Sharp asymptotics for the KPP's equation with geometric and irregular initial condition.

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Abstract. In this article sharp asymptotics for the solution of non homogeneous Kolmogorov Petrovskii Piskunov equation depending on a small parameter are considered when the initial condition is the characteristic function of a set $A \in \mathbb{R}^d$. We show how to extend the Ben Arous and Rouault's result that deal with $d = 1$ and initial condition $A = \{x \leq 0\}$. The dependance of the asymptotics on the geometry of the boundary of A is precisely described for the problem with constraint.

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1 Introduction.

The aim of this article is to obtain sharp asymptotics ($\epsilon \to 0$) for the solution of non homogeneous Kolmogorov Petrovskii Piskunov [5] (in short KPP) equation when the variables are in $\mathbb{R}^d$ i.e.

$$\begin{cases} \dfrac{\partial u^\epsilon}{\partial t}(t,x) &= \dfrac{\epsilon^2}{2}\Delta u + \dfrac{c(x)}{\epsilon^2}u^\epsilon(1-u^\epsilon) \\ u^\epsilon(0,x) &= \mathbf{1}_A(x) \end{cases} \tag{1}$$

where $x \in \mathbb{R}^d$, c is C^3 non negative function on $\mathbb{R}^d$ such that $c(x) \leq const(1 + \|x\|)$, for some generic strictly positive constant $const$. Actually we are extending both Ben Arous-Rouault's [2] results , that deal with $d = 1$ and initial condition $A = \{x \leq 0\}$, and Freidlin's results [4] that give only logarithmic asymptotics. From these previous studies we know that all asymptotics are related to the path ϕ maximizing the rate function of this large deviation problem with $\phi_0 = x$ and $\phi_T \in A$. If ϕ_T belongs to the interior of A the problem is equivalent to a problem without constraint which is solved in [1]. For related results we can quote [3] where sharp asymptotics are given for smooth initial conditions in different frameworks, and the travelling wave for a step initial distribution is studied but without the speed of convergence when ϵ goes to 0. In this article we are focusing on the case $\phi_T \in \partial A$ the boundary of A and we explain how the geometry of ∂A interferes with the constant describing the asymptotics. From this point of view Theorem 1 shows that the sharp asymptotic for the linearized equation associated to (1) involves the secund fundamental form of ∂A at ϕ_T. Furthermore the constant which describes the asymptotics of the non linear problem depends only on ϕ_T and on the normal vector at ϕ_T pointing outside A. Besides this geometric part the main tool is the Laplace transform in the Wiener space and we refer to [2] as long as higher dimension and geometry do not require more delicate techniques. For instance many arguments of the one dimensional case can be extended to greater dimension if we let $A = \{f(x) \leq 0\}$ where the assumptions concerning the function $f : \mathbb{R}^d \longmapsto \mathbb{R}$ are given in the next section.

2 Statement of the results.

Let us recall the definitions of the actions related to this large deviation problem :

$$R_{0,t}(\psi) = \int_0^t [c(\psi_s) - \frac{1}{2}\|\dot{\psi}_s\|^2] ds \qquad\qquad R^*_{o,t}(\psi) = \min_{0\leq a\leq t} R_{o,a}(\psi) \tag{2}$$

and the potentials

$$\begin{aligned} V(T,x) &= sup(R_{0,T}(\psi)\ ; \psi_0 = x, f(\psi_T) \leq 0) \qquad (3)\\ V^*(T,x) &= sup(R^*_{0,T}(\psi)\ ; \psi_0 = x, f(\psi_T) \leq 0). \qquad (4)\end{aligned}$$

The asymptotic behavior of the solution of the linear equation

$$\begin{cases} \dfrac{\partial v^\epsilon}{\partial t}(t,x) &= \dfrac{\epsilon^2}{2}\Delta v + \dfrac{c(x)}{\epsilon^2} v^\epsilon \\ v^\epsilon(0,x) &= \mathbf{1}_A(x) \end{cases} \tag{5}$$

is described by R and V. Under the hypothesis :

H 1 *the maximum in V is attained at a unique path ϕ, and this maximum is non degenerate*

sharp asymptotics for v^ϵ are given by the Theorem 1. At first we need some smoothness assumption for f in a neighborhood of ϕ_T to express the Euler equation associated to V. Hence

H 2 *ϕ_T is assumed to be on ∂A the boundary of A which is a C^2 manifold of dimension $d-1$ imbedded in $\mathbb{R}^d$ in some neighborhood of ϕ_T.*

Let us introduce some notations concerning the local geometry of ∂A at point ϕ_T. In the neighborhood N of ϕ_T where ∂A is smooth we can assume $f(x) = \epsilon(x) d(x, \partial A)$ where $\epsilon(x)$ is 1 outside A and -1 inside. Hence $\nabla f(x)$ is always a unitary vector on N and it is the normal vector pointing outside A at each point of ∂A. Moreover the Hessian $f"(\phi_T)$ is given by the opposite of the secund fundamental form α of ∂A at ϕ_T i.e.

$$f"(\phi_T)(v,w) = -\alpha_{\phi_T}(\pi(v), \pi(w)) \tag{6}$$

where π is the orthogonal projection of $\mathbb{R}^d$ onto $T_{\phi_T}\partial A$. When **H 1**, **H 2** are fulfilled ϕ satisfies

$$\begin{cases} \ddot{\phi} = -c'(\phi) \\ p\ := -\dot{\phi}_T = \lambda.\nabla f(\phi_T), \quad \mathit{with} \quad \lambda \geq 0 \\ f(\phi_T) = 0 \quad \mathit{or} \quad \dot{\phi}_T = 0 \end{cases} \tag{7}$$

where the last condition is known as complementary slackness. In the following theorem we only consider the problem with the constraint $f(\phi_T) = 0$ since the other instance is already solved in [1].

