha} \leq \frac{1}{k^d}\|f_0\|_{j+\alpha}$.*

2.1 A McKean-Vlasov model

In this section, we deal with a mollified version of the nonlinear stochastic differential equation (0.1) :

$$\begin{cases} \bar{Z}_t = \zeta + \int_0^t \sigma(W * P_s(\bar{Z}_s)).dB_s + \int_0^t b(W * P_s(\bar{Z}_s))ds \\ P \ \text{ is the law of } \bar{Z} \end{cases} \tag{2.1}$$

were W is a probability density on $\mathbb{R}^d$ bounded by M_w and Lipschitz continuous with constant K_w. Although the coefficients are not linear in the measure, this equation can be treated like in the classical McKean-Vlasov framework (McKean [8], Sznitman [14] or Léonard [7]).

Proposition 2.2 *There is existence and uniqueness, trajectorial and in law for (2.1). Moreover, if for some $j \geq 2$, $[\mathbf{Hyp'_j}]$ holds, then the law P of the solution $\bar{Z}$ belongs to $\tilde{\mathcal{P}}(\Omega)$ and admits a function $p \in H^{\frac{j+\alpha}{2},j+\alpha}$ with $\|p\|_{\frac{j+\alpha}{2},j+\alpha} \leq 1$ as a measurable version for its densities. The function p is a solution of the Cauchy problem*

$$\frac{\partial p}{\partial t} = L_{W*p}^* p \ \text{ on } \ [0,T] \times \mathbb{R}^d \quad \text{and} \quad p(0,x) = f_0(x) \tag{2.2}$$

Proof of Proposition 2.2 : The proof for existence and uniqueness is just a generalization of the one given by Sznitman [14] Theorem 1.1 p.172 and is based on a fixed point theorem for the mapping $\psi : \mathcal{P}(\Omega) \to \mathcal{P}(\Omega)$ which associates with m the law of the unique strong solution of the stochastic differential equation

$$Z_t^m = \zeta + \int_0^t \sigma(W * m_s(Z_s^m)).dB_s + \int_0^t b(W * m_s(Z_s^m))ds$$

and the topology of weak convergence on $\mathcal{P}(\Omega)$ which is metrisable for the Kantorovitch-Rubinstein or Vaserstein metric. The fixed-point of ψ is denoted by P.
Let us suppose that [$\mathbf{Hyp'_j}$] holds for some $j \geq 2$. To obtain the regularity properties of P, we study a sequence of fixed-point iterations $(\psi^n(m))_n$ where m is a probability measure in $\tilde{\mathcal{P}}(\Omega)$ with time-independent densities $p^0(s,x) = h(x)$ such that $\|h\|_{j+\alpha} \leq 1$. Clearly, the mapping $\phi : H^{\frac{j+\alpha}{2}, j+\alpha} \to H^{\frac{j+\alpha}{2}, j+\alpha}$ which associates with g the function $\phi(g)(t,x) = W * g(t,.)(x)$ is nonexpansive. Hence $\|\phi(p^0)\|_{\frac{j+\alpha}{2}, j+\alpha} \leq 1$. As $\psi(m)$ is the law of the solution of the linear stochastic differential equation (1.2) for the particular choice $q = \phi(p^0)$, by Proposition 1.1, we conclude that $\psi(m)$ belongs to $\tilde{\mathcal{P}}(\Omega)$ and admits a measurable version of the densities $p^1 \in H^{\frac{j+\alpha}{2}, j+\alpha}([0,T] \times \mathbb{R}^d)$ with $\|p^1\|_{\frac{j+\alpha}{2}, j+\alpha} \leq 1$.
By induction, for any $n \in \mathbb{N}$, $\psi^n(m)$ belongs to $\tilde{\mathcal{P}}(\Omega)$ and admits a measurable version of the densities $p^n \in H^{\frac{j+\alpha}{2}, j+\alpha}$ with $\|p^n\|_{\frac{j+\alpha}{2}, j+\alpha} \leq 1$.
Combining Ascoli's theorem and a diagonal extraction process, we obtain a subsequence $(p^{n'})_{n'}$ such that $p^{n'}$ converges uniformly on compact sets together with its derivatives to a function p and its derivatives. Clearly, $p \in H^{\frac{j+\alpha}{2}, j+\alpha}$ and $\|p\|_{\frac{j+\alpha}{2}, j+\alpha} \leq 1$. As $\psi^{n'}(m)$ converges weakly to P, p is a measurable version of the densities for P.
Applying Itô's formula and taking expectations, we obtain that p is a weak solution of (2.2). As $p \in H^{\frac{j+\alpha}{2}, j+\alpha}$, this function is actually a classical solution of (2.2). ∎

Like in the classical McKean-Vlasov framework, it is possible to construct a sequence of weakly interacting particle systems that approximate the solution of (2.1). Let B^i, $i \in \mathbb{N}^*$ be a sequence of independent $\mathbb{R}^d$-valued Brownian motions and ζ^i, $i \in \mathbb{N}^*$ be a sequence of random variables IID with law $f_0(x)dx$ independent of the Brownian motions. The particle system of order n is the unique strong solution of

$$Z_t^{i,n} = \zeta^i + \int_0^t \sigma\left(\frac{1}{n}\sum_{j=1}^n W(Z_s^{i,n} - Z_s^{j,n})\right).dB_s^i + \int_0^t b\left(\frac{1}{n}\sum_{j=1}^n W(Z_s^{i,n} - Z_s^{j,n})\right)ds, \ 1 \leq i \leq n$$

On the same probability space we define $\bar{Z}^i$ to be the solution of the nonlinear equation

$$\begin{cases} \bar{Z}_t^i = \zeta^i + \int_0^t \sigma(W * P_s(\bar{Z}_s^i)).dB_s^i + \int_0^t b(W * P_s(\bar{Z}_s^i))ds \\ P \text{ is the law of } \bar{Z}^i \end{cases}$$

given by Proposition 2.2.

Proposition 2.3 *For any $i \in \mathbb{N}^*$, for any $n \geq i$,*

$$\mathbb{E}\left(\sup_{t\leq T}|Z_t^{i,n} - \bar{Z}_t^i|^2\right) \leq \frac{CM_w^2}{nK_w^2}\exp(CK_w^2); \qquad \mathbb{E}\left(\sup_{t\leq T}|Z_t^{i,n} - \bar{Z}_t^i|^4\right) \leq \frac{CM_w^4}{n^2K_w^4}\exp(CK_w^4) \quad (2.3)$$

where C is a real constant independent of W.

Remark 2.4 *These bounds obviously imply propagation of chaos : for any $k \in \mathbb{N}^*$, the law of the susbsystem $(Z^{1,n}, \ldots, Z^{k,n})$ converges weakly to $P^{\otimes k}$ where P is the law of the solution of (2.1).*

Proof of Proposition 2.3 : Our proof is an easy adaptation of the one given by Sznitman [14] Theorem 1.4 p.174 but as we need to precise the dependence on W, we present the calculations. In the following, K and K' are real constants which may change from line to line. Using Burkholder inequality, we get that for any $t \leq T$,

$$\mathbb{E}\Big(\sup_{s\leq t}|Z^{i,n}_s - \bar{Z}^i_s|^2\Big) \leq K\mathbb{E}\Big(\int_0^t \Big(\frac{1}{n}\sum_{j=1}^n W(Z^{i,n}_r - Z^{j,n}_r) - W(\bar{Z}^i_r - Z^{j,n}_r)\Big)^2 dr$$
$$+ \int_0^t \Big(\frac{1}{n}\sum_{j=1}^n W(\bar{Z}^i_r - Z^{j,n}_r) - W(\bar{Z}^i_r - \bar{Z}^j_r)\Big)^2 dr$$
$$+ \int_0^t \Big(\frac{1}{n}\sum_{j=1}^n W(\bar{Z}^i_r - \bar{Z}^j_r) - W * P_r(\bar{Z}^i_r)\Big)^2 dr\Big)$$

By exchangeability of the couples $(Z^{i,n}, \bar{Z}^i)$, $1 \leq i \leq n$, we get

$$\mathbb{E}\Big(\sup_{s\leq t}|Z^{i,n}_s - \bar{Z}^i_s|^2\Big) \leq KK_w^2\int_0^t \mathbb{E}\Big(|Z^{i,n}_r - \bar{Z}^i_r|^2\Big)dr$$
$$+ K'\int_0^t \sum_{j,k=1}^n \mathbb{E}\Big(\Big(W(\bar{Z}^i_r - \bar{Z}^j_r) - W * P_r(\bar{Z}^i_r)\Big)\Big(W(\bar{Z}^i_r - \bar{Z}^k_r) - W * P_r(\bar{Z}^i_r)\Big)\Big)dr$$

When $j \neq k$, either $j \neq i$ or $k \neq i$. Suppose that $j \neq i$. As the law of $\bar{Z}^j_r$ is P_r and this variable is independent of the couple $(\bar{Z}^i_r, \bar{Z}^k_r)$,

$$\mathbb{E}\Big(\Big(W(\bar{Z}^i_r - \bar{Z}^j_r) - W * P_r(\bar{Z}^i_r)\Big)\Big(W(\bar{Z}^i_r - \bar{Z}^k_r) - W * P_r(\bar{Z}^i_r)\Big)\Big) =$$
$$\mathbb{E}\Big(\mathbb{E}\Big(W(\bar{Z}^i_r - \bar{Z}^j_r) - W * P_r(\bar{Z}^i_r)\Big|\bar{Z}^i_r, \bar{Z}^k_r\Big)\Big(W(\bar{Z}^i_r - \bar{Z}^k_r) - W * P_r(\bar{Z}^i_r)\Big)\Big) = 0$$

Hence

$$\mathbb{E}\Big(\sup_{s\leq t}|Z^{i,n}_s - \bar{Z}^i_s|^2\Big) \leq KK_w^2\int_0^t \mathbb{E}\Big(|Z^{i,n}_r - \bar{Z}^i_r|^2\Big)dr + \frac{K'M_w^2 t}{n}$$

If $\phi(t) = \mathbb{E}\Big(\sup_{s\leq t}|Z^{i,n}_s - \bar{Z}^i_s|^2\Big) + \frac{K'M_w^2}{nKK_w^2}$, we have

$$\forall t \leq T,\ \phi(t) \leq \frac{K'M_w^2}{nKK_w^2} + KK_w^2\int_0^t \phi(r)dr$$

By Gronwall's lemma, we conclude

$$\phi(t) \leq \frac{K'M_w^2}{nKK_w^2}\exp(KK_w^2 T)$$

The second inequality in (2.3) is obtained by similar calculations. ∎

2.2 Approximation of the nonlinear stochastic differential equation (0.1) for regular initial data

In this section, we suppose that $[\mathbf{Hyp'_j}]$ holds for some $j \geq 2$. We need this restrictive assumption which implies compactness (as seen in the proof of Proposition 2.2) to prove the propagation of chaos result. But it also enables us to obtain a new existence result for (0.1) without hypothesis (1.9).

