1 Introduction

The problem that we are interested in in this work is to understand the asymptotical behavior, that is the thermodynamic limit, of a classical mean field model perturbated by a small term of Sherrington-Kirkpatrick type. Such a question arises in a natural way ...

More precisely, let us consider N independent and identically distributed random variables $\sigma_i \in [-1,1]$ with common distribution ρ. Denote by $\sigma = (\sigma_1, \dots, \sigma_N) \in [-1,1]^N$ the configuration of the system, by $\mathbf{P}_\sigma$ the product measure $\rho^{\otimes N}$ and by $\mathbf{E}_\sigma$ the expectation w.r.t. $\mathbf{P}_\sigma$. Let us consider a smooth function $f : [-1,1] \to \mathbb{R}$. By classical mean field system, we mean a spin system ruled by the following hamiltonian:

$$H_N^f(\sigma) = Nf\left(\frac{\sigma.\mathbf{1}}{N}\right)$$

where we have set $\sigma.\mathbf{1} = \sum_{i=1}^N \sigma_i$ the scalar product in $\mathbb{R}^N$ of σ and $\mathbf{1} = (1, \dots, 1) \in \mathbb{R}^N$. The random perturbation that we consider is given by a sequence $(J_{i,j})_{1\le i<j}$ of independent and identically distributed random variables with common standard gaussian distribution $\mathcal{N}(0,1)$ through the hamiltonian:

$$SK(\sigma) = N^{-1/2} \sum_{1\le i<j\le N} J_{i,j}\sigma_i\sigma_j$$

We recognize here the usual Sherrington-Kirkpatrick hamiltonian introduced in [SK75]. Let us now consider for $\alpha \in (1/2, 1)$ the following hamiltonian:

$$H_N^{f,\alpha}(\sigma) = H_N^f(\sigma) + N^{-\alpha/2} SK(\sigma)$$

We are more specifically interested in the the partition function $Z_N(\beta)$ at the inverse temperature $\beta > 0$ given by:

$$Z_{N,\beta} = \mathbf{E}_\sigma \exp \beta H_N^{f,\alpha}(\sigma)$$

and in the Gibbs measure $G_{N,\beta}$ given by:

$$G_{N,\beta}(d\sigma_1, \dots, d\sigma_N) = \frac{\exp \beta H_N^{f,\alpha}(\sigma)}{Z_{N,\beta}} \mathbf{P}_\sigma(d\sigma_1, \dots, d\sigma_N)$$

The behaviour of $Z_{N,\beta}$ is well known when $\alpha = \infty$, that is without the SK term. We aim at comparing our situation with this classical situation.

In the first part, we shall briefly summarize the results concerning the classical situation and state our results. In the second part, we shall exhibit a short list of examples showing that all the situations that we consider are likely to happen. In the third part, we shall prove the claimed results.

2 Statement of the results

2.1 General results about classical mean field models

The main tool that is to be used is the theory of large deviation as developped in [Ell85], [DS89] or [DZ93]. Let us introduce some usual notations:

$$\Lambda(\lambda) = \ln \int \exp(\lambda t)\, \rho(dt) \text{ and } I(m) = \sup_{\lambda\in\mathbb{R}} \{\lambda m - \Lambda(\lambda)\} \tag{1}$$

Function I is called the Cramer transform of ρ. It is known to be lower semi-continuous with compact level sets, that is for every $L > 0$, the set $\{m : I(m) \le L\}$ is a compact subset of $\mathbb{R}$. From Varadhan's theorem (see [DS89], theorem 2.1.10), one deduces:

$$\lim_{N\to\infty} \frac{1}{N} \ln Z_{N,\beta}^f = \sup_{m\in[-1,1]} \{\beta f(m) - I(m)\} =: \Delta$$

It is easy to show that the supremum is attained on a finite set $\mathcal{K}(\beta)$. Furthermore, if $m \in \mathcal{K}(\beta)$ then function I is of class $\mathcal{C}^\infty$ in a neighborhood of m and (see [Ell85])

$$\Lambda'[\beta f'(m)] = m \tag{2}$$

Hence, $I(m) = \beta f'(m)m - \Lambda[\beta f'(m)]$. We assume that all points in $\mathcal{K}(\beta)$ are non-degenerate that is:

$$\forall m \in \mathcal{K}(\beta), \quad \beta f''(m) - I''(m) < 0$$

Denote by $\tilde{\mu}_m$ the tilted measure at m, that is the measure with density w.r.t. ρ:

$$\frac{d\tilde{\mu}_m}{d\rho}(\sigma) \propto \exp(\beta f'(m)\sigma)$$

Equation (2) can then also be written:

$$\int_{[-1,1]} \sigma \, \tilde{\mu}_m(d\sigma) = m$$

Let us at last denote by $\tilde{\gamma}_m$ the variance of $\tilde{\mu}_m$. It is easy to see that $\tilde{\gamma}_m = (I''(m))^{-1}$ hence the non-degeneracy assumption yields $\beta f''(m)\tilde{\gamma}_m < 1$. The result concerning the behaviour of $Z^f_{N,\beta}$ is:

$$\lim_{N\to\infty} e^{N\Delta} Z^f_{N,\beta} = \sum_{m\in\mathcal{K}(\beta)} (1 - \beta f''(m_k)\tilde{\gamma}_m)^{-1/2}$$

We set $c_m = (1 - \beta f''(m)\tilde{\gamma}_m)^{-1/2} / \left[\sum_{m'\in\mathcal{K}(\beta)} (1 - \beta f''(m')\tilde{\gamma}_{m'})^{-1/2}\right]$. The asymptotical behaviour of the Gibbs measure (see [Ell85]) is given by:

$$G^f_{N,\beta} \Longrightarrow \sum_{m\in\mathcal{K}(\beta)} c_m \delta_m^{\otimes\infty} \text{ and } G^f_{N,\beta} \circ \left(\frac{\sigma.\mathbf{1}}{N}\right)^{-1} \Longrightarrow \sum_{m\in\mathcal{K}(\beta)} c_m \delta_m$$

2.2 Results about the randomly perturbated model

The behavior of the perturbated model may be rather different from what has been described in the previous section. Let us consider the following sets:

$$\widetilde{\mathcal{K}}(\beta) = \operatorname{argmax}\{\tilde{\gamma}_m(2m^2 + \tilde{\gamma}_m) : m \in \mathcal{K}(\beta)\}$$
$$\mathcal{K}^+(\beta) = \operatorname{argmax}\left\{m^2 : m \in \widetilde{\mathcal{K}}(\beta)\right\} \text{ and } \mathcal{K}^-(\beta) = \operatorname{argmin}\left\{m^2 : m \in \widetilde{\mathcal{K}}(\beta)\right\}$$

Clearly to $\mathcal{K}^+(\beta)$ and $\mathcal{K}^-(\beta)$ belong at most 2 points. Two situations may then occur:

1. $\mathcal{K}^+(\beta) = \mathcal{K}^-(\beta) = \{m\}$ or $\{-m, m\}$,
2. $\mathcal{K}^+(\beta) = \{m^+\}$ or $\{-m^+, m^+\}$ and $\mathcal{K}^-(\beta) = \{m^-\}$ or $\{-m^-, m^-\}$ with $|m^+| > |m^-|$.

Theorem 2.1
The following asymptotical expansion holds in probability:

$$e^{-N\Delta} Z^{f,\alpha}_{N,\beta} = (1 + o_{\mathbb{P}}(1)) \sum_{m\in\mathcal{K}^+(\beta)\cup\mathcal{K}^-(\beta)} \frac{\exp\left[\frac{\beta^2}{4}\tilde{\gamma}_m(2m^2 + \tilde{\gamma}_m)N^{1-\alpha} + \frac{\beta m^2 S}{2N^{(\alpha+1)/2}}\right]}{\sqrt{1 - \beta f''(m)\tilde{\gamma}_m}}$$

From this expression of the partition function, one can guess the asymptotical behavior of the Gibbs measure. Let us consider again both situations that have been introduced previously:

1. If $\mathcal{K}^+(\beta) \neq \mathcal{K}^-(\beta)$, set for $m \in \mathcal{K}^+(\beta)$

$$d_m = (1 - \beta f''(m)\tilde{\gamma}_m)^{-1/2} / \left[\sum_{q\in\mathcal{K}^+(\beta)} (1 - \beta f''(q)\tilde{\gamma}_q)^{-1/2}\right]$$

2. If $\mathcal{K}^+(\beta) \neq \mathcal{K}^-(\beta)$, set for $m \in \mathcal{K}^+(\beta)$

$$d^+_m = (1 - \beta f''(m)\tilde{\gamma}_m)^{-1/2} / \left[\sum_{q\in\mathcal{K}^+(\beta)} (1 - \beta f''(q)\tilde{\gamma}_q)^{-1/2}\right]$$

and for $m \in \mathcal{K}^-(\beta)$ $\quad d^-_m$ accordingly.

