On the long time behaviour of the solution to the two-fluids incompressible Navier-Stokes equations [1]

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January 29, 1998

Abstract

We devote this work to the long time behaviour of the solution to the incompressible Navier-Stokes equations for two viscous immiscible fluids contained in a bounded domain and subjected only to gravity forces. When there is surface tension at the interface or not, for the model linearized around the steady-state of minimal energy or for the standard nonlinear model, we investigate the following question. Do the equations reproduce the behaviour expected from experiment, namely a convergence to zero of the velocity field, and a convergence of the interface to its stable position. Our results show a wide variety of behaviours, depending on the case considered.

Résumé

Nous consacrons cette étude au comportement en temps long de la solution des Equations de Navier-Stokes pour deux fluides incompressibles visqueux immiscibles remplissant un domaine borné et soumis à l'influence de la gravité seulement. En présence ou non de tension de surface à l'interface, pour le modèle linéarisé autour de l'état stationnaire d'énergie minimale ou pour le modèle non linéaire initial, nous cherchons à savoir si les équations reproduisent ou non le comportement attendu pour ce système dissipatif au vu de l'expérience : une convergence vers zéro du champ de vitesse et un retour à l'équilibre de l'interface. Les résultats obtenus font apparaître une grande variété de comportements suivant les cas considérés.

[1] To appear in *Differential and Integral Equations*.

Contents

1 Introduction

We present in this article a study of the long time behaviour of the time-dependent Navier-Stokes equations for two incompressible immiscible fluids in various settings. The main question under consideration here is the following : assume that the forces and the boundary conditions are such that for any steady state, both fluids are at rest (the velocity is zero all over the domain), then can one show that the viscous dissipation drives the system to such a steady state as time goes to infinity ? Intuitively, if for instance the only forces are due to the gravity, and if the two fluids are of different densities, it is expected that the system goes, as time goes to infinity, to the situation when the two fluids are at rest, separated by a flat interface, the heaviest fluid below this interface, and the lightest above. One of the goal of this article is to investigate in what sense this simple intuitive expectation (and observation) is satisfied mathematically. In other words, we aim at studying in what sense the Navier-Stokes equations do reproduce the physical reality on that particular point.

More mathematically, consider (u, p, ρ) a solution to

$$\begin{aligned} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \Delta u &= -\nabla p + \rho f_m + f_v, \\ \partial_t \rho + \operatorname{div}(\rho u) &= 0, \\ \operatorname{div} u &= 0, \end{aligned} \tag{1.1}$$

with the no-slip boundary condition

$$u = 0 \text{ on } \partial\Omega, \tag{1.2}$$

and with the initial data

$$u(\cdot, t = 0) = u_0(\cdot), \tag{1.3}$$

and

$$\rho(\cdot, t = 0) = \rho^0(\cdot) = \begin{cases} \rho_1 > 0, & \text{constant in } \Omega_1, \\ \rho_2 > 0, & \text{constant in } \Omega_2, \end{cases} \tag{1.4}$$

for a partition (Ω_1, Ω_2) of the domain Ω where the system (1.1) is set. We do not detail in this introduction the sense in which (u, p, ρ) is a solution to (1.1), but we will make it precise below. Possibly, we shall add to the right-hand side of (1.1) a term modelling the effect of the surface tension at the interface between the two fluids. Let us assume then that the given massic forces f_m and volumic forces f_v are such that any steady-state solution of (1.1) consists of some piecewise constant density $\rho \in \{\rho_1, \rho_2\}$ and of the zero velocity field $u \equiv 0$. Can we say something on the behaviour of (u, ρ) for t going to infinity ?

1.1 Position of the problem

It is first of all to be remarked that we cannot hope to solve the question of the long time behaviour of the solution to the two-fluids Navier-Stokes equations in a very general setting, for arbitrary forces f_m and f_v, since, even for the one fluid case, this question is an extremely difficult one.

Let us briefly overview the main results known to this day on this subject.

As far as the long time behaviour of the Navier-Stokes equations (and more generally of any dissipative system of infinite size) is concerned, the main body of the theory is due to R. Temam and coworkers (*see* R. Temam [46, 45], and P. Constantin, C. Foias, B. Nicolaenko, R. Temam [12]). Globally speaking, the long time behaviour of these equations is finite dimensional in two dimensions without restriction and in three dimensions at least for flows which remain smooth. In fact, as will become clear in the examples below and in the whole sequel, the determination of the long time behaviour of the solutions is closely related to the existence of regular solutions for all time.

In two dimensions, the solution is regular and therefore many things are known. If the force is time independent, there exists an attractor, and its Haussdorf dimension is finite. This attractor is all the more regular as the force is (e.g. $\mathcal{C}^\infty$ if the force is $\mathcal{C}^\infty$). In the space periodic case, it is even possible to show that there exists an inertial manifold. An upper bound on the finite dimension of the attractor is related to the Reynolds number of the problem (*see e.g.* A. Miranville & X. Wang [32] and references therein). Most of these results apply to the MHD system (M. Sermange, R. Temam [34]). In three dimensions, it is only known that the functional invariant sets bounded in L^2 are of finite Haussdorf dimension, but no existence of attractor (which would exist if the solutions were regular for all time) have been established to this day in the generic case.

In very particular situations, it is possible to improve these general results by proving the convergence of the flow to some stationary state. Such kind of results is in fact expected from experiment. When the body force the fluid is subjected to is large (and even if it is stationary) there are some situations where the flow remains turbulent and time dependent for long times (for instance, it tends to a time-periodic solution). But when the force is small, there are many situations where the flow converges as time goes to infinity to the state where the fluid is at rest. Let us examine now the mathematical counterpart of this experimental observation.

The first result in this spirit concerns the case of one homogeneous fluid enclosed in a fixed box and goes back to Leray. In two or three dimensions with homogeneous Dirichlet boundary conditions, when there is no body force, the only steady state is the fluid at rest and the time dependent flow converges to it in H^1 as time goes to infinity. This result has in particular been extended in the following two directions : if the force f and the data (initial velocity and boundary conditions) are small enough then the flow remains regular for all time (even in three dimensions) and the speed of convergence toward the steady state can be evaluated (*see* C. Guillope [24], J.G Heywood [25, 26], C. Foias, J.C. Saut[18]); if the initial velocity is large but when the force is gradient-like it is possible to show that the flow becomes smooth after a finite time, then remains smooth and converges to the steady state (*see* J.G Heywood [26]). Some analogous results are available under convenient hypotheses in the unbounded case (*see* G.P. Galdi, J.G Heywood, Y. Shibata [20], W. Borchers & T. Miyakawa [10] and references therein).

Let us now leave the case of one fluid in a fixed domain and deal with the case of one fluid enclosed within a free surface or the case of two fluids. There again, most studies deal with situations when there exist regular solutions which is mostly the case when the data are small and the evolution is not far from equilibrium : let us mention here the works by V.A. Solonnikov [38, 40, 37] and by J.T Beale [5, 7]. The basic result is the convergence to the steady state as time goes to infinity. Let us also mention for the sake of completeness the work by A. Tani and N. Tanaka [43], the works in the irrotational inviscid case J.T Beale, T.Y Hou, J.S Lowengrub [6], T.Y Hou, Z.H Teng, P. Zhang [27] and also a connected work by H. Beirao da Veiga [8].

In the case we deal with in this paper, this is therefore only under very restrictive assumptions one can hope to settle this question. All the situations we shall consider below share the same following feature : there is uniqueness of the stationary velocity field (but not necessarily of the stationary interface).

Our study is actually motivated by the examination of the question of the long time behaviour of the solution in a more complicated situation (arising in the modelling of many industrial problems of metal processing), namely the situation when the two incompressible fluids are in addition two electrically conductor fluids, confined in a bounded domain, initially disposed as two horizontal layers separated by a regular interface, and when the motion of these two fluids is governed by a system of equations consisting of the Navier-Stokes equations coupled with the Maxwell equations (namely the magnetohydrodynamic equations). The massic force term f_m in (1.1) is then only due to the gravity, while the volumic force term f_v is the Lorentz force

$$f_v = \operatorname{curl} B \times B, \tag{1.5}$$

the magnetic field B being solution to the Maxwell equations in a more or less simplified form (*see* J.-F. Gerbeau, C. Le Bris [23]). This article is to be seen as a first step toward the study of this system. Many studies have already dealt with this question : *see* among other references J. Descloux, Y. Jaccard, M.V. Romerio [15], P. Maillard, M.V. Romerio [31]. So far as we know, most studies treat the linearized case (expansion of the solution in the neighborhood of the zero steady state solution when the initial data is a small perturbation). In view of all the difficulties of the generic case explained above, it must of course not be surprising for the reader that the somewhat practical studies we indicate here focus on this simplified linearized setting. Many cases of magnetic and electric fields are considered, in various geometries, in two or three dimensions, under various assumptions of symmetry. The emphasis is layed on the behaviour of the velocity and of the electromagnetic field, and the conclusion provided by these studies is mainly that, under convenient assumptions, the velocity goes to zero, in a more or less strong sense, while the electromagnetic field tends to some well identified limit. Unfortunately, nothing (or almost nothing) is known about the behaviour of the interface separating the two fluids (*see* for instance Remark 3.4 in [15]). We believe that some information on this behaviour could be useful, in particular if one has in mind questions of stability of such two-fluids systems. From a rigorous viewpoint, it is indeed not clear (and it will indeed be illustrated in the sequel) that the interface goes to some equilibrium shape if the topology for which the velocity goes to zero is too weak ; moreover, if the interface does converge, one has to identify its limit. Some pathological situations have to be ruled out (*see* Figure 1). In order to investigate this question, we first go back from the sophisticated magnetohydrodynamic problem to the more basic problem of the two fluids subjected only to the gravity (*see* [21] for some magnetohydrodynamics case). The question is then : to what extent does the hydrodynamic equation reproduce the behaviour expected from experiment ?

Let us end this paragraph by making a comment on the numerical counterpart of the questions we address here. Checking that the mathematical model does reproduce the reality in very simple situations might be of primary interest when instabilities are observed in the numerical experimentations for most difficult cases (or even for these simple cases). Indeed, one must then settle the following question : are such instabilities due to the numerical approximation, or to the continuous mathematical model *per se* ? In addition, knowing that the mathematical model have the good dissipativity properties helps in the process of designing numerical algorithms that also share the same properties. On this latter point, we refer the reader to the work of F. Armero and J.C. Simo [3]. In this reference, one may also find an enlightening presentation of the theoretical concepts of attractors and related notions, that we have hardly approached above, precisely in the context of MHD equations (but in the *one* fluid case).

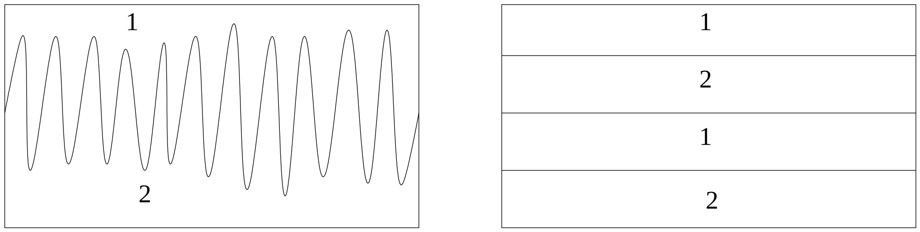

Figure 1: *Two examples of situations when the velocity goes to 0 but when the interface does not converge to the equilibrium expected from experiment. On the left, the interface oscillates more and more (think to sin(tx) as t grows) ; on the right, the interface is flat but has split into three pieces that separate fluids 1 and 2 alternatively.*

1.2 Summary of our results

Paradoxically, simple as it might seem, the question *To what extent does the hydrodynamic equation reproduce the behaviour expected from experiment ?* has not been addressed before in this framework, to our knowledge at least. Because, as shown above, this is a situation extensively addressed in the literature devoted to applications, and because it is a case that exhibits very particular properties, we first consider this question in the linearized setting. This is the purpose of Section 3. We shall detail in particular there the role played by the surface tension. The system that we shall consider there (*see* below in Section 2 how we derivate it from (1.1)) is the following :

$$\begin{cases} \rho_0 \dfrac{\partial u}{\partial t} - \Delta u = -\nabla p - (\psi - \gamma \Delta_{x,y}\psi)\delta_{z=0} e_z, \\ \operatorname{div} u = 0, \text{ in } \Omega, \\ \dfrac{\partial \psi}{\partial t} - u_z = 0 \text{ on } z = 0, \end{cases} \tag{1.6}$$

with the boundary conditions

$$u(\cdot, t) = 0 \quad \text{on} \quad \partial\Omega, \text{ for all } t, \tag{1.7}$$

In this system, (x, y, z) denote the three coordinates, z being along the vertical direction. The density ρ_0 is the steady-state density (consisting of the two layers of fluids separated by a flat interface at $z = 0$), the field $u(x, y, z, t)$ is the linearized velocity field, the field p is likewise the linearized pressure field, the function $\psi(x, y, t)$ defines the position of the interface in this linearized setting through the equation $z = \psi(x, y, t)$ (*see* Figure 2). The measure $\delta_{z=0}$ is the measure of unit charge supported by the 2-dimensional plane $z = 0$. In addition, in this introduction, all constants have been set to one, except the coefficient γ related to the surface tension that therefore vanishes in the case when there is no surface tension (Subsection 3.1).

Basically, the main results of these sections are the following ones, that we state here in a heuristic way.

Basic Results in the linear setting

(i) linear case without surface tension i.e. $\gamma = 0$

> *As time goes to infinity, the velocity* u *goes to 0 in* H^1*, and the shape* ψ *of the interface goes to 0 in* $H^{-\varepsilon}$ *(for all* $\varepsilon > 0$*), and in weak-*L^2*. It is not known whether that latter convergence holds true in* L^2*.*

(ii) linear case with surface tension i.e. $\gamma > 0$

> *As time goes to infinity, the velocity* u *goes to 0 in* H^1*. The shape* ψ *of the interface goes, in* $H^{1-\varepsilon}$ *for all* $\varepsilon > 0$*, to some interface* ψ_∞ *solution of the steady-state equation with zero velocity field. If in addition, the velocity* u *is assumed to remain regular for all time, then* ψ_∞ *corresponds to the unique steady-state with zero velocity field sharing the same boundary condition as the initial data* $\psi|_{t=0}$*.*

These results will be made precise below (*see* Propositions 1 and 2), but let us already make a few comments.

Consider first the case (i) without surface tension. It is to be mentionned that in this case, we can prove the existence of a solution with the regularity that allows one to make all the manipulations needed to prove part (i) of Proposition 1. In addition, despite the somewhat weak result of convergence of the interface given in part (i) of Proposition 1 (oscillations may appear), it remains that, in some weak sense at least, the fluid does return to its stable steady-state in this linearized setting. Therefore we may conclude, in a very rough way at least, that the physical behaviour is obtained. We shall see below that this is out of our reach in the analogous case in the nonlinear setting.

In the case with surface tension, the situation is less simple. We are able to show an existence result for a reasonably regular solution of the equations. We can show that the velocity goes to zero as time goes to infinity, and we can identify the set of all possible limits for the shape of the interface ψ. This set consists of all steady-states ψ_∞ associated to a zero velocity field. Unfortunately, without any additionnal assumption, we are not able to bootstrap enough regularity on the velocity field to identify in this set the limit of ψ (recall the link mentionned above between existence of regular solutions and behaviour of the solutions at infinity). If we assume some better regularity for the velocity field, then we are able to completely determine the behaviour of the interface as time goes to infinity. It turns out then that the behaviour obtained is at least surprising from a physical viewpoint (*see* the details in Section 3).

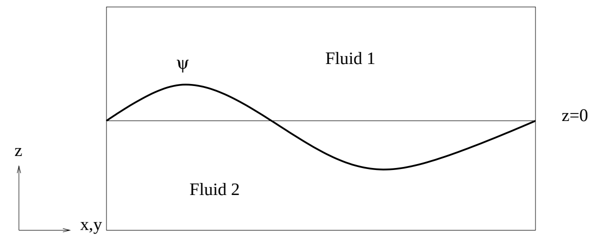

Figure 2: *The linearized case. The interface is defined by the equation* $z = \psi(x, y, t)$*, the density* ρ_0 *is constant on both sides of the plane* $z = 0$ *(*ρ_1 *above,* $\rho_2 > \rho_1$ *below). The question is : does* $\psi(\cdot, t) \longrightarrow 0$ *as* $t \longrightarrow +\infty$ *?*

The strategy to obtain the behaviours at infinity of the solution basically follows the same patterns in case (i) and in case (ii), nevertheless each of these two cases requires very special techniques that differ from one case to the other. That is the reason why we present both settings here.

Once we have treated the linearized setting, we go back, in Section 4 to the nonlinear equations (1.1).

The situation is radically different, the drastic difference lying basically in two facts. Of course the nonlinear setting leads to well known difficulties : some compactness is required in order to determine the behaviour of the nonlinear terms, and obtaining such a compactness through *a priori* estimates is a real difficulty. But mostly for the aspects we are interested in here, the difference with the linearized case is primarily due to the tremendous difference between the number of steady-state solutions in each case. Indeed, in the linearized case ((1.6) with $\gamma = 0$), it is easy to see that if $(u = 0, \psi)$ is a steady-state solution, then $-\nabla p + \psi \delta_{z=0} e_z = 0$ thus ψ is a constant. Therefore, if one wants to reach such a steady state through an evolution for which $\int_\Sigma \psi \, d\sigma = 0$, one necessarily obtains $\psi = 0$. Therefore, the only steady state that can be reached with $u \longrightarrow 0$, is $\psi = 0$.