Let $(\epsilon_n)_n$ be a sequence of positive numbers converging to 0. We set $V^n(.) = \frac{1}{\epsilon_n^d} V(\frac{\cdot}{\epsilon_n})$. By Proposition 2.2, there is existence and uniqueness for the nonlinear stochastic differential equations

$$\begin{cases} \bar{Y}^n_t = \zeta + \int_0^t \sigma(V^n * P^n_s(\bar{Y}^n_s)).dB_s + \int_0^t b(V^n * P^n_s(\bar{Y}^n_s))ds \\ P^n \ \text{ is the law of } \bar{Y}^n \end{cases} \tag{2.4}$$

and $\forall n$, P^n admits a measurable version of the densities p^n in $H^{\frac{j+\alpha}{2}, j+\alpha}$ with $\|p^n\|_{\frac{j+\alpha}{2}, j+\alpha} \leq 1$. We set $q^n(t,x) = V^n * p^n(t,.)(x)$.

Proposition 2.5 *Under $[\mathbf{Hyp'_j}]$ for some $j \geq 2$, there is existence for the nonlinear stochastic differential equation (0.1). When (1.8) also holds, the solution is unique and if it is denoted by $\bar{X}$,*

$$\mathbb{E}\left(\sup_{t \leq T} |\bar{Y}^n_t - \bar{X}_t|^4 \right) \leq K \epsilon_n^{4\beta} \quad \textit{with } \ \beta = \alpha, 1, 2 \ \textit{ respectively for } \ j = 2, 3, > 3 \tag{2.5}$$

where K is a real constant independent of n.

The proof of the proposition is based on the following lemma which states existence for the Cauchy problem (1.6) under $[\mathbf{Hyp'_j}]$ and compares the solution with p^n under the additional assumption (1.8).

Lemma 2.6 *If $[\mathbf{Hyp'_j}]$ holds for some $j \geq 2$, then the Cauchy problem (1.6) admits a solution $p \in H^{\frac{j+\alpha}{2}, j+\alpha}$ with $\|p\|_{\frac{j+\alpha}{2}, j+\alpha} \leq 1$. If moreover (1.8) holds, then*

$$\begin{gathered} \sup_{[0,T]\times\mathbb{R}^d} |p - p^n| \leq C\epsilon_n^\beta; \qquad \sup_{[0,T]\times\mathbb{R}^d} |p - q^n| \leq C\epsilon_n^\beta \\ \textit{where } \ \beta = \alpha, 1, 2 \ \textit{ respectively for } \ j = 2, 3, > 3 \end{gathered} \tag{2.6}$$

Proof of Lemma 2.6 : First, under different asumptions on $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$, we upper-bound the rate of convergence of $f_k(t,x) = V^k * f(t,.)(x)$ to $f(t,x)$.

If $\|f\|_{1+\frac{\alpha}{2},2+\alpha} \leq 1$, as $\int_{\mathbb{R}^d} yV(y)dy = 0$,

$$\begin{aligned}
|f_k(t,x) - f(t,x)| &= \left| \int_{\mathbb{R}^d} (f(t,x-\epsilon_k y) - f(t,x))V(y)dy \right| \\
&= \left| \int_{\mathbb{R}^d} \Big(-\epsilon_k y.\nabla_x f(t,x) + \epsilon_k^2 \sum_{i,j=1}^d y_i y_j \int_0^1 (1-\theta) \frac{\partial^2 f}{\partial x_i \partial x_j}(t, x-\theta\epsilon_k y) d\theta \Big) V(y)dy \right| \\
&\leq C(V)\epsilon_k^2
\end{aligned}$$

If $\|f\|_{\frac{1+\alpha}{2},1+\alpha} \leq 1$, then

$$\begin{aligned}
|f_k(t,x) - f(t,x)| &= \int_{\mathbb{R}^d} |f(t,x-\epsilon_k y) - f(t,x)|V(y)dy \leq \sup_{[0,T]\times\mathbb{R}^d} |\nabla_x f(t,x)| \int_{\mathbb{R}^d} \epsilon_k |y| V(y)dy \\
&\leq C(V)\epsilon_k
\end{aligned}$$

If $\|f\|_{\frac{\alpha}{2},\alpha} \leq 1$, then

$$\begin{aligned}
|f_k(t,x) - f(t,x)| &= \int_{\mathbb{R}^d} |f(t,x-\epsilon_k y) - f(t,x)|V(y)dy \\
&\leq \|f\|_{\frac{\alpha}{2},\alpha} \int_{|\epsilon_k y|\leq 1} \epsilon_k^\alpha |y|^\alpha V(y)dy + 2 \sup_{[0,T]\times\mathbb{R}^d} |f| \int_{|\epsilon_k y|>1} \epsilon_k^\alpha |y|^\alpha V(y)dy \\
&\leq C(V)\epsilon_k^\alpha
\end{aligned}$$

As $\sup_n \|p^n\|_{\frac{j+\alpha}{2},j+\alpha} \leq 1$, we deduce

$$\begin{aligned}
&\forall n, \quad \sup_{[0,T]\times\mathbb{R}^d} |p^n - q^n| \leq C\epsilon_n^2 \\
&\quad \forall i, \quad \sup_{[0,T]\times\mathbb{R}^d} \left| \frac{\partial p^n}{\partial x_i} - \frac{\partial q^n}{\partial x_i} \right| \leq C\epsilon_n^\gamma \quad \text{with } \gamma = 1 \text{ or } \gamma = 2 \text{ resp. for } j = 2 \text{ or } j > 2 \\
&\quad \forall i,j \quad \sup_{[0,T]\times\mathbb{R}^d} \left| \frac{\partial^2 p^n}{\partial x_i \partial x_j} - \frac{\partial^2 q^n}{\partial x_i \partial x_j} \right| \leq C\epsilon_n^\beta \quad \text{with } \beta = \alpha, 1, 2 \text{ resp. for } j = 2, 3, > 3
\end{aligned} \tag{2.7}$$

Combining Ascoli's theorem and a diagonal extraction process it is possible to obtain from $(p^n)_n$ a subsequence $(p^k)_k$ such that p^k converges uniformly on compact sets together with its derivatives (the first order time derivative and the first and second order space derivatives) to a function p and its derivatives. The norm of this function in $H^{\frac{j+\alpha}{2},j+\alpha}$ is smaller than 1. By (2.7), we deduce that q^k and its first and second order space derivatives converge to p and its derivatives uniformly on compact sets. As by (2.2), p^k solves the Cauchy problem

$$\frac{\partial p^k}{\partial t} = L^*_{q^k} p^k \quad \text{on} \quad [0,T]\times\mathbb{R}^d \quad \text{and} \quad p^k(0,x) = f_0(x),$$

taking the limit $k \to +\infty$ we obtain that p solves (1.6).

To prove (2.6) we are going to express the difference $p - p^n$ as the solution of a linear partial differential equation (with coefficients depending on p, p^n and q^n).

$$\begin{aligned}
\frac{\partial(p-p^n)}{\partial t} &= \frac{1}{2} \sum_{i,j=1}^d (a_{ij}(p) + a'_{ij}(p)p) \frac{\partial^2(p-p^n)}{\partial x_i \partial x_j} + \sum_{i=1}^d \Big(\sum_{j=1}^d a'_{ij}(p) \frac{\partial p}{\partial x_j} - b_i(p) \Big) \frac{\partial(p-p^n)}{\partial x_i} \\
&\quad + \Big(\frac{1}{2} \sum_{i,j=1}^d a''_{ij}(p) \frac{\partial p}{\partial x_i} \frac{\partial p}{\partial x_j} - \sum_{i=1}^d b'_i(p) \frac{\partial p}{\partial x_i} \Big)(p - p^n)
\end{aligned}$$

$$+\frac{1}{2}\sum_{i,j=1}^{d}(a_{ij}(p)-a_{ij}(q^n))\frac{\partial^2 p^n}{\partial x_i\partial x_j}+\frac{1}{2}\sum_{i,j=1}^{d}\left(a'_{ij}(p)p\frac{\partial^2 p^n}{\partial x_i\partial x_j}-a'_{ij}(q^n)p^n\frac{\partial^2 q^n}{\partial x_i\partial x_j}\right)$$
$$+\sum_{i=1}^{d}\left(\sum_{j=1}^{d}\left(a'_{ij}(p)\frac{\partial p}{\partial x_j}-a'_{ij}(q^n)\frac{\partial q^n}{\partial x_j}\right)-(b_i(p)-b_i(q^n))\right)\frac{\partial p^n}{\partial x_i}$$
$$+\left(\frac{1}{2}\sum_{i,j=1}^{d}\left(a''_{ij}(p)\frac{\partial p}{\partial x_i}\frac{\partial p}{\partial x_j}-a''_{ij}(q^n)\frac{\partial q^n}{\partial x_i}\frac{\partial q^n}{\partial x_j}\right)-\sum_{i=1}^{d}\left(b'_i(p)\frac{\partial p}{\partial x_i}-b'_i(q^n)\frac{\partial q^n}{\partial x_i}\right)\right)p^n \tag{2.8}$$

Let us modify the four last terms of the right-hand-side in such a way that the differences $(p-p^n)$, $\frac{\partial(p-p^n)}{\partial x_i}$, (p^n-q^n), $\frac{\partial(p^n-q^n)}{\partial x_i}$ and $\frac{\partial^2(p^n-q^n)}{\partial x_i\partial x_j}$ appear.
For instance, we set $G(\theta)=a'_{ij}(q^n+\theta(p-q_n))\,(p^n+\theta(p-p^n))\left(\frac{\partial^2 q_n}{\partial x_i\partial x_j}+\theta\frac{\partial^2(p^n-q^n)}{\partial x_i\partial x_j}\right)$ and make the following computation for the fifth term :

$$a'_{ij}(p)p\frac{\partial^2 p^n}{\partial x_i\partial x_j}-a'_{ij}(q^n)p^n\frac{\partial^2 q^n}{\partial x_i\partial x_j}=\int_0^1 G'(\theta)d\theta$$
$$=((p-p^n)+(p^n-q^n))\int_0^1 a''_{ij}(q^n+\theta(p-q^n))(p^n+\theta(p-p^n))\left(\frac{\partial^2 q^n}{\partial x_i\partial x_j}+\theta\frac{\partial^2(p^n-q^n)}{\partial x_i\partial x_j}\right)d\theta$$
$$+(p-p^n)\int_0^1 a'_{ij}(q^n+\theta(p-q^n))\left(\frac{\partial^2 q^n}{\partial x_i\partial x_j}+\theta\frac{\partial^2(p^n-q^n)}{\partial x_i\partial x_j}\right)d\theta$$
$$+\frac{\partial^2(p^n-q^n)}{\partial x_i\partial x_j}\int_0^1 a'_{ij}(q^n+\theta(p-q^n))(p^n+\theta(p-p^n))d\theta$$