Theorem 1 (linear case) *Under the assumptions* **H 1** *and* **H 2** *if* $f(\phi_T)=0$ *:*

$$\frac{1}{\epsilon}\exp(-\frac{V(T,x)}{\epsilon^2})v^\epsilon(T,x)\underset{\epsilon\to 0}{\longrightarrow} \tag{8}$$
$$\frac{1}{\|p\|(2\pi T)^{d/2}}\int_{<\nabla f(\phi_T)>^\perp} C_b(\phi)\exp(-\frac{\|b\|^2}{2T}+\frac{\|p\|}{2}\alpha_{\phi_T}(b,b))db$$

with

$$C_b(\phi)=\mathbb{E}(\exp(\frac{1}{2}\int_0^T c''(\phi_s)(W_s^0+\frac{sb}{T})^2ds)).$$

The limit in (1) is always finite if ϕ is a non degenerated maximum of $V(T,x)$.

Please note that $c''(\phi_s)$ stands for the Hessian of c at point ϕ_s and that this quadratic form is applied to $(W_s^0+\frac{sb}{T},W_s^0+\frac{sb}{T})$ written symbolically $(W_s^0+\frac{sb}{T})^2$ where W_s^0 is a Brownian bridge on $[0,T]$. We remark that $C_b(\phi)$ is the natural extension of the constant $C_0(\phi)$ in [2] where various interpretations of this constant are given. Moreover the limit in (1) can be written as

$$\frac{1}{\|p\|\sqrt{2\pi T}}\mathbb{E}_{W_{\partial A}}[\exp(\frac{1}{2}\int_0^T c''(\phi_s)(W_{\partial A}(s))^2ds+\frac{\|p\|}{2}\alpha_{\phi_T}(W_{\partial A}(T))^2)]$$

where $W_{\partial A}$ is Brownian bridge with the end point on the tangent space of ∂A at ϕ_T. To describe the asymptotics of u^ϵ the counterpart of the hypotheses (H3)and (H4) in [2] is

H 3 $\{(\phi,a)\in H^x\times[0,T]\ ;f(\phi_T)\leq 0$ *and* $R_{0,a}(\phi)=R^*_{o,T}(\phi)=V^*(T,x)\}$ *is a singleton* (ϕ,T) *with* $V^*(T,x)<0$ *and* $c(\phi_T)-\frac{1}{2}\|\dot{\phi}_T\|^2<0$.

As in [2] this hypothesis summarizes the fact that the path ϕ leading the asymptotics for the linear problem runs always ahead of the front of the non linear problem. It is crucial to use the boundary layer techniques and it naturally involves global geometric conditions on A which seem hard to precise at least when c is not constant. For examples where the hypotheses **H 1**, **H 2** and **H 3** are fulfilled one can refer to Remark 4 in [2] and the discussion of the hypothesis (N) in [4]. To find the constant for the sharp asymptotics of the non linear problem the following homogeneous KPP equation is introduced

$$\begin{cases}\frac{\partial\tilde u}{\partial s}(s,\xi) &= \frac{1}{2}\Delta\tilde u+c(\phi_T)\tilde u(s,\xi)(1-\tilde u(s,\xi))\\ \tilde u(0,\xi) &= \mathbf{1}_{\mathbf{n}.\xi\leq 0}\end{cases} \tag{9}$$

where $\mathbf{n}=\nabla f(\phi_T)$. Actually $\tilde u(s,\xi)$ does only depend on the normal component of ξ, it is nothing but $u(s,\xi.\mathbf{n})$ where $u(s,x)$ is the unique solution of

$$\begin{cases}\frac{\partial u}{\partial s}(s,x) &= \frac{1}{2}\frac{\partial^2 u}{\partial x^2}(s,x)+c(\phi_T)u(s,x)(1-u(s,x))\\ u(0,x) &= \mathbf{1}_{x\leq 0}.\end{cases} \tag{10}$$

Then the following functional of $\tilde u$:

$$G(p,y)=\mathbb{E}[\exp(-c(\phi_T)\int_0^\infty \tilde u(s,[B_s+\|p\|s+y]\mathbf{n})ds)]$$

is introduced, where B is a standard Wiener process. We remark that this constant depends only on ϕ_T and $\mathbf{n}$ which is a consequence of the boundary layer techniques.

Theorem 2 *Let (T,x) be such that $V^*(T,x)<0$. Under the assumptions* **H 1**, **H 2** *and* **H 3**

$$\frac{1}{\epsilon}\exp(-\frac{V^*(T,x)}{\epsilon^2})u^\epsilon(T,x)\underset{\epsilon\to 0}{\longrightarrow} \tag{11}$$
$$\frac{1}{(2\pi T)^{d/2}}\int_{<\nabla f(\phi_T)>^\perp} db\, C_b(\phi).\exp(-\frac{\|b\|^2}{2T})\int_{-\infty}^{\frac{1}{2}\alpha_{\phi_T}(b,b)} G(p,y)e^{\|p\|y}dy.$$

3 Proof for linear regime

Applying Feynman-Kac formula we get :

$$v^\epsilon(T,x) = \mathbb{E}[1(f(X_T^\epsilon \leq 0))\exp(\frac{F(X^\epsilon)}{\epsilon^2})] \tag{12}$$

$$u^\epsilon(T,x) = \mathbb{E}[1(f(X_T^\epsilon \leq 0))\exp(\frac{F(X^\epsilon) - F_1(X^\epsilon;0,T)}{\epsilon^2})] \tag{13}$$

with $X_s^\epsilon = x + \epsilon W_s$ and

$$F(\psi) = \int_0^T c(\psi_s)ds, \qquad F_1(\psi;t_1,t_2) = \int_{t_1}^{t_2} c(\psi_s)u^\epsilon(T-s,\psi_s)ds. \tag{14}$$

To localize around ϕ let us denote $Z_s^\epsilon = \phi_s + \epsilon W_s$, $U^\epsilon(t_1,t_2) = \int_{t_1}^{t_2}[\int_0^1(1-v)c''(\phi_s + \epsilon W_s)dv]W_s^2 ds$ and $\Gamma_\rho(t_1,t_2) = \{\max(|\epsilon W_s'|, \|\epsilon W_s''\| < \rho; \forall s \in [t_1,t_2]\}$ where we have used the following convention : if $x \in \mathbb{R}^d$ we denote $x \cdot \mathbf{n}$ by x' and the projection $\pi(x)$ by x'', the norm $\max(|x'|, \|x''\|)$ will be denoted by $|||x|||$.