On the contrary, in the nonlinear setting (1.1), there are infinitely many steady state solutions with $u = 0$. Indeed, when there is no surface tension, it is easy to see that the flat interface may be splitted in many pieces (possibly infinitely many), giving rise to stratified steady states as shown in Figure 1.

The difficulty would not be that great if these steady states were in some way quantized. Now it turns out that they form a continuum of energy in the neighbourhood of the steady state of minimal energy.

When there is surface tension, it will be shown in Section 4 that infinitely many shapes of interface are convenient, also forming a continuum of energy near the minimal energy steady-state. We have seen that in the linearized setting, such a set of steady-states also exists. But the difficulty is now that identifying the limit would require a regularity that seems out of reach (so far as we know) for the nonlinear equations.

The results we have obtained in the nonlinear setting are the following proposition (once more we state the results in a rather schematic way).

Basic Results in the nonlinear setting

(i) nonlinear case without surface tension i.e. $\gamma = 0$

As time goes to infinity, the velocity u *goes to 0 in* H^1 *in some weak sense* (see *(4.34)), and the density* ρ *goes to a density* $\rho_\infty = \rho_\infty(z)$ *in some weak sense. In two dimensions,* $u(\cdot, t)$ *goes to zero in* $\mathbb{H}^{1-\varepsilon}$, $\forall \varepsilon > 0$.

(ii) nonlinear case with surface tension i.e. $\gamma > 0$

Under some reasonable assumption of regularity, the velocity u *goes to 0 in* H^1*, as time goes to infinity, in the same sense as in case (i), the density* ρ *goes to a density* ρ_∞ *in a stronger sense* (see *(4.46)).*

In both cases, we are not able to identify ρ_∞*, and we exhibit an infinity of steady-states, possible limits whose energy is arbitrarily close to the solution with minimal energy.*

We now turn to the detailed statements and proofs of the results announced above.

2 Preliminaries

2.1 Derivation of the linearized equations

The derivation of the linearized Navier-Stokes equations for two fluids is somewhat standard and we only reproduce it here for the sake of self consistency.

Our starting point is the incompressible Navier-Stokes equations for two immiscible fluids of constant positive densities ρ_1, ρ_2 :

$$\begin{aligned} \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \eta \Delta u &= -\nabla p + \rho f_m + f_v, \\ \partial_t \rho + \operatorname{div}(\rho u) &= 0, \\ \operatorname{div} u &= 0. \end{aligned} \tag{2.1}$$

Remark 2.1 *For the sake of simplicity, we suppose that the viscosity* η *is constant over the domain, and we therefore set it to 1 henceforth. We could as well consider a variable viscosity of the form* $\eta(\rho)$ *which gives rise to a term* $div(\eta \nabla^s u)$ *in the equation (2.1) instead of the term* $\eta \Delta u$*, where* $\nabla^s u$ *denotes the symmetrized gradient of* u*. All the results we obtain in this work hold true* mutatis mutandis *except the somewhat technical result of regularity* *established at the end of Section 3.1.2 and, more important, the results of Section 4.3.2 where the hypothesis of constant η enables us to improve the regularity of the flow in two dimensions. As mentioned there, the results of Section 4.3.2 may however be extended to the case of slight variable viscosities, in the sense of B. Desjardins (*see *[16]).*

We assume that the given forces are such that $(u = 0, p = p_0, \rho = \rho_0)$ is a steady state solution. We also assume that the domain Ω has Lipschitz continuous boundary Ω which will allow us to make use in the sequel of all the standard theorems of regularity and trace for convenient Sobolev exponents. We now derivate the linearized equations in the neighborhood of this steady-state solution. For the sake of simplicity, we assume in this derivation that the massic forces are only due to the gravity, but it is a straightforward modification of the following argument to extend this linearization procedure to the case when there are other massic forces. We denote by ρ_0 the steady state solution for the density :

$$\rho(\cdot, t=0) = \rho_0(\cdot) = \begin{cases} \rho_1 > 0, & \text{constant in } \Omega_1, \\ \rho_2 > 0, & \text{constant in } \Omega_2, \end{cases} \tag{2.2}$$

where the partition Ω_1, Ω_2 is entirely fixed by the given forces the system is subjected to. In the purely gravitational case that we consider here, Ω_1 and Ω_2 are the two subdomains separated by the flat horizontal interface at $z = 0$ as shown in Figure 2.

Let us consider a "small" constant $\varepsilon > 0$ which defines the size of the perturbation, and denote by $(\varepsilon u_\varepsilon, \rho_0 + \varepsilon\rho_\varepsilon, p_0 + \varepsilon p_\varepsilon)$ the solution to the above Navier-Stokes equations. Neglecting the terms of second order or more with respect to ε, we obtain

$$\begin{aligned} \varepsilon\rho_0\partial_t u_\varepsilon - \varepsilon\Delta u_\varepsilon &= -\nabla(p_0 + \varepsilon p_\varepsilon) + (\rho_0 + \varepsilon\rho_\varepsilon) f_m + f_v, \\ \varepsilon\partial_t\rho_\varepsilon + \varepsilon \operatorname{div}(\rho_0 u_\varepsilon) &= 0, \\ \operatorname{div} u_\varepsilon &= 0, \end{aligned}$$

which, in view of the stationary equation

$$0 = -\nabla p_0 + \rho_0 f_m + f_v \tag{2.3}$$

may be written

$$\begin{aligned} \varepsilon\rho_0\partial_t u_\varepsilon - \varepsilon\Delta u_\varepsilon &= -\varepsilon\nabla p_\varepsilon + \varepsilon\rho_\varepsilon f_m, \\ \partial_t\rho_\varepsilon + \operatorname{div}(\rho_0 u_\varepsilon) &= 0, \\ \operatorname{div} u_\varepsilon &= 0. \end{aligned} \tag{2.4}$$

Let us now define the function ψ such that the shape of the perturbed interface (with respect to the steady-state flat horizontal interface $z = 0$) is given by the equation $z = \varepsilon\psi(x, y, t)$ at any time t. We assume that

$$\int_{\{z=0\}} \psi(\cdot, t = 0) = 0. \tag{2.5}$$

We have

$$\rho_0 + \varepsilon\rho_\varepsilon(x, y, z, t) = \begin{cases} \rho_1, & \text{if } z > \varepsilon\psi(x, y, t), \\ \rho_2, & \text{if } z < \varepsilon\psi(x, y, t), \end{cases}$$

and therefore

$$\rho_\varepsilon = \begin{cases} 0, & \text{if } z > max(0, \varepsilon\psi(x, y, t)) \text{ or } z < min(0, \varepsilon\psi(x, y, t)), \\ \frac{1}{\varepsilon}(\rho_2 - \rho_1), & \text{if } 0 < z < \varepsilon\psi(x, y, t)), \\ \frac{1}{\varepsilon}(\rho_1 - \rho_2), & \text{if } \ \varepsilon\psi(x, y, t) < z < 0. \end{cases} \tag{2.6}$$

In the sense of distributions, we see that, as ε goes to 0, the function ρ_ε goes to the distribution m defined for any arbitrary $\varphi \in \mathcal{D}(\Omega)$ by

$$< m, \varphi >= (\rho_2 - \rho_1) \int_{z=0} \psi\varphi, \tag{2.7}$$

which is in fact a bounded measure on Ω, supported on the plane $z = 0$.

Therefore, from the equations (2.4), we obtain the linearized equations

$$\begin{aligned} \rho_0 \partial_t u - \Delta u &= -\nabla p + m f_m, \\ \operatorname{div} u &= 0, \ in\ \Omega \\ \partial_t m - (u \cdot \nabla \rho_0) &= 0. \end{aligned}$$

In the purely gravitational case, the massic forces are $f_m = -e_z$ (we set the gravitational constant to unity), and we consider the perturbations with respect to the standard steady-state where the heaviest fluid fills in the zone below the plane $z = 0$. Then, the gradient of ρ_0 is the measure $\nabla\rho_0 = -(\rho_2 - \rho_1)e_z$ concentrated on the plane $z = 0$. For the sake of simplicity, we henceforth normalize the jump of densities $\rho_2 - \rho_1$ to unity, and denote by $\psi\delta_{z=0}$ the measure m. The linearized equations therefore read

$$\rho_0 \partial_t u - \Delta u = -\nabla p - \psi\delta_{z=0} e_z, \tag{2.8}$$

$$\operatorname{div} u = 0, \ in\ \Omega \tag{2.9}$$

$$\partial_t \psi - u \cdot e_z = 0, \ on\ \{z = 0\} \tag{2.10}$$

We shall deal with this system in Subsection 3.1 below.

Note that the interface consisting in the $\mathcal{C}^\infty$ set $\{z = 0\}$, we are allowed to apply the classical trace theorems $W^{k,p}(\Omega) \longrightarrow W^{r,s}(\{z = 0\})$ for any convenient k, p, r, s with k and r as large as we wish.

Remark 2.2 *In the spirit of Remark 2.1, let us mention that if the viscosity* η *depends on the density through a law* $\eta = \eta(\rho)$*, the equation* $\rho_0\partial_t u - \Delta u = -\nabla p - \psi\delta_{z=0}e_z$ *must be replaced by* $\rho_0\partial_t u - \eta(\rho_0)\Delta u = -\nabla p - \psi\delta_{z=0}e_z$.

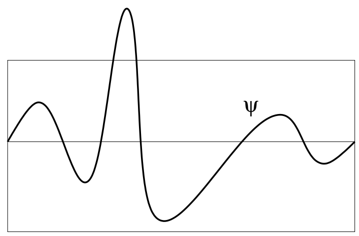

Figure 3: *In the linearized case, some part of the curve* $z = \psi(x, y, t)$ *may* a priori *ly outside the domain* Ω.

2.2 About the surface tension

The argument of the previous subsection is now modified in order to take the surface tension into account. Going back to our starting point (2.1), we add to the right-hand side a term modelling the surface tension at the interface between the two fluids. Let us denote the normal to the interface by n (say from fluid 1 to fluid 2 to fix the ideas). The term $\mathcal{T}$ modelling the surface tension may then be written as follows : it is the distribution defined, for any test velocity w, by

$$< \mathcal{T}, w >= \int_\Gamma \gamma \, \mathcal{C} \, w \cdot n, \tag{2.11}$$

where the coefficient γ denotes as in the introduction the amplitude of the surface tension, and where $\mathcal{C}$ denotes the local mean curvature of the interface oriented with n. This may also be written as follows

$$< \mathcal{T}, w >= - \int_\Gamma \gamma (\mathrm{div}\, n) w \cdot n. \tag{2.12}$$

It is important to note that the above expression does not depend of course on the orientation of the interface : it is quadratic with respect to n. It depends only on its local mean curvature.

Let us now argue as in Subsection 2.1 above : we linearize the equations in the neighborhood of the steady-state solution for the purely gravitational case (note that in this setting the presence of the surface tension term does not modify the steady-state, since the interface is flat for this steady-state). The perturbed interface is then defined by the equation $z = \varepsilon\psi(x, y, t)$. It is standard to compute the normal vector to such a surface, namely

$$n = \frac{1}{\sqrt{1 + \varepsilon^2((\partial_x \psi)^2 + (\partial_y \psi)^2)}} (-\varepsilon \partial_x \psi e_x - \varepsilon \partial_y \psi e_y + e_z), \tag{2.13}$$

and the corresponding curvature

$$\mathcal{C} = \frac{1}{2} \frac{\varepsilon \partial^2_{xx} \psi (1 + (\varepsilon \partial_y \psi)^2) + \varepsilon \partial^2_{yy} \psi (1 + (\varepsilon \partial_x \psi)^2) - 2\varepsilon^3 \partial^2_{xy} \psi \partial_x \psi \partial_y \psi}{(1 + \varepsilon^2((\partial_x \psi)^2 + (\partial_y \psi)^2))^{3/2}} \tag{2.14}$$

If we now argue as we did in the previous subsection, and follow our linearization process, we only keep in the expression of $\mathcal{C}n$ the term depending linearly on the parameter ε. This yields the following value of the linearized surface tension term $\mathcal{T}_0$:

$$< \mathcal{T}_0, w >= \frac{1}{2}\gamma \int_{\{z=0\}} \Delta\psi \, w \cdot e_z \tag{2.15}$$

We henceforth set the coefficient $\frac{1}{2}\gamma$ to 1. Therefore, the linearized equations in this setting are

$$\begin{aligned} \rho_0 \partial_t u - \Delta u &= -\nabla p - (\psi - \Delta\psi)\delta_{z=0} e_z, \\ \operatorname{div} u &= 0, \ \textit{in } \Omega \\ \partial_t \psi - u \cdot e_z &= 0, \ \textit{on } \{z=0\} \end{aligned} \tag{2.16}$$

Of course, the same calculation holds *mutatis mutandis* in 2 dimensions, where the interface is only a curve $z = \psi(x,t)$.

The study of the long time behaviour of the solution to (2.16) is the purpose of Section 3.2 below.

3 The linearized case

3.1 The linearized case without surface tension

In this subsection, we study the system

$$\begin{cases} \rho_0 \partial_t u - \Delta u = -\nabla p - \psi \delta_{z=0} e_z, \ \text{in } \Omega, \\ \operatorname{div} u = 0, \ \text{in } \Omega, \\ \partial_t \psi - u_z = 0, \quad \text{on} \quad \Sigma = \{z=0\}, \end{cases} \tag{3.1}$$

The next paragraph deals with *a priori* estimates. For the moment, we establish them formally, or at least under the assumption that u and ψ are sufficiently regular solutions of (3.1). All the manipulations we make will be justified (up to mild modifications if necessary) in the sequel.

3.1.1 A priori estimates

We begin with a basic remark : the last equation of (3.1) and the incompressibility constraint yield

$$\frac{d}{dt} \int_{z=0} \psi \, d\sigma = \int_{z=0} u_z \, d\sigma = \int_{\Omega_2} \operatorname{div} u \, dx = 0,$$

thus, in view of (2.5),

$$\int_{z=0} \psi \, d\sigma = 0, \tag{3.2}$$

which means nothing but the mass conservation of each fluid.

First estimate Multiplying the first equation of (3.1) by u and integrating over the domain, we obtain

$$\int_\Omega \rho_0 \partial_t u \cdot u\, dx - \int_\Omega \Delta u \cdot u\, dx = -\int_{z=0} \psi u \cdot e_z\, d\sigma,$$

that is, using the third equation of (3.1),

$$\int_\Omega |\nabla u|^2\, dx + \frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0 u^2\, dx = -\int_{z=0} \psi \partial_t \psi\, d\sigma. \tag{3.3}$$

We therefore obtain the standard first energy estimate

$$\int_\Omega |\nabla u|^2\, dx + \frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0 u^2\, dx + \int_{z=0} \psi^2\, d\sigma\Big) = 0. \tag{3.4}$$

From this estimate, we deduce first that

$$\sup_{t\in[0,\infty)} ||u||_{\mathbb{L}^2(\Omega)} \leq C_1, \tag{3.5}$$

$$\sup_{t\in[0,\infty)} ||\psi||_{L^2(\Sigma)} \leq C_1. \tag{3.6}$$

Then, integrating (3.4) in time from 0 to ∞, we obtain

$$\int_0^{+\infty} ||\nabla u||^2_{\mathbb{L}^2(\Omega)}\, dt < +\infty. \tag{3.7}$$

Second estimate We differentiate the first equation of (3.1) in the t variable, we multiply it by $\partial_t u$ and integrate over Ω :

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0 (\partial_t u)^2\, dx + \int_\Omega |\nabla \partial_t u|^2\, dx = -\int_{z=0} \partial_t \psi \partial_t u \cdot e_z\, d\sigma. \tag{3.8}$$

Derivating the third equation of (3.1) with respect to time, we have

$$\partial^2_{tt}\psi - \partial_t u \cdot e_z = 0, \tag{3.9}$$

and thus we obtain the second energy estimate :

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \rho_0 (\partial_t u)^2\, dx + \int_{z=0} (\partial_t \psi)^2\, d\sigma\right) + \int_\Omega |\nabla \partial_t u|^2\, dx = 0 \tag{3.10}$$

We easily deduce from (3.10) that

$$\sup_{t\in[0,\infty)} ||\partial_t u||_{\mathbb{L}^2(\Omega)} \leq C^{st}, \tag{3.11}$$

and

$$\sup_{t\in[0,\infty)} ||\partial_t \psi||_{L^2(\Sigma)} \leq C^{st}. \tag{3.12}$$

Then, by integration in t from 0 to ∞ of (3.10), we obtain :

$$\int_0^{+\infty} ||\nabla \partial_t u||^2_{\mathbb{L}^2(\Omega)}\, dt < \infty. \tag{3.13}$$

3.1.2 Questions of existence and regularity

We suppose for the moment that $u_0 \in \mathbb{L}^2(\Omega)$ and $\psi_0 \in L^2(\Sigma)$. With the first *a priori* estimate, it is straightforward to prove the existence of a solution (u, ψ) satisfying $u \in L^\infty(0, T; \mathbb{L}^2(\Omega)) \cap L^2(0, T; \mathbb{H}^1_0(\Omega))$ and $\psi \in L^\infty(0, T; L^2(\Sigma))$ for all arbitrary time T. It suffices for instance to consider a Galerkin approximation of a weak form of system (3.1), prove the analogous estimate of (3.4) for the finite dimensional solution, and then pass to the limit. We leave this standard point to the reader. We just emphasize that such a solution satisfies $\partial_t u \in L^2(0, T; V')$, where V' denotes the dual of $\{v \in \mathbb{H}^1_0(\Omega), \operatorname{div} v = 0\}$. Indeed, we have for $v \in L^2(0, T; \mathbb{H}^1_0(\Omega))$ with $\operatorname{div} v = 0$,

$$\begin{aligned} \int_0^T \left| \int_\Omega \partial_t u.v \, dx \right| dt &\leq \int_0^T \left(\int_\Omega |\nabla u|.|\nabla v| \, dx + \int_\Sigma |\psi||v_z| \, d\sigma \right) dt \\ &\leq C^{st}(||\nabla u||_{L^2(0,T;\mathbb{L}^2(\Omega))} + ||\psi||_{L^2(0,T;\mathbb{L}^2(\Sigma))})||v||_{L^2(0,T;\mathbb{H}^1(\Omega))} \end{aligned}$$

The right-hand side of this inequality is easily controlled by (3.6) and (3.7).