The coefficients behind (p^n-q^n), $(p-p^n)$ and $\frac{\partial^2(p^n-q^n)}{\partial x_i\partial x_j}$ in the right-hand-side are bounded on $[0,T]\times\mathbb{R}^d$ uniformly in n.
Treating the fourth, the sixth and the seventh term of the right-hand-side of (2.8) in the same way, we obtain

$$\frac{\partial(p-p^n)}{\partial t}=\frac{1}{2}\sum_{i,j=1}^{d}(a_{ij}(p)+a'_{ij}(p)p)\frac{\partial^2(p-p^n)}{\partial x_i\partial x_j}+\sum_{i=1}^{d}B_i^n\frac{\partial(p-p^n)}{\partial x_i}+C^n(p-p^n)+f^n$$

where

$$f^n=\sum_{i,j=1}^{d}\bar{A}^n_{ij}\frac{\partial^2(p^n-q^n)}{\partial x_i\partial x_j}+\sum_{j=1}^{d}\bar{B}^n_i\frac{\partial(p^n-q^n)}{\partial x_i}+\bar{C}^n(p^n-q^n)$$

and the coefficients $\bar{A}^n_{ij}$, B^n_i, $\bar{B}^n_i$, C^n and $\bar{C}^n$ are bounded on $[0,T]\times\mathbb{R}^d$ uniformly in n.
If (1.8) holds, it is possible to apply Theorem 2.5 p.18 [6], to obtain

$$\sup_{[0,T]\times\mathbb{R}^d}|p-p^n|\le C(T,\sigma,b)\sup_{[0,T]\times\mathbb{R}^d}|f^n|$$

By (2.7), $\sup_{[0,T]\times\mathbb{R}^d}|f^n|\le C(T,\sigma,b,V)\epsilon_n^\beta$ with $\beta=\alpha,1,2$ respectively for $j=2,3,>3$. Hence (2.6) holds. ∎

Proof of Proposition 2.5 : We suppose that $[\mathbf{Hyp'_j}]$ holds for some $j\ge 2$. By Lemma 2.6 the Cauchy problem (1.6) admits a solution p in $H^{\frac{j+\alpha}{2},j+\alpha}([0,T]\times\mathbb{R}^d)$. Existence of a solution

for the nonlinear equation (0.1) is deduced like in the proof of Proposition 1.3.
Now, we also assume that (1.8) holds. By Proposition 1.2, we deduce that (0.1) admits a unique solution. If this solution is denoted by $\bar{X}$, using Burkholder inequality, we get that $\mathbb{E}(\sup_{s\le t}|\bar{Y}^n_s - \bar{X}_s|^4)$ is less than

$$K\int_0^t \mathbb{E}\Big(|\sigma(q^n(s,\bar{Y}^n_s)) - \sigma(p(s,\bar{Y}^n_s))|^4 + |\sigma(p(s,\bar{Y}^n_s)) - \sigma(p(s,\bar{X}_s))|^4 + |b(q^n(s,\bar{Y}^n_s)) - b(p(s,\bar{Y}^n_s))|^4 + |b(p(s,\bar{Y}^n_s)) - b(p(s,\bar{X}_s))|^4\Big)ds$$

As σ and b are Lipschitz continuous, for any $t\le T$,

$$\mathbb{E}\Big(\sup_{s\le t}|\bar{Y}^n_s - \bar{X}_s|^4\Big) \le K\left(T\sup_{[0,T]\times\mathbb{R}^d}|q^n - p|^4 + \sup_{[0,T]\times\mathbb{R}^d}|\nabla_x p|^4\int_0^t \mathbb{E}(|\bar{Y}^n_s - \bar{X}_s|^4)ds\right)$$

By (2.6) and Gronwall's lemma, we obtain (2.5). ∎

We are going to approximate the solution of (0.1) by the moderately interacting particle systems (0.2) :

$$X^{i,n}_t = \zeta^i + \int_0^t \sigma\Big(\frac{1}{n}\sum_{j=1}^n V^n(X^{i,n}_s - X^{j,n}_s)\Big).dB^i_s + \int_0^t b\Big(\frac{1}{n}\sum_{j=1}^n V^n(X^{i,n}_s - X^{j,n}_s)\Big)ds,\ 1\le i\le n$$

We suppose that (1.8) holds and define $\bar{X}^i$ to be the solution of the nonlinear equation

$$\begin{cases}\bar{X}^i_t = \zeta^i + \int_0^t \sigma(p(s,\bar{X}^i_s)).dB^i_s + \int_0^t b(p(s,\bar{X}^i_s))ds\\ p\in C_b^{1,2}([0,T]\times\mathbb{R}^d)\ \text{ is a measurable version of the densities for the law of } \bar{X}^i\end{cases}$$

given by Proposition 2.5.

Theorem 2.7 *Assume that for some $j\ge 2$, $[\mathbf{hyp'_j}]$ and (1.8) hold. If ϵ_n converges to zero slowly enough to ensure that*

$$\lim_n \frac{\epsilon_n^2}{n}\exp\left(\frac{CK_v^2}{\epsilon_n^{2d+2}}\right) = 0$$

where the constant C is given by (2.3), then

$$\lim_{n\to+\infty}\mathbb{E}\Big(\sup_{t\le T}|X^{i,n}_t - \bar{X}^i_t|^2\Big) = 0,$$

which implies the propagation of chaos and the convergence in law of the empirical measures $\mu_n = \frac{1}{n}\sum_{i=1}^n \delta_{X^{i,n}}$ to P, the law of $\bar{X}^i$.

Proof : The probability density V^n is bounded by M_v/ϵ_n^d and admits K_v/ϵ_n^{d+1} as a Lipschitz continuity constant. Once this remark is made, it is enough to associate Proposition 2.3 and Proposition 2.5 to obtain

$$\mathbb{E}\Big(\sup_{t\le T}|X^{i,n}_t - \bar{X}^i_t|^2\Big) \le K\left(\epsilon_n^{2\beta} + \frac{\epsilon_n^2}{n}\exp\left(\frac{CK_v^2}{\epsilon_n^{2d+2}}\right)\right)\quad \text{with}\quad \beta = \alpha, 1, 2 \ \text{ resp. for }\ j = 2, 3, > 3$$

The conclusion follows obviously. ∎

Remark 2.8 *In a similar way, if we assume that* $[\mathbf{hyp'_4}]$ *and (1.8) hold and* $d=1$*, we obtain*

$$\mathbb{E}\left(\sup_{t\leq T}|X_t^{i,n}-\bar{X}_t^i|^4\right)\leq K\left(\epsilon_n^8+\frac{\epsilon_n^4}{n^2}\exp\left(\frac{CK_v^4}{\epsilon_n^8}\right)\right)$$

We want to have the best convergence rate as possible for the left-hand-side. So we choose ϵ_n *to be the unique solution of*

$$\exp\left(\frac{CK_v^4}{\epsilon_n^8}\right)=n^2\epsilon_n^4 \tag{2.9}$$

Then we obtain

$$\mathbb{E}\left(\sup_{s\leq T}|X_s^{i,n}-\bar{X}_s^i|^4\right)\leq K\epsilon_n^8 \tag{2.10}$$

3 The fluctuation result

In this part we consider the case of the dimension one (for simplicity). We assume that (1.8) and $[\mathbf{hyp'_4}]$ hold, that σ and b are bounded together with their partial derivatives up to order 4 and that $\int_{\mathbb{R}}|x|^8 f_0(x)dx<+\infty$ i.e. ζ admits an eighth order moment.
We are interested in the behaviour of the fluctuations associated with the convergence in law of the empirical measures μ^n of the system $(X^{i,n})$ to the law P of $\bar{X}^i$. We suppose that ϵ_n solves (2.9). By (2.10), it appears that the presumed rate of convergence is ϵ_n^2. Let us denote by a_n the number $\frac{1}{\epsilon_n^2}$. We now study the process η^n defined for every t and every function ϕ by

$$<\eta_t^n,\phi>=a_n(<\mu_t^n,\phi>-<p_t,\phi>).$$

For each Brownian motion B^i, $i\in\mathbb{N}^*$, we consider a nonlinear process similar to (2.4)

$$\begin{cases}\bar{Y}_t^{i,n}=\zeta^i+\int_0^t\sigma(V^n*P_s^n(\bar{Y}_s^{i,n})).dB_s^i+\int_0^t b(V^n*P_s^n(\bar{Y}_s^{i,n}))ds\\ P^n\ \text{ is the law of } \bar{Y}^{i,n}\end{cases}$$

Under our assumptions, $\forall n$, P^n admits a measurable version of the densities p^n in $H^{\frac{4+\alpha}{2},4+\alpha}$ with $\|p_n\|_{\frac{4+\alpha}{2},4+\alpha}\leq 1$.

3.1 A few pathwise estimations

Lemma 3.1 *Let* $\Phi:[0,T]\times\mathbb{R}\to\mathbb{R}$ *be a function continuous and bounded together with its first order spatial derivative. We have*

$$\forall\beta>0,\ \sup_{[0,T]\times\mathbb{R}}\mathbb{E}\left((V^n*(\Phi_s(\mu_s^n-p_s^n))(x))^2\right)\leq K_{1,\beta}\ \epsilon_n^\beta \tag{3.1}$$

$$\forall\beta>0,\ \forall s\in[0,T],\ \mathbb{E}\left(<\mu_s^n,(V^n*(\Phi_s(\mu_s^n-p_s^n))(.))^2>\right)\leq K_{2,\beta}\ \epsilon_n^\beta \tag{3.2}$$

$$\sup_{[0,T]\times\mathbb{R}}\mathbb{E}\left((V^n*\mu_s^n(x)-p_s(x))^2\right)\leq K_1\epsilon_n^4 \tag{3.3}$$

$$\forall s\in[0,T],\ \mathbb{E}\left(<\mu_s^n,(V^n*\mu_s^n(.)-p_s(.))^2>\right)\leq K_2\epsilon_n^4 \tag{3.4}$$

where the real constants $K_{1,\beta}$, $K_{2,\beta}$, K_1 and K_2 do not depend on n.

Proof : We only prove the second and the forth inequalities. The first and the third are obtained in a similar way but the calculations are easier.
We recall that V^n is bounded by $\frac{M_v}{\epsilon_n}$ and Lipschitz continuous with constant $\frac{K_v}{\epsilon_n^2}$.