The result is the following:

Theorem 2.2

1. *If $\mathcal{K}^+(\beta) = \mathcal{K}^-(\beta)$, then in probability*

$$G_{N,\beta}^{f,\alpha} \Longrightarrow \sum_{m\in\mathcal{K}^+(\beta)} d_m \delta_m^{\otimes\infty}$$

2. *If $\mathcal{K}^+(\beta) \neq \mathcal{K}^-(\beta)$, then in distribution*

$$\frac{1}{N}\sum_{n=1}^{N} G_{n,\beta}^{f,\alpha} \Longrightarrow \left(\int_0^1 \mathbf{1}_{B_{t^2}>0}\, dt\right) \sum_{m\in\mathcal{K}^+(\beta)} d_m^+ \delta_m^{\otimes\infty} + \left(\int_0^1 \mathbf{1}_{B_{t^2}\leq 0}\, dt\right) \sum_{m\in\mathcal{K}^-(\beta)} d_m^- \delta_m^{\otimes\infty}$$

Let us comment on this theorem. The second situation is clearly different from the "deterministic" one. In the first one, the result may differ from the deterministic one according to whether $\mathcal{K}(\beta) = \mathcal{K}^+(\beta)$ or not. We shall see in the forthcoming examples that both cases are likely to occur.

To emphasize on the significance of this theorem, we should notice that for $m \in \mathcal{K}(\beta)$ one has:

$$\tilde{\gamma}_m[2m^2 + \tilde{\gamma}_m] = \langle \tilde{\mu}_m, \sigma^2\rangle^2 - \langle \tilde{\mu}_m, \sigma\rangle^4 = \langle \tilde{\mu}_m \otimes \tilde{\mu}_m, \sigma^2\tau^2\rangle - \langle \tilde{\mu}_m \otimes \tilde{\mu}_m, \sigma\tau\rangle^2$$

Hence maximizing $\tilde{\gamma}_m[2m^2 + \tilde{\gamma}_m]$ means in reality maximizing the variance of the order parameter.

3 Some examples

3.1 The generalized Curie-Weiss model

We refer to the paper by Eisele-Ellis [EE88] and add to the Sherrington-Kirkpatrick term. In this situation, all the functions and distributions that are considered are even or symmetric. Function f is strictly increasing on $[0, L[$ with $L > 0$ being finite or infinite. According to theorem 1.2 in [EE88], and depending on β, the supremum $\beta f(x) - I(x)$ is attained either at 0 or at $\pm m(\beta)$ with $m(\beta) > 0$. In either situation, and as function I is even, one has $\mathcal{K}(\beta) = \mathcal{K}^+(\beta) = \mathcal{K}^-(\beta)$. As a conclusion, nothing differs from the deterministic case.

3.2 Selection by the random model among the critical points

On the upper part of figure 1 is drawn the shape of $\tilde{\gamma}_x(\tilde{\gamma}_x + 2x^2) = 1 - x^4$ for the usual Ising model, that is $\rho = (\delta_1 + \delta_{-1})/2$. On the lower part of the figure, one can see an example of function $I - \beta f$ leading to $\mathcal{K}^+(\beta) = \mathcal{K}^-(\beta) = \{0\} \subsetneq \mathcal{K}(\beta) = \{-1/2, 0, 1/2\}$. In this precise situation, one has $G_{N,\beta}^{f} \Rightarrow c\delta_0 + (1-c)(\delta_1 + \delta_{-1})/2$ for a $c > 0$ whereas $G_{N,\beta}^{f,\alpha} \Rightarrow \delta_0$.

3.3 Existence of metastates

On the lower part of figure 2 is now drawn the shape of $\tilde{\gamma}_x(\tilde{\gamma}_x + 2x^2)$ for a non-GHS *a priori* measure $\rho = a\delta_0 + (1-a)(\delta_1 + \delta_{-1})/2$, with $a = 0.9$. On the upper part, one can see an exemple of function $I - \beta f$ leading to $\mathcal{K}^+(\beta) = \{-m^+, m^+\}$ and $\mathcal{K}^(\beta) = \{-m^-, m^-\}$, with $0 < m^- < m^+$. On the other hand, one has $\mathcal{K}(\beta) = \{-m^+, -m^-, m^-, m^+\}$. In the classical situation there exists a real number $c > 0$ such that:

$$G_{N,\beta}^{f} \Longrightarrow c(\delta_{-m^+}^{\otimes\infty} + \delta_{m^+}^{\otimes\infty})/2 + (1-c)(\delta_{-m^-}^{\otimes\infty} + \delta_{m^-}^{\otimes\infty})/2$$

whereas for the perturbated model:

$$\frac{1}{N}\sum_{n=1}^{N} G_{n,\beta}^{f} \Longrightarrow \left(\int_0^1 \mathbf{1}_{B_{t^2}>0}\, dt\right)(\delta_{-m^+}^{\otimes\infty} + \delta_{m^+}^{\otimes\infty})/2 + \left(\int_0^1 \mathbf{1}_{B_{t^2}\leq 0}\, dt\right)(\delta_{-m^-}^{\otimes\infty} + \delta_{m^-}^{\otimes\infty})/2$$

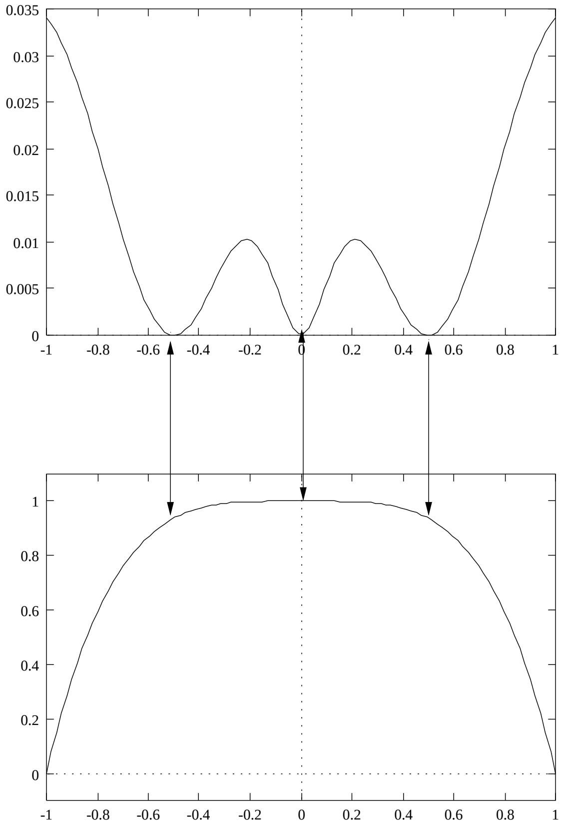

Figure 1: Selection of a critical point

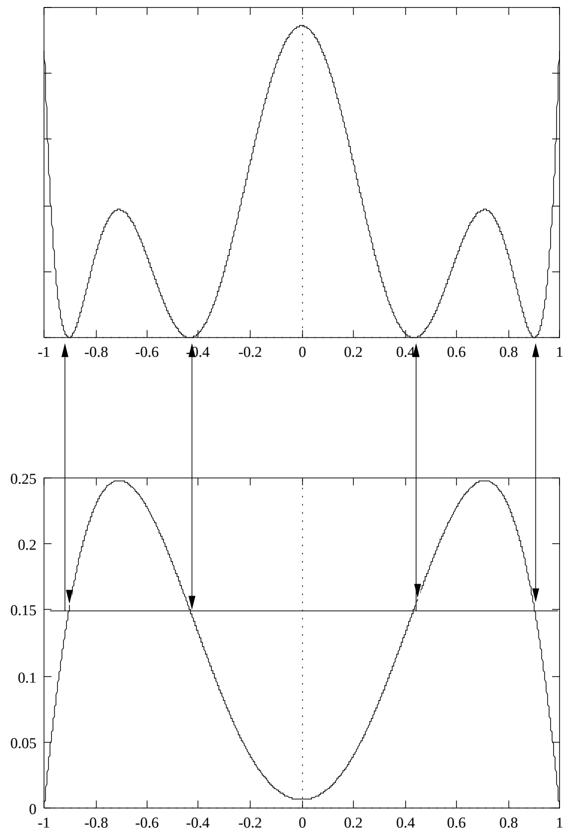

Figure 2: Existence of metastates

4 Proofs

Our main result is theorem 2.1. Its proof relies on precise asymptotic expansions of the hamiltonian in a neighborhood of each critical point. For sake of clarity, we shall set $g(x) = \beta f(x)$.

4.1 Preliminary calculations

The point of the proof is to reduce the complexity of the Sherrington-Kirkpatrick hamiltonian by conditioning. We first introduce the conditioning variables.