A similar argument proves that $\partial_t \psi \in L^2(0, T; L^2(\Sigma))$. Indeed, for any $\phi \in L^2(0, T; L^2(\Sigma))$, we have

$$\begin{aligned} \int_0^T \left| \int_\Sigma \partial_t \psi.\phi \, ds \right| dt &\leq \int_0^T \int_\Sigma |u_z|.|\phi| \, ds \\ &\leq C^{st}||\nabla u||_{L^2(0,T;\mathbb{L}^2(\Omega))}||\phi||_{L^2(0,T;\mathbb{L}^2(\Sigma))}. \end{aligned}$$

This regularity results yield (*see* R. Temam [44] for example)

$$u \in \mathcal{C}([0, T]; \mathbb{L}^2(\Omega)), \forall T > 0 \quad \text{and} \quad \frac{1}{2}\frac{d}{dt}\int_\Omega |u|^2 \, dx = \int_\Omega \partial_t u.u \, dx, \tag{3.14}$$

$$\psi \in \mathcal{C}([0, T]; L^2(\Omega)), \forall T > 0 \quad \text{and} \quad \frac{1}{2}\frac{d}{dt}\int_{z=0} \psi^2 \, d\sigma = \int_{z=0} \partial_t \psi \psi \, d\sigma. \tag{3.15}$$

Remark 3.1 *In the case when $u_0 \in \mathbb{L}^2(\Omega)$ and $\psi_0 \in H^{1/2}(\Sigma)$, it is possible to show that $\psi \in \mathcal{C}^{1/2}(0, T, H^{1/2}(\Sigma))$ and $\partial_t \psi \in L^2(0, +\infty, H^{1/2}(\Sigma))$.*

In the same spirit, assuming now that $u_0 \in \mathbb{H}^2(\Omega)$, $\operatorname{div} u_0 = 0$, and $\psi_0 \in H^{1/2}(\Sigma)$, we can prove with the second estimate that we have a solution (u, ψ) satisfying $\partial_t u \in L^2(0, T; \mathbb{H}^1(\Omega)) \cap L^\infty(0, T; \mathbb{L}^2(\Omega))$ and $\partial_t \psi \in L^2(0, T; L^2(\Sigma))$. Thus, we have in particular $\partial_t \nabla u \in L^2(0, T; \mathbb{L}^2(\Omega))$ and $\nabla u \in L^2(0, T; \mathbb{L}^2(\Omega))$, therefore

$$u \in \mathcal{C}([0, T]; \mathbb{H}^1_0(\Omega)), \forall T > 0 \quad \text{and} \quad \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 \, dx = \int_\Omega \partial_t \nabla u.\nabla u \, dx. \tag{3.16}$$

Actually, we also have $u \in \mathcal{C}^{1/2}([0, T]; \mathbb{H}^1_0(\Omega))$ since $u \in H^1(0, T; \mathbb{H}^1_0(\Omega))$.

This gives a rigorous sense to all the manipulations we made above. Therefore, the solution we have obtained satisfies the energy equality (3.4). Likewise, it satisfies the second energy estimate (3.10), this time as an inequality (since we have no compactness on $\partial_t u$ in $L^2(0,T;\mathbb{H}^1(\Omega))$) :

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \rho_0(\partial_t u)^2\,dx + \int_\Sigma (\partial_t\psi)^2\,d\sigma\right) + \int_\Omega |\nabla\partial_t u|^2\,dx \leq 0 \qquad \text{in } \mathcal{D}'(0,\infty)$$

and

$$\frac{1}{2}\int_\Omega \rho_0(\partial_t u)^2\,dx + \frac{1}{2}\int_\Sigma (\partial_t\psi)^2\,d\sigma + \int_0^T \int_\Omega |\nabla\partial_t u|^2\,dx \leq C^{st} \qquad \text{for all } T<\infty.$$

The two previous estimates (3.4) and (3.10) suffice to prove the existence results and to study the long time behaviour of the solution. Nevertheless, we end this paragraph by establishing an estimate that shows that the regularity of the solutions obtained with the two previous estimates can be improved locally in time, but (so far as we know) it does not give more information on the long time behaviour.

We multiply the first equation of (3.1) by $-\Delta u$ and integrate over Ω :

$$-\int_\Omega \rho_0\partial_t u\cdot\Delta u\,dx + \int_\Omega |\Delta u|^2\,dx = \int_{z=0}\psi\Delta u\cdot e_z\,d\sigma. \tag{3.17}$$

This estimate is somewhat even more formal that the two preceeding ones. Indeed, the function $-\Delta u$ does not vanish on the boundary $\partial\Omega$ even in a weak sense, contrarily to u itself and $\partial_t u$. Therefore *stricto sensu* the pressure term does not disappear. But we recall that all the estimates we do here on the continuous solution u for the sake of simplicity have to be made at first on the discrete Galerkin solution (*see* Section 3.1.2). It is standard in the study of Navier-Stokes to obtain regularity results on the solution by formally multiplying the equation by Δu : the rigorous counterpart of this formal argument is to use a special basis for the Galerkin approximation, that is a basis of functions $w_j \in (H_0^1(\Omega))^N$ satisfying $-\Delta w_j + \nabla p_j = \lambda_j w_j$, $\operatorname{div} w_j = 0$: in this latter context, the pressure term does disappear, which gives a sense to (3.17).

We now estimate the first term of (3.17) :

$$-\int_\Omega \rho_0\partial_t u\cdot\Delta u\,dx = \frac{1}{2}\frac{d}{dt}\int_\Omega \rho_0|\nabla u|^2\,dx + \int_{z=0}\nabla u\cdot e_z\cdot\partial_t u\,d\sigma, \tag{3.18}$$

where the last term is estimated as follows

$$\begin{aligned}\left|\int_{z=0}\nabla u\cdot e_z\cdot\partial_t u\,d\sigma\right| &\leq \|\nabla u.e_z\|_{\mathbb{L}^2(\{z=0\})}\|\partial_t u\|_{L^2(\{z=0\})}\\ &\leq C^{st}\|\nabla u.e_z\|_{\mathbb{H}^{1/2}(\{z=0\})}\|\partial_t u\|_{\mathbb{H}^{1/2}(\{z=0\})}\\ &\leq C^{st}\|u\|_{\mathbb{H}^2(\Omega)}\|\nabla\partial_t u\|_{\mathbb{L}^2(\Omega)}\end{aligned}$$

Hence, controlling the H^2 norm by elliptic regularity,

$$\Big|\int_{z=0} \nabla u\cdot e_z\cdot \partial_t u\, d\sigma\Big| \le C^{st}\|\Delta u\|_{\mathbb{L}^2(\Omega)}\|\nabla\partial_t u\|_{\mathbb{L}^2(\Omega)}. \tag{3.19}$$

On the other hand, we now estimate the right-hand side of (3.17). We denote by ψ_0 the initial position of the interface, we recall that we suppose that $\int_{z=0}\psi_0\, d\sigma = 0$. For the sake of simplicity, we suppose also that $\psi_0\in H_0^1(\Sigma)$. The third equation of (3.1) with the no-slip condition yields $\psi(t)|_{\partial\Omega\cap\{z=0\}} = 0$.

We denote by the subscript x, y the differential operators on $z = 0$: for instance, $\Delta_{x,y}$ is the Laplace-Beltrami operator $\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ on the plane $z = 0$, and likewise $\text{div}_{x,y} u$ is the function $\frac{\partial}{\partial x}u\cdot e_x + \frac{\partial}{\partial y}u\cdot e_y$. Using the fact that $\text{div}\, u = 0$ we transform the first term of the right-hand side of (3.17) as follows :

$$\begin{aligned}\int_{z=0}\psi\Delta u\cdot e_z\, d\sigma &= \int_{z=0}\psi\Delta_{x,y}u\cdot e_z\, d\sigma - \int_{z=0}\psi \text{div}_{x,y}\partial_z u\, d\sigma,\\ &= -\int_{z=0}\nabla\psi\cdot\nabla_{x,y}u\cdot e_z\, d\sigma + \int_{z=0}\nabla\psi\cdot\partial_z u\, d\sigma\end{aligned}$$

where we have integrated by parts using $\psi(t)|_{\partial\Omega\cap\{z=0\}} = 0$.

Thus we obtain

$$\begin{aligned}\int_{z=0}\psi\Delta u\cdot e_z\, d\sigma &= -\frac{1}{2}\frac{d}{dt}\int_{z=0}|\nabla\psi|^2\, d\sigma + \int_{z=0}\nabla\psi\cdot\partial_z u\, d\sigma\\ &\le -\frac{1}{2}\frac{d}{dt}\int_{z=0}|\nabla\psi|^2\, d\sigma + C^{st}\|\partial_z u\|_{\mathbb{L}^2(\{z=0\})}\|\nabla\psi\|_{L^2(\{z=0\})},\\ &\le -\frac{1}{2}\frac{d}{dt}\int_{z=0}|\nabla\psi|^2\, d\sigma + C^{st}\|\Delta u\|_{\mathbb{L}^2(\Omega)}\|\nabla\psi\|_{L^2(\{z=0\})},\end{aligned}$$

which we insert, together with (3.18) and (3.19), in (3.17) to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Big(\int_\Omega\rho_0|\nabla u|^2\, dx &+ \int_{z=0}|\nabla\psi|^2\, d\sigma\Big) + \int_\Omega|\Delta u|^2\, dx\\ &\le C^{st}(\|\Delta u\|_{\mathbb{L}^2(\Omega)}\|\partial_t\nabla u\|_{\mathbb{L}^2(\Omega)} + \|\Delta u\|_{\mathbb{L}^2(\Omega)}\|\nabla\psi\|_{L^2(\{z=0\})}).\end{aligned}$$

and finally the following third energy estimate

$$\begin{aligned}\frac{d}{dt}\Big(\int_\Omega\rho_0|\nabla u|^2\, dx &+ \int_{z=0}|\nabla\psi|^2\, d\sigma\Big) + \int_\Omega|\Delta u|^2\, dx\\ &\le C^{st}\|\partial_t\nabla u\|_{L^2}^2 + C^{st}\|\nabla\psi\|_{L^2(\{z=0\})}^2.\end{aligned} \tag{3.20}$$

This third estimate yields $u\in L^\infty(0,T;\mathbb{H}_0^1(\Omega))\cap L^2(0,T;\mathbb{H}^2(\Omega))$, $\psi\in L^\infty(0,T;H^1(\Sigma))$ and $\partial_t\psi\in L^2(0,T;H^{3/2}(\Sigma))\cap L^\infty(0,T;H^{1/2}(\Sigma))$, thus $\psi\in \mathcal{C}^{1/4}(0,T;H^1(\Sigma))$, for any arbitrary time T.

Unfortunately, the new bounds obtained with this estimate depend on the time T. Therefore, it improves the regularity results but it does not provide, so far as we know, any additional information on the behaviour of (u,ψ) as time goes to infinity.

3.1.3 Long time behaviour

Under the assumptions $u_0 \in \mathbb{H}^2(\Omega)$, $\operatorname{div} u_0 = 0$, and $\psi_0 \in H_0^{1/2}(\Sigma)$, we have built above a solution that satisfies in particular $u \in \mathcal{C}(0, +\infty; \mathbb{H}^1(\Omega))$, $\psi \in \mathcal{C}(0, +\infty; \mathbb{L}^2(\Sigma))$, $\partial_t u \in \mathbb{L}^2_{loc}(0, +\infty; \mathbb{H}^1(\Omega))$ and the two energy inequalities

$$\int_\Omega |\nabla u|^2 \, dx + \frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0 u^2 \, dx + \int_{z=0} \psi^2 \, d\sigma\Big) \le 0.$$

$$\frac{1}{2}\frac{d}{dt}\left(\int_\Omega \rho_0 (\partial_t u)^2 \, dx + \int_\Sigma (\partial_t \psi)^2 \, d\sigma\right) + \int_\Omega |\nabla \partial_t u|^2 \, dx \le 0 \qquad \text{in } \mathcal{D}'(0, \infty)$$

Only with these properties, we are now able to determine the behaviour of (u, ψ) as time goes to infinity.

Behaviour of the velocity The second estimate allows us to establish the behaviour of u in $\mathbb{H}^1(\Omega)$ as t goes to infinity. Indeed,

$$\begin{aligned} \left|\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 \, dx\right| &\le \int_\Omega |\nabla \partial_t u| |\nabla u| \, dx \\ &\le ||\nabla \partial_t u||_{\mathbb{L}^2(\Omega)} ||\nabla u||_{\mathbb{L}^2(\Omega)}. \end{aligned}$$

Relations (3.7) and (3.13) imply that the right-hand side of this inequality belongs to $L^1(0, \infty)$. Therefore, together with (3.7), we deduce that the function $t \longrightarrow \int_\Omega |\nabla u|^2 \, dx$ belongs to $W^{1,1}(0, \infty)$. This yields

$$\lim_{t \longrightarrow +\infty} \|\nabla u\|_{\mathbb{L}^2(\Omega)} = 0. \tag{3.21}$$

Let us indicate here that we give another proof of this assertion in Appendix A.

Behaviour of the interface So far, we have established convergence results about the velocity field u and its derivatives. Let us now use these informations in order to examine the behaviour of the shape of the interface ψ.

First, using (3.4), we see that

$$t \longrightarrow \int_\Omega \rho_0 u^2 \, dx + \int_{z=0} \psi^2 \, d\sigma \ \text{ is a nonincreasing continuous function of time } t. \tag{3.22}$$

Thus, this quantity has a limit, denoted by α, as t goes to infinity.

$$\alpha = \lim_{t \longrightarrow +\infty} \left[\int_\Omega \rho_0 u^2 \, dx + \int_{z=0} \psi^2 \, d\sigma\right]. \tag{3.23}$$

It follows from (3.21) that we have

$$\lim_{t \longrightarrow +\infty} ||\psi||_{L^2(\Sigma)} = \alpha. \tag{3.24}$$

Since (3.6) holds, it is clear that $\psi(\cdot,t)$ lives in a bounded set of $L^2(\{z=0\})$ and therefore that, up to an extraction in time, it is weakly convergent in this space. We are going to see that actually the whole sequence converges to 0 for the weak topology of L^2.

For this purpose, we prove the following convergence result :

$$\lim_{t\longrightarrow+\infty}\|\psi\|_{H^{-1}(\{z=0\})}=0. \tag{3.25}$$

Let us introduce, for any time t the function $\varphi(x,y,t)\in H_0^1(\{z=0\}\cap\Omega)$, such that

$$-\Delta\varphi+\varphi=\psi. \tag{3.26}$$

It is standard that $\|\varphi\|_{H^1(\Sigma)}=\|\psi\|_{H^{-1}(\Sigma)}$. Next,

$$\begin{aligned}\frac{1}{2}\left|\frac{d}{dt}\|\varphi\|^2_{H^1(\Sigma)}\right| &\le \|\varphi\|_{H^1(\Sigma)}\|\partial_t\varphi\|_{H^1(\Sigma)}\\ &= \|\psi\|_{H^{-1}(\Sigma)}\|\partial_t\psi\|_{H^{-1}(\Sigma)}\\ &= \|\psi\|_{H^{-1}(\Sigma)}\|u\cdot e_z\|_{H^{-1}(\Sigma)}\end{aligned}$$

By a standard result for trace of divergence free fields,

$$\|u\cdot e_z\|_{H^{-1}(\Sigma)}\le C^{st}\|u\|_{\mathbb{L}^2(\Omega)} \tag{3.27}$$

It follows from the above two estimates that

$$\frac{1}{2}\left|\frac{d}{dt}\|\varphi\|^2_{H^1}\right|\le C^{st}\|\psi\|_{H^{-1}}\|u\|_{\mathbb{L}^2(\Omega)}. \tag{3.28}$$

We denote by $\dot{H}^{-1}(\Sigma)$ the quotient space $H^{-1}(\Sigma)/\mathbb{R}$. By definition of the $\dot{H}^{-1}$ norm, we have

$$\|\psi\|_{\dot{H}^{-1}(\Sigma)}=\sup_{f\in H_0^1(\Sigma),\int_\Sigma f\,d\sigma=0}\frac{<\psi,f>}{\|f\|_{H_0^1(\{z=0\})}}$$

Using the fact that for any function $f\in H_0^1(\Sigma)$ such that $\int_\Sigma f\,d\sigma=0$, there exists a divergence free field $w\in\mathbb{H}^{3/2}(\Omega)$ such that $w=0$ on $\partial\Omega$, $w\cdot e_z=f$ on Σ, and $\|w\|_{\mathbb{H}^{3/2}(\Omega)}\le C^{st}\|f\|_{H_0^1(\Sigma)}$, we may then write

$$\|\psi\|_{\dot{H}^{-1}(\Sigma)}\le C^{st}\sup\frac{<\psi,w\cdot e_z>}{\|w\|_{\mathbb{H}^{3/2}(\Omega)}}. \tag{3.29}$$

If we now turn to the linearized equation (3.1), we see that, for any arbitrary time t,

$$\begin{aligned}\|\psi\|_{\dot{H}^{-1}(\Sigma)} &\le C^{st}\sup\frac{<\rho_0\partial_t u-\Delta u,w>}{\|w\|_{\mathbb{H}^{3/2}(\Omega)}}\\ &\le C^{st}(\|\partial_t u\|_{\mathbb{L}^2(\Omega)}+\|\nabla u\|_{\mathbb{L}^2(\Omega)}).\end{aligned}$$

Using (3.7) and (3.13), this yields

$$\|\psi\|_{\dot{H}^{-1}(\Sigma)} \in L^2(0,+\infty). \tag{3.30}$$

Let us next insert this information in (3.28) and use (3.5). This yields

$$\frac{d}{dt}\|\psi\|^2_{\dot{H}^{-1}(\Sigma)} \in L^1(0,+\infty). \tag{3.31}$$

The two assertions (3.31) and (3.30) together imply that

$$\|\psi\|^2_{\dot{H}^{-1}(\Sigma)} \in W^{1,1}(0,+\infty). \tag{3.32}$$

Thus, ψ tends to 0 in $\dot{H}^{-1}(\Sigma)$. This yields that ψ tends to a constant in $H^{-1}(\Sigma)$ and this constant is zero in view of (3.2). Therefore (3.25) holds.