$$\begin{aligned}
&\mathbb{E}\Big(< \mu_s^n, (V^n * (\Phi_s(\mu_s^n - p_s^n))(.))^2 > \Big) \\
&\quad = \frac{1}{n}\sum_{i=1}^n \mathbb{E}\Bigg(\Big(\frac{1}{n}\sum_{j=1}^n (V^n(X_s^{i,n} - X_s^{j,n})\Phi_s(X_s^{j,n}) - < p_s^n, \Phi_s(.)V^n(X_s^{i,n} - .) >)\Big)^2\Bigg) \\
&\quad \leq \frac{3}{n}\sum_{i=1}^n \mathbb{E}\Bigg[\Big(\frac{1}{n}\sum_{j=1}^n (V^n(X_s^{i,n} - X_s^{j,n})\Phi_s(X_s^{j,n}) - V^n(\bar{Y}_s^{i,n} - \bar{Y}_s^{j,n})\Phi_s(\bar{Y}_s^{j,n}))\Big)^2 \\
&\qquad\qquad + \Big(\frac{1}{n}\sum_{j=1}^n (V^n(\bar{Y}_s^{i,n} - \bar{Y}_s^{j,n})\Phi_s(\bar{Y}_s^{j,n}) - < p_s^n, \Phi_s(.)V^n(\bar{Y}_s^{i,n} - .) >)\Big)^2 \\
&\qquad\qquad + < p_s^n, \Phi_s(.)V^n(\bar{Y}_s^{i,n} - .) - \Phi_s(.)V^n(X_s^{i,n} - .) >^2 \Bigg] \\
&\quad \leq \frac{K}{n}\sum_{i=1}^n \Bigg[\frac{1}{n}\sum_{j=1}^n \Big(\frac{\sup|\Phi|^2}{\epsilon_n^4}\mathbb{E}(|X_s^{i,n} - \bar{Y}_s^{i,n}|^2 + |X_s^{j,n} - \bar{Y}_s^{j,n}|^2) + \frac{\sup|\frac{\partial\Phi}{\partial x}|^2}{\epsilon_n^2}\mathbb{E}(|X_s^{j,n} - \bar{Y}_s^{j,n}|^2)\Big) \\
&\qquad\qquad + \frac{\sup|\Phi|^2}{n\epsilon_n^2} + \frac{\sup|\Phi|^2}{\epsilon_n^4}\mathbb{E}(|X_s^{i,n} - \bar{Y}_s^{i,n}|^2)\Bigg]
\end{aligned}$$

as the variables $\bar{Y}_s^{i,n}$ are independent and their common law has a density equal to p_s^n. By Proposition 2.3, replacing M_w and K_w by M_v/ϵ_n and K_v/ϵ_n^2 in (2.3), we deduce

$$\mathbb{E}\Big(< \mu_s^n, (V^n * (\Phi_s(\mu_s^n - p_s^n))(.))^2 > \Big) \leq K\left(\frac{1}{\epsilon_n^4} + \frac{1}{\epsilon_n^2}\right)\frac{\epsilon_n^2}{n}\exp\left(\frac{CK_v^2}{\epsilon_n^4}\right) + \frac{K}{n\epsilon_n^2}$$

Taking into account the definition of ϵ_n (2.9), we conclude

$$\forall\beta > 0,\ \forall s \in [0,T],\ \mathbb{E}\Big(< \mu_s^n, (V^n * (\Phi_s(\mu_s^n - p_s^n))(.))^2 > \Big) \leq K_{2,\beta}\ \epsilon_n^\beta$$

By this inequality in the case $\Phi := 1$ and $\beta = 4$ and the results given in Lemma 2.6, we obtain

$$\begin{aligned}
\sup_{s\in[0,T]} \mathbb{E}\Big(< \mu_s^n, (V^n * \mu_s^n(.) - p_s(.))^2 > \Big) &\leq 2 \sup_{s\in[0,T]} \mathbb{E}\Big(< \mu_s^n, (V^n * \mu_s^n(.) - V^n * p_s^n(.))^2 > \Big) \\
&\quad + 2 \sup_{[0,T]\times\mathbb{R}} |V^n * p^n - p|^2 \\
&\leq K_2\epsilon_n^4
\end{aligned}$$

which puts an end to the proof. ∎

Let us now prove that uniformly in t and n, $\mathbb{E}(\|\eta_t^n\|_{-2,2}^2)$ is finite.

Proposition 3.2

$$\sup_n \sup_{t\in[0,T]} \mathbb{E}\Big(\|\eta_t^n\|_{-2,2}^2 \Big) < +\infty.$$

Proof : Let us first remark that, as σ and b are bounded and $\mathbb{E}(|\zeta|^8) < +\infty$,

$$\sup_n \sup_{1\le i\le n} \mathbb{E}\Big(\sup_{s\le T} |X_s^{i,n}|^8 \Big) < +\infty; \quad \sup_i \mathbb{E}\Big(\sup_{s\le T} |\bar{X}_s^i|^8 \Big) < +\infty \tag{3.5}$$

For every function ϕ in $W_0^{2,2}$, we write $< \eta_t^n, \phi >= S_t^n(\phi) + T_t^n(\phi)$, where

$$S_t^n(\phi) = \frac{a_n}{n}\sum_{i=1}^n (\phi(X_t^{i,n}) - \phi(\bar{X}_t^i)) \quad ; \quad T_t^n(\phi) = \frac{a_n}{n}\sum_{i=1}^n (\phi(\bar{X}_t^i) - < p_t, \phi >).$$

Let us consider a complete orthonormal system (ϕ_k) in $W^{2,2}$. Since the variables $(X_t^{i,n}, \bar{X}_t^i)$ are exchangeable,

$$\begin{aligned}
\mathbb{E}\Big(\sum_{k\ge1} S_t^n(\phi_k)^2 \Big) &\le \mathbb{E}\Big(\frac{a_n^2}{n} \sum_{i=1}^n \sum_{k\ge1} (\phi_k(X_t^{i,n}) - \phi_k(\bar{X}_t^i))^2 \Big) \le a_n^2 \mathbb{E}\Big(\|D_{X_t^{1,n}\bar{X}_t^1}\|_{-2,2}^2 \Big) \\
&\le K a_n^2 \mathbb{E}\Big(|X_t^{1,n} - \bar{X}_t^1|^4 \Big)^{1/2} \mathbb{E}\Big(1 + |X_t^{1,n}|^8 + |\bar{X}_t^1|^8 \Big)^{1/2} \quad \text{by (0.5)}
\end{aligned}$$

By (2.10) and (3.5), we deduce that $\sup_n \sup_{t\in[0,T]} \mathbb{E}(\sum_{k\ge1} S_t^n(\phi_k)^2) < +\infty$. Moreover, since the variables $\bar{X}_t^i$ are independent with law $p_t(x)dx$,

$$\begin{aligned}
\mathbb{E}\Big(\sum_{k\ge1} T_t^n(\phi_k)^2 \Big) &= \frac{a_n^2}{n^2} n \sum_{k\ge1} \mathbb{E}\Big((\phi_k(\bar{X}_t^1) - < p_t, \phi_k >)^2 \Big) \le \frac{a_n^2}{n} \mathbb{E}\Big(\sum_{k\ge1} D^2_{\bar{X}_t^1}(\phi_k) \Big) \\
&\le K \frac{a_n^2}{n} \mathbb{E}\Big(1 + |\bar{X}_t^1|^4 \Big) \quad \text{by (0.6)}
\end{aligned}$$

and $\sup_n \sup_{t\in[0,T]} \mathbb{E}(\sum_{k\ge1} T_t^n(\phi_k)^2) < +\infty$. As $\|\eta_t^n\|_{-2,2}^2 \le 2\sum_{k\ge1}(S_t^n(\phi_k)^2 + T_t^n(\phi_k)^2)$, the conclusion holds. ∎

3.2 The tightness result

In order to prove the tightness of the laws of the fluctuation processes η^n, we study the semi-martingale representation of these processes. Applying Itô's formula, we obtain that η^n satisfies the following martingale property: for every $\phi \in C_b^2(\mathbb{R})$,

$$M_t^n(\phi) = \langle \eta_t^n, \phi \rangle - \langle \eta_0^n, \phi \rangle - \int_0^t A_s^n \phi ds,$$

is a real continuous martingale with quadratic variation process

$$< M^n(\phi) >_t = \frac{a_n^2}{n} \int_0^t < \mu_s^n, \phi'^2(.)\sigma^2(V^n * \mu_s^n(.)) > ds$$

where

$$A_s^n \phi = a_n \Bigg(< \mu_s^n, b(V^n * \mu_s^n(.))\phi'(.) > - < p_s, b(p_s(.))\phi'(.) > \\ + \frac{1}{2} < \mu_s^n, \sigma^2(V^n * \mu_s^n(.))\phi''(.) > - \frac{1}{2} < p_s, \sigma^2(p_s(.))\phi''(.) > \Bigg)$$

Proposition 3.3 *For every integer n, the process (M_t^n) is a strongly continuous martingale in $W_0^{-2,2}$, and for $\{\phi_k\}_{k\geq 1}$ a complete orthonormal system in $W_0^{2,2}$,*

$$\sup_n \frac{n}{a_n^2} \sum_{k\geq 1} \mathbb{E}\Big(\sup_{t\leq T} (M_t^n(\phi_k))^2 \Big) < +\infty \qquad (3.6)$$

which implies that $\sup_n \frac{n}{a_n^2}\mathbb{E}(\sup_{t\leq T} \|M_t^n\|_{-2,2}^2) < +\infty$ and that the $C([0,T], W_0^{-2,2})$-valued variables M^n converge to 0 in L^2.

Proof : Let $\{\phi_k\}_{k\geq 1}$ be a complete orthonormal system in $W_0^{2,2}$ of C^∞ functions with compact support. By Doob's inequality, $\sum_{k\geq 1} \mathbb{E}(\sup_{t\leq T}(M_t^n(\phi_k))^2)$ is bounded by

$$\begin{aligned} K \sum_{k\geq 1} \mathbb{E}((M_T^n(\phi_k))^2) &= \frac{Ka_n^2}{n} \sum_{k\geq 1} \mathbb{E}\Big(\int_0^T < \mu_s^n, {\phi'_k}^2(.)\sigma^2(V^n * \mu_s^n(\cdot)) > ds \Big) \\ &\leq \frac{Ka_n^2}{n} \sum_{k\geq 1} \int_0^T \mathbb{E}\Big(< \mu_s^n, {\phi'_k}^2(.) > \Big) ds = \frac{Ka_n^2}{n} \mathbb{E}\Big(\int_0^T \|H_{X_s^{1,n}}\|_{-2,2}^2 ds \Big) \\ &\leq \frac{Ka_n^2}{n} \mathbb{E}\Big(\sup_{s\leq T} (1 + |X_s^{1,n}|^4) \Big) \quad \text{by (0.7)} \end{aligned}$$

By (3.5), we conclude that (3.6) holds.
We still have to prove the continuity of M^n. Let $\epsilon > 0$. By (3.6), there exists a positive number N_0 (depending on ω) such that $\sum_{k>N_0} \sup_{t\leq T}(M_t^n(\phi_k))^2 < \frac{\varepsilon}{6}$ a.s.. Let $\{t_m\}_{m\geq 1}$ be a sequence in $[0,T]$ such that (t_m) tends to t when m tends to infinity.

$$\begin{aligned} \|M_{t_m}^n - M_t^n\|_{-2,2}^2 &= \sum_{k\geq 1} (M_{t_m}^n(\phi_k) - M_t^n(\phi_k))^2 \\ &\leq \sum_{k=1}^{N_0} (M_{t_m}^n(\phi_k) - M_t^n(\phi_k))^2 + 2 \sum_{k>N_0} \{(M_{t_m}^n(\phi_k))^2 + (M_t^n(\phi_k))^2\} \\ &\leq \sum_{p=1}^{N_0} \frac{\varepsilon}{3N_0} + \frac{4\varepsilon}{6} = \varepsilon. \end{aligned}$$

The majoration of the first term if m is sufficiently large is due to the continuity of the process $M_t^n(\phi_k)$, for every $k \geq 1$. Thus the mapping $t \mapsto M_t^n$ is continuous in $W_0^{-2,2}$. ∎

To study the drift term we transform $A_s^n\phi$ where $\phi \in C_b^2(\mathbb{R})$.