4.1.1 Gaussian conditional calculations

We summarize in this section the results of some easy calculations with gaussian random vectors. We set:

$$S_i^N = \sum_{j=1}^{i-1} J_{j,i} + \sum_{j=i+1}^{N} J_{i,j} \text{ and } S^N = \sum_{i=1}^{N} S_i^N = 2 \sum_{1\leq i<j\leq N} J_{i,j}$$

Denote by $\mathbf{S}^N$ vector $\mathbf{S}^N = (S_1^N, \dots, S_N^N) \in \mathbb{R}^N$. For clarity, we shall drop as often as possible superscript N. Let us first describe the covariance structure of $\mathbf{S}^N$ and the consequences of it.

$$\forall i = 1, \dots, N,\ \mathbb{E}[(S_i^N)^2] = N-1 \quad \text{and} \quad \forall i \neq j,\ \mathbb{E} S_i^N S_j^N = 1$$

If we consider a family $(\hat{S}_i, 1 \leq i \leq N, \xi)$ of iid $\mathcal{N}(0,1)$ random variables then we have in distribution:

$$\mathbf{S}^N \stackrel{\mathcal{D}}{=} (\sqrt{N-2}\hat{S}_i + \xi)_{1\leq i\leq N} \tag{3}$$

As a consequence, there exists a random constant K_ω such that:

$$\mathbb{P}\left\{\forall N \geq 1,\ \max_{1\leq i\leq N} |S_i^N| \leq K_\omega \sqrt{N \ln N}\right\} \tag{4}$$

Equality (3) also yields to:

$$\mathbb{P}\left\{\lim_{N\to\infty}\frac{1}{N^2}\sum_{i=1}^{N}(S_i^N)^2=1\right\}=1 \text{ and } \frac{1}{N^2}\sum_{i=1}^{N}S_i^2=1+o_{a.s.}(N^{-1/2+\varepsilon}) \tag{5}$$

Let us now describe the conditional distribution of the $J_{i,j}$'s given $\mathbf{S}^N$ which is known to be gaussian. The conditional expectation is given by:

$$\mathbb{E}[J_{i,j}|\mathbf{S}^N]=\frac{S_i^N+S_j^N}{N-2}-\frac{S^N}{(N-1)(N-2)}$$

And the conditional variance by:

$$V_{ij,pq}=\mathbb{E}\left[\left(J_{i,j}-\mathbb{E}[J_{i,j}|\mathbf{S}^N]\right)\left(J_{p,q}-\mathbb{E}[J_{p,q}|\mathbf{S}^N]\right)|\mathbf{S}^N\right]$$

with

$$V_{ij,ij}=1-\frac{2}{N-1}\quad V_{ij,ip}=-\frac{N-3}{(N-1)(N-2)}\quad V_{ij,pq}=\frac{2}{(N-1)(N-2)}$$

4.1.2 Conditional integration of the partition function

Let us consider V a subinterval of $[-1,1]$ and the truncated partition function Z_N^V defined by:

$$Z_N^V=\mathbf{E}_\sigma\left\{\exp[H_{N,\beta}^{f,\alpha}(\sigma)]\mathbf{1}_{\frac{\sigma.\mathbf{1}}{N}\in V}\right\}$$

We have:

$$\begin{aligned}\mathbb{E}[Z_N^V|\mathbf{S}]=\mathbf{E}_\sigma\exp\Bigg\{&\frac{\beta}{N^{(\alpha+1)/2}}\sum_{i<j}\left(\frac{S_i+S_j}{N-2}-\frac{S}{(N-1)(N-2)}\right)\sigma_i\sigma_j\\&+Ng\left(\frac{\sigma.\mathbf{1}}{N}\right)+\frac{\beta^2}{2N^{\alpha+1}}\sum_{i<j,p<q}V_{ij,pq}\sigma_i\sigma_j\sigma_p\sigma_q\Bigg\}\mathbf{1}_{\frac{\sigma.\mathbf{1}}{N}\in V}\end{aligned}$$

Hence, we have to expand the covariance term, that is:

$$V_N(\sigma)=\sum_{i<j,p<q}V_{ij,pq}\sigma_i\sigma_j\sigma_p\sigma_q$$

We shall introduce a more general expression $V_N(\sigma,\tau)$ as a function of two 'replicas' σ and τ of the system, and expand it up to the order $O(N)$ using the fact that the spins are bounded:

$$V_N(\sigma,\tau)=\sum_{i<j,p<q}V_{ij,pq}\sigma_i\sigma_j\tau_p\tau_q=\frac{1}{2}(\sigma.\tau)^2-\frac{1}{2N}(\sigma.\mathbf{1})(\tau.\mathbf{1})(\sigma.\tau)+\frac{1}{2N^2}(\sigma.\mathbf{1})^2(\tau.\mathbf{1})^2+O(N) \tag{6}$$

4.1.3 Change of variables and upper bounds

Let us now introduce two vectors $\mathbf{m}=(m_1,\dots,m_N)$ and $\boldsymbol{\mu}=(\mu_1,\dots,\mu_N)$ that will appear as centerings a bit later on. We set $\eta_i=\sigma_i-m_i$. Vectors $\mathbf{m}$ and $\boldsymbol{\mu}$ may be random but in this case they are a measurable function of $\mathbf{S}^N$. Let us define the random variable $\hat{H}_N(\sigma)$ by:

$$\frac{\beta^2}{4N^{\alpha+1}}\left[(\sigma^2.\mathbf{1})^2+\frac{1}{N^2}(\sigma.\mathbf{1})^4-\frac{2}{N}(\sigma^2.\mathbf{1})(\sigma.\mathbf{1})^2\right]=\frac{\beta^2N^{1-\alpha}}{4}\left[\frac{1}{N}\boldsymbol{\mu}.\mathbf{1}-\left(\frac{\mathbf{m}.\mathbf{1}}{N}\right)^2\right]^2+\hat{H}_N(\sigma)$$

Since the spins are bounded, there exists a deterministic constant $C>0$ such that the following upper bound holds:

$$|\hat{H}_N(\sigma)|\le\frac{C}{N^{\alpha-1/2}}\left(\frac{|\eta.\mathbf{1}|}{\sqrt{N}}+\left|\frac{\sum_{i=1}^N(\sigma_i^2-\mu_i)}{\sqrt{N}}\right|\right) \tag{7}$$

We shall denote by $\overline{\mathbf{m}}$ the empirical mean of $\mathbf{m}$ that is $\overline{\mathbf{m}} = (\mathbf{m}.\mathbf{1})/N$. Making use of our centerings $\mathbf{m}$ and $\boldsymbol{\mu}$ we get:

$$(\sigma.\mathbf{1})(\sigma.\mathbf{S}) = (\eta.\mathbf{1})(\eta.\mathbf{S}) + N\overline{\mathbf{m}}(\sigma.\mathbf{S}) + (\sigma.\mathbf{1})(\mathbf{m}.\mathbf{S}) - N\overline{\mathbf{m}}(\mathbf{m}.\mathbf{S})$$

$$\frac{S}{N^2}\sum_{i<j}\sigma_i\sigma_j = \frac{S}{2N^2}(\eta.\mathbf{1})^2 + \frac{\overline{\mathbf{m}}S}{N}(\eta.\mathbf{1}) + \frac{\overline{\mathbf{m}}^2 S}{2} - \frac{S}{2N^2}(\sigma^2.\mathbf{1})$$

Furthermore

$$\frac{\beta}{N^{(\alpha+1)/2}}\sum_{i<j}\left(\frac{S_i+S_j}{N-2} - \frac{S}{(N-1)(N-2)}\right)\sigma_i\sigma_j$$

$$= \frac{\beta}{N^{(\alpha+1)/2}}\left(\frac{1}{N}(\sigma.\mathbf{1})(\sigma.\mathbf{S}) - \frac{S}{N^2}\sum_{i<j}\sigma_i\sigma_j\right) + o_{p.s.}(1)$$

Thus we can define a random variable $\tilde{H}_N(\sigma)$ by:

$$\tilde{H}_N(\eta) = \frac{\beta}{N^{(\alpha+1)/2}}\frac{(\eta.\mathbf{1})(\eta.\mathbf{S})}{N} - \frac{\beta S}{2N^{2+(\alpha+1)/2}}(\eta.\mathbf{1})^2 - \frac{\beta\overline{\mathbf{m}}S}{2N^{1+(\alpha+1)/2}}(\eta.\mathbf{1})$$