Remark 3.2 *The same argument as above, but with some technical modifications, shows that the convergence of ψ to 0 also holds in $H^{-1/2}$.*

In view of (3.25) and of the L^2 bound on ψ (3.6), it is straightforward to see that ψ converges weakly to 0 in L^2.

From (3.25) and (3.6), we also deduce by interpolation that, for all $\varepsilon > 0$, we have

$$\lim_{t\longrightarrow+\infty} \|\psi\|_{H^{-\varepsilon}(\{z=0\})} = 0. \tag{3.33}$$

We now collect in the following Proposition the information we have obtained in this Section on the behaviour as time goes to infinity of the solution to (3.1).

Proposition 1
In the linearized case without surface tension (3.1), the behaviour as time goes to infinity of a solution u, ψ satisfying the estimates recalled at the beginning of Section 3.1.3, is the following :

(i) the velocity field u goes to 0 in $\mathbb{H}^1_0(\Omega)$.

(ii) the shape ψ of the interface goes to 0 in $H^{-\varepsilon}$ (for all $\varepsilon > 0$) and in weak-L^2.

In addition,

(iii) $\|\psi\|_{L^2(\Sigma)}$ has a limit as t goes to infinity,

(iv) $\displaystyle\int_0^\infty \|u\|^2_{\mathbb{H}^1(\Omega)} + \|\partial_t u\|^2_{\mathbb{H}^1(\Omega)}\, dt < +\infty.$

Remark 3.3 *In this setting we cannot control (as far as we know at least) the L^∞ norm of ψ. In other words, nothing seems to ensure that the graph of ψ, which models the interface, does not go out of the domain Ω (*see *Figure 3). This is of course in contradiction with the intuition ! It is not clear whether it must be interpreted as the possibility of some explosion of the system within a finite time or not.*

Remark 3.4 *It is not known whether the convergence of ψ to 0 holds true for the strong topology of L^2. This is of course a very interesting open question. Typically, say in 2 dimensions to fix the ideas, one must show that some oscillation of the form $\psi(x,t) = sin(x\sqrt{t})$ cannot occur.*

3.2 The linearized case with surface tension

In this subsection, we consider the linearized equations in presence of surface tension (2.16). We essentially follow the same scheme as in Section 3.1 : we first establish formally *a priori* bounds, next we study the long time behaviour.

3.2.1 A priori estimates

First estimate Let us first multiply the equation

$$\rho_0 \partial_t u - \Delta u = -\nabla p + (\Delta\psi - \psi)\delta_{z=0} e_z \tag{3.34}$$

by u and integrate over the domain Ω. All the terms, except the surface tension term in $\Delta\psi$, have already been treated in Subsection 3.1 above (*see* equations (3.3) and (3.4)). Therefore we concentrate on that latter term. We have, integrating by parts,

$$\begin{aligned}
\int_\Sigma \Delta\psi u \cdot e_z \, d\sigma &= \int_\Sigma \Delta\psi \partial_t \psi \, d\sigma \\
&= -\int_\Sigma \nabla\psi . \nabla\partial_t\psi \, d\sigma + \int_{\Sigma\cap\partial\Omega} \nabla\partial_t\psi . n_\Sigma . \partial_t\psi \, d\lambda \\
&= -\frac{1}{2}\frac{d}{dt}\int_\Sigma |\nabla\psi|^2 \, d\sigma.
\end{aligned}$$

Hence, the first estimate (3.4) of Subsection 3.1 is replaced here by

$$\int_\Omega |\nabla u|^2 \, dx + \frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0 u^2 \, dx + \int_\Sigma (\psi^2 + |\nabla\psi|^2) \, d\sigma\Big) = 0. \tag{3.35}$$

It is immediate to deduce from this equality the following three estimates :

$$\int_0^{+\infty} \|\nabla u\|^2_{\mathbb{L}^2(\Omega)} dt < +\infty, \tag{3.36}$$

$$\sup_{t\in[0,\infty)} \|u\|^2_{\mathbb{L}^2(\Omega)} \leq C^{st}, \tag{3.37}$$

$$\sup_{t\in[0,\infty)} \|\psi\|^2_{H^1(\Sigma)} \leq C^{st}. \tag{3.38}$$

Moreover

$$t \longrightarrow \|\sqrt{\rho_0} u\|^2_{\mathbb{L}^2(\Omega)} + \|\psi\|^2_{H^1(\Sigma)} \text{ is a nonincreasing function of time } t. \tag{3.39}$$

Remark 3.5 *When Ω is a domain of $\mathbf{R}^2$, the set $\Sigma = \{z = 0\}$ is one dimensional, and thus $H^1(\Sigma) \hookrightarrow L^\infty(\Sigma)$. Therefore the estimate (3.38) yields a control of $\|\psi\|_{L^\infty(\Sigma)}$. If the initial data is small enough, this ensures that the graph of ψ remains inside Ω (compare with Remark 3.3).*

We now turn to another energy estimate anologous to (3.10).

Second estimate We differentiate (3.34) with respect to the time, we multiply it by $\partial_t u$ and we integrate over Ω. We only treat the term related to the tension surface since the others have already been computed in the second estimate of Section 3.1.1.

$$\begin{aligned}\int_\Sigma \Delta\partial_t\psi\partial_t u \cdot e_z \, d\sigma &= \int_\Sigma \Delta\partial_t\psi\partial_{tt}\psi \, d\sigma \\ &= -\int_\Sigma \nabla\partial_t\psi . \nabla\partial_{tt}\psi \, d\sigma + \int_{\Sigma\cap\partial\Omega} \nabla\partial_t\psi . n_\Sigma . \partial_{tt}\psi \, d\lambda \\ &= -\frac{1}{2}\frac{d}{dt}\int_\Sigma |\nabla\partial_t\psi|^2 \, d\sigma.\end{aligned}$$

We deduce the second estimate :

$$\int_\Omega |\nabla\partial_t u|^2 \, dx + \frac{1}{2}\frac{d}{dt}\Big(\int_\Omega \rho_0|\partial_t u|^2 \, dx + \int_\Sigma (|\partial_t\psi|^2 + |\nabla\partial_t\psi|^2) \, d\sigma\Big) = 0. \quad (3.40)$$

It is immediate to deduce from this estimate the following estimates :

$$\int_0^{+\infty} \|\nabla\partial_t u\|^2_{\mathbb{L}^2(\Omega)} dt < +\infty, \quad (3.41)$$

$$\sup_{t\in[0,\infty)} \|\partial_t u\|^2_{\mathbb{L}^2(\Omega)} \leq C^{st}, \quad (3.42)$$

$$\sup_{t\in[0,\infty)} \|\partial_t \psi\|^2_{H^1(\Sigma)} \leq C^{st}. \quad (3.43)$$

3.2.2 Questions of existence and regularity

As in Section 3.1.2, we can prove under the assumptions that $u_0 \in \mathbb{L}^2(\Omega)$ and $\psi_0 \in \mathbb{H}^1(\Sigma)$, that the above energy estimate (3.35) yields the existence of a solution $u \in L^2(0,T;\mathbb{H}^1_0(\Omega)) \cap L^\infty(0,T;\mathbb{L}^2(\Omega))$, $\psi \in L^\infty(0,T;H^1(\Sigma))$, on any finite time interval $(0,T)$.

Likewise, assuming the required regularity $u_0 \in \mathbb{H}^2(\Omega)$, $\operatorname{div} u_0 = 0$, and $\psi_0 \in H_0^{5/2}(\Sigma)$, on the initial data, the estimate (3.40) yields the existence of a solution satisfying $\partial_t u \in L^2(0,T;\mathbb{H}^1_0(\Omega)) \cap L^\infty(0,T;\mathbb{L}^2(\Omega))$, $\partial_t\psi \in L^\infty(0,T;H^1(\Sigma))$, on any finite time interval $(0,T)$.

In particular, it is worth noticing that such a solution satisfies :

$$u \in \mathcal{C}(0,T;\mathbb{H}^1_0(\Omega)), \text{ for any } T>0, \tag{3.44}$$

$$\psi \in \mathcal{C}(0,T;H^1(\Sigma)), \text{ for any } T>0, \tag{3.45}$$

$$\partial_t\psi \in \mathcal{C}(0,T;H^{1/2}(\Sigma)), \text{ for any } T>0. \tag{3.46}$$

Remark 3.6 *In fact, some better regularity is available : we also have* $u \in \mathcal{C}^{1/2}(0,T;\mathbb{H}^1_0(\Omega))$*,* $\psi \in Lip(0,T;H^1(\Sigma))$*,* $\partial_t\psi \in \mathcal{C}^{1/2}(0,T;H^{1/2}(\Sigma))$*.*

3.2.3 Long time behaviour

Behaviour of the velocity The arguments to study the long time behaviour of the velocity in this setting are those already used in the case without surface tension. We have :

$$\begin{aligned} \left|\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx\right| &\leq \int_\Omega |\nabla\partial_t u||\nabla u|\,dx \\ &\leq ||\nabla\partial_t u||_{\mathbb{L}^2(\Omega)}||\nabla u||_{\mathbb{L}^2(\Omega)}. \end{aligned}$$

Relations (3.36) and (3.41) imply that the right-hand side of this inequality belongs to $L^1(0,\infty)$. Therefore, together with (3.36), we deduce that the function $t \longrightarrow \int_\Omega |\nabla u|^2\,dx$ belongs to $W^{1,1}(0,\infty)$. This yields

$$\lim_{t\longrightarrow+\infty} \|\nabla u\|_{\mathbb{L}^2(\Omega)} = 0. \tag{3.47}$$

Behaviour of the interface According to the continuity of u and ψ (*see* (3.44,3.45)), we deduce from the linearized Navier-Stokes equations that $\partial_t u \in \mathcal{C}(0,\infty;\mathbb{H}^{-3/2}(\Omega))$.

Properties (3.36) and (3.41) yields that there exists a sequence $(t_n)_{n\in\mathbb{N}}$, $t_n > 0$, $\lim_{n\to+\infty} t_n = +\infty$ such that $||\partial_t u(t_n)||_{\mathbb{H}^{-3/2}(\Omega)}$ and $||u(t_n)||_{\mathbb{H}^1(\Omega)}$ both go to zero as $n\to\infty$. Extracting a subsequence if necessary, we may assume, in view of (3.38), that $\psi(t_n)$ converges weakly in H^1 to some ψ_∞. Therefore we can pass to the limit in

$$\rho_0\partial_t u(t_n) - \triangle\, u(t_n) = -\nabla p(t_n) + (\triangle\,\psi(t_n) - \psi(t_n))\delta_\Sigma e_z$$

which gives $(-\triangle\,\psi_\infty + \psi_\infty)e_z\delta_\Sigma = -\nabla p_\infty$. The left-hand side of this equality only depends on (x,y) whereas the right-hand side only depends on z. Therefore, the left-hand side is a constant.

Briefly speaking, we have proved that, up to an extraction, ψ converges as t goes to infinity to a function ψ_∞ defined on Σ solution of

$$\left\{\begin{aligned} -\triangle\,\psi_\infty + \psi_\infty &= \alpha \qquad \text{on } \Sigma, \\ \int_\Sigma \psi_\infty\,d\sigma &= 0, \end{aligned}\right. \tag{3.48}$$

where α is some unknown constant. It is worth noticing that there exists an infinity of ψ_∞ which satisfy (3.48), each of them being associated to *one* Dirichlet boundary condition on $\partial\Sigma$, and that the energy of these steady-states related to a fluid at rest describe a continuum. This statement can straightforwardly be checked in two dimensions ($\Omega \subset \mathbf{R}^2$) when the partial differential equation of (3.48) becomes an ordinary differential equation :

$$-\psi''_\infty + \psi_\infty = \alpha. \tag{3.49}$$

The solutions are of the form $\psi_\infty = ae^x + be^{-x} + \alpha$ where the three constants (a, b, α) are related together by the only condition that $\int_\Sigma \psi_\infty = 0$, namely

$$a(e^L - 1) - b(e^{-L} - 1) + \alpha L = 0, \tag{3.50}$$

where L is the length of Σ. The energy associated to such a solution is

$$\int_\Sigma (\psi'_\infty)^2 + (\psi_\infty)^2 = \int_\Sigma (ae^x - be^{-x})^2 + (ae^x + be^{-x} + \alpha)^2, \tag{3.51}$$

which can be shown to be arbitrarily close to the zero energy of the interface with minimum energy $\psi_\infty \equiv 0$, even under the condition (3.50). We leave to the reader the analogous proof in three dimensions.

Assuming a little more regularity on u we can improve this result. Indeed, if we suppose that $u \in \mathcal{C}(0, T; \mathbb{H}^{1+\varepsilon}(\Omega)$ for any $T > 0$ (compare with (3.44)), the function $u_z(t)|_\Sigma$ belongs to $\mathbb{H}^{1/2+\varepsilon}(\Sigma)$ and thus has a trace on $\partial\Sigma$ (other assumptions than the $\mathbb{H}^{1+\varepsilon}(\Omega)$ regularity are possible, namely any regularity $W^{r,s}$ which allows to define a trace on $\partial\Sigma$). Therefore, in this case $\partial_t\psi(t)|_{\partial\Sigma} = 0$ for $t \geq 0$. In particular $\psi_\infty|_{\partial\Sigma} = \psi_0|_{\partial\Sigma}$, where $\psi_0 = \psi|_{t=0}$. Then we claim that the limit ψ_∞ is now precisely identified as the unique solution of

$$\begin{cases} -\triangle \psi_\infty + \psi_\infty = \alpha & \text{on } \Sigma, \\ \psi_\infty = \psi_0 & \text{on } \partial\Sigma, \\ \displaystyle\int_\Sigma \psi_\infty \, d\sigma = 0. & \end{cases} \tag{3.52}$$

In other words, the possible indetermination of the limit ψ_∞ has disappeared, because the linearized system has kept memory of the boundary value of the initial data $\psi|_{t=0}$.

We now prove the convergence in time to ψ_∞. For ease of notation, we introduce the functions $\overline{\psi}$ and h defined on $\Sigma \times (0, T)$ by $\overline{\psi} = \psi - \psi_\infty$ and $h = -\triangle \overline{\psi} + \overline{\psi}$. It is worth noticing that $\overline{\psi}(t)$ vanishes on $\partial\Sigma$ for any time t and $h\delta_\Sigma e_z = \triangle u - \partial_t u - \nabla p - \alpha$. We denote by $\dot{H}^{-1}(\Sigma)$ the quotient space $H^{-1}(\Sigma)/\mathbb{R}$. By definition,

$$||h||_{\dot{H}^{-1}(\Sigma)} = \sup_{\phi \in H_0^1(\Sigma), \int_\Sigma \phi \, d\sigma = 0, \phi \neq 0} \frac{|< h, \phi >|}{||\phi||_{H_0^1(\Sigma)}}$$

For $\phi \in H_0^1(\Sigma)$ with $\int_\Sigma \phi \, d\sigma = 0$, there exists $w \in \mathbb{H}_0^1(\Omega)$ (even in $\mathbb{H}^{3/2}(\Omega) \cap \mathbb{H}_0^1(\Omega)$) such that $\operatorname{div} w = 0$ and $w.e_z|_\Sigma = f$. Thus,

$$\begin{aligned} ||h||_{\dot{H}^{-1}(\Sigma)} &\leq C^{st} \sup_{w \in \mathbb{H}_0^1(\Omega), divw=0} \frac{|<h, w>|}{||w||_{\mathbb{H}^1(\Omega)}} \\ &\leq C^{st} \sup_{w \in \mathbb{H}_0^1(\Omega), divw=0} \frac{|< \triangle u - \partial_t u, w>|}{||w||_{\mathbb{H}_0^1(\Omega)}} \\ &\leq C^{st}||u||_{\mathbb{H}_0^1(\Omega)} + C^{st}||\partial_t u||_{\mathbb{H}^{-1}(\Omega)} \end{aligned}$$

In view of (3.41) and (3.36), this proves that $h \in L^2(0, \infty; \dot{H}^{-1}(\Sigma))$. It is straightforward to check that $||\overline{\psi}||_{H^1(\Sigma)} = ||h||_{\dot{H}^{-1}(\Sigma)}$. Indeed, on the one hand we note that $||\overline{\psi}||_{H^1(\Sigma)} = ||h||_{H^{-1}(\Sigma)}$ since $\overline{\psi}$ vanishes on $\partial\Sigma$, on the other hand we have

$$\begin{aligned} ||h||_{\dot{H}^{-1}(\Sigma)} &= \sup_{\phi \in H_0^1(\Sigma), \int_\Sigma \phi \, d\sigma = 0, \phi \neq 0} \frac{|<h, \phi>|}{||\phi||_{H_0^1(\Sigma)}} \\ &\leq \sup_{\phi \in H_0^1(\Sigma), \phi \neq 0} \frac{|<h, \phi>|}{||\phi||_{H^1(\Sigma)}} = ||h||_{H^{-1}(\Sigma)} \end{aligned}$$

and

$$\begin{aligned} ||h||_{\dot{H}^{-1}(\Sigma)} &= \sup_{\phi \in H^1(\Sigma), \int_\Sigma \phi \, d\sigma = 0, \phi \neq 0} \frac{|<h, \phi>|}{||\phi||_{H^1(\Sigma)}} \\ &\geq \frac{|< -\triangle \overline{\psi} + \overline{\psi}, \overline{\psi}>|}{||\overline{\psi}||_{H^1(\Sigma)}} = ||\overline{\psi}||_{H^1(\Sigma)} \end{aligned}$$

Therefore,

$$\overline{\psi} \in L^2(0, \infty; H_0^1(\Sigma)). \tag{3.53}$$

We have moreover

$$\begin{aligned} \left| \frac{d}{dt} ||\overline{\psi}||^2_{H^1(\Sigma)} \right| &\leq ||\overline{\psi}||_{H^1(\Sigma)} ||\partial_t \overline{\psi}||_{H^1(\Sigma)} \\ &\leq C^{st} ||\overline{\psi}||_{H^1(\Sigma)} ||u||_{H_0^1(\Omega)} \end{aligned}$$

This inequality together with (3.53) proves that $||\overline{\psi}||^2_{H^1(\Sigma)} \in W^{1,1}(0, +\infty)$. In particular, $\lim_{t \longrightarrow +\infty} ||\psi - \psi_\infty||_{H^1(\Sigma)} = 0$. Therefore, using (3.45) and (3.39) we deduce by interpolation that

$$\lim_{t \longrightarrow +\infty} ||\psi - \psi_\infty||_{H^{1-\varepsilon}} = 0. \tag{3.54}$$

Proposition 2
In the linearized case with surface tension (3.34), the behaviour of u, ψ (satisfying the two estimates of Section 3.2.1 and the regularity mentionned in Section 3.2.2) as time goes to infinity is the following :

(i) the velocity field u belongs to $\mathcal{C}(0,+\infty;\mathbb{H}^1_0(\Omega))$ and goes to 0 in $\mathbb{H}^1_0(\Omega)$.