$$
\begin{aligned}
A_s^n\phi = a_n\Big(& < \mu_s^n - p_s, b(p_s(\cdot))\phi'(.) > + < \mu_s^n, (b(V^n * \mu_s^n(\cdot)) - b(p_s(\cdot)))\phi'(.) > \\
& + \frac{1}{2} < \mu_s^n - p_s, \sigma^2(p_s(\cdot))\phi''(.) > + \frac{1}{2} < \mu_s^n, (\sigma^2(V^n * \mu_s^n(\cdot)) - \sigma^2(p_s(\cdot)))\phi''(.) > \Big) \\
& = < \eta_s^n, b(p_s(\cdot))\phi'(.) > + < \eta_s^n, \frac{\sigma^2}{2}(p_s(\cdot))\phi''(.) > \\
& + a_n < \mu_s^n, \phi'(\cdot)(V^n * \mu_s^n(\cdot) - p_s(\cdot)) \int_0^1 b'(\tau V^n * \mu_s^n(\cdot) + (1-\tau)p_s(\cdot))d\tau > \\
& + a_n < \mu_s^n, \frac{\phi''(\cdot)}{2}(V^n * \mu_s^n(\cdot) - p_s(\cdot)) \int_0^1 (\sigma^2)'(\tau V^n * \mu_s^n(\cdot)) + (1-\tau)p_s(\cdot))d\tau > \\
& = < \eta_s^n, L_s\phi > + < Z_s^n, \phi > .
\end{aligned}
$$

with

$$
L_s\phi(x) = b(p_s(x))\phi'(x) + \frac{\sigma^2}{2}(p_s(x))\phi''(x), \tag{3.7}
$$

$$
\begin{aligned}
< Z_s^n, \phi > = a_n < \mu_s^n, (V^n * \mu_s^n(\cdot) - p_s(\cdot)) \Big(& \phi'(\cdot) \int_0^1 b'(\tau V^n * \mu_s^n(\cdot) + (1-\tau)p_s(\cdot))d\tau \\
& + \frac{\phi''(\cdot)}{2} \int_0^1 (\sigma^2)'(\tau V^n * \mu_s^n(\cdot)) + (1-\tau)p_s(\cdot))d\tau \Big) >
\end{aligned} \tag{3.8}
$$

Proposition 3.4 *For every s, the operator L_s is a linear continuous mapping from $W_0^{4,1}$ into $W_0^{2,2}$, and for all $\phi \in W_0^{4,1}$,*

$$
\|L_s\phi\|_{2,2}^2 \quad \leq \quad K_1\|\phi\|_{4,1}^2. \tag{3.9}
$$

For every n, s and ω, the operator Z_s^n is a linear continuous operator from $W_0^{4,1}$ into $\mathbb{R}$, and

$$
\mathbb{E}(\|Z_s^n\|_{-4,1}^2) \quad \leq \quad K_2 < +\infty. \tag{3.10}
$$

The constants K_1 and K_2 are independent of n and $s \leq T$.

Proof : The upperbound is clear for $L_s\phi$, since p belongs to $H^{\frac{4+\alpha}{2}, 4+\alpha}([0,T]\times\mathbb{R})$, and then to $C_b^2([0,T]\times\mathbb{R})$.
For Z_s^n, we observe that as $\|\phi\|_{C^{2,1}} \leq K\|\phi\|_{4,1}$ (by (0.3)),

$$
\mathbb{E}(< Z_s^n, \phi >^2) \quad \leq \quad a_n^2\|\phi\|_{4,1}^2 K_{b,\sigma}\mathbb{E}\Big(\int (1+|y|^2)\mu_s^n(dy)\Big)\mathbb{E}\Big(\int (V^n * \mu_s^n(y) - p_s(y))^2\mu_s^n(dy)\Big).
$$

By (3.4) and (3.5), we conclude that (3.10) holds. ∎

To prove the tightness of η^n in $C([0,T], W_0^{-4,1})$, we use the Hilbert semimartingale decomposition of η^n in $W_0^{-4,1}$

$$
\eta_t^n = \eta_0^n + \int_0^t (L_s)^*\eta_s^n ds + \int_0^t Z_s^n ds + M_t^n. \tag{3.11}
$$

where $(L_s)^*$ is the adjoint of the operator L_s.

Lemma 3.5 *The integrals $\int_0^t (L_s)^*\eta_s^n ds$ and $\int_0^t Z_s^n ds$ are defined as Bochner integrals in $W_0^{-4,1}$.*

Proof : As $W_0^{-4,1}$ is separable, following Yoshida [15] p.132, it is enough to check that :
1) $\forall \phi \in W_0^{4,1}$, the mappings $s \to < (L_s)^*\eta_s^n, \phi > = < \eta_s^n, L_s\phi >$ and $s \to < Z_s^n, \phi >$ are measurable
2)a.s., $\int_0^t \|(L_s)^*\eta_s^n\|_{-4,1} ds < +\infty$ and $\int_0^t \|Z_s^n\|_{-4,1} ds < +\infty$.
Condition 1) is obviously satisfied.
By (3.9) we obtain

$$\int_0^T \|(L_s)^*\eta_s^n\|_{-4,1} ds \leq K_1 \int_0^T \|\eta_s^n\|_{-2,2} ds$$

By Proposition 3.2, $\mathbb{E}\left(\int_0^T \|\eta_s^n\|^2_{-2,2} ds \right) < +\infty$ which implies that a.s., $\int_0^T \|\eta_s^n\|_{-2,2} ds < +\infty$. Hence condition 2) holds for the first integral. For the second integral, we remark that, a.s. $\int_0^T \|Z_s^n\|_{-4,1} ds < +\infty$, as by (3.10), $\mathbb{E}\left(\int_0^T \|Z_s^n\|^2_{-4,1} ds \right) < +\infty$. ∎

Proposition 3.6

$$\sup_n \mathbb{E}\left(\sup_{t \leq T} \|\eta_t^n\|^2_{-4,1} \right) < +\infty \tag{3.12}$$

The trajectories of η^n are a.s. strongly continuous in $W_0^{-4,1}$.

Proof : By the semimartingale decomposition of η^n (3.11),

$$\|\eta_t^n\|^2_{-4,1} \leq 4\left(\|\eta_0^n\|^2_{-4,1} + t \int_0^t (\|(L_s)^*\eta_s^n\|^2_{-4,1} + \|Z_s^n\|^2_{-4,1}) ds + \|M_t^n\|^2_{-4,1} \right)$$

Taking (3.9) and (3.10) into account, we deduce

$$\begin{aligned} \mathbb{E}\left(\sup_{t \leq T} \|\eta_t^n\|^2_{-4,1} \right) &\leq 4\mathbb{E}\left(\|\eta_0^n\|^2_{-4,1} + T \int_0^T (\|(L_s)^*\eta_s^n\|^2_{-4,1} + \|Z_s^n\|^2_{-4,1}) ds + \sup_{t \leq T} \|M_t^n\|^2_{-4,1} \right) \\ &\leq 4\left(\mathbb{E}(\|\eta_0^n\|^2_{-4,1}) + K_1 T^2 \sup_{s \leq T} \mathbb{E}(\|\eta_s^n\|^2_{-2,2}) + K_2 T^2 + \mathbb{E}\left(\sup_{t \leq T} \|M_t^n\|^2_{-4,1} \right) \right) \end{aligned}$$

Propositions 3.2 and 3.3 and the continuous embedding of $W_0^{-2,2}$ into $W_0^{-4,1}$ imply that (3.12) holds.
The Bochner integrals $\int_0^t (L_s)^*\eta_s^n ds$ and $\int_0^t Z_s ds$ are strongly continuous in $W_0^{-4,1}$ ([15] Corollary 1 p.133). Moreover, by Proposition 3.3 and the continuous embedding from $W_0^{-2,2}$ into $W_0^{-4,1}$, the process M^n is a.s. strongly continuous in $W_0^{-4,1}$. The decomposition (3.11) of η^n allows to conclude that this process is a.s. strongly continuous. ∎

We are now able to prove

Theorem 3.7 *The sequence of the laws of $(\eta^n)_{n\geq 1}$ is tight in $C([0,T],W_0^{-4,1})$.*

Proof : By Proposition 3.3 and the continuous embedding from $W_0^{-2,2}$ into $W_0^{-4,1}$, we know that the processes M^n considered as $C([0,T],W_0^{-4,1})$ valued variables converge to 0 in L^2. As $C([0,T],W_0^{-4,1})$ endowed with the sup norm is a Polish space, we deduce that the sequence of the laws of $(M^n)_{n\geq 1}$ is tight in $C([0,T],W_0^{-4,1})$. Therefore it is enough to prove the tightness of the laws of the drift terms $D_t^n=\eta_0^n+\int_0^t(L_s)^*\eta_s^n ds+\int_0^t Z_s^n ds$ to conclude. Let us now recall the criterion that we will use :

A sequence of (Ω^n,F_t^n)-adapted processes $(Y^n)_{n\geq 1}$ with paths in $C([0,T],H)$ where H is a Hilbert space is tight if both of the following conditions hold:
I: *There exists a Hilbert space H_0 such that $H_0\hookrightarrow_{H.S.} H$ and such that for all $t\leq T$,*

$$\sup_n \mathbb{E}(\|Y_t{}^n\|_{H_0}^2)<+\infty.$$

II: *(Aldous condition) For every $\gamma_1,\gamma_2>0$ there exists $\delta>0$ and an integer n_0 such that for every (F_t^n)-stopping time $\tau_n\leq T$,*

$$\sup_{n\geq n_0}\sup_{\theta\leq\delta} P(\|Y_{\tau_n}^n-Y_{\tau_n+\theta}^n\|_H\geq\gamma_1)\leq\gamma_2.$$

As $W_0^{-2,2}\hookrightarrow_{H.S.} W_0^{-4,1}$ and $\|D_t^n\|_{-2,2}^2\leq 2(\|\eta_t^n\|_{-2,2}^2+\|M_t^n\|_{-2,2}^2)$, Propositions 3.2 and 3.3 imply that condition I holds for $(D^n)_{n\geq 1}$.
Let $\gamma_1>0$, $0\leq\theta\leq\delta$ and $\tau_n\leq T$ be a stopping time. By Chebychev inequality,

$$P\Big(\|D_{\tau_n+\theta}^n-D_{\tau_n}^n\|_{-4,1}\geq\gamma_1\Big)\leq\frac{1}{\gamma_1^2}\mathbb{E}\left(\left\|\int_{\tau_n}^{\tau_n+\theta}((L_s)^*\eta_s^n+Z_s^n)ds\right\|_{-4,1}^2\right)$$

By Proposition 3.4 and 3.2 $$\leq\frac{1}{\gamma_1^2}\left(2\theta^2\left(K_1\sup_n\sup_{t\in[0,T]}\mathbb{E}\Big(\|\eta_s^n\|_{-2,2}^2\Big)+K_2\right)\right)\leq\frac{K\delta^2}{\gamma_1^2}$$

The right-hand-side is arbitrarily small uniformly in n for δ small and condition II holds which puts an end to the proof. ∎

3.3 Characterization of the limit values

If we consider equation

$$\eta_t^n=\eta_0^n+\int_0^t(L_s)^*\eta_s^n ds+\int_0^t Z_s^n ds+M_t^n$$

it appears that as $n\to+\infty$, it is not possible to close the equation at the limit in $W_0^{-4,1}$ because of the unboundedness of the operator L_s in $W_0^{4,1}$. But this operator is bounded from $W_0^{6,1}$ to $W_0^{4,1}$. Therefore, we are going to obtain a limit equation in $W_0^{-6,1}$.

