Mathematical study of a coupled system arising in Magnetohydrodynamics

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1 Introduction

This work deals with the mathematical study of a system of partial differential equations related to a magnetohydrodynamic (MHD) problem. The MHD equations we consider modelize the behaviour of an homogeneous incompressible conducting viscous fluid subjected to a Lorentz force due to the presence of a magnetic field. More precisely, we study a coupling between the *transient* Navier-Stokes equations and the *stationary* Maxwell equations. This model can be considered for example in industrial situations when the magnetic phenomena are known to reach their steady state "infinitely" faster than the hydrodynamics phenomena.

Many mathematical works have been devoted to the study of MHD problems. We only present here some of them briefly and we refer to J.-F. Gerbeau, C. Le Bris [5] and A.J. Meir, P.G. Schmidt [9] for some more detailed overviews.

The coupling between the transient Navier-Stokes equations and the transient Maxwell equations (without displacement current) has been studied in G. Duvaut, J.-L. Lions [3] and in M. Sermange, R. Temam [11]. Numerical methods conserving the dissipative properties of the continuum system in 2D are presented in F. Armero, J.C. Simo [1]. Less numerous works have been devoted to the fully stationnary MHD equations, namely a coupling between two elliptic partial differential equations (see for example M.D. Gunzburger, A.J. Meir, J.S. Peterson [6], J.-M. Domingez de la Rasilla [2]). Finally, let us mention an interesting alternative viewpoint which consists in considering the electrical current rather than the magnetic field as the main electromagnetic unknown (see A.J. Meir, P.G. Schmidt [8, 9]).

In the present work, the equations related to the velocity field are the transient Navier-Stokes equations whereas those related to the magnetic field are elliptic (see (2.1)-(2.8)). The difficulty is that the ellipticity of the equation on B depends on the velocity field u. Briefly speaking, if the velocity becomes too large, the system may become ill-posed.

Under restrictive assumptions upon the physical data, we can however prove that a strong solution exists and is unique at least on a time interval $[0, T^*]$ for some time T^* depending on the data (see Section 4, Theorem 1). For this purpose, we give in Section 2 a presentation of the equations and the functional spaces, and we establish in Section 3 some preliminary existence and regularity results upon the magnetic equation.

As soon as the magnetic operator is no longer invertible – which may occur if the velocity becomes too large – we show in Section 5 that we can construct two distincts solutions to the system.

This latter observation shows that the model we study here should be used only with great care in numerical simulations.

2 Equations and function spaces

2.1 The transient/stationary model

Let Ω be a simply-connected, fixed bounded domain in $\mathbb{R}^3$ enclosed in a $\mathcal{C}^\infty$ boundary Γ. We shall denote by n the outward-pointing normal to Ω. The transient/stationary problem we shall consider is the following : find two vector-valued functions, the velocity u and the magnetic field b, and a scalar function p, defined on $\Omega \times [0, T]$, such that

$$\partial_t u + u.\nabla u - \eta \triangle u = f - \nabla p + \operatorname{curl} b \times b \qquad \text{in } \Omega, \tag{2.1}$$

$$\operatorname{div} u = 0 \qquad \text{in } \Omega, \tag{2.2}$$

$$\frac{1}{\sigma} \operatorname{curl} (\operatorname{curl} b) = \operatorname{curl} (u \times b) \qquad \text{in } \Omega, \tag{2.3}$$

$$\operatorname{div} b = 0 \qquad \text{in } \Omega, \tag{2.4}$$

with the following initial and boundary conditions :

$$u = 0 \qquad \text{on } \Gamma, \tag{2.5}$$

$$b.n = q \qquad \text{on } \Gamma, \tag{2.6}$$

$$\operatorname{curl} b \times n = k \times n \qquad \text{on } \Gamma, \tag{2.7}$$

$$u|_{t=0} = u_0 \qquad \text{in } \Omega. \tag{2.8}$$

2.2 Functional setting

For $m \geq 0$, we denote as usual by $H^m(\Omega)$ the Sobolev space

$$H^m(\Omega) = \{u \in L^2(\Omega); D^\gamma u \in L^2(\Omega), \forall \gamma, |\gamma| \leq m\},$$

where $\gamma = (\gamma_1, \gamma_2, \gamma_3)$ is a multi-index and $|\gamma| = \gamma_1 + \gamma_2 + \gamma_3$. The norm associated with $H^m(\Omega)$ that we will use is :

$$||u||_{H^m(\Omega)} = \left(\sum_{|\gamma|=0}^{m} ||D^\gamma u||^2_{L^2(\Omega)} \right)^{1/2}.$$

The subspace of $H^1(\Omega)$ consisting of functions vanishing on $\partial\Omega$ is denoted as usual by $H_0^1(\Omega)$.

We shall denote respectively $(L^p(\Omega))^3$ and $(H^m(\Omega))^3$ by $\mathbb{L}^p(\Omega)$ and $\mathbb{H}^m(\Omega)$ or, when there is no ambiguity, by $\mathbb{L}^p$ and $\mathbb{H}^m$.

We shall use the Sobolev inequality : for $2 \leq p \leq 6$,

$$||f||_{\mathbb{L}^p(\Omega)} \leq c_0 ||f||_{\mathbb{H}^1(\Omega)}. \tag{2.9}$$

Let $T > 0$ and let X be a Banach space. The space $L^p(0, T; X)$, $1 \leq p \leq \infty$ is the space of classes of L^p functions from $[0, T]$ into X. We recall that this is a Banach space for the norm

$$\left(\int_0^T ||u(t)||_X^p \, dt \right)^{1/p} \text{ if } 1 \leq p < \infty, \qquad \operatorname*{ess\,sup}_{t \in [0,T]} ||u(t)||_X \text{ if } p = \infty.$$

