Identifying the multifractional function of a Gaussian process.

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Abstract

Gaussian processes that are multifractional are studied in this paper. By multifractional processes we mean that they behave locally like a fractional Brownian motion, but the fractional index is no more a constant: it is a function. We introduce estimators of this multifractional function based on discrete observations of one sample path of the process and we study their asymptotical behaviour as the mesh decreases to zero.

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1 Introduction

The fractional Brownian motion introduced in Mandelbrot and Van Ness (1968) provides a very powerful model in applied mathematics. The fractional (or Hölder) index α measures the smoothness of the sample paths of the fractional Brownian motion and the problem of identifying this scalar index has been widely investigated by many authors: Istas and Lang (1997), Benassi *et al.* (1997b), Benassi *et al.* (1996), Hall *et al.* (1994), Hall and Wood (1993). However fractional Brownian motion satisfies a strong stationarity condition on the increments that is too restrictive for some applications. Then the natural idea is to replace the fractional index α by a so-called multifractional function $\alpha(t)$ that depends on the time. These multifractional models have been used since a while in turbulence analysis (Frisch and Parisi (1985)) and more recently for edges detection in image analysis (Levy-Vehel and Mignot (1994), Canus and Levy-Vehel (1996), Levy-Vehel (1996)). Two generalizations of the fractional Brownian motion have been proposed independently by Benassi *et al.* (1997a) and Peltier and Levy-Vehel (1996) to conveniently model these phenomenons. They are both called multifractional Brownian motions.

In this paper multifractional Gaussian processes that are generalizations of multifractional Brownian motions are introduced. Our aim is to identify the multifractional function of multifractional Gaussian processes from the observations of the process X at N sampling instants. Peltier and Levy-Vehel (1996) describes a method to estimate the multifractional function of a multifractional Brownian motion. As far as we know the convergence of their estimator has not been studied. Their estimator is based on the average variation of the sampled process. Using the method of Benassi *et al.* (1996) for filtered white noises we introduce generalized quadratic variations because they allow Gaussian limiting distribution with $\sqrt{N}$-rate of convergence. To estimate the multifractional function generalized quadratic variations are now considered in a neighborhood of point t where identification of function $\alpha(t)$ is performed. We therefore have to choose a convenient neighborhood in terms of the number N of sampling points to build an estimator $\widehat{\alpha}(t)$. This estimator is strongly consistent and we study asymptotically the mean square error. Since the method of Benassi *et al.* (1996) are used to show the convergence of this new estimator we focus on the localization problem in this paper. A major advantage of this technique is that it can be applied to a generalization of multifractional Brownian motion that we call multifractional Gaussian processes (mGp). Let $\sigma(t,s) = \mathbb{E}(X(t) - X(s))^2$ be the variance of the increments of X. The mGp X satisfies a quadratic mean Hölder condition of index $\alpha(t)$ at point t, i.e. there exists a limit to $\sigma(t, t+h)|h|^{-2\alpha(t)}$ as $h \to 0$. In this framework the limit which is called the normalized variance depends both on the multifractional function $\alpha(t)$ and on another functional parameter $a_\infty(t)$ which was already introduced for filtered white noises.

The paper is organized as follows : in section 2 multifractional Gaussian processes are introduced. They are shown to behave locally around a point t as a fractional Brownian motion with index $\alpha(t)$. Section 3 is devoted to the definition of the estimator and the proof of the convergence.

2 Multifractional Gaussian processes and local scaling

Let us first derive from the definition of multifractional Brownian motion and of filtered white noises in Benassi *et al.* (1996) the definition of multifractional Gaussian processes (mGp).

Definition 1 *Multifractional Gaussian processes (mGp)* $(X_t)_{t\in[0,1]}$ *are real Gaussian processes whose covariance* Σ *are of the form*

$$\Sigma(t,s) \quad = \quad \int_{\mathbb{R}} f(t,\lambda)\overline{f(s,\lambda)}d\lambda \ ,$$

where

$$f(t,\lambda) \quad = \quad \frac{(e^{it\lambda}-1)a(t,\lambda)}{|\lambda|^{1/2+\alpha(t)}} \ . \tag{1}$$

Moreover it is assumed

A 1 *Smoothness of the process.*

Function α *is* C^1 *with* $0<\alpha(t)<1 \quad \forall t\in[0,1]$.