This random variable satisfies:

$$\frac{\beta}{N^{(\alpha+1)/2}}\sum_{i<j}\left(\frac{S_i+S_j}{N-2} - \frac{S}{(N-1)(N-2)}\right)\sigma_i\sigma_j = \frac{\beta\overline{\mathbf{m}}(\sigma.\mathbf{S})}{N^{(\alpha+1)/2}} + \frac{\beta(\sigma.\mathbf{1})(\mathbf{m}.\mathbf{S})}{N^{(\alpha+3)/2}} + \tilde{H}_N(\sigma) + o_{p.s.}(1)$$

and there exists a random constant K_ω such that:

$$\mathbb{P}\left\{\forall N\geq 1,\ |\tilde{H}_N| \leq \frac{K_\omega}{N^{\alpha/2}}\left[\left(\frac{\eta.\mathbf{1}}{\sqrt{N}}\right)^2 + \left(\frac{\eta.\mathbf{S}}{N}\right)^2\right] + \frac{\beta|S|}{N^{1+\alpha/2}}\left|\frac{\eta.\mathbf{1}}{\sqrt{N}}\right|\right\} = 1 \tag{8}$$

Using now vector $\mathbf{m}$ one can write:

$$Ng(\frac{\sigma.\mathbf{1}}{N}) = N(g(\overline{\mathbf{m}}) - \overline{\mathbf{m}}g'(\overline{\mathbf{m}})) + g'(\overline{\mathbf{m}})\sigma.\mathbf{1} + N\left(g(\frac{\eta.\mathbf{1}}{N} + \overline{\mathbf{m}}) - g(\overline{\mathbf{m}}) - g'(\overline{\mathbf{m}})\frac{\eta.\mathbf{1}}{N}\right)$$

Hence we have:

$$\begin{aligned}
&\mathbb{E}[Z_N^V|\mathbf{S}] = \\
&= (1+o_{p.s.}(1))\exp\left\{\frac{\beta^2N^{1-\alpha}}{4}\left[\frac{1}{N}\boldsymbol{\mu}.\mathbf{1} - \left(\frac{\mathbf{m}.\mathbf{1}}{N}\right)^2\right]^2 - \frac{\beta\overline{\mathbf{m}}(\mathbf{m}.\mathbf{S})}{N^{(\alpha+1)/2}} - \frac{\beta\overline{\mathbf{m}}^2S}{2N^{(\alpha+1)^2}} + N(g(\overline{\mathbf{m}}) - \overline{\mathbf{m}}g'(\overline{\mathbf{m}}))\right\} \\
&\qquad \mathbf{E}_\sigma\exp\left\{\sum_{i=1}^N\left(g'(\overline{\mathbf{m}}) + \frac{\beta\overline{\mathbf{m}}S_i}{N^{(\alpha+1)/2}} + \frac{\beta(\mathbf{m}.\mathbf{S})}{N^{(\alpha+3)/2}}\right)\sigma_i\right. \\
&\qquad\qquad \left. + N\left(g(\frac{\eta.\mathbf{1}}{N} + \overline{\mathbf{m}}) - g(\overline{\mathbf{m}}) - g'(\overline{\mathbf{m}})\frac{\eta.\mathbf{1}}{N}\right) + \tilde{H}_N(\sigma) + \hat{H}_N(\sigma)\right\}\mathbf{1}_{\frac{\sigma.\mathbf{1}}{N}\in V}
\end{aligned} \tag{9}$$

4.2 Truncation of the partition function and asymptotic expansion

Let us choose so small a $\delta > 0$ that the intervals $[m-\delta, m+\delta]$, for $m \in \mathcal{K}(\beta)$, are disjoint sets. Let us now define

$$Z_N^\delta = \sum_{m\in\mathcal{K}(\beta)}\mathbf{E}_\sigma\left[\exp(H_{N,\beta}^{f,\alpha}(\sigma))\mathbf{1}_{\left|\frac{\sigma.\mathbf{1}}{N} - m\right|\leq\delta}\right]$$

We are going to make an asymptotic expansion of Z_N^δ and then prove rigorously than Z_N may be approximated by Z_N^δ. So, let us choose $m_* \in \mathcal{K}(\beta)$ and consider $V = [m_* - \delta, m_* + \delta]$. Equation (9) suggests that we should define 'effective' magnetic fields h_i by:

$$h_i = g'(\overline{\mathbf{m}}) + \frac{\beta\overline{\mathbf{m}}S_i}{N^{(\alpha+1)/2}} + \frac{\beta(\mathbf{m}.\mathbf{S})}{N^{(\alpha+3)/2}} \quad i = 1,\ldots,N \tag{10}$$

Using these magnetic fields, one can define a new probability $\mathbf{P}_\sigma^h$ by its density w.r.t. $\mathbf{P}_\sigma$:

$$\frac{d\mathbf{P}_\sigma^h}{d\mathbf{P}_\sigma} \propto \exp \sum_{i=1}^N h_i \sigma_i$$

We now can rewrite expression (9) as follows:

$$\begin{aligned}
&\mathbb{E}[Z_N^{[m_*-\delta, m_*+\delta]} | \mathbf{S}] = (1 + o_{p.s.}(1)) \\
&\times \exp\left\{ \frac{\beta^2 N^{1-\alpha}}{4} \left[\frac{1}{N}\boldsymbol{\mu}.\mathbf{1} - \left(\frac{\mathbf{m}.\mathbf{1}}{N}\right)^2 \right]^2 - \frac{\beta \overline{\mathbf{m}}(\mathbf{m}.\mathbf{S})}{N^{(\alpha+1)/2}} - \frac{\beta \overline{\mathbf{m}}^2 S}{2N^{(\alpha+1)^2}} + N(g(\overline{\mathbf{m}}) - \overline{\mathbf{m}} g'(\overline{\mathbf{m}})) + \sum_{i=1}^N \Lambda(h_i) \right\} \\
&\qquad \times \mathbf{E}_\sigma^h \exp\left\{ N\left(g(\frac{\eta.\mathbf{1}}{N} + \overline{\mathbf{m}}) - g(\overline{\mathbf{m}}) - g'(\overline{\mathbf{m}})\frac{\eta.\mathbf{1}}{N}\right) + \tilde{H}_N(\eta) + \hat{H}_N \right\} \mathbf{1}_{|\frac{\sigma.\mathbf{1}}{N} - m_*| \le \delta}
\end{aligned}$$

The point is now to prove that one may use in the last expectation the central limit theorem. This will be essentially possible if the η_i are centered. We prove in the next lemma that one can choose such an $\mathbf{m}$ that this condition is fulfilled.

Lemma 4.1
There exists a δ_0 independent of $\mathbf{S}$ such that for any $\delta \le \delta_0$, there exists N_ω such that for $N \ge N_\omega$ the following non-linear system admits a unique solution $\mathbf{m} = \mathbf{m}(\mathbf{S})$ with $|\overline{\mathbf{m}} - m_*| \le \delta$:

$$m_i = \Lambda'\left(g'(\overline{\mathbf{m}}) + \frac{\beta \overline{\mathbf{m}} S_i}{N^{(\alpha+1)/2}} + \frac{\beta}{N^{(\alpha+3)/2}}(\mathbf{m}.\mathbf{S}) \right), \ 1 \le i \le N \tag{11}$$

Moreover, there exists a random constant K_ω such that

$$\mathbb{P}\left\{ \forall N \ge N_\omega,\ \max_{1\le i\le N} |m_i - m_*| \le \frac{K_\omega \sqrt{\ln N}}{N^{\alpha/2}} \right\} = 1$$

Proof : Let us define for $(x, \mu) \in [-1,1] \times \mathbb{R}$ the following function:

$$\Phi_{N,x}(\mu) = \frac{1}{N^{3/2}} \sum_{i=1}^N S_i \Lambda'\left(g'(x) + \frac{\beta x S_i}{N^{(\alpha+1)/2}} + \frac{\beta}{N^{\alpha/2}}\mu \right)$$

One has:

$$|\Phi_{N,x}(\mu) - \Phi_{N,x}(\mu')| \le \frac{\beta}{N^{(\alpha+3)/2}} \sup_{\xi\in\mathbb{R}} \Lambda''(\xi) \sum_{i=1}^N |S_i|.|\mu - \mu'| \le \frac{\beta K_\omega \sqrt{\ln N}}{N^{\alpha/2}}.|\mu - \mu'|$$

Hence, for $N \ge N_\omega$ which depends on ω but not on x, function $\Phi_{N,x}$ is a contraction and thus admits a unique fixed point $\mu_N(x)$. Function μ_N is smooth. One can easily check the following bound:

$$\forall N \ge N_\omega, \quad \sup_{x\in[-1,1]} \{|\mu_N(x)| + |\mu_N'(x)|\} \le K_\omega' \sqrt{\ln N} \tag{12}$$

Let us now recall that $g''(m_*) - I''(m_*) < 0$, that $I''(x) = 1/\Lambda''[(\Lambda')^{-1}(x)]$ and that $m_* = \Lambda'(g'(m_*))$. As a consequence, in a neighborhood $[m_* - \delta_0, m_* + \delta_0]$ of m_*, one has $\Lambda''(g'(x))g''(x) - 1 < 0$. Hence function $\Delta(x) = x - \Lambda'(g'(x))$ is strictly increasing on $[m_* - \delta_0, m_* + \delta_0]$ and equals 0 at m_*. Consider function Δ_N defined by:

$$\Delta_N(x) = x - \frac{1}{N}\sum_{i=1}^N \Lambda'\left(g'(x) + \frac{\beta x S_i}{N^{(\alpha+1)/2}} + \frac{\beta}{N^{\alpha/2}}\mu_N(x) \right)$$