The following trace spaces will also be needed :

$$H^{1/2}(\Gamma) = \{v|_\Gamma, v \in H^1(\Omega)\},$$

$$\mathbb{H}^{1/2}(\Gamma) = \{v|_\Gamma, v_i \in H^{1/2}(\Gamma), i = 1, .., 3\},$$

$$\mathbb{H}^{-1/2}(\Gamma) = (\mathbb{H}^{1/2}(\Gamma))'.$$

They are equipped with the norms

$$||q||_{H^{1/2}(\Gamma)} = \inf_{w \in H^1(\Omega), w|_\Gamma = q} ||w||_{H^1(\Omega)},$$

$$||g||_{\mathbb{H}^{1/2}(\Gamma)} = \inf_{w \in \mathbb{H}^1(\Omega), w|_\Gamma = g} ||w||_{\mathbb{H}^1},$$

$$||k||_{\mathbb{H}^{-1/2}(\Gamma)} = \sup_{g\in\mathbb{H}^{1/2}(\Gamma), g\neq 0} \frac{< k, g >}{||g||_{\mathbb{H}^{1/2}(\Gamma)}}.$$

We denote by $\mathcal{C}_c^\infty(\Omega)$ (resp. $\mathcal{C}_c^\infty(\overline{\Omega})$) the space of real functions infinitely differentiable with compact support in Ω (resp. $\overline{\Omega}$). We introduce the spaces

$$\mathcal{V} = \{v \in (\mathcal{C}_c^\infty(\Omega))^3, \operatorname{div} v = 0\},$$

$$V = \{v \in \mathbb{H}_0^1(\Omega), \operatorname{div} v = 0\},$$

$$\mathcal{W} = \{C \in (\mathcal{C}_c^\infty(\overline{\Omega}))^3, \operatorname{div} C = 0, C.n|_{\partial\Omega} = 0\},$$

$$W = \{C \in \mathbb{H}^1(\Omega), \operatorname{div} C = 0, C.n|_{\partial\Omega} = 0\},$$

$$H = \{v \in \mathbb{L}^2(\Omega), \operatorname{div} v = 0, v.n|_{\partial\Omega} = 0\}.$$

The space V (resp. W) is the closure of $\mathcal{V}$ (resp. $\mathcal{W}$) in $\mathbb{H}_0^1(\Omega)$ (resp. $\mathbb{H}^1(\Omega)$). H is the closure of $\mathcal{V}$ (and $\mathcal{W}$) in $\mathbb{L}^2(\Omega)$. Let us recall that $u.n$ makes sense in $H^{-1/2}(\partial\Omega)$ as soon as $u \in \mathbb{L}^2(\Omega)$ satisfies $\operatorname{div} u = 0$. For $v \in V$ and $C \in W$ we denote

$$||v||_V = \left(\int_\Omega |\nabla v|^2 \, dx\right)^{1/2},$$

$$||C||_W = \left(\int_\Omega |\operatorname{curl} C|^2 \, dx\right)^{1/2}.$$

One can establish that $||.||_V$ (resp. $||.||_W$) defines a norm (resp. W) which is equivalent to that induced by $\mathbb{H}^1(\Omega)$ on V (resp. W) (cf. G. Duvaut and J.-L. Lions [4]). Thus we have for $B \in W$:

$$||B||_{\mathbb{H}^1(\Omega)} \leq d_1 ||B||_W.$$

For $2 \leq p \leq 6$, this inequality together with the Sobolev imbedding (2.9) imply that, for $B \in W$

$$||B||_{\mathbb{L}^p(\Omega)} \leq d_2 ||B||_W.$$

As well, Poincaré inequality and (2.9) imply that, for $u \in V$

$$||u||_{\mathbb{L}^p(\Omega)} \leq d_3 ||u||_V.$$

2.3 Regularity of the data

We shall suppose in the sequel that

$$u_0 \in \mathbb{H}_0^1(\Omega) \cap \mathbb{H}^2(\Omega), \text{ with } \operatorname{div} u_0 = 0, \tag{2.10}$$

$$q \in \mathcal{C}(0,T; H^{3/2}(\Gamma)), \tag{2.11}$$

$$k \in \mathcal{C}(0,T; \mathbb{H}^{1/2}(\Gamma)), \tag{2.12}$$

$$f \in L^\infty(0,T; \mathbb{L}^2(\Omega)). \tag{2.13}$$

From a physical viewpoint, it is natural to assume that k is the trace on Γ of the gradient of the electrical potential :

$$k = \sigma \nabla \phi|_\Gamma. \tag{2.14}$$

3 Preliminary results

First of all, we notice that we can split the magnetic field $b(t) \in \mathbb{H}^1(\Omega)$ satisfying (2.6) and (4.14) into the sum of a function $B^d(t)$ that satisfies (2.6) and a function $B(t) \in W$. Indeed, we have :

Lemma 3.1 *Let $q \in \mathcal{C}(0,T; H^{k-1/2}(\Omega))$ for $k = 1$ or $k = 2$, there exist $B^d \in \mathcal{C}(0,T; \mathbb{H}^k(\Omega))$ and a constant d_4 such that*

$$B^d.n = q \quad \text{on } [0,T] \times \Gamma \text{ and } \quad ||B^d||_{\mathcal{C}(0,T;\mathbb{H}^k(\Omega))} \leq d_4 ||q||_{\mathcal{C}(0,T;H^{k-1/2}(\Omega))}.$$

Moreover, we can impose that

$$div\, B^d(t) = 0 \qquad and \qquad curl\, B^d(t) = 0 \qquad for\ t \in [0,T]. \diamondsuit$$

Proof. It suffices to define B^d as follows : $B^d(t) = \nabla\phi(t)$ where $\phi(t)$ is a solution of the Neumann problem

$$\begin{cases} -\triangle\, \phi &= 0 \qquad \text{in } \Omega \\ \dfrac{\partial \phi}{\partial n} &= q(t) \qquad \text{on } \Gamma. \diamondsuit \end{cases}$$

Let $B(t) = b(t) - B^d(t)$. We replace the original problem (2.1)-(2.8) with the following one :

$$\begin{aligned}
\partial_t u + u.\nabla u - \eta \triangle u &= f - \nabla p + \mathrm{curl}\, B \times B + \mathrm{curl}\, B \times B^d \text{ in } \Omega & (3.1)\\
\mathrm{div}\, u &= 0 \qquad \text{in } \Omega, & (3.2)\\
\frac{1}{\sigma}\mathrm{curl}\,(\mathrm{curl}\, B) &= \mathrm{curl}\,(u \times B) + \mathrm{curl}\,(u \times B^d) \qquad \text{in } \Omega, & (3.3)\\
\mathrm{div}\, B &= 0 \qquad \text{in } \Omega, & (3.4)
\end{aligned}$$

with the following initial and boundary conditions :

$$\begin{aligned}
u &= 0 \qquad \text{on } \Gamma, & (3.5)\\
B.n &= 0 \qquad \text{on } \Gamma, & (3.6)\\
\mathrm{curl}\, B \times n &= k \times n \qquad \text{on } \Gamma, & (3.7)\\
u|_{t=0} &= u_0 \qquad \text{in } \Omega, & (3.8)
\end{aligned}$$

We now proceed to establish a preliminary existence and uniqueness result for the magnetic problem and two estimates which will be needed in the next section.