(ii) the shape ψ of the interface belongs to $\mathcal{C}(0,+\infty;H^1(\Sigma))$; there exists a sequence $t_n \to +\infty$ such that, in weak $-$ H^1, $\psi(\cdot,t_n) \to \psi_\infty$ solution of (3.48).

In addition,

(iii) $\|\psi\|_{H^1(\Sigma)}$ has a limit as t goes to infinity,

(iv) $\displaystyle\int_0^\infty \|u\|^2_{\mathbb{H}^1(\Omega)} + \|\partial_t u\|^2_{\mathbb{H}^1(\Omega)}\,dt < +\infty.$

If we assume that the velocity u remains more regular, say $\mathcal{C}(0,\infty;\mathbb{H}^{1+\varepsilon}(\Omega))$, then (ii) may be improved into

(v) Denote by ψ_∞ the unique solution of (3.52) then ψ goes to ψ_∞ in $H^{1-\varepsilon}$, for all $\varepsilon > 0$, thus in L^p, for all $1 \le p < +\infty$. In 2 dimensions, this also implies in particular that $\sup_\Sigma |\psi - \psi_\infty|$ goes to 0.

This Proposition deserves some comments.

Remark 3.7 *We do not know whether the additional assumption of global regularity of u is automatically satisfied by the solution or not. But we need it in order to show (v).*

Remark 3.8 *Other types of "weak" convergence than (ii) can be proved. We refer the reader to the nonlinear case below.*

Remark 3.9 *The result (v) is somewhat puzzling. Indeed, assume that the flow remains regular, and suppose (just to fix the ideas) that Ω is a cylinder. Take an initial data $\psi(t=0)$ such that its boundary value is not a constant (and in particular it is not zero). If the coefficient of surface tension is small enough, it is expected that the limit ψ_∞ of ψ will be the state of minimal energy $\psi_\infty \equiv 0$, or at least a state (meniscus-like) not too far from this state (remark that for the model we deal with in this article ψ_∞ is the state of minimal energy, whereas from experiment, it is known that it is the meniscus which minimizes the energy; this is related to the modelling of the surface tension we have chosen and to the question of boundary conditions). Considering the case we deal with, the state is at least expected to be radially symmetric, thus have a constant boundary value. This* cannot *be the case ! Note in addition that the initial state may be chosen arbitrarily close to the expected limit, in such a way that we do not theoretically leave the setting of a* small perturbation *problem. The result (v) suggests the following alternative in such a situation : either the flow becomes singular at some time (in the sense that it is not more regular than H^1) or we may conclude that the linearized model* does not *reproduce the physical observation.*

Remark 3.10 *There exists an infinity of steady-states with zero velocity field and since they form a continuum of energy it is not possible to discriminate* *among them in (ii). Of course, if the system is in such a steady state at $t = 0$, it remains there. A similar situation will be observed in the nonlinear case.*

Remark 3.11 *All the difficulties we experiment in the treatment of the boundary value $\partial_t \psi = 0$ have their numerical counterpart. The macroscopic non-slip condition $u = 0$ on the boundary is obviously not true on the microscopic scale and one must find numerical tricks to artificially move the interface on the boundary of the domain.*

4 The nonlinear case

We now return to the nonlinear case, that is equations (1.1). As we will see below, and as we announced in the introduction, the situation is radically different from the situation encountered in Section 3 for the linearized case. Let us begin with a heuristic argument that shows what we may expect in this case.

4.1 A heuristic argument

We begin with a very simple heuristic argument that shows that we expect that the velocity field vanishes as time goes to infinity. Multiplying by u the Navier-Stokes equation (1.1) :

$$\partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) - \Delta u = -\nabla p - \rho e_z, \tag{4.1}$$

we obtain the standard energy estimate (we skip the details of the computations that will be made precisely below in the next two sections)

$$\frac{d}{dt}\Big(\frac{1}{2}\int_\Omega \rho u^2\,dx + \int_\Omega \rho z\,dx\Big) + \int_\Omega |\nabla u|^2\,dx = 0. \tag{4.2}$$

It follows that

$$\int_0^{+\infty} \|\nabla u\|^2_{\mathbb{L}^2(\Omega)}\,dt < +\infty. \tag{4.3}$$

This suggests that, in a formal sense at least, u goes to zero as time goes to infinity. We deduce then, in some way that obviously has to be made precise (we recall that we are only making here a formal argument), that $\partial_t(\rho u)$ also goes to zero. We then recover with the Navier-Stokes equation (4.1) that

$$-\nabla p - \rho e_z \longrightarrow 0,$$

as t goes to infinity. This means that $-\nabla(p + \rho z) + z\nabla\rho$ goes to zero, which can be expressed as follows : $\mathrm{curl}\,(z\,\nabla\rho) \longrightarrow 0$, or also $\nabla\rho \times e_z$ goes to zero, which means that ρ becomes a function of z as time goes to infinity. If we admit, relying upon some common sense, that no mixing of the two fluids happens in the limit $t \longrightarrow +\infty$, this implies that the interface between the two fluids is made of planes, which are parallel to the (O, x, y) plane, and which separate two consecutive layers of fluids. It is then to be remarked that nothing tells us that the interface is made of only **one** plane (*see* Section 4.2.1).

Let us now continue our formal argument by adding to the Navier-Stokes equation a term due the presence of surface tension.

$$\partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) - \Delta u = -\nabla p - \rho e_z + \mathcal{T}, \tag{4.4}$$

The energy estimate then becomes (*see* the details below)

$$\frac{d}{dt}\Big(\frac{1}{2}\int_\Omega \rho u^2\,dx + \int \rho z\,dx + \frac{1}{2}L(\Gamma)\Big) + \int_\Omega |\nabla u|^2\,dx = 0, \tag{4.5}$$

where $L(\Gamma)$ denotes the length of the interface Γ between the two fluids. The same argument as above shows that u and $\partial_t(\rho u)$ go to 0 as time goes to infinity. Next, with the Navier-Stokes equation we recover

$$-\nabla p - \rho e_z - (\mathrm{div}\, n)\, n \longrightarrow 0,$$

where the normal vector n is also $n = \nabla\rho$. We therefore have

$$-\nabla(p + \rho z) + (z - \mathrm{div}\, n)\nabla\rho \longrightarrow 0.$$

As above, it implies, taking for instance the curl of the above expression, that the quantity $z - \mathrm{div}\, n$ is constant along the connected components of the interface (assuming that $\nabla\rho$ is normal to the interface, namely there is non homogeneization in the fluids).

Of course, $z = 0$ (and thus $\nabla\rho = e_z$) is a solution to the equation giving the position of the interface at the equilibrium, but there exists a lot of other solutions (*see* Section 4.2.2 below).

In both cases (with or without surface tension) the above heuristic argument shows that the situation is the following :

1. It is reasonably easy to show that the velocity field u goes to zero, at least in a weak sense, as time goes to infinity.

2. As well, we can prove that ρ converges to some limit ρ_∞ (in a weak sense also), which is a solution to the Navier-Stokes equation with zero velocity field.

3. Only an argument based upon energetic considerations could possibly help us to discriminate between all the solutions ρ_∞ of the Navier-Stokes equations with zero velocity field (in fact we shall see below that such an argument unfortunately cannot help us to conclude).

Therefore, before turning to the rigorous proofs of convergence of u and ρ to their limit, it is important to deal with the solutions $(u = 0, \rho)$ of the Navier-Stokes equations with or without surface tension.

4.2 An infinity of steady states

4.2.1 Without surface tension

As claimed above, as t goes to infinity, one can prove that the fluid velocity goes to zero and the density is a function of z. Nevertheless, we are not able to prove that the situation shown on the right-hand side of Figure 1 (several layers of the two fluids) cannot occur (it is in fact even worse than that, since there might exist an infinite superposition of layers, in the sense that the two fluids might mix with each other in the limit, but let us leave apart this situation that we shall detail in the sequel). Moreover, it is easy to check that the energy of such a pathological state may be arbitrarily close to the minimal energy of the system (when the heaviest fluid is below the flat interface, and the lightest above; a situation that we henceforth denote by the density ρ_0). Indeed, if suffices to swap in the minimal energy steady-state an arbitrary thin layer of the heaviest fluid with a layer of the lightest one. We then obtain a steady-state (namely a zero velocity field and flat interfaces between the two fluids) with an energy arbitrarily close to the minimal one.

4.2.2 With surface tension

In presence of surface tension, we have explained above that we expect to reach, as t goes to infinity, a state with zero velocity and an interface satisfying $z - \operatorname{div} n = C^{st}$ along each connected component.

As in the previous case, we do not know if the interface remains connected. Nevertheless, even if one would able to prove that the interface is a connected graph, we now show that there is still an infinity of steady-states.

For the sake of simplicity, we restrict ourselves to the the case when $\Omega \subset \mathbb{R}^2$. Then, equation $z - \operatorname{div} n = C^{st}$ reads

$$z - \frac{1}{2}\frac{z''}{(1+z'^2)^{3/2}} = C^{st}.$$

We consider a case with a zero right-hand side. Integrating this equation we have

$$z^2 + \frac{1}{\sqrt{1+z'^2}} = C^{st}. \tag{4.6}$$

We assume that the constant is 1 and that the interface is a graph described by a one-to-one function $z = z(x)$. Even under these restrictive hypotheses,

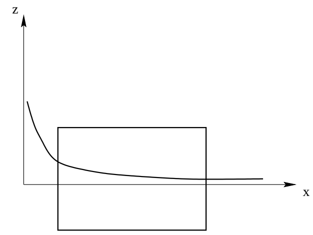

Figure 4: The window represents a domain $\Omega \subset \mathbb{R}^2$, the curve is an interface solution of the steady state equations with surface tension and a zero velocity field.

there is still an infinity of solutions. Indeed, the functions

$$x = \frac{1}{\sqrt{2}} \left(\text{Argch} \frac{\sqrt{2}}{z} - 2\sqrt{1 - \frac{z^2}{2}} \right) + x_0.$$

where x_0 is a constant, are solution of (4.6) (*see* L. Landau, E. Lifchitz [28]). The curve is ploted on Figure 4 where the window represents the domain Ω. Notice that the window has to be translated along the z axis in order to satisfy the mass conservation. Moreover, it can be translated along the x axis by fixing the constant x_0. It should be noted that the energy of the system tends to the minimal energy (flat interface) as the window is translated on the right.

Thus, we have outlined a proof of the existence of steady-states with zero velocity and a non-flat interface with an energy arbitrarily close to the minimal energy.

Remark 4.1 *An analogous situation has been encountered in Section 3.2 : the lack of information on the position of the interface on $\partial\Omega$ prevents us from identifying a unique steady-state. Nevertheless, to obtain this information in the linear case, it was sufficient to assume a slightly better regularity on u (namely $H^{1+\varepsilon}$) whereas in this nonlinear case, the regularity required to give a sense to $\partial_t \rho = -div(\rho u)$ on $\partial\Omega$ seems definitely out of reach.*

The consequence of the existence of infinitely many steady states $(u = 0, \rho)$ forming a continuum of energy above the state of minimal energy $(u = 0, \rho_0)$ is the following. Even if we were able to prove that the convergence of $(u(t, x), \rho(t, x))$ to $(u = 0, \rho_\infty = \rho(x))$ holds in a (reasonable) strong sense, we could not prove that $\rho_\infty = \rho_0$, thereby recovering with the mathematical model the behaviour expected from common sense.

Therefore we continue our study of the nonlinear case in the following spirit : we show in the next two sections how the convergences stated in a heuristic way in Section 4.1 for u and ρ can be made precise. For this purpose, we show *some* convergences for u and ρ. We do not pretend that these convergences cannot be improved, but in view of the above remark on the number of possible limits, we have chosen to present some convergences that can be established reasonably easily. It is likely that intricate arguments might lead to better convergences. They will however not allow to circumvent the main difficulty : it cannot be shown that the only limit is $(u = 0, \rho_0)$.

4.3 The nonlinear case without surface tension

4.3.1 A priori estimates, the general case

First, we observe that for any $\beta \in \mathcal{C}^1([0,\infty);\mathbb{R})$, we have

$$\partial_t(\beta(\rho)) + \mathrm{div}\,(u\beta(\rho)) = \beta'(\rho)\{\partial_t\rho + u.\nabla\rho\} = 0.$$

This yields (*see* P.-L. Lions [30] for details)

$$||\rho(t)||_{L^\infty(\Omega)} = ||\rho^0||_{L^\infty(\Omega)}, \forall t \geq 0, \tag{4.7}$$

and more precisely

$$\mathrm{meas}\{x \in \Omega, \rho(x) = \rho_i\}, i = 1, 2 \text{ is independent of } t \geq 0. \tag{4.8}$$

Next, we multiply the Navier-Stokes equation

$$\partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) - \Delta u = -\nabla p - \rho e_z$$

by u and integrate over the domain. In doing so, we obtain

$$\frac{1}{2}\int_\Omega \partial_t(\rho u)\cdot u\,dx + \int_\Omega \mathrm{div}\,(\rho u \otimes u)\cdot u\,dx + \int_\Omega |\nabla u|^2\,dx = -\int_\Omega \rho u\cdot e_z\,dx. \tag{4.9}$$

It is standard to compute the first two terms. We have

$$\frac{1}{2}\int_\Omega \partial_t(\rho u)\cdot u\,dx = \frac{1}{2}\frac{d}{dt}\int_\Omega \rho u^2\,dx + \frac{1}{2}\int_\Omega (\partial_t\rho)u^2\,dx \tag{4.10}$$

$$\int_\Omega \mathrm{div}\,(\rho u \otimes u)\cdot u\,dx = -\frac{1}{2}\int_\Omega \rho u\cdot\nabla(|u|^2)\,dx \tag{4.11}$$

Adding (4.10) to (4.11) and making use of the equation of mass conservation in (1.1), we obtain

$$\frac{1}{2}\int_\Omega \rho\partial_t(|u|^2)\,dx + \int_\Omega \mathrm{div}\,(\rho u \otimes u)\cdot u\,dx = \frac{1}{2}\frac{d}{dt}\int_\Omega \rho u^2\,dx. \tag{4.12}$$

For the right-hand side of (4.9), we write

$$\int_\Omega \rho u\cdot e_z\,dx = \int_\Omega \rho u\cdot\nabla(z)\,dx = -\int_\Omega z\mathrm{div}\,(\rho u)\,dx = \int_\Omega z\partial_t\rho\,dx = \frac{d}{dt}\int_\Omega \rho z\,dx. \tag{4.13}$$

Inserting (4.12) and (4.13) into (4.9), we obtain the first energy estimate

$$\frac{d}{dt}\Big(\frac{1}{2}\int_\Omega \rho u^2\,dx + \int_\Omega \rho z\,dx\Big) + \int_\Omega |\nabla u|^2\,dx = 0. \tag{4.14}$$

From this energy estimate, we deduce that in particular

$$\int_0^{+\infty} \|u\|^2_{\mathbb{H}^1(\Omega)}\,dt < +\infty, \tag{4.15}$$

and

$$\sup_{t\in[0,\infty)} ||u||_{\mathbb{L}^2(\Omega)} \leq +\infty. \tag{4.16}$$

4.3.2 A priori estimates, the bidimensional case

The rest of the argument depends on the dimension of the space. Moreover, in this Section, it is necessary to assume that the viscosity is constant over the domain (or to suppose at least that it is slightly variable, using the results of B. Desjardins [16]).