A 2 *Modulation.*

Function $a(t,\lambda)$ *is defined from* $[0,1]\times\mathbb{R}$ *to* $\mathbb{R}$ *and satisfies:*

$$a(t,\lambda) = a_\infty(t) + R(t,\lambda) \ , \tag{2}$$

where a_∞ *is* C^1 *on* $[0,1]$ *with,* $\forall t\in[0,1]$, $a_\infty(t)\neq 0$ *and* $R\in C^{1,2}([0,1]\times\mathbb{R})$ *is negligible at high frequency i.e.* $\exists\eta>0$ *such that* $\forall i=0$ *to* $1,\quad j=0$ *to* 2

$$\left|\frac{\partial^{i+j}}{\partial t^i\partial\lambda^j}R(t,\lambda)\right| \quad \le \quad \frac{C}{|\lambda|^{\eta+j}} \ .$$

To motivate this definition recall that multifractional Brownian motion is presented in Benassi *et al.* (1997a) as

$$X_t = \int_{\mathbb{R}} \frac{e^{it\lambda}-1}{|\lambda|^{1/2+\alpha(t)}}W(d\lambda)$$

where $W(d\lambda)$ is a Brownian measure on $L^2(\mathbb{R})$, hence it corresponds to

$$f(t,\lambda) = \frac{e^{it\lambda}-1}{|\lambda|^{1/2+\alpha(t)}}.$$

Filtered white noises are other instances of mGp which are associated to a function

$$f(t,\lambda) = \frac{a(t,\lambda)(e^{it\lambda}-1)}{|\lambda|^{1/2+\alpha}}.$$

We refer to Benassi *et al.* (1996) where filtered white noises are introduced to understand the meaning of the function $a(t,\lambda)$. Although filtered white noises have no stationary increments as multifractional Brownian motion they have still a constant index α. The most appealing fact is that identification techniques close to those of Benassi *et al.* (1996) used for filtered white noises are applied to mGp.

The aim is to obtain a process that behaves locally at point t like an $\alpha(t)-$fractional process. The next proposition shows that mGps satisfy a multifractional framework as defined in the introduction.

Proposition 1 *Let us assume X is a mGp and $\sigma(t,s)=\mathbb{E}(X(t)-X(s))^2$ then*

$$\lim_{h\to 0}\sigma(t,t+h)|h|^{-2\alpha(t)} \quad = \quad 4a_\infty^2(t)\int_{\mathbb{R}}\frac{sin^2 u/2}{|u|^{1+2\alpha(t)}}du\ .$$

Proof of Proposition 1

Let $T=a(t+h,\lambda)|\lambda|^{\alpha(t)}(e^{i(t+h)\lambda}-1)-a(t,\lambda)(e^{it\lambda}-1)|\lambda|^{\alpha(t+h)}$,

so that $\sigma(t,t+h)=\int_{\mathbb{R}}\frac{|T|^2}{|\lambda|^{2\alpha(t)+2\alpha(t+h)+1}}d\lambda$. By Taylor expansions

$$\begin{aligned}|\lambda|^{\alpha(t+h)} &= |\lambda|^{\alpha(t)}+hLog|\lambda|\alpha'(c)|\lambda|^{\alpha(c)}\ ,\\ a(t+h,\lambda) &= a(t,\lambda)+ha'(\theta,\lambda)\ ,\end{aligned}$$

where $c,\theta\in]t,t+h[$ and $a'(t,\lambda)$ stands for $\frac{\partial}{\partial t}a(t,\lambda)$. Let $T=T_1+T_2+T_3$ with

$$\begin{aligned}T_1 &= a(t,\lambda)|\lambda|^{\alpha(t)}e^{it\lambda}(e^{ih\lambda}-1)\ ,\\ T_2 &= -a(t,\lambda)|\lambda|^{\alpha(c)}\alpha'(c)(e^{it\lambda}-1)hLog|\lambda|\ ,\\ T_3 &= ha'(\theta,\lambda)(e^{i(t+h)\lambda}-1)|\lambda|^{\alpha(t)}\ .\end{aligned}$$

Define $I_{i,j}=\int_{\mathbb{R}}\frac{T_i\overline{T_j}}{|\lambda|^{2\alpha(t)+2\alpha(t+h)+1}}d\lambda$, $i,j=1,2,3$, so that $\sigma(t,t+h)=\sum_{i,j=1}^{3}I_{i,j}$. First consider $I_{1,1}=4\int_{\mathbb{R}}\frac{sin^2(h\lambda/2)a^2(t,\lambda)}{|\lambda|^{2\alpha(t+h)+1}}d\lambda$. By the change of variable $u=h\lambda$,