We define the convex set

$$\begin{aligned}
\mathcal{K} = \{v \in L^2(0,T;V), \quad & \sup_{t\in[0,T]} ||v(t)||_V \le M,\\
& ||v||_{L^2(0,T;\mathbb{H}^2(\Omega))} \le M,\\
& ||\partial_t v||_{L^2(0,T;\mathbb{L}^2(\Omega))} \le M\}.
\end{aligned}$$

The values of the constant M and T will be fixed later. We only suppose here that

$$M < \frac{1}{d_2 d_3 \sigma}. \tag{3.9}$$

Let us note that $v \in \mathcal{K}$ implies $v \in \mathcal{C}(0,T;V)$. For $v \in \mathcal{K}$, we consider the following problem : find $B \in \mathcal{C}(0,T;W)$

$$\frac{1}{\sigma}\mathrm{curl}\,(\mathrm{curl}\, B) = \mathrm{curl}\,(v \times B) + \mathrm{curl}\,(v \times B^d) \qquad \text{in } \Omega \times [0,T], \tag{3.10}$$

$$\mathrm{div}\, B = 0 \qquad \text{in } \Omega \times [O,T], \tag{3.11}$$

with the following boundary conditions :

$$B.n = 0 \qquad \text{on } \Gamma \times [0,T], \tag{3.12}$$

$$\mathrm{curl}\, B \times n = k \times n \qquad \text{on } \Gamma \times [0,T]. \tag{3.13}$$

Proposition 1
For $v \in \mathcal{K}$ and M satisfying hypothesis (3.9), the problem (3.10)-(3.13) has a unique solution $B \in \mathcal{C}(0,T;W)$. Moreover, we have the following estimate :

$$\sup_{t\in[0,T]} ||B(t)||_W \leq \frac{\alpha_1 + \beta_1 ||v||_{L^\infty(0,T;V)}}{1-\gamma_1 ||v||_{L^\infty(0,T;V)}}, \tag{3.14}$$

where α_1, β_1 and γ_1 are some constants defined below. $\diamondsuit$

Proof.
• **Existence and uniqueness.** We define on $W \times W$ the bilinear form

$$a_v(C_1, C_2) = \frac{1}{\sigma}\int_\Omega \text{curl}\, C_1.\text{curl}\, C_2\, dx - \int_\Omega (v(t)\times C_1).\text{curl}\, C_2\, dx$$

and $h_v(t) \in W'$ such that, for $C \in W$,

$$< h_v(t), C >= \int_\Omega v(t)\times B^d(t).\text{curl}\, C\, dx + < k(t)\times n, C >_\Gamma$$

First, let us prove that problem (3.10)-(3.13) is equivalent to find $B \in \mathcal{C}(0,T;W)$ such that

$$a_v(B(t), C) = < h_v(t), C > \tag{3.15}$$

for all $C \in W$.

Let $B \in \mathcal{C}(0,T;W)$ which satisfies (3.15). Integrating by part, we have :

$$\int_\Omega \frac{1}{\sigma}\text{curl}\,(\text{curl}\, B) - \text{curl}\,(v(t)\times(B+B^d)).C\, dx = < \frac{1}{\sigma}(k\times n - \text{curl}\, B\times n), C >_\Gamma$$

for all $C \in W$. First we deduce that

$$< \text{curl}\, B\times n, C >_\Gamma = < k(t)\times n, C >_\Gamma$$

which yields (3.13).

Moreover, since $\text{div}\, C = 0$, there exists p such that

$$\frac{1}{\sigma}\text{curl}\,(\text{curl}\, B) - \text{curl}\,(v(t)\times(B+B^d)) = \nabla p.$$

The function p satisfies :

$$\left\{\begin{array}{rcl} -\triangle\, p &=& 0 \text{ in } \Omega \\ \dfrac{\partial p}{\partial n} &=& \dfrac{1}{\sigma}\text{curl}\,(\text{curl}\, B).n - \text{curl}\,(v(t)\times(B+B^d)).n \end{array}\right.$$

It is straightforward to check that the normal component of curl $(v\times(B+B^d))$ contains only tangential derivatives of v. Thus, using $v|_\Gamma = 0$, curl $(v\times(B+B^d)).n = 0$.

Moreover, $(\mathrm{curl}\,\mathrm{curl}\,B).n = -\partial_{t_1}((\mathrm{curl}\,B\times n).t_2) - \partial_{t_2}((\mathrm{curl}\,B\times n).t_1)$, where ∂_{t_1} and ∂_{t_2} denote the tangential derivatives. Then, hypothesis (2.14) yields $(\mathrm{curl}\,\mathrm{curl}\,B).n = 0$.

Therefore $\dfrac{\partial p}{\partial n} = 0$, which proves $p = C^{st}$ and (3.10).

Conversely, we easily check that a solution of (3.10)- (3.13) satisfies (3.15).

Moreover, $a_v(.,.)$ is continuous and coercive on $W\times W$. Indeed :

$$\begin{aligned}|a_v(C_1,C_2)| &\leq \frac{1}{\sigma}||\mathrm{curl}\,C_1||_{\mathbb{L}^2(\Omega)}||\mathrm{curl}\,C_2||_{\mathbb{L}^2(\Omega)}\\ &\quad+||v(t)||_{\mathbb{L}^6(\Omega)}||C_1||_{\mathbb{L}^3(\Omega)}||\mathrm{curl}\,C_2||_{\mathbb{L}^2(\Omega)}\\ &\leq (\frac{1}{\sigma}+d_2d_3M)||C_1||_W||C_2||_W\end{aligned}$$

and

$$\begin{aligned}|a_v(C,C)| &\geq \frac{1}{\sigma}||\mathrm{curl}\,C||^2_{\mathbb{L}^2} - ||v||_{\mathbb{L}^6}||C||_{\mathbb{L}^3}||\mathrm{curl}\,C||_{\mathbb{L}^2}\\ &\geq (\frac{1}{\sigma}-d_2d_3||v||_V)||C||^2_W\\ &\geq (\frac{1}{\sigma}-d_2d_3M)||C||^2_W\end{aligned}$$

Therefore, the Lax-Milgram Theorem implies that the variational problem (3.15) has a unique solution $B(t)\in W$.