Lemma 1 *Under assumptions* **H 1**, **H 2**, $\forall\rho > 0$, $\exists\zeta > 0$ *such that*

$$v^\epsilon(T,x) = exp(\frac{V(T,x)}{\epsilon^2})\mathbb{E}[1(f(Z_T^\epsilon \leq 0) \cap \Gamma_\rho(0,T))\exp(\frac{p\cdot W_T}{\epsilon} + U^\epsilon(0,T))] + O(\exp(\frac{V(T,x)-\zeta}{\epsilon^2})) \tag{15}$$

Under assumptions **H 1**, **H 2** *and* **H 3**, $\exists\rho_1 > 0$, $\forall\alpha \in (0,1)$, $\forall\rho \leq \rho_1$, $\exists\zeta > 0$ *such that*

$$u^\epsilon(T,x) = exp(\frac{V^*(T,x)}{\epsilon^2})u_3^\epsilon(T,x,\alpha) + O(\exp(\frac{V^*(T,x)-\zeta}{\epsilon^2})) \tag{16}$$

where

$$u_3^\epsilon(T,x,\alpha) = \mathbb{E}[1(f(Z_T^\epsilon \leq 0) \cap \Gamma_{\epsilon^{1-\alpha}}(0,T))\exp(\frac{p\cdot W_T}{\epsilon} + U^\epsilon(0,T) - \frac{F_1(Z^\epsilon,0,T)}{\epsilon^2})].$$

Proof
The proofs of lemmas 3.5 and 3.9 in [2] are essentially unchanged, since we have just transformed $\mathbf{1}_{X_T^\epsilon \leq 0}$ into $\mathbf{1}_{f(X_T^\epsilon)\leq 0}$ to get (12) and (13), all the arguments concerning the Wiener Space remain true.□
Proof of th 1
Using a Taylor expansion of U^ϵ, we deduce from (15):

$$v^\epsilon(T,x) = exp(\frac{V(T,x)}{\epsilon^2})Z_\rho(1+o(1))$$

where $Z_\rho = \mathbb{E}(\mathbf{1}(f(Z_T^\epsilon \leq 0) \cap \Gamma_\rho(0,T))\exp(\frac{p\cdot W_T}{\epsilon} + U^0(0,T)))$. Since the Brownian bridge $W_s^0 = (W_s - \frac{s}{T}W_T)_{0\leq s\leq T}$ is independent of W_T

$$Z_\rho = \int_{-\infty}^{+\infty}\int_{w''\in<\mathbf{n}>^\perp}\frac{dw'dw''}{(2\pi T)^{d/2}}\mathbf{1}(f(\phi_T + \epsilon[w'\mathbf{n} + w'']) \leq 0, \epsilon|||W_T||| \leq \rho)$$
$$\exp(-(\frac{(w')^2 + \|w''\|^2}{2T}) + \frac{w'\|p\|}{\epsilon})$$
$$\mathbb{E}(\mathbf{1}(\epsilon|||W_.^0 + \frac{.}{T}(w'\mathbf{n} + w'')|||_\infty \leq \rho)\exp(\frac{1}{2}\int_0^T c''(\phi_s)(W_s^0 + \frac{s}{T}(w'\mathbf{n} + w''))^2 ds))$$

where we have written $W_T = w'\cdot\mathbf{n}+w''$. Hence if $w'\to\epsilon w'$ we get $f(\phi_T+\epsilon^2[w'\mathbf{n}+w''])=\epsilon^2(w'+\frac{f''(\phi_T)(w'')^2}{2})+o(\epsilon^2)$ because of $\nabla f(\phi_T)=\mathbf{n}$ and of (6). Then

$$Z_\rho = \int_{w'\in\mathbf{R}}\int_{w''\in<\mathbf{n}>^\perp}\frac{\epsilon dw'dw''}{(2\pi T)^{d/2}}\mathbf{1}(w'+\frac{f''(\phi_T)(w'')^2}{2}+o(1)\le 0)$$
$$\exp(-(\frac{\epsilon^2(w')^2+\|w''\|^2}{2T})+w'\|p\|)$$
$$\mathbb{E}(1(\epsilon\|W^0_. + \frac{\cdot}{T}(\epsilon w'\mathbf{n}+w'')\|_\infty\le\rho)\exp(\frac{1}{2}\int_0^T c''(\phi_s)(W^0_s+\frac{s}{T}(\epsilon w'\mathbf{n}+w''))^2ds))$$

which leads us to

$$\sim\int_{w'+\frac{1}{2}f''(\phi_T)(w'')^2\le 0}\int_{w''\in<\mathbf{n}>^\perp}\frac{\epsilon dw'dw''}{(2\pi T)^{d/2}}\exp(-\frac{\|w''\|^2}{2T}+w'\|p\|)$$
$$\mathbb{E}(\exp(\frac{1}{2}.\int_0^T c''(\phi_s)(W^0_s+\frac{s}{T}w'')^2ds))$$

$$=\int_{w''\in<\mathbf{n}>^\perp}\frac{\epsilon dw''}{(2\pi T)^{d/2}}\exp(-\frac{\|w''\|^2}{2T}+\frac{\|p\|}{2}\alpha_{\phi_T}(b,b))$$
$$\mathbb{E}(\exp(\frac{1}{2}.\int_0^T c''(\phi_s)(W^0_s+\frac{s}{T}w'')^2ds))$$