$I_{1,1}=4|h|^{2\alpha(t+h)}\int_{\mathbb{R}}\frac{sin^2(u/2)a^2(t,u/h)}{|u|^{2\alpha(t+h)+1}}du$. By Lebesgue Theorem, as $h\to 0$,

$I_{1,1}\sim 4a_\infty^2(t)|h|^{2\alpha(t)}\int_{\mathbb{R}}\frac{sin^2 u/2}{|u|^{2\alpha(t)+1}}du$.

Let us now consider $I_{2,2}=4h^2{\alpha'}^2(c)\int_{\mathbb{R}}\frac{Log^2|\lambda|sin^2(t\lambda/2)a^2(t,\lambda)}{|\lambda|^{2\alpha(t+h)+2\alpha(t)-2\alpha(c)+1}}d\lambda$. By the change of variable $u=h\lambda$,

$$I_{2,2} \quad = \quad 4{\alpha'}^2(c)|h|^{2\alpha(t+h)+2\alpha(t)-2\alpha(c)+2}\int_{\mathbb{R}}\frac{Log^2|u|sin^2(\frac{tu}{2h})a^2(t,u/h)}{|u|^{2\alpha(t+h)+2\alpha(t)-2\alpha(c)+1}}du \tag{3}$$

$$-4{\alpha'}^2(c)|h|^{2\alpha(t+h)+2\alpha(t)-2\alpha(c)+2}Log^2|h|\int_{\mathbb{R}}\frac{sin^2(\frac{tu}{2h})a^2(t,u/h)}{|u|^{2\alpha(t+h)+2\alpha(t)-2\alpha(c)+1}}du\ . \tag{4}$$

The computations for (3) and (4) are similar. Therefore, we only consider (3) and split the integral by integrating first on $|u| \leq h$ and then on $|u| > h$:

$$\begin{aligned} J_{2,2} &= \int_{|u|\leq|h|} \frac{Log^2|u|sin^2(\frac{tu}{2h})a^2(t,u/h)}{|u|^{2\alpha(t+h)+2\alpha(t)-2\alpha(c)+1}}du \\ &+ \int_{|u|>|h|} \frac{Log^2|u|sin^2(\frac{tu}{2h})a^2(t,u/h)}{|u|^{2\alpha(t+h)+2\alpha(t)-2\alpha(c)+1}}du \ . \end{aligned}$$

Using assumption **A**2

$$\begin{aligned} \int_{|u|\leq|h|} \frac{Log^2|u|sin^2(\frac{tu}{2h})a^2(t,u/h)}{|u|^{2\alpha(t+h)+2\alpha(t)-2\alpha(c)+1}}du &\leq \frac{t^2}{4h^2}\sup_{t,\lambda}|a(t,\lambda)| \int_{|u|\leq|h|} \frac{Log^2|u|u^2}{|u|^{2\alpha(t+h)+2\alpha(t)-2\alpha(c)+1}}du \\ &\leq CLog^2|h||h|^{-(2\alpha(t+h)+2\alpha(t)-2\alpha(c))} \ , \end{aligned}$$

where C is a constant that may change from an occurrence to another. The second integral

$$\begin{aligned} \int_{|u|>|h|} \frac{Log^2|u|sin^2(\frac{tu}{2h})a^2(t,u/h)}{|u|^{2\alpha(t+h)+2\alpha(t)-2\alpha(c)+1}}du &\leq C\int_{|u|>|h|} \frac{Log^2|u|}{|u|^{2\alpha(t+h)+2\alpha(t)-2\alpha(c)+1}}du \\ &\leq CLog^2|h||h|^{-(2\alpha(t+h)+2\alpha(t)-2\alpha(c))} \ . \end{aligned}$$

To sum up $I_{2,2} \leq CLog^2|h|h^2$. With closely related arguments, one proves that $I_{3,3} \leq O(h^2)$. We now consider $I_{i,j}$, $i,j = 1,2,3$ $i \neq j$. By Cauchy-Schwarz, one has $|I_{i,j}|^2 \leq I_{i,i}\ I_{j,j}$. Hence $I_{1,1}$ is preponderant and Proposition 1 is proved.□