The continuity in time of B^d and v implies that $B\in\mathcal{C}(0,T;W)$.

- **Estimate in $L^\infty(0,T;W)$.** Taking $C = B(t)$ in (3.15), we have

$$\begin{aligned}\frac{1}{\sigma}\int_\Omega |\mathrm{curl}\,B|^2\,dx = &\int_\Omega v\times(B+B^d).\mathrm{curl}\,B\,dx + \frac{1}{\sigma} < k\times n, B > .\\ \leq\ & ||v||_{\mathbb{L}^6}\left(||B||_{\mathbb{L}^3}+||B^d||_{\mathbb{L}^3}\right)||B||_W + \frac{1}{\sigma}||k||_{\mathbb{H}^{-1/2}}||B||_{\mathbb{H}^{1/2}}.\end{aligned}$$

Thus

$$||B||_W \leq d_2d_3\sigma||v||_V||B||_W + c_1\sigma||v||_V||q||_{H^{1/2}} + d_1||k||_{\mathbb{H}^{-1/2}}.$$

We deduce the estimate :

$$\sup_{t\in[0,T]}||B(t)||_W \leq \frac{c_1\sigma||q||_{L^\infty(0,T;H^{1/2})}||v||_{L^\infty(0,T;V)} + d_1||k||_{L^\infty(0,T;\mathbb{H}^{-1/2})}}{(1-d_2d_3\sigma||v||_{L^\infty(0,T;V)})}.$$

For simplicity, we introduce the constants

$$\alpha_1 = d_1 ||k||_{L^\infty(0,T;\mathbb{H}^{-1/2})},$$

$$\beta_1 = c_1 \sigma ||q||_{L^\infty(0,T;H^{1/2})},$$

$$\gamma_1 = d_2 d_3 \sigma,$$

which gives (3.14). $\diamondsuit$

In the next section, the vector field B defined above will appear on the right hand side of the Navier-Stokes equation in the Lorentz force curl $B \times B$. We see that we need an estimate on u in $L^\infty(0,T;\mathbb{H}^1(\Omega))$ in order to prove the coercivity of problem (3.10)-(3.13). Such a control on u is typically obtained with strong solutions of Navier-Stokes equations. To define strong solutions, the force term in Navier-Stokes equations has to belong to $L^\infty(0,T;\mathbb{L}^2(\Omega))$ (see R. Temam [12]). In this scope, the estimate on B in $L^\infty(0,T;W)$ is not sufficient. That is why we establish now a "better" estimate on B. First, we need the following proposition which is a straightforward extension (in the non homogeneous case) of Proposition 2.1 of Saramito [10] (see also Lemma 2.1 and Remark 2.3 of [10]).

Proposition 2
Let m be a nonegative integer and $1 < p < \infty$. Let $g \in \mathbb{W}^{m,p}(\Omega)$, with $div\, g = 0$ and $g.n = 0$ on Γ, $k \in \mathbb{W}^{m+1-1/p,p}(\Gamma)$, $q \in \mathbb{W}^{m+2-1/p,p}(\Gamma)$.

Then, there exists a unique $B \in \mathbb{W}^{m+2,p}(\Omega)$ such that

$$\left\{ \begin{array}{rcll} curl\,(curl\, B) & = & g & in\ \Omega, \\ div\, B & = & 0 & in\ \Omega, \\ B.n & = & q & on\ \Gamma, \\ curl\, B \times n & = & k \times n & on\ \Gamma, \end{array} \right.$$

and

$$||B||_{\mathbb{W}^{m+2,p}(\Omega)} \leq c_2 (||g||_{\mathbb{W}^{m,p}(\Omega)} + ||k||_{\mathbb{W}^{m+1-1/p,p}(\Gamma)} + ||q||_{W^{m+2-1/p,p}(\Gamma)}). \diamondsuit$$

Proposition 3
Under hypothesis (3.9), the solution of problem (3.10)-(3.13) given by Proposition 1 satisfies

$$||B||_{L^\infty(0,T;\mathbb{W}^{1,3}(\Omega))} \leq \alpha_2 + \gamma_2 ||v||_{L^\infty(0,T;V)} \frac{\alpha_1 + \beta_1 ||v||_{L^\infty(0,T;V)}}{1 - \gamma_1 ||v||_{L^\infty(0,T;V)}} + \beta_2 ||v||_{L^\infty(0,T;V)} \tag{3.16}$$

where α_2, β_2 and γ_2 are some constants defined below. $\diamondsuit$

Proof.
Let g be defined by

$$g = \sigma \mathrm{curl}\,(v \times (B + B^d)) = \sigma(B.\nabla v - v.\nabla B + B^d.\nabla v - v.\nabla B^d).$$

We have $\mathrm{div}\, g = 0$, $g.n = 0$ on Γ (because $v = 0$ on Γ and the normal component of $\mathrm{curl}\,(v \times B)$ contains only tangential derivatives of v, as said above). Moreover :

$$\begin{aligned} ||g||_{\mathbb{L}^{3/2}} &\leq \sigma \left(||B||_{\mathbb{L}^6} + ||B^d||_{\mathbb{L}^6} \right) ||\nabla v||_{\mathbb{L}^2} + \sigma ||v||_{\mathbb{L}^6} \left(||\nabla B||_{\mathbb{L}^2} + ||\nabla B^d||_{\mathbb{L}^2} \right) \\ &\leq \sigma ||v||_V (d_2 ||B||_W + d_1 ||B^d||_{\mathbb{H}^1(\Omega)}) + \sigma d_3 ||v||_V (d_1 ||B||_W + ||B^d||_{\mathbb{H}^1(\Omega)}) \\ &\leq \sigma c_3 ||v||_V (||B||_W + ||q||_{H^{1/2}(\Gamma)}) \end{aligned}$$