We now assume for the rest of this Subsection 4.3.2 that the domain Ω is a subset of $\mathbf{R}^2$. We multiply (4.1) by $\partial_t u$ and integrate over the domain

$$\int_\Omega \rho(\partial_t u)^2\,dx + \int_\Omega \rho u\cdot\nabla u\partial_t u\,dx + \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2\,dx = -\int_\Omega \rho e_z\cdot\partial_t u\,dx. \tag{4.17}$$

The point is to treat the Navier term in the left-hand side. We have

$$\Big|\int_\Omega \rho u\cdot\nabla u\partial_t u\,dx\Big| \leq \|\rho\|_{L^\infty}\|u\cdot\nabla u\|_{L^2}\|\partial_t u\|_{L^2} \leq \|\rho\|_{L^\infty}\|\nabla u\|_{L^4}\|u\|_{L^4}\|\partial_t u\|_{L^2}.$$

We use the following inequality (of Gagliardo-Nirenberg type)

$$\|\nabla u\|_{L^4} \leq C^{st}\|\nabla u\|_{L^2}^{1/2}\|u\|_{H^2}^{1/2}, \tag{4.18}$$

which yields

$$\Big|\int_\Omega \rho u\cdot\nabla u\partial_t u\,dx\Big| \leq \|\rho\|_{L^\infty}\|\nabla u\|_{L^2}^{1/2}\|u\|_{L^4}\|u\|_{H^2}^{1/2}\|\partial_t u\|_{L^2}. \tag{4.19}$$

Considering now (4.1), we remark that

$$\|-\Delta u + \nabla p\|_{L^2} \leq C^{st}(\|\partial_t u\|_{L^2} + \|u\cdot\nabla u\|_{L^2} + \|\rho\|_{L^2}),$$

which, by standard elliptic regularity for the Stokes equation, yields

$$\|u\|_{H^2} \leq C^{st}(\|u\|_{L^2} + \|\partial_t u\|_{L^2} + \|u\cdot\nabla u\|_{L^2} + \|\rho\|_{L^2}).$$

If we use again (4.18) we know for any $\varepsilon > 0$ the existence of a constant C_ε such that

$$\|u\cdot\nabla u\|_{L^2} \leq \|u\|_{L^4}\|\nabla u\|_{L^4} \leq C^{st}\|u\|_{L^4}\|\nabla u\|_{L^2}^{1/2}\|u\|_{H^2}^{1/2} \leq \varepsilon\|u\|_{H^2}+C_\varepsilon\|u\|_{L^4}^2\|\nabla u\|_{L^2}.$$

Taking ε small enough, we have

$$\frac{1}{2}\|u\|_{H^2} \leq C^{st}(\|u\|_{L^2} + \|\partial_t u\|_{L^2} + \|u\|_{L^4}^2\|\nabla u\|_{L^2} + \|\rho\|_{L^2}). \tag{4.20}$$

Inserting this latter estimate into (4.19), we obtain

$$\begin{aligned}\left|\int_\Omega \rho u\cdot\nabla u\partial_t u\,dx\right| \;\leq\; & C^{st}\|\rho\|_{L^\infty}\|\nabla u\|_{L^2}^{1/2}\|u\|_{L^4}\|\partial_t u\|_{L^2}(\|u\|_{L^2}^{1/2} + \|\partial_t u\|_{L^2}^{1/2}\\ & +\|u\|_{L^4}\|\nabla u\|_{L^2}^{1/2} + \|\rho\|_{L^2}^{1/2}).\end{aligned}$$

Since we know that $\|u\|_{L^2} + \|\rho\|_{L^\infty}$ is bounded by a constant, we have

$$\begin{aligned}\left|\int_\Omega \rho u\cdot\nabla u\partial_t u\,dx\right| \;\leq\; & C^{st}\|\nabla u\|_{L^2}^{1/2}\|u\|_{L^4}\|\partial_t u\|_{L^2} + C^{st}\|\nabla u\|_{L^2}^{1/2}\|u\|_{L^4}\|\partial_t u\|_{L^2}^{3/2}\\ & +C^{st}\|\nabla u\|_{L^2}\|u\|_{L^4}^2\|\partial_t u\|_{L^2}.\end{aligned}$$

If we use now

$$\|u\|_{L^4} \leq C^{st}\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}, \tag{4.21}$$

and note again that $\|u\|_{L^2}$ is bounded, we may bound the first term as follows

$$\|\nabla u\|_{L^2}^{1/2}\|u\|_{L^4}\|\partial_t u\|_{L^2} \leq C^{st}\|\partial_t u\|_{L^2}\|\nabla u\|_{L^2} \leq \varepsilon\|\partial_t u\|_{L^2}^2 + C_\varepsilon\|\nabla u\|_{L^2}^2.$$

The second term is estimated by an interpolation inequality

$$\|\nabla u\|_{L^2}^{1/2}\|u\|_{L^4}\|\partial_t u\|_{L^2}^{3/2} \leq \varepsilon\|\partial_t u\|_{L^2}^2 + C_\varepsilon\|\nabla u\|_{L^2}^2\|u\|_{L^4}^4,$$

and the third term is estimated likewise by

$$\|\nabla u\|_{L^2}\|u\|_{L^4}^2\|\partial_t u\|_{L^2} \leq \varepsilon\|\partial_t u\|_{L^2}^2 + C_\varepsilon\|\nabla u\|_{L^2}^2\|u\|_{L^4}^4.$$

Therefore, we have

$$\left|\int_\Omega \rho u\cdot\nabla u\partial_t u\,dx\right| \leq \varepsilon\|\partial_t u\|_{L^2}^2 + C_\varepsilon(\|\nabla u\|_{L^2}^2 + \|u\|_{L^4}^4\|\nabla u\|_{L^2}^2). \tag{4.22}$$

Now that we have controlled the Navier term, we turn to the right-hand side of (4.17)

$$\begin{aligned}\int_\Omega \rho e_z\cdot\partial_t u\,dx &= \frac{d}{dt}\int_\Omega \rho e_z\cdot u\,dx - \int_\Omega \partial_t\rho e_z\cdot u\,dx\\ &= -\frac{d}{dt}\int_\Omega z\nabla\rho\cdot u\,dx + \int_\Omega \mathrm{div}\,(\rho u)e_z\cdot u\,dx\\ &= \frac{d^2}{dt^2}\int_\Omega \rho z\,dx - \int_\Omega \rho u\cdot\nabla(e_z\cdot u)\,dx.\end{aligned}$$

Hence

$$-\int_\Omega \rho e_z \cdot \partial_t u \, dx \le -\frac{d^2}{dt^2}\int_\Omega \rho z \, dx + C^{st}\|\rho\|_{L^\infty}\|\nabla u\|^2_{L^2}.$$

Inserting this latter estimate together with estimate (4.22) into (4.17), we obtain, for a small constant $\alpha > 0$,

$$\alpha\|\partial_t u\|^2_{L^2} + \frac{1}{2}\frac{d}{dt}(\|\nabla u\|^2_{L^2} + \frac{d}{dt}\int_\Omega \rho z \, dx) \le C^{st}(\|\nabla u\|^2_{L^2} + \|u\|^4_{L^4}\|\nabla u\|^2_{L^2}). \tag{4.23}$$

This estimate may be written

$$\|\partial_t u\|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{2}\frac{d}{dt}(\|\nabla u\|^2_{\mathbb{L}^2(\Omega)} + \frac{d}{dt}\int_\Omega \rho z \, dx) \le C^{st}\|\nabla u\|^2_{\mathbb{L}^2(\Omega)} + C^{st}\|u\|^4_{\mathbb{L}^4(\Omega)}||\nabla u||^2_{\mathbb{L}^2(\Omega)}, \tag{4.24}$$

or also

$$\|\partial_t u\|^2_{L^2} + \frac{1}{2}\frac{d}{dt}(\|\nabla u\|^2_{\mathbb{L}^2(\Omega)} + \frac{d}{dt}\int_\Omega \rho z \, dx) \le f(t) + g(t)\|\nabla u\|^2_{\mathbb{L}^2(\Omega)}, \tag{4.25}$$

where the nonnegative functions $f(t)$ and $g(t)$ are both $L^1(0, +\infty)$ since

$$f(t) = C^{st}\|\nabla u\|^2_{\mathbb{L}^2(\Omega)} \tag{4.26}$$

$$g(t) = C^{st}\|u\|^4_{\mathbb{L}^4(\Omega)} \le C^{st}\|u\|^2_{\mathbb{L}^2(\Omega)}\|\nabla u\|^2_{\mathbb{L}^2(\Omega)} \le C^{st}\|\nabla u\|^2_{\mathbb{L}^2(\Omega)}. \tag{4.27}$$

We finally obtain the *a priori* estimates by a Gronwall type argument : let us introduce

$$y(t) = \exp\Big(-2\int_0^t g(s)ds\Big)\|\nabla u\|^2_{\mathbb{L}^2(\Omega)}.$$

Inequality (4.25) yields

$$\frac{1}{2}y'(t) \le f(t) - \frac{d^2}{dt^2}\int_\Omega \rho z \, dx.$$

Integrating this inequality in the t variable and using

$$\left|-\frac{d}{dt}\int_\Omega \rho z \, dx\right| = \left|-\int_\Omega \rho e_z \cdot u \, dx\right| \le C^{st}\|\rho\|_{L^\infty(\Omega)}\|u\|_{\mathbb{L}^2(\Omega)} \le C^{st}.$$

we obtain that $y \in L^\infty(0,\infty)$ which implies

$$\sup_{t\in[0,+\infty)} \|\nabla u\|_{\mathbb{L}^2(\Omega)} < \infty. \tag{4.28}$$

Finally, integrating (4.25) we obtain

$$\int_0^\infty \|\partial_t u\|^2_{\mathbb{L}^2(\Omega)} \, dt < +\infty. \tag{4.29}$$

We deduce from (4.20) that for any $T > 0$,

$$||u||_{L^2(0,T;\mathbb{H}^2(\Omega))} < +\infty. \tag{4.30}$$

This new bound is not uniform in T but it allows to prove that u is a strong solution for all $t \in (0, +\infty)$ and that $u \in \mathcal{C}(0, \infty; \mathbb{H}^1_0(\Omega))$.

4.3.3 Questions of existence and regularity

The general case The first existence results in the setting of the Navier-Stokes equations with a free surface are local in time existence results due to V.A. Solonnikov [35] and to J.T.Beale [4]. Global existence for small initial data and $f \equiv 0$ is due to V. Solonnikov, [38] (bounded case) and to Tani & Tanaka [43] (unbounded case). In our case when two fluids are present, an existence result of weak solutions is due to A. Nouri, F. Poupaud [33], a global in time existence result of strong solutions for small data is announced in N. Tanaka [41] (bounded case, with an initial data consisting of a bubble of the first fluid enclosed in the second fluid), but the most exhaustive work to this day is due to P.-L. Lions. It is proved in P.-L. Lions [30] that there exists a weak solution to the system (1.1) defined on $[0, +\infty)$ satisfying for any time $T > 0$:

$$u \in L^2(0, T; \mathbb{H}^1_0(\Omega)) \cap L^\infty(0, T; \mathbb{L}^2(\Omega)), \tag{4.31}$$

$$\rho \in L^\infty((0, T) \times \Omega) \cap \mathcal{C}(0, T; L^p(\Omega)), 1 \leq p < \infty, \tag{4.32}$$

together with the energy inequality

$$\frac{d}{dt}\Big(\frac{1}{2}\int_\Omega \rho u^2\, dx + \int_\Omega \rho z\, dx\Big) + \int_\Omega |\nabla u|^2\, dx \leq 0 \tag{4.33}$$

Remark 4.2 *Let us make a few remarks on the regularity of the flow. For the standard one fluid Navier-Stokes equation, it is well known that a global strong solution exists in 3D if the initial data and the forces are "small enough". As far as the body force term is concerned, "small" means small in a functional space of the type $L^p((0, \infty), X(\Omega))$ for some functional space $X(\Omega)$ and some $p < +\infty$ (*see *R. Temam [44]) or even in $L^\infty((0, \infty), X(\Omega))$ (*see *H. Fujita & T. Kato [19]). As we have mentioned above, such results of regularity have been extended for some small special initial data in the two-fluids case by N. Tanaka [41] only for a force that is small in $L^p((0, \infty), X(\Omega))$ for some $p < +\infty$. The result does not cover the case of some force in $L^\infty((0, \infty), \mathbb{L}^q(\Omega))$ that does not vanish as $t \longrightarrow +\infty$ in any weak sense, say for instance a force constant in time, or also the force we deal with here, namely $-\rho e_z$, whose $\mathbb{L}^q$ norm is a constant. Indeed, in our context, the body force term is "small" in $L^\infty((0, \infty), \mathbb{L}^q(\Omega))$ as soon as the densities of the two fluids are close to each other : it suffices to replace the term of $-\rho g e_z$ by $(\rho - \rho_1) g e_z$ and to add the term $\rho_1 g z$ to the pressure p. It is of course not small in any $L^p((0, \infty), X(\Omega))$ for $p < +\infty$ since it even does not belong to such a space. However, the result by Tanaka suffices to show that, given some arbitrary time T, the solution remains regular on $(0, T)$ if the initial data u_0 and the difference of densities $\delta\rho$ are both small enough.*

We suspect it is possible to improve this result in the following way. It is known that we also have global regularity for two fluids in 2D under the *additional assumption that the viscosity is constant all over the domain, and then no matter how large the force is (see S.N. Antontsev, A.V. Kazhikov, V.N. Monakhov [2]). In view of all these results, it sounds reasonable to believe that the following regularity result holds : in 3D, for two fluids sharing the same viscosity, under the hypothesis that the initial velocity is small and that the body force is small in some $L^\infty((0,\infty), X(\Omega))$, the flow remains regular for all time. To the best of our knowledge, such a result has not been proven yet. We will approach this question in a subsequent work ([22]) since it would provide a regularity result for small data in the setting we work in.*

Furthermore, continuing our formal analysis of open questions that should be relevant in our context, we even believe that in the very special case we are interested in here, the regularity results can be extended. Noticing that the term $(\rho - \rho_1)ge_z$ does not modify the first energy estimate (that holds for the zero force case), one should be able to show (at least) the following property : given an arbitrary density difference $\delta\rho = \rho_2 - \rho_1$, then if the initial velocity u_0 is small enough and the initial state is not far from equilibrium, the flow remains regular for all time. Since we have chosen to focus in this article on the long time behaviour we will not present here the investigation of this question and refer the reader to [22] where we hope to settle all these regularity issues.

The bidimensional case In the bidimensional case and when the viscosity η is supposed to be a positive constant, it is proved in S.N. Antontsev, A.V. Kazhikov, V.N. Monakhov [2] that there exists a global in time regular solution (*see* also P.-L. Lions [30]). More precisely, we have for any time $T > 0$,

$$u \in L^2(0,T;\mathbb{H}^2(\Omega)) \cap \mathcal{C}([0,T];\mathbb{H}^1(\Omega)),$$

and

$$\partial_t u \in L^2((0,T)\times\Omega).$$

4.3.4 Long time behaviour

Let $(t_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of positive reals such that $\lim_{n\to+\infty} t_n = +\infty$. We define the sequences ρ_n and u_n by $\rho_n(x,t) = \rho(x,t+t_n)$ and $u_n(x,t) = u(x,t+t_n)$ (in the sense of distributions).