Besides the Proposition 1 that states a quadratic mean Hölder condition for mGp there is another theoretical reason to consider mGp as a natural multifractional generalization of fractional Brownian motion. The distributions of mGp satisfy a local scaling property which is similar to the one satisfied by the multifractional Brownian motion. If mGp is localized around t by a scaling factor ϵ and if a convenient renormalization is applied to the increments of this process then it asymptotically converges in distribution to a fractional Brownian motion with index $\alpha(t)$ when the scaling factor goes to zero. Let us write

$$\sigma^2(t) = 4a_\infty^2(t)\int_{\mathbb{R}} \frac{sin^2 u/2}{|u|^{1+2\alpha(t)}}du$$

which is called the normalized variance, the following proposition expresses the local scaling property.

Proposition 2 *Let X be a mGp with multifractional function α and normalized variance σ,*

$$\lim_{\epsilon\to 0^+}\left(\frac{X(t+\epsilon u)-X(t)}{\epsilon^{\alpha(t)}}\right)_{u\in\mathbb{R}} \overset{(d)}{=} \sigma(t)\left(B_{\alpha(t)}(u)\right)_{u\in\mathbb{R}}$$

where $B_{\alpha(t)}$ is a fractional Brownian motion with index $\alpha(t)$. The convergence is a convergence in distribution on the space of continuous functions endowed with the topology of the uniform convergence on compact sets.

Proof

Proposition 1 yields the convergence of the finite dimensional distribution of the process $\left(\frac{X(t+\epsilon u)-X(t)}{\epsilon^{\alpha(t)}}\right)_{u\in\mathbb{R}}$ to those of $\left(B_{\alpha(t)}(u)\right)_{u\in\mathbb{R}}$. To have the convergence in distribution for the topology of the uniform convergence on compact sets a tightness result is required. Because of Proposition 1 for integer $p=2$ and for every compact K there exists a finite constant $C(t,p)$ such that

$$\forall u,\, v \in K \quad \mathbb{E}\left\|\frac{X(t+\epsilon u)-X(t+\epsilon v)}{\epsilon^{\alpha(t)}}\right\|^p \leq C(t,p)|u-v|^{\alpha(t)p}.$$

Since the processes are Gaussian this inequality can extended to p large enough to get $\alpha(t)p>1$. Hence one can classically deduce that the sequence of the laws of $\left(\frac{X(t+\epsilon u)-X(t)}{\epsilon^{\alpha(t)}}\right)_{\epsilon>0}$ is relatively compact.

3 Identification result

First the estimator is introduced then the convergence is proved in Theorem 1. The Process X is observed at sampling points $\frac{p}{N}, p=0,\ldots,N$. For convenience, one assumes N even. The localized generalized quadratic variations are now precisely described. For any $t\in]0,1[$, $\varepsilon>0$ and $N>0$, we define an (ε,N)-neighborhood of t by

$$\mathcal{V}_{\varepsilon,N}(t) \quad = \quad \left\{p\in\mathbb{Z},\, \left|\frac{p}{N}-t\right| \leq \varepsilon\right\} \ .$$

As pointed out in Istas and Lang (1997) the generalized variations have to be the sum of discrete second derivatives (i.e. $X(\frac{p+1}{N})-2X(\frac{p}{N})+X(\frac{p-1}{N})$) to have $\sqrt{N}-$rate of convergence for our estimator. Hence the localized generalized variation is defined by

$$\mathbf{V}_{\varepsilon,N}(t) \quad = \quad \sum_{p\in\mathcal{V}_{\varepsilon,N}(t)} \left(X(\frac{p+1}{N})-2X(\frac{p}{N})+X(\frac{p-1}{N})\right)^2 \ . \tag{5}$$

There is an edge problem in the definition of (5): $\mathbf{V}_{\varepsilon,N}(t)$ is defined for t such that $\varepsilon\leq t\leq 1-\varepsilon-K/N$. As $N\to\infty$, ε will decrease to 0. Therefore, for any $t\in]0,1[$, $\mathbf{V}_{\varepsilon,N}(t)$ is defined for N large enough.