Thus, Proposition 2 with $m = 0, p = 3/2$ yields

$$\begin{aligned} ||B||_{\mathbb{W}^{2,3/2}(\Omega)} &\leq c_2 (||g||_{\mathbb{L}^{3/2}} + ||k||_{\mathbb{W}^{1/3,3/2}(\Gamma)} + ||q||_{W^{4/3,3/2}(\Gamma)}) \\ &\leq c_2 \left[\sigma c_3 ||v||_V (||B||_W + ||q||_{H^{1/2}(\Gamma)}) \right. \\ &\qquad \left. + ||k||_{\mathbb{W}^{1/3,3/2}(\Gamma)} + ||q||_{W^{4/3,3/2}(\Gamma)} \right] \end{aligned}$$

We deduce :

$$\begin{aligned} \sup_{t \in [0,T]} ||B||_{\mathbb{W}^{2,3/2}(\Omega)} \leq c_2 ||k||_{L^\infty(0,T;W^{1/3,3/2})} + c_4 \sigma ||v||_{L^\infty(0,T;V)} (\sup_{t \in [0,T]} ||B||_W + \\ + ||q||_{L^\infty(0,T;H^{1/2})}) + c_2 ||q||_{L^\infty(0,T;W^{4/3,3/2})}. \end{aligned}$$

Finally, we use (3.14) and the Sobolev inequality

$$||f||_{\mathbb{W}^{1,3}(\Omega)} \leq d_5 ||f||_{\mathbb{W}^{2,3/2}(\Omega)},$$

and we introduce some constants for ease of notation :

$$\begin{aligned} \alpha_2 &= c_2 d_5 (||q||_{L^\infty(0,T;W^{4/3,3/2})} + ||k||_{L^\infty(0,T;W^{1/3,3/2})}), \\ \beta_2 &= c_4 d_5 \sigma ||q||_{L^\infty(0,T;H^{1/2})}, \\ \gamma_2 &= c_4 d_5 \sigma ||q||_{L^\infty(0,T;H^{1/2})}, \end{aligned}$$

which gives (3.16). $\diamondsuit$

4 An existence and uniqueness result for small data

Let $M > 0$, we define

$$\Theta(M) = \alpha_0 + c_5 \left(\alpha_2 + \gamma_2 M \frac{\alpha_1 + \beta_1 M}{1 - \gamma_1 M} + \beta_2 M\right)\left(1 + \frac{\alpha_1 + \beta_1 M}{1 - \gamma_1 M}\right),$$

where $\alpha_0 = ||f||_{L^\infty(0,T;\mathbb{L}^2(\Omega))}$ and the constants α_i, β_i and γ_i are defined in the previous section. We also define the functions μ_1, μ_2 and μ_3 by

$$\mu_1(M)^2 = 4 \max\left(||u_0||_V^2, \frac{2}{c_7 \eta^2}\Theta(M)^2\right), \tag{4.1}$$

$$\mu_2(M)^2 = \frac{c_8}{\eta}\left(||u_0||_V^2 + \frac{2T}{\eta}\Theta(M)^2 + \mu_1(M)^3\right), \tag{4.2}$$

$$\mu_3(M) = \alpha_0 + c_9 \mu_2(M) + c_{10}\mu_1(M)\mu_2(M). \tag{4.3}$$

The constants c_5,...,c_{10} appear in the following proof and do not depend on the physical data.

Theorem 1
As soon as the physical data u_0, $1/\eta$, σ, f, q, k, are "small enough" (in a sense made precise below), there exists a time $T^* > 0$ such that the MHD problem (2.1)-(2.8) has a unique solution on $[0, T^*]$. This solution satisfies $u \in L^2(0, T^*; \mathbb{H}^2(\Omega)) \cap L^\infty(0, T^*; \mathbb{H}_0^1(\Omega))$ and $b \in \mathcal{C}(0, T^*; \mathbb{H}^1(\Omega)) \cap L^\infty(0, T^*; \mathbb{H}^2(\Omega))$.

Proof.
• **Existence.**
In the previous Theorem, "small enough" means that the data are such that the following property holds :

$$\text{There exists } 0 < M < 1/\gamma_1 \text{ such that } \mu_i(M) \leq M, i = 1, 2, 3. \tag{4.4}$$

Note that it is indeed possible to choose the physical data such that (4.4) is satisfied : a straightforward calculus shows that

$$\Theta'(0) = c_5(1 + \alpha_1)(\alpha_1\gamma_2 + \beta_2) + c_5\alpha_2(\alpha_1\gamma_1 + \beta_1),$$

thus, q, k and σ can be set small enough such that $0 < \Theta'(0) < 1$ and therefore one can choose $M > 0$ small enough such that $\Theta(M) < M$. In view of definitions (4.1)-(4.3) of μ_i, $i = 1, 2, 3$, it is a simple matter to check by an analogous calculus that (4.4) holds as soon as u_0, f, $1/\eta$ are small enough too.

We define the time T^* by $T^* = \min(T, 3/(4c_6\mu_1^2(M)))$, we choose $M > 0$ such that (4.4) holds and we define $\mathcal{K}$ by

$$\begin{aligned}\mathcal{K} = \{v \in L^2(0,T^*;V), \quad & \sup\nolimits_{t\in[0,T^*]} ||v(t)||_V \leq M,\\ & ||v||_{L^2(0,T^*;\mathbb{H}^2(\Omega))} \leq M,\\ & ||\partial_t v||_{L^2(0,T^*;\mathbb{L}^2(\Omega))} \leq M\}.\end{aligned}$$

The set $\mathcal{K}$ is clearly convex. Moreover, in view of a classical compactness result (see for instance R. Temam [12], Theorem 2.1), $\mathcal{K}$ is a compact set of the Banach space $L^2(0,T^*;V)$. For $\overline{u} \in \mathcal{K}$, we use Proposition 1 to define B as the unique solution of

$$\left\{\begin{array}{rcl} \dfrac{1}{\sigma}\mathrm{curl}\,(\mathrm{curl}\,B) & = & \mathrm{curl}\,(\overline{u}\times B) + \mathrm{curl}\,(\overline{u}\times B^d) \qquad \text{in } \Omega,\\ \mathrm{div}\,B & = & 0 \qquad \text{in } \Omega,\\ B.n & = & 0 \qquad \text{on } \Gamma,\\ \mathrm{curl}\,B\times n & = & k \qquad \text{on } \Gamma. \end{array}\right. \tag{4.5}$$