We explain now why this last integral converges. Since ϕ is the maximum of $R_{0,T}(\psi)=\int_0^T[c(\psi_s)-\frac{1}{2}\|\dot\psi_s\|^2]ds$ under the constraint $f(\psi_T)\le 0$, the Lagrangian $L(\psi)=R_{0,T}(\psi)-\lambda f(\psi_T)$, $\lambda\ge 0$ is critical at ϕ, i.e. $\forall h\in H\ DL(\phi)(h)=0$, where $H=\{h\ :h(0)=0,\int_0^T\|\dot h\|_s^2ds\}$ is the Cameron Martin space

$$\begin{aligned}DL(\phi)(h) &= \int_0^T c'(\phi_s)h_sds-\int_0^T\dot\phi_s\dot h_sds-\lambda f'(\phi_T)h_T\\ &= \int_0^T c'(\phi_s)h_sds+\int_0^T\ddot\phi_sh_sds-\dot\phi_Th_T-\lambda f'(\phi_T)h_T.\end{aligned}$$

Hence $\lambda=\|p\|$ and $c'(\phi_s)=-\ddot\phi_s$. The second order condition for a maximum is here

$$\forall h\in H\quad 0\ge D^2L(\phi)(h)=\int_0^T c''(\phi_s)h_s^2ds-\int_0^T\|\dot h_s\|^2ds-\|p\|f''(\phi_T)(h_T)^2.$$

This last inequality shows that the integral is well defined. □

4 The non linear regime

To study the asymptotics of $u_3^\epsilon(T,x,\alpha)$, a so called boundary layer is introduced and for $a\in(0;1)$, $T_\epsilon=T-\epsilon^a$, Lemma 2 shows that the contribution on $[0;T_\epsilon]$ comes only from the linear part. If

$$Q^\epsilon=\frac{1}{2}\int_0^{T_\epsilon}c''(\phi_s)W_s^2ds$$

and

$$u_4^\epsilon(T,x,\alpha)=\mathbb{E}[\mathbf{1}(f(Z_T^\epsilon\le 0)\cap\Gamma_{\epsilon^{1-\alpha}}(0,T))\exp(\frac{p\cdot W_T}{\epsilon}+Q^\epsilon-\frac{F_1(Z^\epsilon;T_\epsilon,T)}{\epsilon^2})].$$

Lemma 2 *For all $\alpha\in(0;1-a)$, $u_4^\epsilon-u_3^\epsilon=O(\epsilon^{1+a-2\alpha}),(\epsilon\to 0)$.*

The proof is postponed to section 5.
To take into account the coordinates W'' in the study of u_4^ϵ, we need a conditional expectation with respect to the tangent component, hence g^ϵ becomes

$$g^\epsilon(W'';y,z) \ := \mathbb{E}[\mathbf{1}(\Gamma_{\epsilon^{1-\alpha}}(T_\epsilon,T))\exp(-\frac{F_1(Z^\epsilon,T_\epsilon,T)}{\epsilon^2})/(W'';W'_T=\epsilon y, W'_{T_\epsilon}=z)].$$

Lemma 3

$$u_4^\epsilon(T,x,\alpha,a) = \mathbb{E}[\mathbf{1}(f(Z_T^\epsilon \le 0)\cap\Gamma_{\epsilon^{1-\alpha}}(0,T_\epsilon))\exp(\frac{p\cdot W_T}{\epsilon}+Q^\epsilon)g^\epsilon(W'';\frac{W'_T}{\epsilon};W'_{T_\epsilon})]$$

Proof : As W'' is assumed to be constant by conditioning, we can apply the Markov property for W'.□

A more accurate localization will be done by considering

$$G_1^\epsilon = \{(W'_T \ge -\epsilon^{1-\gamma})\cap|W'_{T_\epsilon}-\frac{T_\epsilon}{T}W_T^1|\le\epsilon^{1-\gamma}\}$$

and by introducing

$$u_5^\epsilon(T,x,\alpha,a) = \mathbb{E}[\mathbf{1}(G_1^\epsilon\cap(f(Z_T^\epsilon)\le 0)\cap\Gamma_{\epsilon^{1-\alpha}}(0,T_\epsilon))\exp(\frac{p\cdot W_T}{\epsilon}+Q^\epsilon)g^\epsilon(W'';\frac{W'_T}{\epsilon},W'_{T_\epsilon})]$$

Lemma 4 *For all γ in $(1-a/2;2-a)$, we have*
$u_4^\epsilon - u_5^\epsilon = O(\exp(-const.\epsilon^{2-a-2\gamma})), (\epsilon\to 0)$.

Proof : Since

$$\begin{aligned} u_4^\epsilon - u_5^\epsilon \le & \mathbb{E}[\mathbf{1}(f(Z_T^\epsilon \le 0)\cap\Gamma_{\epsilon^{1-\alpha}}(0,T_\epsilon)).1(W'_T < -\epsilon^{1-\gamma})\exp(\frac{\|p\|.W'_T}{\epsilon}+Q^\epsilon)] \\ & +\mathbb{E}[e^{Q^\epsilon}\mathbf{1}(\Gamma_{\epsilon^{1-\alpha}}(0,T_\epsilon)\cap|W'_{T_\epsilon}-\frac{T_\epsilon}{T}|>\epsilon^{1-\gamma})] \end{aligned}$$

we use the same proof as in [2] by replacing W by W'. □

Let $\tilde{Q}^\epsilon = \frac{1}{2}\int_0^{T_\epsilon} c''(\phi_s)(W'_s - \frac{s}{T_\epsilon}W'_{T_\epsilon};(W'')_s^2)ds$
and let

$$u_6^\epsilon(T,x,\alpha,a) = \mathbb{E}[\mathbf{1}(G_1^\epsilon\cap(f(Z_T^\epsilon)\le 0))\exp(\frac{p\cdot W_T}{\epsilon}+\tilde{Q}^\epsilon)g^\epsilon(W'';\frac{W_T^1}{\epsilon},W'_{T_\epsilon})],$$