An estimator of the multifractional function $\alpha(t)$ is defined by

$$\widehat{\alpha}_{\varepsilon,N}(t) \quad = \quad \frac{1}{2}\left(\log_2\frac{\mathbf{V}_{\varepsilon,N/2}(t)}{\mathbf{V}_{\varepsilon,N}(t)}+1\right) \ .$$

Theorem 1 *Assume* ***A1*** *and* ***A2****. Take* $\varepsilon=N^{-\beta}$ *with* $0<\beta<1/2$ *. Then, as* $N\to\infty$*,*

$$\widehat{\alpha}_{\varepsilon,N}(t) \quad \to \quad \alpha(t) \quad (a.s.)\ .$$

Take $\varepsilon=N^{-\beta}$ *with* $\beta=1/3$

- *if* $\eta \geq 1/3$

$$\mathbb{E}(\widehat{\alpha}_{\varepsilon,N}(t)-\alpha)^2 \quad \leq \quad O(Log^2(N)N^{-2/3})\ .$$

- *if* $\eta < 1/3$

$$\mathbb{E}(\widehat{\alpha}_{\varepsilon,N}(t)-\alpha)^2 \quad \leq \quad O(N^{-2\eta})\ .$$

In the previous result the case $\eta \geq 1/3$ can be understood as the case where the error coming from the localization on $\mathcal{V}_{\varepsilon,N}(t)$ is preponderant whereas the case $\eta < 1/3$ corresponds to the case where the rest term $R(t,\lambda)$ is preponderant.

Since the proof of Theorem 1 is mainly based on technical lemmas that are proved in Benassi *et al.* (1996) some notations are recalled . Then we remark that when the generalized quadratic variations are localized on an open interval non depending on N the infimum of the multifractional function on this open interval is estimated. This remark explains why localized generalized variations work for smooth multifractional functions. At last we have stressed in Lemma 1, 2 and Proposition 3 the arguments concerning the localization.

Classically the study of $\mathbf{V}_{\varepsilon,N}(t)$ requires estimates of its expectation and variance. For each $p,\ p' \in \mathbb{Z}$ and functions $A,\ B$ the following integral is introduced

$$I(A,B)_{p,p'} = \int_{\mathbb{R}} \sum_{k,k'=0}^{K} d_k d_{k'} (e^{i(k+p)u}-1)(e^{i(k'+p')u}-1) A(\frac{k+p}{N}, Nu) B(\frac{k'+p'}{N}, Nu) N du$$

when they are defined. Let

$$S(t,\lambda) = \frac{a_\infty(t)}{|\lambda|^{\alpha(t)+\frac{1}{2}}} + \frac{R(t,\lambda)}{|\lambda|^{\alpha(t)+\frac{1}{2}}},$$

if $d_0 = d_2 = 1$ and $d_1 = -2$ then $X(\frac{p+1}{N}) - 2X(\frac{p}{N}) + X(\frac{p-1}{N}) = \sum_{k=0}^{2} d_k X(\frac{p+k}{N})$ and

$$\begin{aligned}
\mathbb{E}(\mathbf{V}_{\varepsilon,N}(t)) &= \sum_{p \in \mathcal{V}_{\varepsilon,N}(t)} \mathbb{E} \sum_{k,k'=0}^{K} d_k d_{k'} X(\frac{k+p}{N}) X(\frac{k'+p}{N}) && (6)\\
&= \sum_{p \in \mathcal{V}_{\varepsilon,N}(t)} I(S,S)_{p,p}\ . && (7)
\end{aligned}$$

Since X is a Gaussian process

$$\begin{aligned}
\mathrm{var}(\mathbf{V}_{\varepsilon,N}(t))) &= 2 \sum_{p,p' \in \mathcal{V}_{\varepsilon,N}(t)} [\mathbb{E} \sum_{k,k'=0}^{K} d_k d_{k'} X(\frac{k+p}{N}) X(\frac{k'+p'}{N})]^2 && (8)\\
&= 2 \sum_{p,p' \in \mathcal{V}_{\varepsilon,N}(t)} (I(S,S)_{p,p'})^2. && (9)
\end{aligned}$$

Let us recall assumption **A 4** of Benassi *et al.* (1996) which is used to get the asymptotic of $I(A,B)_{p,p'}$ as $N \to \infty$.