According to the estimates (3.14) and (3.16), we have

$$\begin{aligned} ||\mathrm{curl}\,B\times B||_{L^\infty(0,T^*;\mathbb{L}^2)} & \leq ||\mathrm{curl}\,B||_{L^\infty(0,T^*;\mathbb{L}^3(\Omega))}||B||_{L^\infty(0,T^*;\mathbb{L}^6(\Omega))}\\ & \leq c_5||B||_{L^\infty(0,T^*;\mathbb{W}^{1,3})}||B||_{L^\infty(0,T^*;W)}\\ & \leq c_5\left(\alpha_2 + \gamma_2 M\frac{\alpha_1+\beta_1 M}{1-\gamma_1 M} + \beta_2 M\right)\left(\frac{\alpha_1+\beta_1 M}{1-\gamma_1 M}\right), \end{aligned}$$

and

$$\begin{aligned} ||\mathrm{curl}\,B\times B^d||_{L^\infty(0,T^*;\mathbb{L}^2)} & \leq ||\mathrm{curl}\,B||_{L^\infty(0,T^*;\mathbb{L}^3(\Omega))}||B^d||_{L^\infty(0,T^*;\mathbb{L}^6(\Omega))}\\ & \leq c_5||B||_{L^\infty(0,T^*;\mathbb{W}^{1,3})}||B^d||_{L^\infty(0,T^*;\mathbb{H}^1)}\\ & \leq c_5\left(\alpha_2 + \gamma_2 M\frac{\alpha_1+\beta_1 M}{1-\gamma_1 M} + \beta_2 M\right). \end{aligned}$$

Therefore, the force term $F = f + (\mathrm{curl}\,B)\times(B+B^d)$ is in $L^\infty(0,T^*;\mathbb{L}^2(\Omega))$ and

$$\sup_{t\in[0,T]} ||F(t)||_{\mathbb{L}^2(\Omega)} \quad \leq \quad ||f||_{L^\infty(0,T^*;\mathbb{L}^2(\Omega))}$$

$$
\begin{aligned}
& +c_5\left(\alpha_2+\gamma_2 M\frac{\alpha_1+\beta_1 M}{1-\gamma_1 M}+\beta_2 M\right)\left(1+\frac{\alpha_1+\beta_1 M}{1-\gamma_1 M}\right) \\
\leq\ & \Theta(M).
\end{aligned}
$$

Then, it is proved in R. Temam [12, 13] that there exists a unique solution $u\in L^2(0,T^*;\mathbb{H}^2(\Omega))\cap L^\infty(0,T^*;\mathbb{H}^1_0(\Omega))$ to the Navier-Stokes equations

$$
\left\{\begin{array}{rcll}
\partial_t u+u.\nabla u-\eta\triangle u+\nabla p &=& F & \text{in }\Omega,\\
\operatorname{div} u &=& 0 & \text{in }\Omega,\\
u &=& 0 & \text{on }\Gamma,
\end{array}\right. \tag{4.6}
$$

satisfying moreover

$$
\sup_{t\in[0,T^*]}||u(t)||_V^2\leq\mu_1(M)^2\leq M^2,
$$

$$
\begin{aligned}
||u||^2_{L^2(0,T^*;\mathbb{H}^2(\Omega))} &\leq \frac{c_8}{\eta}\left(||u_0||_V^2+\frac{2}{\eta}\int_0^{T^*}||F(t)||^2\,dt+\mu_1(M)^3\right)\\
&\leq \mu_2(M)^2\leq M^2.
\end{aligned}
$$

We then deduce from the Navier-Stokes equations that

$$
\begin{aligned}
||\partial_t u||_{L^2(0,T^*;L^2(\Omega))} &\leq \eta c_9||u||_{L^2(0,T^*;\mathbb{H}^2(\Omega))}+c_{10}||u||_{L^\infty(0,T^*;\mathbb{H}^1(\Omega))}||u||_{L^2(0;T^*;\mathbb{H}^2(\Omega))}\\
&\quad +||f||_{L^\infty(0,T;\mathbb{L}^2(\Omega))}\\
&\leq \mu_3(M)\leq M.
\end{aligned}
$$

We deduce that $u\in\mathcal{K}$. Let us check the continuity in $L^2(0,T;V)$ of $\overline{u}\longrightarrow u$. Let $\overline{u}_n$ be a sequence that goes to $\overline{u}$ in $L^2(0,T;V)$, it defines a sequence B_n, solution of (4.5). The force term corresponding to B_n in the Navier-Stokes equations has the required regularity to define a sequence u_n bounded in $L^2(0,T;\mathbb{H}^2(\Omega))$ and such that $\partial_t u_n$ is bounded in $L^2(0,T;\mathbb{L}^2(\Omega)$. The sequence u_n is therefore compact in $L^2(0,T;\mathbb{H}^1(\Omega)$. The uniqueness of the solution yields that u_n goes to u corresponding to $\overline{u}$.

Thus the application $\overline{u}\longrightarrow u$ maps continuously the convex compact set $\mathcal{K}$ into himself. Therefore, the Schauder theorem ensures that the existence of a fixed point. This yields the existence result.

• **Regularity of b.**

We have just proved that $B\in\mathcal{C}(0,T^*;W)$. We show as in Proposition 3 that $B\in\mathcal{C}(0,T^*;\mathbb{W}^{1,3}(\Omega))$ and therefore we have in particular $B\in L^\infty(0,T^*;\mathbb{L}^q(\Omega)),\forall q>0$. Using for example that $B\in L^\infty(0,T^*;\mathbb{L}^8(\Omega))$, we easily check that the right-hand side of (4.5) belongs to $L^\infty(0,T^*;\mathbb{L}^{8/5}(\Omega))$. Using Proposition 2, we deduce that $B\in L^\infty(0,T^*;W^{2,8/5}(\Omega))$, which implies that $B\in L^\infty(0,T^*;\mathbb{L}^\infty(\Omega))$. The right-hand side of (4.5) is then in $L^\infty(0,T;\mathbb{L}^2(\Omega))$. Applying one more time the regularity result of Proposition 2, we finally conclude that $B\in L^\infty(0,T^*;\mathbb{H}^2(\Omega))$. In view of the regularity of B^d, we deduce that $b\in L^\infty(0,T^*;\mathbb{H}^2(\Omega)$.