Behaviour of the velocity in the general case According to estimate (4.15), we have

$$\lim_{n\to+\infty} \int_{t_n}^{+\infty} \int_\Omega |\nabla u(x,t)|^2\,dx\,dt = 0,$$

therefore

$$u_n \longrightarrow 0 \text{ in } L^2(0,\infty;\mathbb{H}^1(\Omega)) \text{ as } n \to +\infty \tag{4.34}$$

Remark 4.3 *As far as we know, we cannot rigorously improve this convergence since we do not know if $t \to u(t,.)$ is continuous (say with values in $\mathbb{L}^2(\Omega)$).*

If we postulate *that $u \in C(0,+\infty;\mathbb{L}^2(\Omega))$, we can show that $u \longrightarrow 0$ in $\mathbb{L}^2(\Omega)$ as $t \to +\infty$. Indeed, if we go back to (4.14) and use the fact that*

$$\left| -\frac{d}{dt}\int_\Omega \rho z\, dx \right| = \left| -\int_\Omega \rho e_z \cdot u\, dx \right| \leq C^{st}\|\rho\|_{L^\infty(\Omega)}\|u\|_{\mathbb{L}^2(\Omega)} \leq C^{st}.$$

we may write

$$\frac{d}{dt}\frac{1}{2}\int_\Omega \rho u^2\, dx \leq -\int_\Omega |\nabla u|^2\, dx - \frac{d}{dt}\int_\Omega \rho z\, dx \leq C^{st}.$$

Therefore, the nonnegative function $f(t) = \frac{1}{2}\int_\Omega \rho u^2\, dx$ satisfies the two conditions

$$\int_0^{+\infty} f(t)dt < +\infty, \quad \frac{df}{dt} \leq C^{st} \tag{4.35}$$

It follows by a standard argument that f goes to 0 at infinity, that is

$$\lim_{t \longrightarrow +\infty} \|u\|_{L^2} = 0. \diamondsuit \tag{4.36}$$

Behaviour of the velocity in the bidimensional case So far as we know, we cannot say more on the velocity than the convergence (4.34) in three dimensions . On the contrary, in two dimensions, we can go further in the argument. In this case, u is known to belong to $\mathcal{C}([0,T];\mathbb{H}^1(\Omega))$. Estimates (4.15) and (4.29) show that the right hand side of

$$\left| \frac{d}{dt}||u||^2_{\mathbb{L}^2(\Omega)} \right| \leq ||u||_{\mathbb{L}^2(\Omega)}||\partial_t u||_{\mathbb{L}^2(\Omega)}$$

is in $L^1(0,\infty)$. Thus, $||u||^2_{\mathbb{L}^2(\Omega)} \in W^{1,1}(0,\infty)$ and therefore $u \to 0$ in $\mathbb{L}^2(\Omega)$ as $t \to \infty$. Moreover, (4.28) shows that u belongs to $L^\infty(0,\infty;\mathbb{H}^1(\Omega))$. Therefore, by interpolation between $\mathbb{L}^2(\Omega)$ and $\mathbb{H}^1(\Omega)$, we deduce that

$$u \longrightarrow 0 \text{ in } \mathbb{H}^{1-\varepsilon}(\Omega) \text{ as } t \to +\infty, \forall \varepsilon > 0. \tag{4.37}$$

Behaviour of the interface In view of (4.7) the sequence (ρ_n) remains in a bounded set of $L^\infty((0,+\infty)\times\Omega)$. Therefore there exists $\rho_\infty \in L^\infty((0,+\infty)\times\Omega)$ such that

$$\rho_n \rightharpoonup \rho_\infty \qquad \text{in } L^\infty((0,+\infty)\times\Omega) \text{ weak-}* \tag{4.38}$$

We first prove that ρ_∞ does not depend on t. Let $v \in L^2(0,\infty;\mathbb{H}_0^1(\Omega))$, we have

$$\begin{aligned} |<\partial_t\rho_n, v>| &= |-<\operatorname{div}\rho_n u_n, v>| = |\int_\Omega \rho_n u_n.\nabla v\,dx| \\ &\leq C^{st}||\rho_n||_{L^\infty((0,T)\times\Omega)}||u_n||_{L^2(0,T;\mathbb{H}_0^1(\Omega))}||v||_{L^2(0,T;\mathbb{H}_0^1(\Omega))} \end{aligned}$$

which proves in view of (4.34) that

$$\partial_t\rho_n \longrightarrow 0 \text{ in } L^2(0,\infty;\mathbb{H}^{-1}(\Omega)) \text{ as } n\to+\infty.$$

Therefore, since in the sense of distributions $\partial_t\rho_n \rightharpoonup \partial_t\rho_\infty$, we deduce $\partial_t\rho_\infty = 0$.

We now prove that ρ_∞ only depends on the third space variable z. We have

$$-\nabla p_n - \rho_n e_z = \partial_t(\rho_n u_n) + \operatorname{div}(\rho_n u_n\otimes u_n) - \triangle\, u_n$$

Let $v\in\mathcal{C}_0^\infty((0,\infty)\times\Omega)$,

$$\begin{aligned} |<\partial_t(\rho_n u_n), v>| &\leq ||\rho_n||_{L^\infty((0,\infty)\times\Omega)}||u_n||_{L^2((0,\infty)\times\Omega)}||\partial_t v||_{L^2((0,\infty)\times\Omega)}, \\ |<\operatorname{div}(\rho_n u_n\otimes u_n), v>| &\leq C^{st}||\rho_n||_{L^\infty((0,\infty)\times\Omega)}||u_n||^2_{L^2(0,\infty;\mathbb{H}_0^1(\Omega))}||v||_{L^\infty(0,\infty;\mathbb{H}_0^1(\Omega))}, \\ |<-\triangle\, u_n, v>| &\leq ||u_n||_{L^2(0,\infty;\mathbb{H}_0^1(\Omega))}||v||_{L^2(0,\infty;\mathbb{H}_0^1(\Omega))}, \end{aligned}$$

thus the right-hand sides of the these inequalities go to zero as $n\to\infty$ (*see* (4.34) and (4.38)). Therefore

$$-\nabla p_n - \rho_n e_z \longrightarrow 0$$

in the sense of distributions. Thus, $\operatorname{curl}(\rho_\infty e_z) = \nabla\rho_\infty\times e_z = 0$, which proves that $\partial_x\rho_\infty = \partial_y\rho_\infty = 0$. Therefore

$$\rho_\infty = \rho_\infty(z). \tag{4.39}$$

Finally, let us check the global mass conservation. In view of (4.38), we have for arbitary $f\in L^1(\Omega\times(0,+\infty))$

$$\int_0^{+\infty}\int_\Omega \rho_n(x,t)f(x,t)\,dxdt \longrightarrow \int_0^{+\infty}\int_\Omega \rho_\infty(x)f(x,t)\,dxdt.$$

In particular with $f(x,t)=f(t)\in L^1(0,+\infty)$ such that $\int_0^{+\infty} f(t)\,dt = 1$ we have, according to (4.7)

$$\int_0^{+\infty}\int_\Omega \rho_n(x,t)f(t)\,dxdt = \int_0^{+\infty} f(t)\int_\Omega \rho_n(x,t)\,dxdt = \int_\Omega \rho^0(x)\,dx.$$

Thus

$$\int_\Omega \rho_\infty(x)\,dx = \int_\Omega \rho^0(x)\,dx$$

which proves the global mass conservation.

Notice that, according to (4.8), we know that meas $\{x \in \Omega, \rho_n(x) = \rho_i\} = M_i$ is independent of n. Nevertheless, we are not able to prove that meas $\{x \in \Omega, \rho_\infty(x) = \rho_i\} = M_i$. Indeed, to show this property, we need to prove that for any $\beta \in \mathcal{C}^1([0,\infty), \mathbb{R})$

$$\int_\Omega \beta(\rho_n(x,t))\, dx \longrightarrow \int_\Omega \beta(\rho_\infty(x))\, dx,$$

which seems not possible (so far as we know) in view of the weak convergence of ρ_n.

Therefore we cannot prove that homogeneization does not appear in the limit. In other words there may exist some parts of Ω where ρ_∞ has values between ρ_1 and ρ_2. All that we know is that these areas consist of horizontal layers (possibly infinitely thin).

Remark 4.4 *If for some sequence $t_n \longrightarrow +\infty$ we have $\rho(t_n, \cdot) \longrightarrow \rho_\infty(\cdot)$ almost everywhere in Ω, then it is possible to show, using Theorem 2.4 of [30], that*

$$\forall T < \infty, \forall p < \infty, \lim_{n\to\infty} sup_{t\in[0,T]} |\rho(t+t_n,\cdot) - \rho_\infty(\cdot)|_{L^p} = 0, \tag{4.40}$$

which therefore prevents homogeneization.

This shows our claim of Section 4.1, and in view of Section 4.2.1, we cannot say more on ρ_∞.

Proposition 3
In the nonlinear case without surface tension, a solution (ρ, u) satisfying the estimates (4.7), (4.15), (4.16) has the following behaviour as time goes to infinity :

(i) The velocity field u goes to 0 in $\mathbb{H}^1(\Omega)$ in the "weak" sense of (4.34).

If we postulate *that $u \in \mathcal{C}(0,\infty; \mathbb{L}^2(\Omega))$ and that (4.33) holds then u goes to 0 in $\mathbb{L}^2(\Omega)$.*

If $\Omega \subset \mathbb{R}^2$ then $u \in \mathcal{C}(0,\infty; \mathbb{H}^1(\Omega))$ and u goes to 0 in $\mathbb{H}^{1-\varepsilon}(\Omega)$, $\forall \varepsilon > 0$ as $t \to +\infty$.

(ii) The density ρ goes to ρ_∞ in the sense of (4.38) with $\rho_\infty = \rho_\infty(z)$. In other words, the "interface" tends in a weak sense (and up to an extraction in time) to one or several horizontal planes. Homogeneization may appear.

(iii) We are able to exhibit an infinity of steady solutions $(u = 0, \rho_\infty)$ whose energy is arbitrarily close to the minimal energy.

Remark 4.5 *While we do not know much about ρ_∞, it is worth mentionning that, in 2 dimensions, the topology (say number of bubbles to fix the ideas) is preserved by the flow (see [16], [17]).*

Remark 4.6 *In the spirit of Remark 4.2, we would like to indicate here that we believe that under the additional assumption that the data are small (at least initial velocity and difference of densities, but initial velocity is likely to be enough), some better regularity on the flow is available. It might also improve the quality of the convergence to zero of u (as it is well know in the one-fluid case, see Section 1.1. above). Such regularity issues will be investigated in [22]. Once more, we emphasize we have chosen to deal here with* any *initial velocity, and therefore to state the most general result we can prove with the weakest assumptions.*

4.4 The nonlinear case with surface tension

4.4.1 A priori estimates

First of all, notice that the transport equation yields the same properties (4.7) and (4.8) as in the case without surface tension.

$$\rho \in L^\infty(\Omega \times (0,\infty)). \tag{4.41}$$

Let us now establish the energy estimate, analogous to the estimate (4.14). It is obvious that multiplying the Navier-Stokes equation (4.4) by u and integrating over the domain lead to the following assertion

$$\frac{d}{dt}\Big(\frac{1}{2}\int_\Omega \rho u^2\,dx + \int_\Omega \rho z\,dx\Big) + \int_\Omega |\nabla u|^2\,dx = < \mathcal{T}, u > = \int_\Sigma \mathcal{C} u \cdot n,$$

where the curvature $\mathcal{C}$ is oriented along the unit normal n.

We recall that we assume for ease of notation that $\rho_2 - \rho_1 = 1$, thus $n\delta_\Sigma = \nabla\rho/|\nabla\rho| = \nabla\rho$. To compute the right-hand side, at least formally, we suppose that u is smooth enough in order to have $\partial_t\rho = 0$ on $\partial\Omega$. Thus

$$\int_\Sigma \mathcal{C} u{\cdot}n\,d\sigma = -\int_\Omega \mathrm{div}\,(\nabla\rho) u{\cdot}\nabla\rho\,dx = -\int_\Omega \partial_t \nabla\rho.\nabla\rho\,dx = -\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla\rho|^2\,dx.$$

Denoting by $L(\Sigma)$ the length of the interface Σ, we have formally $L(\Sigma) = \int_\Sigma d\sigma = \int_\Sigma |n|^2\,d\sigma = \int_\Omega |\nabla\rho|^2\,dx$. Thus the energy estimate reads

$$\frac{d}{dt}\Big(\frac{1}{2}\int_\Omega \rho u^2\,dx + \int_\Omega \rho z\,dx + \frac{1}{2}L(\Sigma)\Big) + \int_\Omega |\nabla u|^2\,dx = 0. \tag{4.42}$$

From this energy equality, it is straighforward to derive the same estimate as in the nonlinear case without surface tension, namely

$$u \in L^2(0,\infty;\mathbb{H}^1_0(\Omega)) \cap L^\infty(0,\infty;\mathbb{L}^2(\Omega)). \tag{4.43}$$

In addition we obtain here

$$\rho \in L^\infty(0,\infty;BV(\Omega)). \tag{4.44}$$

4.4.2 Questions of existence and regularity

Let us begin with a short overview of the state of the art concerning the existence of solutions in this setting with surface tension. For the case of one fluid with a free surface, local in time existence results can be found in G. Allain [1], V.A. Solonnikov [36], global in time existence results for small initial data and $\vec{f} \equiv \vec{0}$ appeared in V.A. Solonnikov [39] (bounded case), J.T. Beale [5] (unbounded case), and also in A. Tani & N. Tanaka [43], and for small initial data and f not necessarily zero in Tani [42]. For the two fluids case, local in time existence of strong solutions is due to I.V. Denisova [13] and I.V. Denisova, V.A. Solonnikov [14], global in time existence for small data is due to V.A. Solonnikov [37] and also N. Tanaka [41] (for a special initial condition, *see* above)).

As far as we know, no existence result of global weak solution has been established for the multifluids Navier-Stokes equations with surface tension. Thus, we need to assume in the sequel that there exist $(u(x,t), \rho(x,t))$ that are solutions to (4.4) in a formal sense, and that satisfy the *a priori* estimates (4.41), (4.43), (4.44). This regularity implies that $\rho \in \mathcal{C}(0,T;L^p(\Omega))$ for all $1 \leq p < \infty$. Such an assumption seems to us reasonable in view of the manipulations made above and in view of the regularity proved in the case without surface tension. Henceforth, we deal with a solution satisfying all these assumptions.

4.4.3 Longtime behaviour

As in Section 4.3, we define the sequences ρ_n and u_n by $\rho_n(x,t) = \rho(x, t+t_n)$ and $u_n(x,t) = u(x, t+t_n)$ when $(t_n)_{n\in\mathbb{N}}$ is an arbitrary sequence of positive reals such that $\lim_{n\to+\infty} t_n = +\infty$.

Behaviour of the velocity The behaviour of u_n is the same as in the case without surface tension, namely

$$u_n \longrightarrow 0 \text{ in } L^2(0,\infty;\mathbb{H}^1(\Omega)) \text{ as } n \to +\infty.$$

Behaviour of the interface In the sequel, $T > 0$ is fixed. We now show that the presence of surface tension allows us to improve the convergence (4.38) of ρ_n, more precisely we prove that this sequence is in a compact set of $L^p(\Omega \times (0,T))$ for any $p \geq 1$.

Estimates (4.41) and (4.44) show that ρ_n is in a bounded set of the space $L^\infty(0,T; BV(\Omega) \cap L^\infty(\Omega))$. Noticing that $L^\infty(\Omega) \hookrightarrow L^q(\Omega)$ for any $q \geq 1$ and the space of bounded measures $\mathcal{M}_b(\Omega) \hookrightarrow W^{-r,s'}(\Omega)$, with $s' = \frac{s}{s-1}$ for any r, s such that $rs > 3$, we deduce that $\forall\theta$, $0 \leq \theta \leq 1$ (*see* J. Bergh, J. Löfström [9] or J.-L. Lions, E. Magenes [29]),

$$L^\infty(\Omega) \cap BV(\Omega) \hookrightarrow [L^q(\Omega); W^{1-r,s'}(\Omega)]_\theta = W^{(1-r)\theta, \frac{1}{\theta/q + (1-\theta)/s'}}(\Omega).$$

For example, with $q = 6$, $r = 2/3$, $s = 6$ and $\theta = 1/2$ we have

$$L^\infty(\Omega) \cap BV(\Omega) \hookrightarrow H^{1/6}(\Omega).$$

Thus the sequence ρ_n is bounded in $L^2(0, T; H^{1/6}(\Omega)$.

Moreover, the equation $\partial_t \rho_n = -\text{div}\,(\rho_n u_n)$ together with estimates (4.43) and (4.41) show that $\partial_t \rho_n$ is bounded in $L^2(0, T; H^{-1}(\Omega))$.

Thus, ρ_n is bounded in $L^2(0, T; H^{1/6}(\Omega)) \cap H^1(0, T; H^{-1}(\Omega))$. Interpolating between these two spaces, we have for $0 \le \theta \le 1$

$$[L^2(0, T; H^{1/6}(\Omega)), H^1(0, T; H^{-1}(\Omega))]_\theta \hookrightarrow H^{1-\theta}(0, T; H^{7\theta/6-1}(\Omega)).$$

Choosing $6/7 < \theta < 1$, we deduce that ρ_n is bounded in $H^\beta(0, T; H^\gamma(\Omega))$ with $\beta > 0$ and $\gamma > 0$. Therefore, $(\rho_n)_{n\in\mathbb{N}}$ is a compact set of (for example) $L^1(\Omega \times (0, T))$. Since the sequence is bounded in $L^p(\Omega \times (0, T))$, $\forall p \ge 1$, we deduce that $(\rho_n)_{n\in\mathbb{N}}$ is a compact set of $L^p(\Omega \times (0, T))$. Therefore, there exists an extraction of $(\rho_n)_{n\in\mathbb{N}}$ such that

$$\rho_{n'} \longrightarrow \rho_\infty \text{ in } L^p(\Omega \times (0, T)), \forall p \ge 1 \text{ as } n' \to +\infty.$$

Then, we can prove by the arguments used in Section 4.3.4 that $\rho_\infty(x, t) = \rho_\infty(x)$ and the conservation of the global mass.

We next show that there exists a sequence $(s_n)_{n\in\mathbb{N}}$ such that $s_n \in [0, T]$ and $\lim_{n\to+\infty} \sup_{t\in[0,T]} ||\rho_n(\cdot, t + s_n) - \rho_\infty(\cdot)||_{L^p(\Omega)} = 0$, $\forall p \ge 1$

For ease of notation, we define $X_n(t) = ||\rho_n(t)||_{L^p(\Omega)}$ and $X_\infty = ||\rho_\infty||_{L^p(\Omega)}$. We recall that ρ is supposed to be in $C(0, \infty; L^p(\Omega))$, thus $X_n \in C(0, \infty)$. Moreover $X_n \to X_\infty$ as $n \to +\infty$ for the strong topology of $L^p(0, T)$. Thus, there exists a sequence $(s_n)_{n\in\mathbb{N}}$ in $[0, T]$ such that

$$\lim_{n\to+\infty} X_n(s_n) = X_\infty. \tag{4.45}$$

Then, we denote by $(\tilde{u}_n)_{n\in\mathbb{N}}$ and $(\tilde{\rho}_n)_{n\in\mathbb{N}}$ the sequences defined by $\tilde{u}_n(x, t) = u_n(x, t + s_n)$ and $\tilde{\rho}_n(x, t) = \rho_n(x, t + s_n)$. Assertion (4.45) proves the convergence of $\tilde{\rho}_n(\cdot, t = 0)$ to $\rho_\infty(\cdot)$ as $n \to +\infty$ for the strong topology of $L^p(\Omega)$.