Lemma 5

$$u_5^\epsilon = [1+O(\epsilon^{1-\gamma-\alpha})]u_6^\epsilon + O(\exp(-const.\epsilon^{-2\alpha})).$$

Then the limit of g^ϵ is given by :

Lemma 6 $\forall y\in\mathbb{R}$,

$$g^\epsilon(W'';y,z) \longrightarrow \mathbb{E}(\exp(-c(\phi_T)\int_0^T \tilde{u}(s,[B_s+\|p\|s+y]\mathbf{n})ds)) = G(p,y)$$

uniformly for $z\epsilon^{1-a}\to 0$, uniformly in W''.

These two lemmas are proved in section 5.
Proof of theorem 2 Since we have the same relation between the asymptotic behavior of u_6^ϵ and u^ϵ when $\epsilon\to 0$ as in the case $d=1$, the only new feature is the presence of W'' in the definition of u_6^ϵ hence we denote by

$$U_6^\epsilon(W'') = \mathbb{E}_{W'}[\mathbf{1}(G_1^\epsilon\cap(f(Z_T^\epsilon)\le 0))\exp(\frac{p\cdot W_T}{\epsilon}+\tilde{Q}^\epsilon)g^\epsilon(W'';\frac{W'_T}{\epsilon},W'_{T_\epsilon})]$$

and we get $u_6^\epsilon = \mathbb{E}_{W''}[U_6^\epsilon(W'')]$. But by independence of the increments of the Brownian motion W'

$$U_6^\epsilon(W'') = \mathbb{E}_{W'}(e^{\tilde{Q}^\epsilon})\iint \mathbf{1}(-\epsilon^{1-\gamma} \le z_1; |z_2 - \frac{T_\epsilon}{T}z_1| \le \epsilon^{1-\gamma}; f(\phi_T + \epsilon z_1.\mathbf{n} + \epsilon W"_T) \le 0)$$
$$.(2\pi)^{-1}(\epsilon^a T_\epsilon)^{-1/2}\exp(\frac{\|p\|z_1}{\epsilon} - \frac{z_1^2}{2T} - \frac{(z_2 - \frac{T_\epsilon}{T}z_1)^2}{2\epsilon^a\frac{T_\epsilon}{T}})g^\epsilon(W"; \frac{z_1}{\epsilon}, z_2)dz_1dz_2.$$

Let $z_1 = \epsilon y_1$ and $z_2 - \frac{T_\epsilon}{T}z_1 = y_2\epsilon^{a/2}(T_\epsilon/T)^{\frac{1}{2}}$.

$$(2\pi T)^{1/2}\frac{U_6^\epsilon(W'')}{\epsilon\mathbb{E}_{W'}(e^{\tilde{Q}^\epsilon})} = \int_{\mathbb{R}^2}\mathbf{1}(-\epsilon^{-\gamma} \le y_1; f(\phi_T + \epsilon^2 y_1\mathbf{n} + \epsilon W"_T) \le 0\ )$$
$$\mathbf{1}(|y_2| \le \epsilon^{-\frac{a}{2}-\gamma+1}\sqrt{\frac{T_\epsilon}{T}}).g^\epsilon(W"; y_1, z_2)\exp(y_1\|p\| - \frac{\epsilon^2 y_1^2}{2T} - \frac{\|y_2\|^2}{2})dy_1dy_2$$

$$\longrightarrow \int_{y_1 + \frac{1}{2}f"(\phi_T)(W"_T)^2 \le 0} G(p, y_1)\exp(y_1\|p\| - \frac{\|y_2\|^2}{2})dy_1dy_2$$
$$= \int_{-\infty}^{-\frac{1}{2}f"(\phi_T)(W"_T)^2} G(p, y_1)e^{(y_1\|p\|)}dy_1$$

Since $\mathbb{E}_{W'}(e^{\tilde{Q}^\epsilon}) \to C_{W"_T}(\phi)$ the proof of Theorem 2 is complete if a dominated convergence is applied to the expectation with respect of $W"$.□

5 Proofs of the lemmas

5.1 Proof of lemma 2

To prove the Lemma 2 an exponential estimate of $u^\epsilon(s, \zeta)$ is performed where the bounding terms involve $f(\zeta)$ which is related to the distance of ζ to A. The next lemma summarizes the changes performed in Lemma 5.1, 5.2 and 5.3 in [2]. Let us introduce the new definitions for

$$\bar{c}(\eta) = \sup\{c(z); \|z - \phi_T\| \le \eta\}$$

and

$$G_2(\epsilon, \delta, \eta, l) = \{\sqrt{2l[\bar{c}(2\eta) + \eta]}(T - s) \le f(Z_s^\epsilon) \le \frac{\eta}{\sqrt{dl}}$$
$$\text{and } \|Z_s^\epsilon - \phi_T\| \le \eta\ ; \forall s \in [T - \delta; T_\epsilon]\}.$$

Lemma 7 *(i) There exists $\eta_0 > 0$, such that for $0 < \eta < \eta_0$, $\forall l > 1$:*

$$\textit{for } s\sqrt{2l[\bar{c}(2\eta) + \eta]} \le f(\zeta) \le \frac{\eta}{\sqrt{dl}} \quad \textit{and} \quad \|\zeta - \phi_T\| \le \eta$$
$$u^\epsilon(s, \zeta) \quad \le \quad const\exp(-\frac{\eta s}{\epsilon^2}),$$