• **Uniqueness.**

Let (u_1,p_1,B_1) and (u_2,p_2,B_2) two solutions of problem (3.1)-(3.8). We define $u=u_1-u_2$, $B=B_1-B_2$. Combining the equations satisfied by (u_1,B_1) and (u_2,B_2), we have

$$\partial_t u+u.\nabla u_1+u_2.\nabla u-\eta\triangle u+\nabla p=\mathrm{curl}\,B\times B_1+\mathrm{curl}\,B_2\times B, \tag{4.7}$$

$$\frac{1}{\sigma}\mathrm{curl}\,(\mathrm{curl}\,B)=\mathrm{curl}\,(u\times B_1)+\mathrm{curl}\,(u_2\times B)+\mathrm{curl}\,(u\times B^d), \tag{4.8}$$

with $u=0$, $B.n=0$ and $\mathrm{curl}\,B\times n=0$ on the boundary.

Multiplying (4.7) by u, (4.8) by B and integrating we obtain :

$$\begin{aligned}\frac{d}{dt}\int_\Omega |u|^2\,dx+\int_\Omega \eta|\nabla u|^2\,dx+\int_\Omega\frac{1}{\sigma}|\mathrm{curl}\,B|^2\,dx\le\int_\Omega|u.\nabla u.u_1|\,dx+\\ +\int_\Omega|\mathrm{curl}\,B_2\times B.u|\,dx+\int_\Omega|u_2\times B.\mathrm{curl}\,B|\,dx+\int_\Omega|u\times B^d.\mathrm{curl}\,B|\,dx.\end{aligned} \tag{4.9}$$

We estimate the right-hand side of this inequality as follows :

$$\int_\Omega|u.\nabla u.u_1|\,dx\le C_\varepsilon||u_1||^2_{\mathbb{L}^\infty(\Omega)}||u||^2_{\mathbb{L}^2(\Omega)}+\varepsilon||\nabla u||^2_{\mathbb{L}^2(\Omega)},$$

where C_ε and ε are some constant, with ε arbitrarily small.

$$\begin{aligned}\int_\Omega|\mathrm{curl}\,B_2\times B.u|\,dx &\le ||\mathrm{curl}\,B_2||_{\mathbb{L}^4(\Omega)}||B||_{\mathbb{L}^4(\Omega)}||u||_{\mathbb{L}^2(\Omega)}\\ &\le C_\varepsilon||B_2||^2_{\mathbb{H}^2(\Omega)}||u||^2_{\mathbb{L}^2(\Omega)}+\varepsilon||\mathrm{curl}\,B||^2_{\mathbb{L}^2(\Omega)},\end{aligned}$$

$$\int_\Omega|u\times B^d.\mathrm{curl}\,B|\,dx\le C_\varepsilon||B^d||^2_{\mathbb{L}^\infty(\Omega)}||u||^2_{\mathbb{L}^2(\Omega)}+\varepsilon||\mathrm{curl}\,B||^2_{\mathbb{L}^2(\Omega)},$$

$$\int_\Omega|u_2\times B.\mathrm{curl}\,B|\,dx\le C_\varepsilon||u_2||^2_{\mathbb{H}^1(\Omega)}||\mathrm{curl}\,B||^2_{\mathbb{L}^2(\Omega)}+\varepsilon||\mathrm{curl}\,B||^2_{\mathbb{L}^2(\Omega)}.$$

In this last inequality, we estimate $\mathrm{curl}\,B$ with equation (4.8) :

$$\frac{1}{\sigma}\int_\Omega|\mathrm{curl}\,B|^2\,dx\le(||B_1||_{\mathbb{L}^\infty}+||B^d||_{\mathbb{L}^\infty})||\mathrm{curl}\,B||_{\mathbb{L}^2}||u||_{\mathbb{L}^2}+d_2d_3||u_2||_{\mathbb{H}^1(\Omega)}||\mathrm{curl}\,B||^2_{\mathbb{L}^2}.$$

Using $\sup_{t\in[0,T^*]} ||u_2||_{\mathbb{H}^1} \leq M$ and the coercivity assumption $0 < M < 1/\gamma_1$ with $\gamma_1 = d_2 d_3 \sigma$, we deduce that

$$||\mathrm{curl}\, B||_{\mathbb{L}^2} \leq \frac{\sigma}{1-\gamma_1 M}(||B_1||_{\mathbb{L}^\infty} + ||B^d||_{\mathbb{L}^\infty})||u||_{\mathbb{L}^2}. \tag{4.10}$$

Thus,

$$\int_\Omega |u_2 \times B.\mathrm{curl}\, B|\, dx \leq C_\varepsilon \left(\frac{\sigma}{1-\gamma_1 M}\right)^2 (||B_1||_{\mathbb{L}^\infty} + ||B^d||_{\mathbb{L}^\infty})^2 ||u_2||^2_{\mathbb{H}^1} ||u||^2_{\mathbb{L}^2} + \\ + \varepsilon ||\mathrm{curl}\, B||^2_{\mathbb{L}^2}.$$

Gathering these inequalities, estimate (4.9) yields

$$\frac{d}{dt}||u||^2_{\mathbb{L}^2} \leq \phi(t)||u||^2_{\mathbb{L}^2}$$

with $\phi = C_\varepsilon(||u_1||^2_{\mathbb{L}^\infty(\Omega)} + ||B_2||^2_{\mathbb{H}^2(\Omega)} + ||B^d||^2_{\mathbb{L}^\infty(\Omega)} + \left(\frac{\sigma}{1-\gamma_1 M}\right)^2 (||B_1||_{\mathbb{L}^\infty} + ||B^d||_{\mathbb{L}^\infty})^2 ||u_2||^2_{\mathbb{H}^1(\Omega)})$. Note that $\phi \in L^1(0,T^*)$. Therefore, by Gronwall lemma, $u = 0$, and using again (4.10), $B = 0$. This proves the uniqueness of a strong solution (u, b) of (2.1)-(2.8). $\diamondsuit$