Gathering the previous results, we have : $0 \le \tilde{\rho}_n \le C$, $\partial_t \tilde{\rho}_n + \text{div}\,\tilde{u}_n = 0$, $\text{div}\,(\tilde{\rho}\tilde{u}_n) = 0$, $\tilde{\rho}_n|_{t=0} \to \rho_\infty$ in $L^p(\Omega)$ and $\tilde{u}_n \to 0$ in $L^2(0, T; \mathbb{H}^1(\Omega))$. We deduce from these properties (*see* P.-L. Lions [30] Theorem 2.4) that $\tilde{\rho}_n$ converges to ρ_∞ in $\mathcal{C}([0, T], L^p(\Omega))$.

In other words, we have shown that, for $T > 0$, $p \ge 1$ and for any sequences $(t_n)_{n\in\mathbb{N}}$, $t_n \to +\infty$, there exists $(s_n)_{n\in\mathbb{N}}$, $s_n \in [0, T]$ such that, up to an extraction,

$$\lim_{n\to+\infty} \sup_{t\in[0,T]} ||\rho(\cdot, t + t_n + s_n) - \rho_\infty(\cdot)||_{L^q(\Omega)} = 0. \tag{4.46}$$

We finally show that no homogeneization appears. Indeed, for any $\beta \in \mathcal{C}^1([0,\infty),\mathbb{R})$ we have then

$$\int_\Omega \beta(\rho_n(x,t))\,dx \longrightarrow \int_\Omega \beta(\rho_\infty(x))\,dx,$$

thus by regularization we obtain that

$$\text{meas}\,\{x\in\Omega, \rho_\infty(x)=\rho_i\} = \text{meas}\,\{x\in\Omega, \rho(x,t)=\rho_i\} \tag{4.47}$$

which is a constant of the evolution.

We collect the results obtained in this nonlinear case with surface tension in the following final proposition.

Proposition 4
In the nonlinear case with surface tension, assuming the existence of a solution regular enough to give a sense to the surface tension term and satisfying the a priori *estimates (4.41), (4.43) and (4.44), the behaviour of u, ρ as time goes to infinity is the following :*

(i) The velocity field u goes to 0 in $\mathbb{H}^1(\Omega)$ in the same sense as in the case without surface tension (see *(4.34)).*

(ii) The density ρ goes to ρ_∞ in a stronger sense than in the case without surface tension (see *(4.46)). The density ρ_∞ consists only of zones of densities ρ_1 and ρ_2* (see *(4.47)), homogeneization being therefore excluded. In addition, ρ_∞ is such that the quantity $z - \text{div}\, n$ is constant on each connected component of the interface between zones of densities ρ_1 and ρ_2.*

(iii) We do not know whether the limit interface is unique nor connected. Moreover, we are able to exhibit an infinity of steady solutions $(u=0,\rho_\infty)$ whose energy is arbitrarily close to the minimal energy.

5 Final Remarks

We would like to emphasize that most of the above analysis in the purely gravitational case is likely to be extended *mutandis mutandis* to some Magnetohydrodynamics equations, provided the boundary conditions are convenient. The situation we have in mind is the following one : the right-hand side of the two-fluids Navier-Stokes equations contains a Lorentz force term $\text{curl}\, B\times B$ where the evolution of the magnetic field B follows an equation of parabolic type derived from the Maxwell system under convenient simplifying assumptions. In addition, the boundary conditions on the magnetic field are assumed to decay with time. The system under consideration is therefore

$$\begin{aligned} \partial_t\rho + \text{div}\,(\rho u) &= 0,\\ \partial_t(\rho u) + \text{div}\,(\rho u\otimes u) - \text{div}\,(2\eta d(u)) + \nabla p &= -\rho g\vec{e}_z + \text{curl}\, B\times B, \end{aligned}$$

$$\begin{aligned} \operatorname{div} u &= 0, \\ \partial_t B + \operatorname{curl}\left(\frac{1}{\sigma}\operatorname{curl} B\right) &= \operatorname{curl}(u \times B), \\ \operatorname{div} B &= 0, \end{aligned}$$

The well-posedness of this system has been established in [23] (global in time existence of weak solutions), and in [17] (regularity results in the bidimensional case under convenient assumptions on the viscosities and on the electrical conductivities).

We intend to present results on the long time behaviour of this system in a subsequent work (*see* [21]).

Acknowledgments :
The authors would like to thank P.-L. Lions for his help and advice, and B. Desjardins for suggesting many improvements in this work.

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Appendix : alternative proofs

We present in this Appendix two alternative proofs. One of these proofs leads to weaker results than the one presented above but we believe both of them might be useful in other contexts. To simplify the presentation, we only argue in a formal way.

A Convergence in H^1 of u in the linear case without surface tension

We indicate an alternative way to show (3.21) that makes use of another estimate instead of (3.10). Though the argument we shall present is a bit more intricate, we believe it might be useful in other estimate.

First, the consideration of the estimate (3.4) suffices to prove that we have

$$\lim_{t \longrightarrow +\infty} \|u\|_{L^2} = 0. \tag{A.1}$$

Indeed, let us go back to (3.3), and bound from above the right-hand side as follows :

$$\int |\nabla u|^2 + \frac{1}{2}\frac{d}{dt}\int \rho_0 u^2 = -\int_{z=0} \psi \partial_t \psi \leq \|\psi\|_{L^2(\{z=0\})} \|u\|_{L^2(\{z=0\})}.$$

We therefore obtain

$$\int |\nabla u|^2 + \frac{1}{2}\frac{d}{dt}\int \rho_0 u^2 \leq C_0 \|\psi\|_{L^2(\{z=0\})} \|\nabla u\|_{L^2},$$

which in view of (3.5) yields

$$\int |\nabla u|^2 + \frac{1}{2}\frac{d}{dt}\int \rho_0 u^2 \leq C_0 \sqrt{C_1} \|\nabla u\|_{L^2} \leq \frac{1}{2}\int |\nabla u|^2 + C_2,$$

It follows that there exists a constant C_3 such that

$$\frac{d}{dt}\int \rho_0 u^2 \leq C_3, \tag{A.2}$$

We claim that the two assertions (3.7) (A.2) imply (A.1). Indeed, the function $f(t) = \|u\|^2_{L^2}$ is a nonnegative function in $L^1(]0, +\infty[)$ such that its first derivative $f'(t)$ is uniformly bounded from above. Hence it converges to 0 as t goes to infinity (we refer the reader to [15] for a proof of this simple statement).

Next, we multiply the first equation of (3.1) by $\partial_t u$ and integrate over Ω :

$$\int \rho_0 (\partial_t u)^2 + \frac{1}{2}\frac{d}{dt}\int |\nabla u|^2 = -\int_{z=0} \psi \partial_t u \cdot e_z. \tag{A.3}$$

Derivating the second equation of (3.1) with respect to time, we have

$$\partial^2_{tt}\psi - \partial_t u \cdot e_z = 0, \tag{A.4}$$

and thus

$$\int \rho_0 (\partial_t u)^2 + \frac{1}{2}\frac{d}{dt}\int |\nabla u|^2 = -\int_{z=0} \psi \partial^2_{tt}\psi = -\frac{1}{2}\frac{d^2}{dt^2}\int_{z=0} \psi^2 + \int_{z=0} (\partial_t \psi)^2. \tag{A.5}$$

Since

$$\int_{z=0} (\partial_t \psi)^2 = \int_{z=0} (u \cdot e_z)^2,$$

we then obtain an alternative to second energy estimate :

$$\int \rho_0 (\partial_t u)^2 + \frac{1}{2} \frac{d}{dt} \Big(\int |\nabla u|^2 + \frac{d}{dt} \int_{z=0} \psi^2 \Big) = \int_{z=0} (u \cdot e_z)^2. \tag{A.6}$$

By standard trace theorem, the right-hand side of (A.6) is in $L^1(]0, +\infty[)$, thus if we integrate (A.6) from $t = 0$ to $t = T$, for any arbitrary time T, we obtain

$$\begin{aligned} \frac{1}{2} \Big(\int |\nabla u|^2 \quad &+ \quad \frac{d}{dt} \int_{z=0} \psi^2 \Big)(T) + \int_0^T \int \rho_0 (\partial_t u)^2 \\ &\leq \quad C_3 + \frac{1}{2} \Big(\int |\nabla u|^2 + \frac{d}{dt} \int_{z=0} \psi^2 \Big)(0) \\ &= \quad C_4 \end{aligned} \tag{A.7}$$

where the constants C_3 and C_4 do not depend on T. Since

$$\int_0^T \int \rho_0 (\partial_t u)^2 \geq 0,$$

it follows that

$$\int |\nabla u|^2 + \frac{d}{dt} \int_{z=0} \psi^2 \leq C_4, \tag{A.8}$$

at all time. Next, we remark as above that

$$\left| \frac{1}{2} \frac{d}{dt} \int_{z=0} \psi^2 \right| = \left| \int_{z=0} \psi \partial_t \psi \right| \leq C^{te} \|\psi\|_{L^2(\{z=0\})} \|\nabla u\|_{L^2} \leq C_5 \|\nabla u\|_{L^2}. \tag{A.9}$$

Hence, (A.8) yields

$$\|\nabla u\|_{L^2}^2 \leq C_4 + 2C_5 \|\nabla u\|_{L^2},$$

from where we infer

$$\|\nabla u\|_{L^2} \leq C_6, \tag{A.10}$$

for a constant C_6 independant of time. Inserting this estimate (A.10) into (A.9), we deduce

$$\left| \frac{1}{2} \frac{d}{dt} \int_{z=0} \psi^2 \right| \leq C_7. \tag{A.11}$$

Inserting this last estimate into (A.7), we obtain

$$\int_0^{+\infty} \|\partial_t u\|_{L^2}^2 < +\infty. \tag{A.12}$$

We now show that, as t goes to infinity, u goes to 0 in a stronger sense than the L^2 sense given by (A.1).

Using (3.4), we may write

$$\left| \frac{d}{dt} \int_{z=0} \psi^2 \right| \leq 2 \int |\nabla u|^2 + \left| \frac{d}{dt} \int \rho_0 u^2 \right|,$$

thus, by Cauchy-Schwarz inequality,

$$\left|\frac{d}{dt}\int_{z=0}\psi^2\right| \leq 2\int |\nabla u|^2 + \|\rho_0\|_{L^\infty}\left(\int u^2\right)^{1/2}\left(\int (\partial_t u)^2\right)^{1/2}.$$

Therefore

$$\frac{d}{dt}\int_{z=0}\psi^2 \in L^1(]0,+\infty[). \tag{A.13}$$

A straightforward consequence is that

$$\|\nabla u\|_{L^2}^2 + \frac{d}{dt}\int_{z=0}\psi^2 \in L^1(]0,+\infty[). \tag{A.14}$$

In addition, we infer from (A.6) that

$$\begin{aligned}\frac{1}{2}\left|\frac{d}{dt}\left(\int |\nabla u|^2 + \frac{d}{dt}\int_{z=0}\psi^2\right)\right| &= \left|\int_{z=0}(u\cdot e_z)^2 - \int \rho_0(\partial_t u)^2\right| \\ &\leq \int_{z=0}(u\cdot e_z)^2 + \int \rho_0(\partial_t u)^2 \\ &\leq C^{te}\|\nabla u\|_{L^2}^2 + C^{te}\|\partial_t u\|_{L^2}^2.\end{aligned}$$

Hence,

$$\frac{d}{dt}\left(\|\nabla u\|_{L^2}^2 + \frac{d}{dt}\int_{z=0}\psi^2\right) \in L^1(]0,+\infty[), \tag{A.15}$$

which, together with (A.13) and (3.7), yields

$$\|\nabla u\|_{L^2}^2 + \frac{d}{dt}\int_{z=0}\psi^2 \in W^{1,1}(]0,+\infty[). \tag{A.16}$$

As a consequence,

$$\lim_{t\longrightarrow+\infty}\|\nabla u\|_{L^2}^2 + \frac{d}{dt}\int_{z=0}\psi^2 = 0. \tag{A.17}$$

Next, we remark that by a standard trace result (*see* for instance H. Brezis [11]), we have

$$\|w\|_{L^2(\{z=0\})} \leq C^{te}\|w\|_{L^2(\Omega)}^{1/2}\|\nabla w\|_{L^2(\Omega)}^{1/2}, \tag{A.18}$$

for any arbitrary $w \in H^1(\Omega)$. Thus,

$$\begin{aligned}\left|\frac{1}{2}\frac{d}{dt}\int_{z=0}\psi^2\right| &= \left|\int_{z=0}\psi\partial_t\psi\right| \\ &\leq \|\psi\|_{L^2(\{z=0\})}\|\partial_t\psi\|_{L^2(\{z=0\})} \\ &= \|\psi\|_{L^2(\{z=0\})}\|u\cdot e_z\|_{L^2(\{z=0\})} \\ &\leq C^{te}\|\psi\|_{L^2(\{z=0\})}\|u\|_{L^2(\Omega)}^{1/2}\|\nabla u\|_{L^2(\Omega)}^{1/2},\end{aligned}$$

which, in view of (A.1)-(3.24)-(A.10), yields

$$\lim_{t\longrightarrow+\infty}\frac{d}{dt}\int_{z=0}\psi^2 = 0. \tag{A.19}$$

Therefore, we deduce from (A.17) that

$$\lim_{t\longrightarrow+\infty}\|\nabla u\|_{L^2} = 0. \tag{A.20}$$

B Convergence of u in L^2 in the linear case with surface tension

We establish here an estimate which enables us to show that

$$\lim_{t\to\infty} ||u||_{\mathbb{L}^2(\Omega)} = 0 \tag{B.1}$$

instead of (3.47).

This alternative way to proceed is less standard, that is why we present it. Nevertheless, as far as we know, it does not allow one to recover $u \longrightarrow 0$ in $\mathbb{H}^1(\Omega)$ but only in $\mathbb{L}^2(\Omega)$ which is weaker.

Let us introduce the field w in $\mathbb{H}^1_0(\Omega)$ solution to the following Stokes problem on Ω :

$$\begin{cases} -\Delta w = \rho_0 u - \nabla \pi, \\ \operatorname{div} w = 0, \end{cases} \tag{B.2}$$

We then multiply equation (3.34) by $\partial_t w$:

$$\int \rho_0 \partial_t u \cdot \partial_t w + \int \nabla u \cdot \nabla \partial_t w = \int_{z=0} (\psi - \Delta\psi)\partial_t w \cdot e_z. \tag{B.3}$$

We treat the two terms of the left-hand side as follows :

$$\int \rho_0 \partial_t u \cdot \partial_t w = -\int \Delta \partial_t w \cdot \partial_t w = \int |\nabla \partial_t w|^2. \tag{B.4}$$

$$\left| \int \nabla u \cdot \nabla \partial_t w \right| \leq \|\nabla u\|_{L^2} \|\nabla \partial_t w\|_{L^2}. \tag{B.5}$$

Besides,

$$\int_{z=0} (\psi - \Delta\psi)\partial_t w \cdot e_z \tag{B.6}$$

$$\begin{aligned} &= \frac{d}{dt}\left(\int_{z=0} \psi w \cdot e_z + \nabla\psi \cdot \nabla(w \cdot e_z)\right) + \int_{z=0} (u \cdot e_z - \Delta_{x,y} u \cdot e_z) w \cdot e_z \\ &\leq \frac{d}{dt}\left(\int_{z=0} \psi w \cdot e_z + \nabla\psi \cdot \nabla(w \cdot e_z)\right) + C^{st} \|u\|_{H^1(\Omega)} (\|w\|_{H^1(\Omega)} + \|w\|_{H^2(\Omega)}) \\ &\leq \frac{d}{dt}\left(\int_{z=0} \psi w \cdot e_z + \nabla\psi \cdot \nabla(w \cdot e_z)\right) + C^{st} \|u\|^2_{H^1(\Omega)}, \end{aligned} \tag{B.7}$$

using successively trace theorems and elliptic regularity on the Stokes system (B.2). Inserting these three estimates in (B.3), we obtain

$$\int |\nabla \partial_t w|^2 \leq \frac{d}{dt}\left(\int_{z=0} \psi w \cdot e_z + \nabla\psi \cdot \nabla(w \cdot e_z)\right) + \|\nabla u\|_{L^2} \|\nabla \partial_t w\|_{L^2} + C^{st} \|u\|^2_{H^1(\Omega)}$$

thus

$$\frac{1}{2} \int |\nabla \partial_t w|^2 \leq \frac{d}{dt}\left(\int_{z=0} \psi w \cdot e_z + \nabla\psi \cdot \nabla(w \cdot e_z)\right) + C^{st} \|u\|^2_{H^1(\Omega)},$$

which we integrate between 0 and T to obtain

$$\frac{1}{2} \int_0^T \|\nabla \partial_t w\|^2_{L^2} \leq C^{st} \|\psi\|_{H^1} \|u\|_{L^2(\Omega)} + C^{st} \int_0^T \|\nabla u\|^2_{L^2}, \tag{B.8}$$

using again elliptic regularity. In view of (3.7) and (3.37), this yields what will play henceforth the role of the second energy estimate (instead of the estimate (A.12) on $\partial_t u$ derivated from the second energy estimate (3.10) in the case without surface tension)

$$\int \|\partial_t w\|_{H^1}^2 < +\infty. \tag{B.9}$$

It is straightforward to see that this estimate may also be written

$$\int \|\rho_0 \partial_t u\|_{H^{-1}}^2 < +\infty. \tag{B.10}$$

By the same kind of argument as the one used to prove (3.47) and using

$$\left| \frac{d}{dt} \int_\Omega \rho_0 u \, dx \right| \leq ||\rho_0 \partial_t u||_{H^{-1}(\Omega)} ||u||_{\mathbb{H}^1(\Omega)},$$

equation (B.10) yields (B.1).