A 4 $A(t,\lambda) \in C^{1,2}([0,1]\times\mathbb{R}^*)$ *is a function such that,*

$$\left|\frac{\partial^{i+j}}{\partial t^i \partial\lambda^j} A(t,\lambda)\right| \leq \frac{C}{|\lambda|^{\frac{1}{2}+\delta+j}} ,$$

for $i=0$ *to* 1 *and* $j=0$ *to* 2 *with* $1>\delta>0$*.*

Let us define

$$S_0(s,\lambda) = \frac{a_\infty(s)}{|\lambda|^{\alpha(s)+\frac{1}{2}}} \tag{10}$$

and

$$S_1(s,\lambda) = \frac{R(s,\lambda)}{|\lambda|^{\alpha(s)+\frac{1}{2}}}, \tag{11}$$

these two functions satisfy **A 4** when $|\lambda| \to +\infty$ with $\delta < \inf(\alpha(s), s \in [0,1])$ for S_0, and $\delta < \inf(\alpha(s), s \in [0,1]) + \eta$ for S_1. Hence we can upperbound $I_{p,p'}(S_i,S_j)$ as in Lemma 1 of Benassi *et al.* (1996).

Remark 1 *Consider*

$$\widehat{\alpha}(t_0,t_1) = \frac{1}{2}\left(\log_2 \frac{\boldsymbol{V}_{N/2}(t_0,t_1)}{\boldsymbol{V}_N(t_0,t_1)}) + 1\right)$$

where $0 < t_0 < t_1 < 1$ *and where*

$$\boldsymbol{V}_N(t_0,t_1) = \sum_{\{p\in\mathbb{Z},\, t_0 \leq \frac{p}{N} \leq t_1\}} \left(X(\frac{p+1}{N}) - 2X(\frac{p}{N}) + X(\frac{p-1}{N})\right)^2 .$$

One can prove that

$$\lim_{N\to+\infty} \widehat{\alpha}(t_0,t_1) = \inf(\alpha(s),\ s \in (t_0,t_1)) \quad (a.s.).$$

Besides the fact that $\inf(\alpha(s),\ s\in(t_0,t_1))$ *is an important quantity to understand a mGp on a given interval* (t_0,t_1)*(See Proposition 8 in Peltier and Levy-Vehel (1996) for a related result) we prefer a direct estimation of the function* $\alpha(t)$*.*

Let us give localized versions of two Lemmas of Benassi *et al.* (1996)

Lemma 1 *For each* δ *such that* $\alpha(t) > \delta$ *there exists a neighborhood* $\mathcal{V}_t$ *of* t *and* N_0 *such that for* $N \geq N_0$ *and* p/N*,* $p'/N \in \mathcal{V}_t$

$$|I(S_i,S_j)_{p,p'}| \leq \frac{C}{N^{2\delta+(i+j)\eta}(1+(p-p')^2)} \quad \textit{for } i,\ j = 0,\ 1.$$

Proof of Lemma 1

Since α is continuous there exists a neighborhood $\mathcal{V}_t$ of t such that $\inf(\alpha(s), s\in\mathcal{V}_t) > \delta$. Moreover $I(S_i,S_j)_{p,p'}$ depends only on S_i, S_j in a neighborhood of $\frac{p}{N}$ and $\frac{p'}{N}$. Hence Lemma 1 of Benassi *et al.* (1996) can be applied.

Indeed we get a finer estimate for $I(S_0,S_0)_{p,p'}$.

Lemma 2 *For each δ such that $\alpha(t) > \delta > \max(\alpha(t) - \frac{1}{2}, 0)$ there exists a neighborhood $\mathcal{V}_t$ of t such that $\sup(\alpha(s), s \in \mathcal{V}_t) < \delta + 1/2$ and N_0 such that for $N \geq N_0$ p/N, $p'/N \in \mathcal{V}_t$*

$$I(S_0, S_0)_{p,p'} = \frac{a_\infty(p/N)a_\infty(p'/N)}{N^{\alpha(p/N)+\alpha(p'/N)}} F_{\alpha(p/N)+\alpha(p'/N)}(p-p') + O(\frac{1}{N^{2\delta+1}(1+(p-p')^2)}),$$

where

$$F_\gamma(x) = 2\sum_{k,k'=0}^{K} d_k d_{k'} \int_{\mathbb{R}} e^{iux} \frac{\sin^2(\frac{(k-k')u}{2})}{|u|^{\gamma+1}} du.$$

See Lemma 2 of Benassi *et al.* (1996) for a closely related proof.

Then one can get the asymptotic of expectation, variance and a law of the large number for $\mathbf{V}_{\varepsilon,N}(t)$.