Let $A_s\phi(x)=p_s(x)\left(\phi'(x)b'(p_s(x))+\frac{\phi''(x)}{2}(\sigma^2)'(p_s(x))\right)$ and $\mathcal{L}_s=L_s+A_s$.
Since $p\in H^{\frac{4+\alpha}{2},4+\alpha}([0,T]\times\mathbb{R})$, we easily prove that :

Lemma 3.8 *If σ and b belong to C_b^6, then for each s, the operator $\mathcal{L}_s$ is continuous from $W_0^{6,1}$ into $W_0^{4,1}$ and its norm is bounded uniformly in $s \in [0,T]$. Moreover,*

$$\forall \phi \in W_0^{6,1},\ \forall s, s' \in [0,T],\ \|\mathcal{L}_s \phi - \mathcal{L}_{s'}\phi\|_{4,1} \leq K|s-s'|^{\frac{\alpha}{2}}\|\phi\|_{6,1}.$$

We are now ready to obtain the limit equation :

Theorem 3.9 *Let us assume that $\sigma, b \in C_b^6$. Then every limit value of the laws of $(\eta^n)_{n\geq 1}$ (in $\mathcal{P}(C([0,T], W_0^{-4,1}))$) is concentrated on the solutions of the deterministic affine equation*

$$\forall t \in [0,T],\ \xi_t = \int_0^t (\mathcal{L}_s)^* \xi_s ds + \int_0^t G_s ds \quad in \quad W_0^{-6,1} \tag{3.13}$$

where G_s is defined, for every ϕ in $W_0^{6,1}$ by

$$< G_s, \phi > = < p_s, \frac{1}{2}\Big(\int z^2 V(z)dz\Big) p_s''(.)(\phi'(.)b'(p_s(.)) + \frac{\phi''(.)}{2}(\sigma^2)'(p_s(.))) > .$$

Remark 3.10 *Let $\xi \in C([0,T], W_0^{-4,1})$, $\phi \in W_0^{6,1}$ and $s, s' \in [0,T]$. By Lemma 3.8, we obtain*

$$\begin{aligned} | < \xi_s, \mathcal{L}_s\phi > - < \xi_{s'}, \mathcal{L}_{s'}\phi > | &\leq | < \xi_s - \xi_{s'}, \mathcal{L}_s \phi > | + | < \xi_{s'}, (\mathcal{L}_{s'} - \mathcal{L}_s)\phi > | \\ &\leq K(\|\xi_s - \xi_{s'}\|_{-4,1} + \sup_{t\in[0,T]} \|\xi_t\|_{-4,1}|s-s'|^{\frac{\alpha}{2}})\|\phi\|_{6,1} \end{aligned}$$

Hence the mapping $s \to (\mathcal{L}_s)^*\xi_s$ is continuous in $W_0^{-6,1}$ and the integral $\int_0^t (\mathcal{L}_s)^*\xi_s ds$ is defined as a Riemann integral.
By Schwarz inequality, (3.5) and the continuous embedding of $W_0^{6,1}$ into $C^{2,1}$,

$$< G_s, \phi >^2 \leq K < p_s, (1+|x|^2) > \|\phi\|^2_{C^{2,1}} \leq K\|\phi\|^2_{6,1}.$$

Hence $\int_0^t G_s ds$ makes sense as a Bochner integral in $W_0^{-6,1}$.

Proof : We consider a subsequence of η^n converging in law and that we still index by n for simplicity. Let $t \in [0,T]$, η be a variable in $C([0,T], W_0^{-4,1})$ distributed according to the limit law and ϕ be a C^∞ function with compact support in $\mathbb{R}$.
By Lemma 3.8, the function $F_\phi : \xi \in C([0,T], W_0^{-4,1}) \to < \xi_t, \phi > - \int_0^t < \xi_s, \mathcal{L}_s\phi > ds \in \mathbb{R}$ is continuous. Hence the sequence $F_\phi(\eta^n)$ converges in law to $F_\phi(\eta)$.
We have already seen that the martingale part tends to zero. Hence $M_n(\phi)$ converges in law to zero. By the same way, the initial sequence $< \eta_0^n, \phi >$ tends to zero, since the fluctuations of initial independent conditions converge at rate $\sqrt{n}$.
If we prove that $\int_0^t < Z_s^n, \phi > ds - \int_0^t < \eta_s^n, A_s\phi > ds$ converges in law to the deterministic variable $\int_0^t < G_s, \phi > ds$, by the decomposition

$$F_\phi(\eta^n) = < \eta_0^n, \phi > + \int_0^t < Z_s^n, \phi > ds - \int_0^t < \eta_s^n, A_s\phi > ds + M_t^n(\phi)$$

we will deduce that

$$\forall t \in [0,T],\ a.s.,\ <\eta_t, \phi> = \int_0^t <\eta_s, \mathcal{L}_s\phi> ds + \int_0^t <G_s, \phi> ds$$

By continuity, the above equality will hold almost surely for any $t \in [0,T]$. Moreover, choosing ϕ in a sequence dense in $W_0^{6,1}$, and taking limits, we will get

$$a.s., \forall t \in [0,T],\ \forall \phi \in W_0^{6,1},\ <\eta_t, \phi> = \int_0^t <\eta_s, \mathcal{L}_s\phi> ds + \int_0^t <G_s, \phi> ds$$

which is the conclusion of the theorem.

By an easy computation, $<Z_s^n, \phi> - <\eta_s^n, A_s\phi> - <G_s, \phi>$ is equal to $T_1^n(s) + T_2^n(s) + T_3^n(s)$ with

$$\begin{aligned} T_n^1(s) = a_n < \mu_s^n, (V^n * \mu_s^n(\cdot) - p_s(\cdot))\Big(\phi'(\cdot)\Big(\int_0^1 b'(\tau V^n * \mu_s^n(\cdot) + (1-\tau)p_s(\cdot))d\tau - b'(p_s(.))\Big) \\ + \frac{\phi''(\cdot)}{2}\Big(\int_0^1 (\sigma^2)'(\tau V^n * \mu_s^n(\cdot) + (1-\tau)p_s(\cdot))d\tau - (\sigma^2)'(p_s(.))\Big)\Big) > \end{aligned}$$

$$\begin{aligned} T_n^2(s) = &< \mu_s^n, a_n(V^n * p_s(\cdot) - p_s(\cdot))\Big(\phi'(.)b'(p_s(.)) + \frac{\phi''(.)}{2}(\sigma^2)'(p_s(.))\Big) > \\ &- < p_s, \frac{1}{2}\Big(\int z^2 V(z)dz\Big)p_s''(.)\Big(\phi'(.)b'(p_s(.)) + \frac{\phi''(.)}{2}(\sigma^2)'(p_s(.))\Big) > \end{aligned}$$

$$\begin{aligned} T_n^3(s) = &< \mu_s^n, a_n(V^n * \mu_s^n(\cdot) - V^n * p_s(\cdot))\Big(\phi'(.)b'(p_s(.)) + \frac{\phi''(.)}{2}(\sigma^2)'(p_s(.))\Big) > \\ &- < \eta_s^n, p_s(.)\Big(\phi'(.)b'(p_s(.)) + \frac{\phi''(.)}{2}(\sigma^2)'(p_s(.))\Big) > \end{aligned}$$

If we show that $\lim_n \int_0^T \mathbb{E}|T_n^1(s)|ds = \lim_n \int_0^T \mathbb{E}|T_n^2(s)|ds = \lim_n \int_0^T \mathbb{E}|T_n^3(s)|ds = 0$, then the proof will be finished since these limits imply that $\int_0^t < Z_s^n, \phi > ds - \int_0^t < \eta_s^n, A_s\phi > ds$ converges in L^1 to the deterministic variable $\int_0^t < G_s, \phi > ds$ for any $t \in [0,T]$.

Proof of $\lim_n \int_0^T \mathbb{E}|T_n^1(s)|ds = 0$

$$\begin{aligned} T_n^1(s) = a_n < \mu_s^n, (V^n * \mu_s^n(\cdot) - p_s(\cdot))\Big(\phi'(\cdot)\Big(\int_0^1 b'(\tau V^n * \mu_s^n(\cdot) + (1-\tau)p_s(\cdot))d\tau - b'(p_s(.))\Big) \\ + \frac{\phi''(\cdot)}{2}\Big(\int_0^1 (\sigma^2)'(\tau V^n * \mu_s^n(\cdot) + (1-\tau)p_s(\cdot))d\tau - (\sigma^2)'(p_s(.))\Big)\Big) > \end{aligned}$$

As b' and $(\sigma^2)'$ are Lipschitz continuous and ϕ' and ϕ'' are bounded

$$|T_n^1(s)| \le K a_n < \mu_s^n, (V^n * \mu_s^n(.) - p_s(.))^2 >$$

By (3.4), we deduce $\int_0^T \mathbb{E}|T_n^1(s)| \le KT\epsilon_n^2$. Hence the conclusion holds.

Proof of $\lim_n \int_0^T \mathbb{E}|T_n^2(s)|ds = 0$

$$T_n^2(s) = < \mu_s^n, \Big(a_n(V^n * p_s(\cdot) - p_s(\cdot)) - \frac{1}{2}\Big(\int z^2 V(z)dz\Big)p_s''(.)\Big)$$
$$\Big(\phi'(.)b'(p_s(.)) + \frac{\phi''(.)}{2}(\sigma^2)'(p_s(.))\Big) >$$
$$+ < \mu_s^n - p_s, \frac{1}{2}\Big(\int z^2 V(z)dz\Big)p_s''(.)\Big(\phi'(.)b'(p_s(.)) + \frac{\phi''(.)}{2}(\sigma^2)'(p_s(.))\Big) >$$

Let $T_n^{21}(s)$ and $T_n^{22}(s)$ denote the terms in the right hand side.
Since p_s is in C_b^3 uniformly in s and $\int_{\mathbb{R}} zV(z)dz = 0$,

$$\left|V^n * p_s(x) - p_s(x) - \frac{\epsilon_n^2}{2}\Big(\int z^2 V(z)dz\Big)p_s''(x)\right| \leq K\epsilon_n^3 \int |z|^3 V(z)dz$$

The functions b', $(\sigma^2)'$, ϕ' and ϕ'' being bounded, we deduce $\int_0^T \mathbb{E}(|T_n^{21}(s)|)ds \leq K\epsilon_n$ which tends to 0 as n tends to infinity.
The function $y \to p_s''(y)\Big(\phi'(y)b'(p_s(y)) + \frac{\phi''(y)}{2}(\sigma^2)'(p_s(y))\Big)$ is Lipschitz continuous and bounded. Since, by the propagation of chaos result, the sequence $(\mu_s^n(dx))$ converges to $p_s(x)dx$ in probability, $\mathbb{E}|T_n^{22}(s)|$ tends to zero as n tends to infinity. By Lebesgue Theorem, the same is true for $\int_0^T \mathbb{E}|T_n^{22}(s)|ds$. Hence $\lim_n \int_0^T \mathbb{E}|T_n^2(s)|ds = 0$.