(ii) on $G_2 \cap \Gamma_{\epsilon^{1-\alpha}}(0, T)$: $0 \le \epsilon^{-2}F_1(Z_\epsilon; 0, T_\epsilon) \le const.e^{-const.\epsilon^{a-2}}$;

(iii) and for ϵ small enough

$$\mathbf{P}(G_2^c \cap \Gamma_{\rho_2}(0, T)) \le const.e^{-const.\epsilon^{2a-2}}.$$

Proof : For the part (i) as in the case $d = 1$ we are using the Feynman Kac formula and Markov property at $\tau = \inf\{\sigma \leq s, \quad \|\epsilon W_\sigma\| > \eta\}$. Hence for $\sigma < \tau$ and $\|\zeta - \phi_T\| \leq \eta$ we have $0 \leq c(\zeta + \epsilon W_s) \leq \bar{c}(2\eta)$ and

$$u^\epsilon(s,\zeta) \leq \exp(\frac{\bar{c}(2\eta)s}{\epsilon^2})\mathbf{P}(s \leq \tau \cap f(\zeta + \epsilon W_s) < 0) + \mathbf{P}(\tau < s).$$

Since W is a d dimensional Wiener process we get $\mathbf{P}(\tau < s) \leq d\exp(\frac{-\eta^2}{2\epsilon^2 sd})$ and because $s\sqrt{2l[\bar{c}(2\eta)+\eta]} \leq f(\zeta) \leq \frac{\eta}{\sqrt{dl}}$ we have

$$\frac{\bar{c}(2\eta)s}{\epsilon^2} - \frac{\eta^2}{2\epsilon^2 sdl} \leq (\frac{f^2(\zeta)}{2l\epsilon^2 s} - \frac{\eta s}{\epsilon^2}) - \frac{f^2(\zeta)}{2l\epsilon^2 s}$$

and

$$\exp(\frac{\bar{c}(2\eta)s}{\epsilon^2})\mathbf{P}(\tau < s) \leq d\exp(-\frac{\eta s}{\epsilon^2}).$$

Then we set η_0 such that f is C^2 in the ball with center ϕ_T and radius η_0 we are aiming to bound $\mathbf{P}(s \leq \tau \cap f(\zeta + \epsilon W_s) < 0)$. If $\eta \leq \eta_0$ on the event $s \leq \tau$ there exists $\theta \in (0,1)$ such that

$$f(\zeta + \epsilon W_s) = f(\zeta) + \nabla f(\zeta).\epsilon W_s + f''(\zeta + \theta\epsilon W_s).(\epsilon W_s, \epsilon W_s). \tag{17}$$

We can choose η small enough to have $\zeta + \theta\epsilon W_s$ close enough to ϕ_T. Because $f"(\phi_T)$ is given by (6), if K_A is the greater eigenvalue of α_{ϕ_T} for each $m > 0$ and $K > K_A$ there exists $\eta_1 \in (\eta_0, \eta_1)$ such that $\eta \leq \eta_1$ implies that

$$f(\zeta + \epsilon W_s) \geq f(\zeta) + \epsilon W_s^{/} - \epsilon^2(\frac{K}{2}\|W_s^{//}\|^2 + m(W_s^{/})^2) \tag{18}$$

where we have set $W_s^{/} = \nabla f(\zeta).W_s$ and $W_s^{//}$ is a Brownian motion orthogonal to $\nabla f(\zeta)$. Since $W_s^{//}$ is independent of $W_s^{/}$, large deviation for Gaussian random variables gives :

$$\lim_{\epsilon\to 0}\epsilon^2\ln(\mathbf{P}(0 \geq f(\zeta) + \epsilon W_s^{/} - \frac{K}{2}\|\epsilon W_s^{//}\|^2 - m(\epsilon W_s^{/})^2)) = -\inf_{\mathcal{E}^c}\frac{\|x\|^2}{2s}$$

where $x' \in \mathbb{R}$, $x" \in \mathbb{R}^{d-1}$, $x = (x', x")$ and $\mathcal{E}$ is the ellipsoid $\{x' - \frac{K}{2}\|x"\|^2 - m(x')^2 \geq -f(\zeta)\}$. The infimum of $\frac{\|x\|^2}{2s}$ on $\mathcal{E}^c$ is attained at $(\frac{1-\sqrt{1+4mf(\zeta)}}{2m}, 0)$ then for m small enough

$$\forall l > 1 \quad -\inf_{\mathcal{E}^c}\frac{\|x\|^2}{2s} \leq -\frac{f^2(\zeta)}{2ls}$$

and

$$\mathbf{P}(0 \geq f(\zeta) + \epsilon W_s^{/} - \frac{K}{2}\|\epsilon W_s^{//}\|^2 - m(\epsilon W_s^{/})^2) = O(\exp(\frac{-f^2(\zeta)}{2l\epsilon^2 s}))$$

when $\epsilon \to 0$. Because of (18) we get

$$\mathbf{P}(s \leq \tau \cap \{f(\zeta + \epsilon W_s) < 0\}) \leq const\exp(\frac{-f^2(\zeta)}{2l\epsilon^2 s})$$

and

$$\frac{\bar{c}(2\eta)s}{\epsilon^2} - \frac{f^2(\zeta)}{2l\epsilon^2 s} \leq -\frac{\eta s}{\epsilon^2}$$

which concludes the proof of part (i).