Proposition 3 *If $\varepsilon = N^{-\beta}$ with $0 < \beta < 1/2$*

$$\begin{aligned}
\mathbb{E}(\boldsymbol{V}_{\varepsilon,N}(t)) &= a_\infty^2(t)N^{1-2\alpha(t)-\beta}F_{2\alpha(t)}(0) + O(\ln(N)N^{1-2\alpha(t)-2\beta}) + O(N^{1-2\alpha(t)-\beta-\eta}) \\
var(\boldsymbol{V}_{\varepsilon,N}(t)) &= 2\, a_\infty^4(t)\sum_{q=-\infty}^{+\infty} F^2_{2\alpha(t)}(q)N^{1-4\alpha(t)-\beta}(1+o(1)) \\
\lim_{N\to+\infty}\frac{\boldsymbol{V}_{\varepsilon,N}(t)}{\mathbb{E}(\boldsymbol{V}_{\varepsilon,N}(t))} &= 1 \quad (a.s.).
\end{aligned}$$

Proof of Proposition 3

Using Lemmas 1 and 2 the asymptotic of $\mathbb{E}(\mathbf{V}_{\varepsilon,N}(t))$ are estimated as in the proof of Proposition 1 in Benassi *et al.* (1996),

$$\begin{aligned}
\mathbb{E}(\mathbf{V}_{\varepsilon,N}(t)) &= \varepsilon a_\infty^2(t)N^{1-2\alpha(t)}F_{2\alpha}(0) + O(\varepsilon^2\ln(N)N^{1-2\alpha(t)}) \\
&+ O(\varepsilon N^{1-2\underline{\alpha}(\varepsilon)-\eta}) .
\end{aligned}$$

where $\underline{\alpha}(\varepsilon) = \min\{\alpha(t);\ t \in \mathcal{V}_{2\varepsilon,N}(t)\}$. To compute the asymptotic of $\mathrm{var}(\mathbf{V}_{\varepsilon,N}(t))$ Q_N is introduced to truncate the sums that appear in the study of the asymptotic of $\mathrm{var}(\mathbf{V}_{\varepsilon,N}(t))$ as in the proof of Proposition 1 in Benassi *et al.* (1996). Then :

$$\begin{aligned}
\mathrm{var}(\mathbf{V}_{\varepsilon,N}(t)) &= 4\,\varepsilon N^{1-4\alpha(t)}a_\infty^4(t)\sum_{q=-\infty}^{+\infty}F^2_{2\alpha(t)}(q) + O(Q_N\varepsilon^2N^{1-4\alpha(t)}) \\
&+ O(\varepsilon^2N^{1-4\alpha(t)}\ln(N)) + O(Q_N^{-3}N^{1-4\underline{\alpha}(\varepsilon)}) + O(\varepsilon N^{1-4\underline{\alpha}(\varepsilon)-2\eta})
\end{aligned}$$

where $Q_N \to +\infty$ with $Q_N = o(N\varepsilon)$. Since $\varepsilon = N^{-\beta}$ and $N^{-\underline{\alpha}(\varepsilon)} \sim N^{-\alpha(t)}$ as $N \to \infty$, the first two asymptotics are easily deduced. Slight modifications of Proposition 2 in Benassi *et al.* (1996) leads to

$$\mathbb{E}(\mathbf{V}_{\varepsilon,N}(t) - \mathbb{E}(\mathbf{V}_{\varepsilon,N}(t)))^4 = O(\mathrm{var}^2(\mathbf{V}_{\varepsilon,N}(t))),$$

and

$$\frac{\mathrm{var}^2(\mathbf{V}_{\varepsilon,N}(t))}{\mathbb{E}(\mathbf{V}_{\varepsilon,N}(t))^4} = O(N^{2(\beta-1)}).$$

Since $\beta < 1/2$ Borel Cantelli's Lemma applied to $\frac{\mathbf{V}_{\varepsilon,N}(t)}{\mathbb{E}(\mathbf{V}_{\varepsilon,N}(t))}$ proves the almost sure convergence.

*P*roof of Theorem 1

The results on $\mathbf{V}_{\varepsilon,N}(t)$ are then applied to $\widehat{\alpha}_{\varepsilon,N}(t)$ as in Istas and Lang (1997) The rate of convergence of $\mathbb{E}(\widehat{\alpha}_{\varepsilon,N}(t) - \alpha(t))^2$ follows directly from Proposition 3. The choice $\beta = 1/3$ is the optimal choice to balance between the variance and the bias terms.

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