Using bound (12), one can prove that for large enough N function Δ_N is stricly increasing. Function Δ_N converges uniformly on $[m_* - \delta_0, m_* + \delta_0]$ toward Δ, hence $\lim_{N\to\infty} \Delta_N(m_* + \delta) = \Delta(m_* + \delta) > 0$ and $\lim_{N\to\infty} \Delta_N(m_* - \delta) = \Delta(m_* - \delta) >\!<$. As a consequence, for large enough N equation $\Delta_N(x) = 0$

admits a unique solution $m_N \in [m_* - \delta, m_* + \delta]$. Solving the non-linear system of the lemma is clearly equivalent to solving $\Delta_N(x) = 0$ with $x = \overline{\mathbf{m}}$ and afterwards defining m_i by (11). In order to obtain the last bound, one first notices by extracting any converging subsequence that any limit point x of m_N satisfies $\Delta(x) = 0$. Hence m_N converges to m_*. Now:

$$|m_N - m_*| = \left| \frac{1}{N} \sum_{i=1}^{N} \Lambda' \left(g'(m_N) + \frac{\beta m S_i}{N^{(\alpha+1)/2}} + \frac{\beta}{N^{(\alpha+3)/2}} (\mathbf{m}.\mathbf{S}) \right) - \Lambda'(g'(m_*)) \right|$$

$$\leq \frac{1}{N} \sum_{i=1}^{N} \Lambda''(\xi_{i,N}) \left\{ g''(\xi'_{i,N}) |m_N - m_*| + \frac{\beta |m_N S_i|}{N^{(\alpha+1)/2}} + \frac{\beta}{N^{(\alpha+3)/2}} |\mathbf{m}_N.\mathbf{S}| \right\}$$

As $\xi_{i,N}, \xi'_{i,N}$ converges to $g''(m_*)$, for large enough N we have $\frac{1}{N}\sum_{i=1}^{N} \Lambda''(\xi_{i,N}) g''(\xi'_{i,N}) < 1$ and thanks to (4) the bound is clear. We shall drop again subscript N and set $\mathbf{m} = \mathbf{m}_N$. ■

We now have to write down all the asymptotic expansions that we are going to use.

Lemma 4.2
The following asymptotic expansions hold almost surely:

1.
$$\frac{\mathbf{m}.\mathbf{1}}{N} = m_* + O_{a.s.}(N^{-\alpha+\varepsilon})$$

2.
$$\frac{\mathbf{m}.\mathbf{S}}{N^{(\alpha+1)/2}} = \frac{m_* S}{N^{(\alpha+1)/2}} + \Lambda''(g'(m_*))\beta m_* N^{1-\alpha} + o(1)$$

3.
$$\sum_{i=1}^{N} \Lambda(h_i) = N\Lambda(g'(m_*)) + N\Lambda'(g'(m_*)) f''(m_*)(m - m_*) + 2\Lambda'(g'(m_*)) \frac{\beta m_* S}{N^{(\alpha+1)/2}}$$
$$+ \Lambda'(g'(m_*))\Lambda''(g'(m_*))\beta^2 m_* N^{1-\alpha} + \frac{1}{2}\Lambda"(g'(m_*))\beta^2 m_*^2 N^{1-\alpha} + o(1) \quad (13)$$

The proof of this lemma is tedious but straightforward. It only relies on Taylor expansions.

Let us now define vector $\boldsymbol{\mu}$:

$$\mu_i = \mathbf{E}_\sigma^h \sigma_i^2 = \Lambda''(h_i) + [\Lambda'(h_i)]^2$$

With this definition and using the previous asymptotic expansions, it is easy to get:

$$\frac{\beta^2}{4} N^{1-\alpha} \left[\frac{1}{N}\boldsymbol{\mu}.\mathbf{1} - \left(\frac{\mathbf{m}.\mathbf{1}}{N}\right)^2 \right]^2 = \frac{\beta^2}{4} N^{1-\alpha} [\Lambda''(g'(m_*))]^2 + o_{a.s.}(1) \quad (14)$$

As a consequence, we obtain:

$$\frac{\beta^2}{4} N^{1-\alpha} \left[\frac{1}{N}\boldsymbol{\mu}.\mathbf{1} - \left(\frac{\mathbf{m}.\mathbf{1}}{N}\right)^2 \right]^2 - \frac{\beta \overline{\mathbf{m}}(\mathbf{m}.\mathbf{S})}{N^{(\alpha+1)/2}} - \frac{\beta^2 \overline{\mathbf{m}}^2 S}{2N^{(\alpha+1)/2}} + N(g(\overline{\mathbf{m}}) - \overline{\mathbf{m}} g'(\overline{\mathbf{m}})) + \sum_{i=1}^{N} \Lambda(h_i)$$
$$= N(g(m_*) - \Lambda^*(m_*)) + \frac{\beta^2}{4} N^{1-\alpha} [\Lambda''(g'(m_*))]^2 + \frac{\beta m_*^2 S}{2N^{(\alpha+1)/2}} + \frac{1}{2}\Lambda"(g'(m_*))\beta^2 m_*^2 N^{1-\alpha} + o(1)$$

Hence we have:

$$\mathbb{E}[Z_N^{[m_*-\delta, m_*+\delta]} | \mathbf{S}] =$$
$$\exp\left\{ N(g(m_*) - \Lambda^*(m_*)) + \frac{\beta^2}{4} N^{1-\alpha} [\Lambda''(g'(m_*))]^2 + \frac{\beta m_*^2 S}{2N^{(\alpha+1)/2}} + \frac{1}{2}\Lambda"(g'(m_*))\beta^2 m_*^2 N^{1-\alpha} + o_{a.s.}(1) \right\}$$
$$\mathbf{E}_\sigma^h \left[\exp\left\{ N \left(g(\frac{\eta.\mathbf{1}}{N} + \overline{\mathbf{m}}) - g(\overline{\mathbf{m}}) - g'(\overline{\mathbf{m}}) \frac{\eta.\mathbf{1}}{N} \right) + \tilde{H}_N(\sigma) + \hat{H}_N(\sigma) \right\} \mathbf{1}_{|\frac{\sigma.\mathbf{1}}{N} - m_*| \leq \delta} \right]$$

The next lemma shows that we may use the central limit theorem to evaluate the latter expectation:

Lemma 4.3
We have:

$$\lim_{N\to\infty} \mathbf{E}^h_\sigma\left[\exp\left\{N\left(g(\frac{\eta.\mathbf{1}}{N}+\overline{\mathbf{m}})-g(\overline{\mathbf{m}})-g'(\overline{\mathbf{m}})\frac{\eta.\mathbf{1}}{N}\right)+\tilde{H}_N(\eta)+\hat{H}_N\right\}\mathbf{1}_{|\frac{\sigma.\mathbf{1}}{N}-m_*|\leq\delta}\right]=\frac{1}{\sqrt{1-g''(m_*)\tilde{\gamma}_{m_*}}}$$

Proof : A very standard argument on triangular arrays of independent random variables enable to check that under $\mathbf{P}^h_\sigma$ vector $(\eta.\mathbf{1}/\sqrt{N},\eta.\mathbf{S}/N,\sum_{i=1}^N(\sigma_i^2-\mu_i)\sqrt{N})$ satisfies the central limit theorem and converges in distribution to. ■

We have now proved the following proposition:

Proposition 4.4

$$e^{-N\Delta}\mathbb{E}[Z^\delta_N|\mathbf{S}]=(1+o_{p.s.}(1))\sum_{m\in\mathcal{K}(\beta)}\frac{\exp\left[\frac{\beta^2}{4}\tilde{\gamma}_m(2(m^2+\tilde{\gamma}_m)N^{1-\alpha}+\frac{\beta m^2 S}{2N^{(\alpha+1)/2}}\right]}{\sqrt{1-g''(m)\tilde{\gamma}_m}}$$

4.3 Expansion of Z^δ_N

Let us denote by $Z^{m,\delta}_N$ the truncated partition function associated with $[m-\delta,m+\delta]$ for $m\in\mathcal{K}(\beta)$. In this section, we are interested in the asymptotic behavior of $\mathbb{E}[Z^{m,\delta}_N Z^{m',\delta}_N|\mathbf{S}]$.