Proof of $\lim_n \int_0^T \mathbb{E}|T_n^3(s)|ds = 0$
For simplicity, let us denote

$$\psi_s(x) = \phi'(x)b'(p_s(x)) + \frac{\phi''(x)}{2}(\sigma^2)'(p_s(x))$$

$$\begin{aligned}
T_n^3(s) &= \int\int V^n(x-y)\psi_s(x)\mu_s^n(dx)\eta_s^n(dy) - \int p_s(y)\psi_s(y)\eta_s^n(dy) \\
&= \int\int V^n(x-y)\psi_s(x)(\mu_s^n(dx) - p_s^n(x)dx)\eta_s^n(dy) \\
&+ \int\int V^n(x-y)\psi_s(x)(p_s^n(x) - p_s(x))dx\eta_s^n(dy) \\
&+ \Big(\int\int V^n(x-y)\psi_s(x)p_s(x)dx\eta_s^n(dy) - \int p_s(y)\psi_s(y)\eta_s^n(dy)\Big) \\
&= T_n^{31}(s) + T_n^{32}(s) + T_n^{33}(s)
\end{aligned}$$

We set $\bar{V}^n(x) = V^n(-x)$.

$$\begin{aligned}
\mathbb{E}|T_n^{31}(s)| &\leq a_n\mathbb{E}\Big(< \mu_s^n + p_s, |\bar{V}^n * (\psi_s(\mu_s^n - p_s^n))(.)| > \Big) \\
&\leq a_n\Big(\mathbb{E}\Big(< \mu_s^n, |\bar{V}^n * (\psi_s(\mu_s^n - p_s^n))(.)| > \Big) + \sup_{[0,T]\times\mathbb{R}} \mathbb{E}|\bar{V}^n * (\psi_s(\mu_s^n - p_s^n))(x)|\Big)
\end{aligned}$$

The function ψ is continuous and bounded together with its first spatial partial derivative and satisfies the hypothesis made on Φ in Lemma 3.1. Moreover, as $\bar{V}^n$ is bounded and Lipschitz continuous with the same constants as V^n, the proof of Lemma 3.1 shows that (3.1) and (3.2) still hold when V^n is replaced by $\bar{V}^n$. Hence we obtain $\forall \beta > 0$,

$$\int_0^T \mathbb{E}|T_n^{31}(s)|ds \leq K_\beta \epsilon_n^{\frac{\beta}{2}-2}$$

By choosing β greater than 4, we obtain the convergence to zero of $\int_0^T \mathbb{E}|T_n^{31}(s)|ds$.

As ψ_s is equal to 0 outside a compact set which does not depend on $s \in [0,T]$,

$$\begin{aligned}
\int_0^T \mathbb{E}|T_n^{32}(s)|ds &= a_n \int_0^T \mathbb{E}\Big(|\int \psi_s(x) V^n * (\mu_s^n - p_s)(x)(p_s^n(x) - p_s(x))dx| \Big) ds \\
&\leq K_\psi a_n \sup_{[0,T]\times\mathbb{R}} |p_s^n(x) - p_s(x)| \Big(\sup_{[0,T]\times\mathbb{R}} \mathbb{E}|V^n * (\mu_s^n - p_s^n)(x)| + \sup_{[0,T]\times\mathbb{R}} |V^n * (p_s^n - p_s)(x)| \Big) \\
&\leq K_\psi a_n \sup_{[0,T]\times\mathbb{R}} |p_s^n(x) - p_s(x)| \Big(\sup_{[0,T]\times\mathbb{R}} \mathbb{E}|V^n * (\mu_s^n - p_s^n)(x)| + \sup_{[0,T]\times\mathbb{R}} |p_s^n(x) - p_s(x)| \Big)
\end{aligned}$$

By Lemma 2.6 and (3.1) written for $\Phi := 1$ and $\beta = 4$, we obtain, $\int_0^T \mathbb{E}|T_n^{32}(s)|ds \leq K\epsilon_n^2$ which goes to 0 as $n \to +\infty$.
For the third term, an easy computation (using Taylor expansion) gives that

$$\int V^n(x-y)\psi_s(x)p_s(x)dx - \psi_s(y)p_s(y) - \frac{\epsilon_n^2}{2}\int z^2 V(z)dz(p_s(y)\psi_s''(y) + 2p_s'(y)\psi_s'(y) + \psi_s(y)p_s''(y))$$

is smaller than $K\epsilon_n^3 \int_{\mathbb{R}} |z|^3 V(z)dz$. Hence

$$\begin{aligned}
|T_n^{33}(s)| &\leq \left| \frac{\epsilon_n^2}{2} \int z^2 V(z)dz \int (p_s(y)\psi_s''(y) + 2p_s'(y)\psi_s'(y) + \psi_s(y)p_s''(y))\eta_s^n(dy) \right| + K\epsilon_n \\
&= K\Big| < \mu_s^n - p_s, p_s(.)\psi_s''(.) + 2p_s'(.)\psi_s'(.) + \psi_s(.)p_s''(.) > \Big| + K\epsilon_n
\end{aligned}$$

As the function $y \mapsto p_s(y)\psi_s''(y) + 2p_s'(y)\psi_s'(y) + \psi_s(y)p_s''(y)$ is Lipschitz continuous and bounded, the convergence in probability of μ_s^n to p_s implies that

$$\mathbb{E}\Big| < \mu_s^n - p_s, p_s(.)\psi_s''(.) + 2p_s'(.)\psi_s'(.) + \psi_s(.)p_s''(.) > \Big|$$

converges to zero. Hence $\mathbb{E}(\int_0^T |T_n^{33}(s)|ds)$ tends to zero as n tends to infinity.

The proof of Theorem 3.9 is then complete. ∎

The next step consists in proving uniqueness for (3.13). Let ξ^1 and ξ^2 be two solutions in $C([0,T], W_0^{-4,1})$. The difference $\tilde{\xi} = \xi^1 - \xi^2$ is a solution of

$$\tilde{\xi}_t = \int_0^t (\mathcal{L}_s)^* \tilde{\xi}_s ds$$

in $W_0^{-6,1}$. But the operator $(\mathcal{L}_s)^*$ is not bounded in $W_0^{-6,1}$ and Gronwall's arguments do not work to prove $\tilde{\xi}_t = 0$, $\forall t \in [0,T]$. The trick is to use the semi-group associated with the second order operator $\mathcal{L}_s$ to obtain uniqueness. Our approach is very similar to the one developped by Mitoma in [11].

$$\mathcal{L}_s\phi(x) = (b(p_s(x)) + p_s(x)b'(p_s(x)))\phi'(x) + (\sigma^2(p_s(x)) + p_s(x)(\sigma^2)'(p_s(x)))\frac{\phi''(x)}{2}$$

We set $\lambda(s,x) = b(p_s(x)) + p_s(x)b'(p_s(x))$. By (1.8), it is possible to define

$$\gamma(s,x) = \sqrt{\sigma^2(p_s(x)) + p_s(x)(\sigma^2)'(p_s(x))}.$$

In order to ensure that γ is smooth, we have to assume that

$$\exists \mu > 0,\ \forall x \in \mathbb{R},\ \sigma^2(x) + x(\sigma^2)'(x) \geq \mu$$

which is exactly property (1.9).
From now on, we suppose that $\sigma, b \in C_b^{10}$ and that [$\mathbf{hyp_9'}$] and (1.9) hold. The function p belongs to $H^{\frac{9+\alpha}{2},9+\alpha}([0,T]\times\mathbb{R})$ and the functions γ_s and λ_s belong to C_b^9 uniformly for $s \in [0,T]$.
According to Kunita [5] p.227, the flow $(X_{st}(x))_{0\leq s\leq t\leq T}$ defines a C^8 diffeomorphism, where $(X_{st}(x))$ is the unique solution of the Itô stochastic differential equation

$$X_{st}(x) = x + \int_s^t \gamma(r, X_{sr}(x))dB_r + \int_s^t \lambda(r, X_{sr}(x))dr, \quad t \geq s$$

Let $D^j X_{st}(x)$ denote the derivative of order j for $1 \leq j \leq 8$. By [4] p.61,

$$\forall r > 0,\ \forall\, 1 \leq j \leq 8,\ \sup_{x\in\mathbb{R}} \sup_{0\leq s\leq t\leq T} \mathbb{E}\Big(|D^j X_{st}(x)|^r\Big) < +\infty \tag{3.14}$$

Let $\phi \in C_b^2$. Itô's backward formula ([5] p.256) gives

$$\phi(X_{st}(x)) - \phi(x) = \int_s^t \gamma(r, X_{rt}(x))\phi'(X_{rt}(x))DX_{rt}(x)dB_r + \int_s^t \mathcal{L}_r(\phi(X_{rt}))(x)dr$$

By (3.14), the expectation of the above stochastic integral is equal to 0. If we define

$$(U(t,s)\phi)(x) = \mathbb{E}(\phi(X_{st}(x))),$$

taking expectations in Itô's backward formula and using Fubini's theorem, we get

$$(U(t,s)\phi)(x) - \phi(x) = \int_s^t \lambda(r,x)\mathbb{E}\Big(\frac{\partial\phi(X_{rt}(x))}{\partial x}\Big) + \frac{\gamma^2(r,x)}{2}\mathbb{E}\Big(\frac{\partial^2\phi(X_{rt}(x))}{\partial x^2}\Big)dr \tag{3.15}$$

For $k = 1$ or $k = 2$, the variables $\left(\frac{\partial^k}{\partial x^k}\phi(X_{st}(x))\right)_{x\in\mathbb{R}}$ depend continuously on x and are uniformly integrable by (3.14). Hence it is possible to exchange expectations and derivations in the right-hand-side of (3.15) to obtain

$$\forall \phi \in C_b^2,\ \forall 0 \leq s \leq t \leq T,\ \forall x \in \mathbb{R},\ (U(t,s)\phi)(x) - \phi(x) = \int_s^t \mathcal{L}_r(U(t,r)\phi)(x)dr \tag{3.16}$$

We are now going to prove that under our assumptions, for $\phi \in C_b^9$, this equation holds in the Banach space C_b^6.