For the part (ii) we refer the reader to the proof of Lemma 5.3 [2] where we have changed (5.4) in

$$0 \leq \epsilon^{-2} F_1(Z^\epsilon, T-\delta, T(\epsilon)) \leq \frac{const.\bar{c}(\eta)}{\eta} \exp(-\eta \epsilon^{-(2-a)}).$$

To prove (iii) straightforward changes in the proof of Lemma 5.6 [2] (replacing $c(0)$ by $c(\phi_T)$, p by $\|p\|$, $\phi_T - \phi_{T-s}$ by $\|\phi_T - \phi_{T-s}\|$ and Z^ϵ by $Z^\epsilon - \phi_T$) allows us to claim

$$\forall s \in [T-\delta, T(\epsilon)] \quad \mathbf{1}_{\Gamma_{\rho_2}(0,T(\epsilon))} \|Z^\epsilon_s - \phi_T\| \leq \eta$$

for $\delta < \delta_0$ and ρ_2 small enough. Furthermore we get

$$\forall s \in [T-\delta, T(\epsilon)] \quad \mathbf{1}_{\Gamma_{\rho_2}(0,T(\epsilon))} |f(Z^\epsilon_s)| \leq \frac{\eta}{\sqrt{d}}$$

since $f(\phi_T) = 0$ and $\|\nabla f\| = 1$ in a neighborhood of ϕ_T. Hence

$$G_2^c \cap \Gamma_{\rho_2}(0,T) = \{\exists s \in [T-\delta, T(\epsilon)], \quad f(Z^\epsilon_s) < \sqrt{2l[\bar{c}(2\eta)+\eta]}(T-s)\} \cap \Gamma_{\rho_2}(0,T).$$

In **H 3** $c(\phi_T)$ is assumed to be strictly lower than $\frac{1}{2}\|\dot{\phi_T}\|^2$, consequently there exists $l > 1$ such that $c(\phi_T) < \frac{1}{2l}\|\dot{\phi_T}\|^2$. Hence $c_2 = \|p\| - \sqrt{2l[\bar{c}(2\eta)+\eta]} - \delta_0 L$ is strictly positive, we deduce that on $G_2^c \cap \Gamma_{\rho_2}(0,T)$

$$\exists s \in [T-\delta, T(\epsilon)] \quad f(\phi_s + \epsilon W_s) < (\|p\| - c_2 - \delta_0 L)(T-s).$$

Using a Taylor expansion for $s' = T - s \in [\epsilon^a, \delta]$ we get

$$f(\phi_{T-s'} + \epsilon W_{T-s'}) = \nabla f(\phi_T)(\phi_{T-s'} + \epsilon W_{T-s'} - \phi_T) + O(\|\phi_{T-s'} + \epsilon W_{T-s'} - \phi_T\|^2),$$

$\nabla f(\phi_T)(\phi_{T-s'} - \phi_T) = \|p\| s' + O((s')^2)$ and $\nabla f(\phi_T).\epsilon W_{T-s'} = \epsilon W'_{T-s'}$. Therefore on $G_2^c \cap \Gamma_{\rho_2}(0,T)$ $\exists s \in [T-\delta, T(\epsilon)]$ such that $\|p\|(T-s) + \epsilon W'_s + O((T-s)^2) < (\|p\| - c_2 - \delta_0 L)(T-s)$ and $G_2^c \cap \Gamma_{\rho_2}(0,T(\epsilon)) \subset \{\exists s \in [0,T(\epsilon)], \quad \epsilon W'_s < -c_2(T-s)\}$ which gives the inequality (iii).

Proof of lemma 2 : In the end of the proof of Lemma 2 all arguments of the one dimensional case can be used for $d \geq 2$, but we have to show that

$$\mathbf{1}(f(Z^\epsilon_T \leq 0)). \exp \frac{p.W_T}{\epsilon} \leq 1 \qquad (19)$$

instead of $\mathbf{1}(W_T \leq 0)\exp(\frac{p.W_T}{\epsilon}) \leq 1$ to deal with the non linear part that depends on F_1. But (19) is clear since $Z^\epsilon_T = \phi_T + \epsilon W_T$ is inside A and p is pointing outside A.

5.2 Proof of lemma 5

We give the needed modifications of the proof of the corresponding lemma 4.10 in [2]. At first we study the consequences of replacing Q^ϵ by $\tilde{Q}^\epsilon = \frac{1}{2}\int_0^{T_\epsilon} c"(\phi_s).(W'_s - \frac{s}{T_\epsilon}W'_{T_\epsilon}, W"_s)^2 ds$ and u_8^ϵ is introduced as :

$$\mathbb{E}[\mathbf{1}(G_1^\epsilon \cap \Gamma_{\epsilon^{1-\alpha}}[0,T_\epsilon])\mathbf{1}(f(Z^\epsilon_T \leq 0)). \exp(\frac{p.W_T}{\epsilon} + \tilde{Q}^\epsilon) g^\epsilon(W"; \frac{W'_T}{\epsilon}, W_{T_\epsilon})].$$

By a polarization argument

$$Q^\epsilon = \tilde{Q}^\epsilon + \int_0^{T_\epsilon} c"(\phi_s).[(W'_s - \frac{s}{T_\epsilon}W'_{T_\epsilon}, W"_s), (\frac{s}{T_\epsilon}W'_{T_\epsilon}, W"_s)]^2 ds.$$

Then W' is estimated as in the one dimensional case and we get

$$|Q^\epsilon - \tilde{Q}^\epsilon| \leq const(\epsilon^{1-\gamma-\alpha})$$

where the random constant can be easily bounded. Hence

$$u_5 = (1 + O(\epsilon^{1-\gamma-\alpha})u_8.$$

In [2] (6.7) remains true with $\mathbf{1}(f(Z^\epsilon_T \leq 0))$ instead of $\mathbf{1}(W_T \leq 0)$, that implies $u_6 - u_8 = O(-const.\epsilon^{-2\alpha})$. $\square$