The conditional integration leads to a very similar expression with two 'replicas' σ and τ of the system, the only difference being a coupling term that will be proved to have no effect. It may be written:

$$\begin{aligned}H_N(\sigma,\tau)&=\frac{\beta^2}{N^{\alpha+1}}\sum_{i<j,p<q}V_{ij,pq}\sigma_i\sigma_j\tau_p\tau_q\\&=\frac{\beta^2}{N^{\alpha+1}}\left\{\frac{1}{2}(\sigma.\tau)^2-\frac{1}{N}(\sigma.\mathbf{1})(\tau.\mathbf{1})(\sigma.\tau)+\frac{1}{2N^2}(\sigma.\mathbf{1})^2(\tau.\mathbf{1})^2\right\}+o_{p.s.}(1)\end{aligned}$$

Let us introduce the localizations $\mathbf{m}$ and $\boldsymbol{\mu}$ for σ, $\mathbf{m}'$ and $\boldsymbol{\mu}'$ for τ. We set $\eta_i=\sigma_i-m_i$ et $\eta'_i=\tau_i-m_i$. We then make the exponential changes of probability associated with the 'effective' fields $\mathbf{h}$ and $\mathbf{h}'$. Under $\mathbf{P}^h_\sigma\otimes\mathbf{P}^h_\tau$, random vectors η and η' are independent. One then checks that:

$$\begin{aligned}&\frac{\beta^2}{N^{\alpha+1}}\left\{\frac{1}{2}(\sigma.\tau)^2-\frac{1}{N}(\sigma.\mathbf{1})(\tau.\mathbf{1})(\sigma.\tau)+\frac{1}{2N^2}(\sigma.\mathbf{1})^2(\tau.\mathbf{1})^2\right\}\\&\qquad=\frac{\beta^2N^{1-\alpha}}{2}\left[\frac{1}{N}\mathbf{m}.\mathbf{m}'-\left(\frac{\mathbf{m}.\mathbf{1}}{N}\right)\left(\frac{\mathbf{m}'.\mathbf{1}}{N}\right)\right]^2+R_N\end{aligned}$$

with a remainder R_N that can be bounded as follows:

$$|R_N|\leq\frac{K_\omega}{N^{\alpha-1/2}}\left[\left|\frac{\eta.\mathbf{m}'}{\sqrt{N}}\right|+\left|\frac{\eta'.\mathbf{m}}{\sqrt{N}}\right|+\left|\frac{\eta.\eta'}{\sqrt{N}}\right|+\left|\frac{\eta.\mathbf{1}}{\sqrt{N}}\right|+\left|\frac{\eta'.\mathbf{1}}{\sqrt{N}}\right|\right]$$

The empirical covariance term may be estimated:

$$\frac{\beta^2}{N^{\alpha+1}}\frac{N^2}{2}\left[\frac{1}{N}\sum_{i=1}^N\left(m_i-\frac{1}{N}\sum_{i=1}^N m_i\right)\left(m'_i-\frac{1}{N}\sum_{i=1}^N m'_i\right)\right]^2\leq K''_\omega N^{1-3\alpha+\varepsilon}=o_{a.s.}(1)$$

With very similar arguments to what has been used in the previous section, that is central limit theorem and uniform integrability, one gets:

$$\mathbb{E}[Z^{m,\delta}_N Z^{m',\delta}_N|\mathbf{S}]=\mathbb{E}[Z^{m,\delta}_N|\mathbf{S}]\mathbb{E}[Z^{m',\delta}_N|\mathbf{S}](1+o_{a.s.}(1))$$

As a consequence of this result, we get:

$$\mathbb{E}[(Z^\delta_N)^2|\mathbf{S}]=(1+o_{a.s.}(1))\mathbb{E}[Z^\delta_N|\mathbf{S}]^2$$

Hence,

$$\mathbb{P}\left\{|Z_N^\delta - \mathbb{E}[Z_N^\delta|\mathbf{S}]| \geq \varepsilon \mathbb{E}[Z_N^\delta|\mathbf{S}]\Big|\mathbf{S}\right\} \leq \frac{\mathbb{E}\left[Z_N^\delta - \mathbb{E}[Z_N^\delta|\mathbf{S}]|\mathbf{S}\right]^2}{\varepsilon^2 \mathbb{E}[Z_N^\delta|\mathbf{S}]^2} \xrightarrow[N\to\infty]{} 0 \quad a.s.$$

Taking now the expectation w.r.t. $\mathbf{S}$, one gets:

$$\forall \varepsilon > 0, \quad \lim_{N\to\infty} \mathbb{P}\left\{|Z_N^\delta - \mathbb{E}[Z_N^\delta|\mathbf{S}]| \geq \varepsilon \mathbb{E}[Z_N^\delta|\mathbf{S}]\right\} = 0$$

Hence we have proved:

$$e^{-N\Delta} Z_N^\delta = (1 + o_{\mathbb{P}}(1)) \sum_{m\in\mathcal{K}(\beta)} \frac{\exp\left[\frac{\beta^2}{4}\tilde{\gamma}_m(2m^2+\tilde{\gamma}_m)N^{1-\alpha} + \frac{\beta m^2 S}{2N^{(\alpha+1)/2}}\right]}{\sqrt{1-g''(m)\tilde{\gamma}_m}} \tag{15}$$

4.4 Expansion of Z_N

We end this section by proving that we have really made an asymptotic expansion of Z_N. Clearly,

$$\overline{\lim_{N\to\infty}} \frac{1}{N} \ln[\mathbb{E}Z_N - \mathbb{E}Z_N^\delta] \leq \sup\{g(x) - I(x) : \ |x-m| \geq \delta, \, m \in \mathcal{K}(\beta)\} = \Delta_c < \Delta$$

Let us fix $p > 1$ and define $q = p/(p-1)$:

$$Z_N - Z_N^\delta \leq \left(\mathbf{E}_\sigma \exp pNg(\frac{\sigma.\mathbf{1}}{N})\mathbf{1}_{|\frac{\sigma.\mathbf{1}}{N}-m|\geq\delta, \, m\in\mathcal{K}(\beta)}\right)^{1/p} \left(\mathbf{E}_\sigma \exp\frac{\beta q}{N^{(\alpha+1)/2}}\sum_{i<j} J_{i,j}\sigma_i\sigma_j\right)^{1/q}$$

By Markov's inequality:

$$\begin{aligned}\mathbb{P}\left(\mathbf{E}_\sigma \exp\frac{\beta q}{N^{(\alpha+1)/2}}\sum_{i<j} J_{i,j}\sigma_i\sigma_j \geq \exp\frac{\beta^2 q^2}{2}N^{1-\alpha}\right) &\leq \exp[-\frac{\beta^2 q^2}{2}N^{1-\alpha}]\mathbf{E}_\sigma \exp\left(\frac{\beta^2 q^2}{4N^{\alpha+1}}\sum_{i<j}\sigma_i^2\sigma_j^2\right) \\ &\leq \exp[-\frac{\beta^2 q^2}{4}N^{1-\alpha}]\exp[\frac{\beta^2 q^2}{4}N^{1-\alpha}] = \exp -\frac{\beta^2 q^2}{4}N^{1-\alpha}\end{aligned}$$

Borel-Cantelli's lemma shows that there exists Ω_q with full probability such that:

$$\forall \omega \in \Omega_q, \, \exists N(\omega) : \ \forall n \geq N(\omega), \quad \mathbf{E}_\sigma \exp\frac{\beta q}{N^{(\alpha+1)/2}}\sum_{i<j} J_{i,j}\sigma_i\sigma_j < \exp\frac{\beta^2 q^2}{2}N^{1-\alpha}$$

Thus on Ω_q we have:

$$\overline{\lim_{N\to\infty}} \frac{1}{N}\ln[Z_N - Z_N^\delta] \leq \frac{1}{p}\sup\{pg(x) - I(x) : \ \forall m \in \mathcal{K}(\beta), \ |x-m| \geq \delta\}$$

By considering $\cap_{q\in\mathbb{Q}, q>1}\Omega_q$, the previous relation holds almost surely for any $p \in \mathbb{Q}$ such that $p > 1$. As g is bounded,

$$\frac{1}{p}\sup\{pg(x) - I(x) : \forall m \in \mathcal{K}(\beta), \ |x-m| \geq \delta\} \leq \frac{p-1}{p}\|g\|_\infty + \Delta_c$$

Letting now p going to 1, one obtains:

$$\overline{\lim_{N\to\infty}} \frac{1}{N}\ln[Z_N - Z_N^\delta] \leq \Delta_c \quad \mathbb{P}\text{-p.s.}$$

Equation (15) gives for any $\varepsilon > 0$:

$$\begin{aligned}\mathbb{P}\left\{\frac{Z_N - Z_N^\delta}{Z_N^\delta} > \varepsilon\right\} &= \mathbb{P}\left\{\frac{1}{N}\ln Z_N^\delta < -\frac{1}{N}\ln\varepsilon + \frac{1}{N}\ln(Z_N - Z_N^\delta)\right\} \\ &\leq \mathbb{P}\left\{\frac{1}{N}\ln Z_N^\delta < -\frac{1}{N}\ln\varepsilon + \Delta_c + (\Delta - \Delta_c)/2\right\} \\ &\quad + \mathbb{P}\{\frac{1}{N}\ln(Z_N - Z_N^\delta) \geq \Delta_c + (\Delta-\Delta_c)/2\} \xrightarrow[N\to\infty]{} 0\end{aligned}$$

Hence in probability $Z_N = Z_N^\delta(1 + o_{\mathbb{P}}(1))$.