Lemma 3.11 *Assume that $\sigma, b \in C_b^{10}$ and that (1.9) and* $[\mathbf{hyp'_9}]$ *hold. The operator $\mathcal{L}_t$ is a linear operator from C_b^8 into C_b^6 such that*

$$\forall t \in [0,T],\ \|\mathcal{L}_t\phi\|_{C_b^6} \leq K\|\phi\|_{C_b^8} \tag{3.17}$$

$$\forall s,t \in [0,T],\ \|\mathcal{L}_s\phi - \mathcal{L}_t\phi\|_{C_b^6} \leq K\|\phi\|_{C_b^8}|t-s| \tag{3.18}$$

For any $1 \leq j \leq 8$, the operator $U(t,s)$ is a linear operator on C_b^j such that

$$\forall 0 \leq s \leq t \leq T,\ \|U(t,s)\phi\|_{C_b^j} \leq K\|\phi\|_{C_b^j} \tag{3.19}$$

$$\forall 0 \leq s \leq s' \leq t \leq T,\ \|U(t,s)\phi - U(t,s')\phi\|_{C_b^j} \leq K\|\phi\|_{C_b^{j+1}}\sqrt{s'-s} \tag{3.20}$$

Proof : Inequality (3.17) is obvious. As $p \in H^{\frac{9+\alpha}{2}, 9+\alpha}([0,T]\times\mathbb{R})$, this function and its spatial partial derivatives up to order seven admit a continuous and bounded first derivative with respect to the time variable. Inequality (3.18) is easily deduced.
To prove the second part of the Lemma, we set $1 \leq j \leq 8$, $\phi \in C_b^j$ and $1 \leq k \leq j$. We have

$$\frac{\partial^k}{\partial x^k}\phi(X_{st}(x)) = \sum_{l=1}^{k} \sum_{l_1+2l_2+\ldots+kl_k=k} c(L)\phi^{(l)}(X_{st}(x))(DX_{st}(x))^{l_1}(D^2X_{st}(x))^{l_2}\ldots(D^kX_{st}(x))^{l_k}$$

with integer constants $c(L) = c(l, l_1, \ldots, l_k)$. Hence, by (3.14), the variables $\left(\frac{\partial^k}{\partial x^k}\phi(X_{st}(x))\right)_{x\in\mathbb{R}}$ are uniformly integrable. Since they depend continuously on x, we deduce that $U(t,s)\phi \in C_b^j$ with derivative of order k given by $\mathbb{E}\left(\frac{\partial^k}{\partial x^k}\phi(X_{st}(x))\right)$. Moreover, $\left|\frac{\partial^k}{\partial x^k}(U(t,s)\phi)(x)\right|$ is smaller than

$$\sum_{l=1}^{k}\sup_{y\in\mathbb{R}}|\phi^{(l)}(y)| \sum_{l_1+\ldots+kl_k=k} c(L)\mathbb{E}\left|(DX_{st}(x))^{l_1}(D^2X_{st}(x))^{l_2}\ldots(D^kX_{st}(x))^{l_k}\right|$$

and then bounded by $K\|\phi\|_{C_b^k}$. As clearly $\|U(t,s)\phi\|_{C_b^0} \leq \|\phi\|_{C_b^0}$, we deduce that (3.19) holds.

The proof of (3.20) is based on the following estimates given by Mitoma [11], Lemma 3

$$\forall 0 \leq s \leq s' \leq t \leq T,\ \forall x \in \mathbb{R},\ \mathbb{E}|X_{st}(x) - X_{s't}(x)|^2 \leq K(s'-s)$$
$$\forall 1 \leq j \leq 8,\ \mathbb{E}|D^jX_{st}(x) - D^jX_{s't}(x)|^2 \leq K(s'-s) \tag{3.21}$$

and obtained by computations similar to the previous ones. ∎

If $\phi \in C_b^9$, by the previous Lemma, $s \to \mathcal{L}_s(U(t,s)\phi)$ is continuous in C_b^6. Hence $\int_0^t \mathcal{L}_s(U(t,s)\phi)ds$ makes sense as a Riemann integral in C_b^6. Using (3.16), we deduce

$$(U(t,s)\phi) - \phi = \int_s^t \mathcal{L}_r(U(t,r)\phi)dr \ \text{ in } \ C_b^6 \tag{3.22}$$

This equation is the key point in the proof of uniqueness for (3.13).

Proposition 3.12 *Assume that $\sigma, b \in C_b^{10}$ and that (1.9) and* $[\mathbf{hyp'_9}]$ *hold. Then (3.13) has no more than one solution in $C([0,T], W_0^{-4,1})$. Moreover, any such solution ξ is characterized by*

$$\forall t \in [0,T],\ \xi_t = \int_0^t U(t,s)^* G_s ds \quad in \quad C^{-4} \tag{3.23}$$

Remark 3.13 *Let $\phi \in C_b^3$ and $s, r \in [0,T]$.*

$$\begin{aligned} | < G_r - G_s, \phi > | \leq & \Bigg| < p_r, \frac{1}{2} \int |z|^2 V(z) dz \Bigg(p_r''(.) \Big(\phi'(.) b'(p_r(.)) + \frac{\phi''(.)}{2} (\sigma^2)'(p_r(.)) \Big) \\ & \qquad - p_s''(.) \Big(\phi'(.) b'(p_s(.)) + \frac{\phi''(.)}{2} (\sigma^2)'(p_s(.)) \Big) \Bigg) \Bigg| \\ & + \Bigg| < p_s - p_r, \frac{1}{2} \Big(\int |z|^2 V(z) dz \Big) p_s''(.) \Big(\phi'(.) b'(p_s(.)) + \frac{\phi''(.)}{2} (\sigma^2)'(p_s(.)) \Big) > \Bigg| \end{aligned}$$

Since $p \in H^{\frac{9+\alpha}{2}, 9+\alpha}$, the first term of the right-hand-side is smaller than $K\|\phi\|_{C_b^3} |r-s|$. For the second term, we remark that the function $x \to p_s''(x) \Big(\phi'(x) b'(p_s(x)) + \frac{\phi''(x)}{2} (\sigma^2)'(p_s(x)) \Big)$ is bounded by $K\|\phi\|_{C_b^3}$ and Lipschitz continuous with constant $K\|\phi\|_{C_b^3}$. Hence

$$| < G_r - G_s, \phi > | \leq K \Big(|r-s| + d_{FM}(p_s(x)dx, p_r(x)dx) \Big) \|\phi\|_{C_b^3}$$

where d_{FM} denotes the Fortet-Mourier metric on $\mathcal{P}(\mathbb{R})$. Hence the mapping $s \to G_s$ is continuous in C^{-3}. By Lemma 3.11, we deduce that $s \to U(t,s)^* G_s$ is continuous in C^{-4}. Hence $\int_0^t U(t,s)^* G_s ds$ makes sense as a Riemann integral in C^{-4}.

Proof : Let $\xi \in C([0,T], W_0^{-4,1})$ satisfy (3.13) and ϕ belong to C_b^9. As $C_b^6 \hookrightarrow W_0^{6,1}$, by (3.22) we get

$$\begin{aligned} < \xi_t, \phi > &= \int_0^t < (\mathcal{L}_s)^* \xi_s, U(t,s)\phi - \int_s^t \mathcal{L}_r(U(t,r)\phi) dr > ds \\ &+ \int_0^t < G_s, U(t,s)\phi - \int_s^t \mathcal{L}_r(U(t,r)\phi) dr > ds \\ &= \int_0^t (< G_s, U(t,s)\phi > + < (\mathcal{L}_s)^* \xi_s, U(t,s)\phi >) ds \\ &- \int_0^t \int_s^t < (\mathcal{L}_s)^* \xi_s + G_s, \mathcal{L}_r(U(t,r)\phi) > dr ds \\ &= \int_0^t (< G_s, U(t,s)\phi > + < (\mathcal{L}_s)^* \xi_s, U(t,s)\phi >) ds \\ &- \int_0^t \int_0^r < (\mathcal{L}_s)^* \xi_s + G_s, \mathcal{L}_r(U(t,r)\phi) > ds dr \end{aligned}$$

As ξ solves (3.13) and $\mathcal{L}_r(U(t,r)\phi) \in C_b^6 \hookrightarrow W_0^{6,1}$, we have

$$\int_0^r < (\mathcal{L}_s)^* \xi_s + G_s, \mathcal{L}_r(U(t,r)\phi) > ds = < \xi_r, \mathcal{L}_r(U(t,r)\phi) >$$

Hence

$$< \xi_t, \phi > = \int_0^t < G_s, U(t,s)\phi > + < (\mathcal{L}_s)^* \xi_s, U(t,s)\phi >)ds - \int_0^t < \xi_r, \mathcal{L}_r(U(t,r)\phi) > dr$$
$$= \int_0^t < G_s, U(t,s)\phi >$$

Since C_b^9 is dense in C_b^4, we deduce that $\xi_t = \int_0^t U(t,s)^* G_s ds$ in C^{-4}. As C_b^4 is dense in $W_0^{-4,1}$ we conclude that uniqueness holds for (3.13) in $C([0,T], W_0^{-4,1})$. ∎

We are now ready to conclude :

Theorem 3.14 *Assume that $\sigma, b \in C_b^{10}$ and that (1.9) and* [$\mathbf{hyp'_9}$] *hold. Then the variables $\eta^n \in C([0,T], W_0^{-4,1})$ converge in L^1 to the deterministic process η such that the image of η_t by the continuous embedding of $W_0^{-4,1}$ into C^{-4} is given by $\int_0^t U(t,s)^* G_s ds$ for any $t \in [0,T]$.*

Proof : By Theorem 3.7 the laws of the processes $\eta^n \in C([0,T], W_0^{-4,1})$ are tight.
Let η be a variable distributed according to a limit point. By Theorem 3.9 and Proposition 3.12, η is the deterministic process such that $\forall t \in [0,T]$ the image of η_t by the continuous embedding of $W_0^{-4,1}$ into C^{-4} is $\int_0^t U(t,s)^* G_s ds$.
Since the unique limit point is a Dirac probability measure, the whole sequence η^n converges in probability to the process η. As by (3.12), the variables η^n are uniformly integrable, the convergence takes place in L^1. ∎

References

[1] R.A. Adams. *Sobolev Spaces.* Academic Press, 1978.

[2] B. Fernandez and S. Méléard. A hilbertian approach for fluctuations on the mckean-vlasov model. *Stochastic Processes and their Applications*, To appear.

[3] A. Friedman. *Stochastic Differential Equations and Applications*, volume 1. Academic Press, 1975.

[4] I.I. Gihman and A.V. Skorohod. *Stochastic differential equations.* Springer-Verlag, 1972.

[5] H. Kunita. Stochastic differential equations and stochastic flow of diffeomorphisms. In *Ecole d'été de probabilités de Saint-Flour XII - 1982, Lect. Notes in Math. 1097.* Springer-Verlag, 1984.

[6] O.A. Ladyzenskaya, V.A. Solonnikov, and N.N. Ural'ceva. *Linear and Quasilinear Equations of Parabolic Type*, volume 23 of *Translations of Mathematical Monographs.* American Mathematical Society, 1968.

[7] C. Léonard. Une loi des grands nombres pour des systèmes de diffusions avec interaction et à coefficients non bornés. *Ann. Inst. Henri Poincaré*, 22(2):237–262, 1986.

[8] H.P. McKean. Propagation of chaos for a class of non-linear parabolic equations. *In Lecture Series in Differential Equations*, 7:41–57, 1967.

[9] S. Méléard and S. Roelly-Coppoletta. A propagation of chaos result for a system of particles with moderate interaction. *Stochastic Processes and their Applications*, 26:317–332, 1987.

[10] P.A. Meyer. *Probabilités et Potentiel.* Hermann, 1966.

[11] I. Mitoma. An ∞-dimensional inhomogeneous langevin's equation. *Journal of Functional Analysis*, 61:342–359, 1985.

[12] D. Nualart. *The Malliavin Calculus and Related Topics.* Springer-Verlag, 1995.

[13] K. Oelschläger. A law of large numbers for moderately interacting diffusion processes. *Z. Wahrsch. Verw. Geb.*, 69:279–322, 1985.

[14] A.S. Sznitman. Topics in propagation of chaos. In *Ecole d'été de probabilités de Saint-Flour XIX - 1989, Lect. Notes in Math. 1464*. Springer-Verlag, 1991.

[15] K. Yosida. *Functional Analysis.* Springer-Verlag, sixth edition, 1980.