5.3 Proof of lemma 6

Let B be a real valued Wiener process independent of W". The distribution of $\epsilon^{-2}(Z^\epsilon_{T-\epsilon^2 s} - \phi_T)_{0\le s\le \epsilon^{-(2-a)}}$ under $P(./W'_T = \epsilon y, W'_{T_\epsilon} = z)$ is identical to that of $(\xi^\epsilon_s)_{0\le s\le\epsilon^{-(2-a)}}$, where

$$\xi'^{\epsilon}_s = \epsilon^{-2}(\phi'_{T-\epsilon^2 s} - \phi'_T) + y - s\epsilon^{2-a}(B(\epsilon^{-(2-a)}) - z\epsilon^{-1} + y)$$

and where the tangent part $\xi"^{\epsilon}_s$ is defined by :

$$\xi"^{\epsilon}_s = \epsilon^{-2}(\phi"_{T-\epsilon^2 s} - \phi"_T) + \epsilon^{-1}W"_{T-\epsilon^2 s}.$$

Hence we get with the same change of variable argument as in [2]

$$g^\epsilon(W";y,z) = \mathbb{E}[1(\tilde\Gamma_\epsilon)exp(-(\int_0^{\epsilon^{-(2-a)}} \tilde c^\epsilon(\xi^\epsilon_s)\tilde u^\epsilon(s,\xi^\epsilon_s)ds))/W"]$$

where

$$\tilde\Gamma^\epsilon = \{|||\epsilon^2\xi^\epsilon_s + \phi_T - \phi_{T-\epsilon^2 s}||| \le \epsilon^{1-\alpha}, \quad \forall s\in[0,\epsilon^{-(2-a)}]\}$$

and

$$\tilde c^\epsilon(\xi) = c(\epsilon^2\xi + \phi_T), \quad \tilde u^\epsilon(s,\xi) = u(\epsilon^2 s, \epsilon^2\xi + \phi_T).$$

Moreover $\tilde u^\epsilon$ is solution of the P.D.E. :

$$\begin{cases} \dfrac{\partial \tilde u^\epsilon}{\partial s}(s,\xi) & = \frac{1}{2}\Delta\tilde u^\epsilon + \tilde c^\epsilon(\xi)\tilde u^\epsilon(1-\tilde u^\epsilon) \\ \tilde u^\epsilon(0,\xi) & = \mathbf{1}(f(\phi_T + \epsilon^2\xi)\le 0) \end{cases} \tag{20}$$

Hence $\tilde u^\epsilon$ converges uniformly for $s\le s_0$ and $\epsilon^2|\xi|\to 0$ to $\tilde u^\epsilon$ thanks to a standard argument of perturbation applied to the PDE (20). Since $\tilde u(s,\xi)$ depends only on the orthogonal part of ξ, and since for s, y fixed and $z\epsilon^{1-a}\to 0$, ξ'^{ϵ}_s converges a.s. to $B(s) + \|p\|s + y$, and the limit of $\tilde u^\epsilon(s,\xi^\epsilon_s)$ is $\tilde u(B(s)+\|p\|s+y)$ a.s..

To conclude the proof of lemma 6 we only have to upper bound $\tilde c^\epsilon(\xi^\epsilon_s)\tilde u^\epsilon(s,\xi^\epsilon_s)$ in order to have the convergence of $\int_0^{\epsilon^{-(2-a)}} \tilde c^\epsilon(\xi^\epsilon_s)\tilde u^\epsilon(s,\xi^\epsilon_s)ds$ to $c(\phi_T)\int_0^\infty \tilde u(s,B(s)+\|p\|s+y)ds < +\infty$. Because of the definition of $|||\quad|||$ the characteristic function of $\tilde\Gamma^\epsilon$ is the product

$$\begin{aligned}\mathbf{1}(\tilde\Gamma^\epsilon) =& \mathbf{1}(\{\epsilon\|W"_{T-\epsilon^2 s}\| \le \epsilon^{1-\alpha}, \forall s\in[0,\epsilon^{-(2-\alpha)}]\} \\ &.\mathbf{1}(\{|\epsilon\xi'^{\epsilon}_s + (\phi_T - \phi_{T-\epsilon^2 s})'| \le \epsilon^{1-\alpha}, \forall s\in[0,\epsilon^{-(2-\alpha)}]\}\end{aligned}$$

and if $\Omega_1 = \{\frac{B(s)}{s}\to 0\}$ and $\Omega_2 = \{\epsilon^{1+\alpha}sup\{|B(s)|, s\le\epsilon^{2-a}\}\to 0\}$ we get the counterpart of (6.18) which is

$$\mathbf{1}(\tilde\Gamma^\epsilon) = \mathbf{1}(\{\epsilon\|W"_{T-\epsilon^2 s}\|\le\epsilon^{1-\alpha}, \forall s\in[0,\epsilon^{-(2-\alpha)}]\} \quad \text{for} \quad \epsilon<\epsilon_0.$$

Since $\epsilon^2\xi^\epsilon_s + \phi_T$ converges a.s. to ϕ_T,

$$0\le\tilde c(\xi^\epsilon_s)\le\bar c(\eta) \quad \text{for} \quad \epsilon<\epsilon_0$$

where $\bar c(\eta)$ is defined before lemma 7. As in [2] we aim the following inequality : for $s > s_1$ and $\epsilon < \epsilon_3(\omega)$,

$$\mathbf{1}(s\le\epsilon^{-(2-a)}).\tilde u^\epsilon(s,\xi^\epsilon_s) \le 2d\exp(-\eta s)$$

which will be a consequence of lemma 7, if we can prove

$$\frac{f(\epsilon^2\xi^\epsilon_s + \phi_T)}{s} > \sqrt{2l(\bar c(2\eta)+\eta)} \quad \text{for} \quad \epsilon<\epsilon_3 \quad \text{and} \quad s > s_1.$$

But $f(\epsilon^2\xi^\epsilon_s + \phi_T) = \epsilon^2\xi'^{\epsilon}_s + o(\epsilon^2)$, and since ξ'^{ϵ}_s is the exact counterpart of the real valued process denoted by ξ^ϵ_s in [2], there is no further change in their proof.

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