5 Convergence of the Gibbs measure

In order to obtain a more precise result, we are going to study a conditioned Gibbs measure. Let us first choose a $\delta > 0$ such that the intervals $[m-\delta, m+\delta]$, with $m \in \mathcal{K}(\beta)$, are disjoint. Consider now $m \in \mathcal{K}(\beta)$ define as usual the truncated partition function $Z_N^\delta = \mathbf{E}_\sigma \exp H_{N,\beta}^{f,\alpha}(\sigma)\mathbf{1}_{[m-\delta,m+\delta]}$ and the associated Gibbs measure that is in fact $G_{N,\beta}^{f,\alpha}$ given $\frac{\sigma.\mathbf{1}}{N} \in [m-\delta, m+\delta]$ and that we shall denote by Γ_N^δ. We begin by stating a lemma in which we prove that $1/N \ln Z_N^\delta$ is self-averaging:

Lemma 5.1
For any $u > 0$, we have:

$$\lim_{N\to\infty} \frac{1}{N} \ln \mathbb{P}\left\{\left|\frac{1}{N}\ln Z_N^\delta - \frac{1}{N}\ln \mathbb{E}Z_N^\delta\right| \geq u\right\} = -\infty$$

Proof : We may consider that the $J_{i,j}$ are the values at $t = 1$ of independent standard brownian motions $J_{i,j}(t)$. Replacing in Z_N^δ $J_{i,j}$ by $J_{i,j}(t)$ we obtain a process $Z_N(t)$, and a Gibbs measure G_N^t given $\frac{\sigma.\mathbf{1}}{N} \in [m-\delta, m+\delta]$. Usual Ito's formula gives:

$$\frac{1}{N}\ln Z_N^\delta - \frac{1}{N}\mathbb{E}\ln Z_N^\delta = \frac{1}{N}\int_0^1 \sum_{1\leq i<j\leq N} \frac{\beta}{N^{(\alpha+1)/2}} G_N^t\left(\sigma_i\sigma_j \middle| \frac{\sigma.\mathbf{1}}{N} \in [m-\delta, m+\delta]\right)\, dJ_{i,j}(t)$$

Now, for any $u, \theta > 0$ we have:

$$\mathbb{P}\left\{\frac{1}{N}\ln Z_N^\delta - \frac{1}{N}\ln \mathbb{E}Z_N^\delta \geq u\right\} \leq e^{-\theta u}\mathbb{E}\exp\theta\left\{\frac{1}{N}\int_0^1 \sum_{1\leq i<j\leq N} \frac{\beta}{N^{(\alpha+1)/2}} G_N^t\left(\sigma_i\sigma_j \middle| \frac{\sigma.\mathbf{1}}{N} \in [m-\delta, m+\delta]\right)\, dJ_{i,j}(t)\right\}$$

$$= e^{-\theta u}\mathbb{E}\exp\frac{\theta^2}{2}\left\{\frac{1}{N^2}\int_0^1 \sum_{1\leq i<j\leq N} \frac{\beta^2}{N^{\alpha+1}} \left(G_N^t\left(\sigma_i\sigma_j \middle| \frac{\sigma.\mathbf{1}}{N} \in [m-\delta, m+\delta]\right)\right)^2 dt\right\}$$

Since $\left|G_N^t\left(\sigma_i\sigma_j \middle| \frac{\sigma.\mathbf{1}}{N} \in [m-\delta, m+\delta]\right)\right| \leq 1$, we obtain the upper bound:

$$\mathbb{P}\left\{\frac{1}{N}\ln Z_N^\delta - \frac{1}{N}\ln \mathbb{E}Z_N^\delta \geq u\right\} \leq e^{-\theta u + \frac{\theta^2\beta^2}{4N^{\alpha+1}}}$$

Choosing $\theta = 2N^{\alpha+1}u/\beta^2$, we obtain:

$$\mathbb{P}\left\{\frac{1}{N}\ln Z_N^\delta - \frac{1}{N}\ln \mathbb{E}Z_N^\delta \geq u\right\} \leq \exp\left[-\frac{N^{\alpha+1}u^2}{\beta^2}\right]$$

The same result could have been obtained for $\mathbb{P}\left\{\frac{1}{N}\ln Z_N^\delta - \frac{1}{N}\ln \mathbb{E}Z_N^\delta \leq -u\right\}$ with a negative θ. Hence the result of the lemma holds with $\mathbb{E}\ln Z_N^\delta$ instead of $\ln \mathbb{E}Z_N^\delta$. The upper bound that has just been obtained proves that almost surely the difference goes to zero. Since we have already proved that in probability $\ln[Z_N^\delta]/\mathbb{E}Z_N^\delta = o(N)$, it is clear that:

$$\lim_{N\to\infty}\left[\frac{1}{N}\ln \mathbb{E}Z_N^\delta - \frac{1}{N}\mathbb{E}\ln Z_N^\delta\right] = 0$$

Hence we may replace $\mathbb{E}\ln Z_N^\delta$ by $\ln \mathbb{E}Z_N^\delta$ for large enough N and the result is proved. ■

Denote by L_N the empirical measure of (σ,τ), that is $L_N = \frac{1}{N}\sum_{i=1}^N \delta_{\sigma_i,\tau_i}$. Define $q_1(\sigma,\tau)=\sigma$ and $q_2(\sigma,\tau)=\tau$. For any borel subset B of $\mathcal{P}([-1,1]^2)$, we have:

$$
\begin{aligned}
&\mathbb{E}\Gamma_N^\delta \otimes \Gamma_N^\delta(L_N\in B)\\
&= \mathbb{E}\left\{ \frac{\mathbf{E}_{\sigma,\tau} e^{H_N(\sigma)+H_N(\tau)} \mathbf{1}_{\frac{\sigma.\mathbf{1}}{N}\in[m-\delta,m+\delta]} \mathbf{1}_{\frac{\tau.\mathbf{1}}{N}\in[m-\delta,m+\delta]} \mathbf{1}_{L_N\in B}}{\left[\mathbb{E}\mathbf{E}_\sigma e^{H_N(\sigma)}\mathbf{1}_{\frac{\sigma.\mathbf{1}}{N}\in[m-\delta,m+\delta]}\right]^2} \times \frac{\left[\mathbb{E}\mathbf{E}_\sigma e^{H_N(\sigma)}\mathbf{1}_{\frac{\sigma.\mathbf{1}}{N}\in[m-\delta,m+\delta]}\right]^2}{\left[\mathbf{E}_\sigma e^{H_N(\sigma)}\mathbf{1}_{\frac{\sigma.\mathbf{1}}{N}\in[m-\delta,m+\delta]}\right]^2} \right\}\\
&\le \mathbb{P}\left\{ \frac{\left[\mathbb{E}\mathbf{E}_\sigma e^{H_N(\sigma)}\mathbf{1}_{\frac{\sigma.\mathbf{1}}{N}\in[m-\delta,m+\delta]}\right]^2}{\left[\mathbf{E}_\sigma e^{H_N(\sigma)}\mathbf{1}_{\frac{\sigma.\mathbf{1}}{N}\in[m-\delta,m+\delta]}\right]^2} > e^{N\varepsilon} \right\}\\
&\qquad + e^{N\varepsilon}\mathbb{E}\left\{ \frac{\mathbf{E}_{\sigma,\tau} e^{H_N(\sigma)+H_N(\tau)} \mathbf{1}_{\frac{\sigma.\mathbf{1}}{N}\in[m-\delta,m+\delta]} \mathbf{1}_{\frac{\tau.\mathbf{1}}{N}\in[m-\delta,m+\delta]} \mathbf{1}_{L_N\in B}}{\left[\mathbb{E}\mathbf{E}_\sigma e^{H_N(\sigma)}\mathbf{1}_{\frac{\sigma.\mathbf{1}}{N}\in[m-\delta,m+\delta]}\right]^2} \right\}
\end{aligned}
$$

Hence, using the previous lemma, we get:

$$
\begin{aligned}
\overline{\lim_{N\to\infty}} \frac{1}{N} \ln \mathbb{E}\Gamma_N^\delta \otimes \Gamma_N^\delta(L_N\in B) \le \varepsilon + \sup\Big\{ & g(\langle \nu, q_1\rangle) + g(\langle \nu, q_2\rangle) - H(\nu|\rho^{\otimes 2}) :\\
& \langle \nu, q_1\rangle \in [m-\delta,m+\delta], \langle \nu, q_2\rangle \in [m-\delta,m+\delta], \nu\in\overline{B}\Big\} - 2\Delta
\end{aligned}
$$

It is now possible to let ε go to 0. Function $\nu \mapsto g(\langle \nu, q_1\rangle) + g(\langle \nu, q_2\rangle) - H(\nu|\rho^{\otimes 2})$ admits a unique maximum on $\{\nu : \langle \nu, q_1\rangle \in [m-\delta,m+\delta], \langle \nu, q_2\rangle \in [m-\delta,m+\delta]\}$ at $\tilde{\mu}_m\otimes\tilde{\mu}_m$. Standard large deviations arguments then lead to the convergence of L_N to $\tilde{\mu}_m\otimes\tilde{\mu}_m$. Since probability measure $\mathbb{E}\Gamma_N^\delta \otimes \Gamma_N^\delta$ is exchangeable, using Sznitman's results in [Szn], one obtains the propagation of chaos, which means that for any $k\ge 1$, $((\sigma_1,\tau_1),\dots,(\sigma_k,\tau_k))$ converges in distribution to $(\tilde{\mu}_m\otimes\tilde{\mu}_m)^{\otimes k}$. In particular, for any continuous functions ϕ and ψ on $[-1,1]^k$ we have:

$$
\begin{aligned}
\lim_{N\to\infty} \mathbb{E}\left[G_N\left(\phi(\sigma_1,\dots,\sigma_k)\middle| \frac{\sigma.\mathbf{1}}{N}\in[m-\delta,m+\delta]\right)\right]\left[G_N\left(\psi(\tau_1,\dots,\tau_k)\middle| \frac{\tau.\mathbf{1}}{N}\in[m-\delta,m+\delta]\right)\right]\\
= \langle \tilde{\mu}_m^{\otimes k},\phi\rangle\langle \tilde{\mu}_m^{\otimes k},\psi\rangle
\end{aligned}
$$

By taking one function ψ equal to one, one gets:

$$
\lim_{N\to\infty} \mathbb{E}G_N\left(\phi(\sigma_1,\dots,\sigma_k)\middle| \frac{\sigma.\mathbf{1}}{N}\in[m-\delta,m+\delta]\right) = \langle \tilde{\mu}_m^{\otimes k},\phi\rangle
$$

Thus we have proved:

Proposition 5.2
For any $m\in\mathcal{K}(\beta)$ and any $\delta>0$ such that $[m-\delta,m+\delta]$ does not contain any point of $\mathcal{K}(\beta)$ but m, then in $\mathbf{L}^2$, for any $k\ge 1$ and any continuous ϕ on $[-1,1]^k$ we have:

$$
\lim_{N\to\infty} G_N\left(\phi(\sigma_1,\dots,\sigma_k)\middle| \frac{\sigma.\mathbf{1}}{N}\in[m-\delta,m+\delta]\right) = \langle \tilde{\mu}_m^{\otimes k},\phi\rangle
$$

This result describes rather precisely the Gibbs measure. Let us now prove theorem 2.2.

We first assume that $\mathcal{K}^+(\beta)=\mathcal{K}^-(\beta)$. Expression 15 shows that if $m\notin\mathcal{K}^+(\beta)$ then it is clear that $G_N\left(\frac{\sigma.\mathbf{1}}{N}\in[m-\delta,m+\delta]\right)$ goes to zero, whereas if $m\in\mathcal{K}^+(\beta)$ then $G_N\left(\frac{\sigma.\mathbf{1}}{N}\in[m-\delta,m+\delta]\right)$ goes to d_m, all the convergences are meant in the probability sense. Hence:

$$
\begin{aligned}
&\lim_{N\to\infty} G_N(\phi(\sigma_1,\dots,\sigma_k))\\
&= \lim_{N\to\infty} \sum_{m\in\mathcal{K}(\beta)} G_N\left(\phi(\sigma_1,\dots,\sigma_k)\middle| \frac{\sigma.\mathbf{1}}{N}\in[m-\delta,m+\delta]\right) G_N\left(\frac{\sigma.\mathbf{1}}{N}\in[m-\delta,m+\delta]\right) = \sum_{m\in\mathcal{K}^+(\beta)} d_m\langle \tilde{\mu}_m^{\otimes k},\phi\rangle
\end{aligned}
$$

When $\mathcal{K}^+(\beta) \neq \mathcal{K}^-(\beta)$, the Gibbs measure is not converging in the usual sense but in the Cesaro's sense. This is what has been called by Newman 'metastates'. We first notice that we only have to take care of $m \in \mathcal{K}^+(\beta) \cup \mathcal{K}^-(\beta)$.

Our proof relies on Kulske's ideas [Kul96]. For $\varepsilon \in (0,1)$, set

$$\mathcal{H}_N = \{\omega :\ |S^N| \geq N^{1-\delta}\} \text{ and } \mathcal{H} = \left\{\omega :\ \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{\omega \in \mathcal{H}_n^c} = 0\right\}$$

It is proved in [Kul96] that $\mathbb{P}(\mathcal{H}) = 1$. Let us now introduce the relative weights p_N^m, for $m \in \mathcal{K}^+(\beta) \cup \mathcal{K}^-(\beta)$, defined by:

$$p_N^m = \frac{\exp\left[\frac{\beta m^2 S}{2N^{(\alpha+1)/2}}\right] / \sqrt{1-g''(m)\tilde{\gamma}_m}}{\exp\left[\frac{\beta (m^+)^2 S}{2N^{(\alpha+1)/2}}\right] \sum_{q\in\mathcal{K}^+(\beta)} \frac{1}{\sqrt{1-g''(q)\tilde{\gamma}_q}} + \exp\left[\frac{\beta (m^-)^2 S}{2N^{(\alpha+1)/2}}\right] \sum_{q\in\mathcal{K}^-(\beta)} \frac{1}{\sqrt{1-g''(q)\tilde{\gamma}_q}}}$$

It is clear that in probability, $Z_N^\delta/Z_N - p_N^m$ goes to zero. If S^N is large, then p_N^m may be approximated by $d_m^+ \mathbf{1}_{S_N>0}$ if $m \in \mathcal{K}^+(\beta)$ and by $d_m^- \mathbf{1}_{S^N<0}$ if $m \in \mathcal{K}^-(\beta)$. More precisely, let us fix $\varepsilon > 0$. For $N \geq N_\varepsilon$, independent of ω, we have for $m \in \mathcal{K}^+(\beta)$:

$$|p_n^m - d_m^+ \mathbf{1}_{S^n>0}| \leq \varepsilon \mathbf{1}_{\omega\in\mathcal{H}_n} + 2.\mathbf{1}_{\omega\in\mathcal{H}_n^c} \leq \varepsilon + 2.\mathbf{1}_{\omega\in\mathcal{H}_n^c}$$

Hence:

$$\left|\frac{1}{N}\sum_{n=1}^{N} p_n^m - d_m^+ \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{S^n>0}\right| \leq \varepsilon \frac{2}{N}\sum_{n=1}^{N} \mathbf{1}_{\omega\in\mathcal{H}_n^c}$$

Thus, , since $\mathbb{P}(\mathcal{H}) = 1$, we have:

$$\mathbb{P}\left\{\lim_{N\to\infty}\left(\frac{1}{N}\sum_{n=1}^{N} p_n^m - \frac{d_m^+}{N}\sum_{n=1}^{N}\mathbf{1}_{S^n>0}\right) = 0\right\} = 1$$

We deduce from this expression that in probability

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} G_n\left(\frac{\sigma.\mathbf{1}}{N} \in [m-\delta, m+\delta]\right) - \frac{d_m^+}{N}\sum_{n=1}^{N}\mathbf{1}_{S^n>0} = 0$$

Something similar could be proved for $m \in \mathcal{K}^-(\beta)$. Thus, for any $k \geq 1$ and any continuous function ϕ on $[-1,1]^k$ we have in probability:

$$\lim_{N\to\infty}\left[\frac{1}{N}\sum_{n=1}^{N} G_n(\phi(\sigma_1,\ldots,\sigma_k)) - \left(\frac{1}{N}\sum_{n=1}^{N}\mathbf{1}_{S^n>0}\right)\sum_{m\in\mathcal{K}^+(\beta)} d_m^+ \langle \tilde{\mu}_m^{\otimes k}, \phi\rangle - \left(\frac{1}{N}\sum_{n=1}^{N}\mathbf{1}_{S^n<0}\right)\sum_{m\in\mathcal{K}^-(\beta)} d_m^- \langle \tilde{\mu}_m^{\otimes k}, \phi\rangle\right] = 0$$

In order to obtain the claimed result, one just has to notice that $\frac{1}{N}\sum_{n=1}^{N}\mathbf{1}_{S^n<0}$ converges in distribution to $\int_0^1 \mathbf{1}_{B_{t^2}>0}\, dt$.

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