Remark 4.1 *We now sketch an alternative proof of Theorem 1. We only give the main ideas and manipulate the equations in a formal way. We suppose for the sake of simplicity that $curl B \times n = 0$ on the boundary and $f = 0$. The MHD system we are studying may be seen as the singular limit when $\varepsilon \to 0$ of*

$$\begin{aligned} \partial_t u_\varepsilon + u^\varepsilon.\nabla u^\varepsilon - \eta \triangle u^\varepsilon &= -\nabla p + curl\, B^\varepsilon \times B^\varepsilon \quad \text{in } \Omega, && (4.11)\\ div\, u^\varepsilon &= 0 \quad \text{in } \Omega, && (4.12)\\ \varepsilon \partial_t B^\varepsilon + \frac{1}{\sigma} curl\,(curl\, B^\varepsilon) &= curl\,(u^\varepsilon \times B^\varepsilon) \quad \text{in } \Omega, && (4.13)\\ div\, B^\varepsilon &= 0 \quad \text{in } \Omega. && (4.14) \end{aligned}$$

This system has been studied in M. Sermange, R. Temam [11] with $\varepsilon = 1$. The first energy estimate is :

$$\frac{d}{dt}\left(\int_\Omega |u^\varepsilon|^2\, dx + \varepsilon \int_\Omega |B^\varepsilon|^2\, dx\right) + \int_\Omega \eta |\nabla u^\varepsilon|^2 + \frac{1}{\sigma}|curl\, B^\varepsilon|^2\, dx = 0.$$

Unfortunately, the bound on $\partial_t B^\varepsilon$ is not uniform in ε which prevents us to infer any compactness on B^ε in order to treat the non linear term $curl B^\varepsilon \times B^\varepsilon$. We therefore argue as follows.

Derivating with respect to the time, multiplying the Maxwell equation by $\partial_t B^\varepsilon$ and integrating yield :

$$\begin{aligned}
\frac{\varepsilon}{2}\frac{d}{dt}\int_\Omega |\partial_t B^\varepsilon|^2\,dx + \frac{1}{\sigma}\int_\Omega |curl\,\partial_t B^\varepsilon|^2\,dx = \\
= \int_\Omega \partial_t u^\varepsilon \times B^\varepsilon . curl\,\partial_t B^\varepsilon + \int_\Omega u^\varepsilon \times \partial_t B^\varepsilon . curl\,\partial_t B^\varepsilon\,dx \\
\leq \frac{1}{2}\int_\Omega |curl\,\partial_t B^\varepsilon|^2\,dx + C^{st}(||\partial_t u^\varepsilon . B^\varepsilon||^2_{\mathbb{L}^2} + ||u^\varepsilon||^2_{\mathbb{L}^3}||\partial_t B^\varepsilon||^2_{\mathbb{L}^6}),
\end{aligned}$$

and thus

$$\frac{\varepsilon}{2}\frac{d}{dt}\int_\Omega |\partial_t B^\varepsilon|^2\,dx + (\frac{1}{\sigma} - C^{st}||u^\varepsilon||_V)\int_\Omega |curl\,\partial_t B^\varepsilon|^2\,dx \leq C^{st}||\nabla\partial_t u^\varepsilon||^2_{\mathbb{L}^2}||curl\,B^\varepsilon||^2_{\mathbb{L}^2}.$$

which shows that we have $\nabla\partial_t B^\varepsilon$ bounded in $L^2(0,T;\mathbb{H}^1)$ uniformly *with respect to ε as soon as $||u^\varepsilon||_V$ is small enough. This gives some compactness on B^ε and thus allows us to complete this proof.*$\diamondsuit$

5 Remark on the non-uniqueness

It has been proven in the previous section that the MHD problem (3.1)-(3.8) has a unique solution for small data, at least on an interval $[0,T^*]$, $T^* > 0$. The idea of the proof has been to ensure the coercivity of equation (3.3) by controlling the $H^1(\Omega)$ norm of u on $[0,T^*]$. We exhibit in this section an example (due to P.-L. Lions [7]) of non uniqueness in the case when the operator $T_u : B \to \mathrm{curl}\,(\mathrm{curl}\,B) - \mathrm{curl}\,(u \times B)$ is not invertible.

From now on, we assume for simplicity that $k = 0$, $q = 0$, thus we deal with the homogeneous boundary conditions on Γ :

$$\left\{\begin{array}{rcl} u &=& 0, \\ B.n &=& 0, \\ \mathrm{curl}\,B \times n &=& 0. \end{array}\right.$$

Let us assume that for some t_0 and some $\tilde{u} = \tilde{u}(t_0,x)$ the operator T_u : $B \to \mathrm{curl}\,(\mathrm{curl}\,B) - \mathrm{curl}\,(u \times B)$ is not invertible. There exists a divergence-free field $\tilde{B} \not\equiv 0$ satisfying

$$\mathrm{curl}\,(\mathrm{curl}\,\tilde{B}) = \mathrm{curl}\,(\tilde{u} \times \tilde{B}).$$

Note that such a $\tilde{u}$ is necessarilly "large enough", otherwise, $T_{\tilde{u}}$ would be coercive. If we consider the force $\tilde{f} = \tilde{u}.\nabla\tilde{u} - \eta\triangle\,\tilde{u} - \mathrm{curl}\,\tilde{B}\times\tilde{B}$, then $(\tilde{u},\tilde{B})$ is a (stationary) solution to

$$\left\{\begin{array}{rcl} \partial_t u + u.\nabla u - \eta\triangle\,u + \nabla p &=& \tilde{f} + \mathrm{curl}\,B\times B,\\ \mathrm{div}\,u &=& 0,\\ \mathrm{curl}\,(\mathrm{curl}\,B) &=& \mathrm{curl}\,(u\times B),\\ \mathrm{div}\,B &=& 0. \end{array}\right. \tag{5.1}$$

Next, we define u' as the solution of

$$\left\{\begin{array}{rcl} \partial_t u + u.\nabla u - \eta\triangle\,u + \nabla p &=& \tilde{f},\\ \mathrm{div}\,u &=& 0. \end{array}\right.$$

with the "initial" condition $u'(t_0,.) = \tilde{u}(t_0,.)$.

We finally observe that $(\tilde{u},\tilde{B})$ and $(u',0)$ are different (since $\tilde{B}\not\equiv 0$) while they both satisfy (5.1) on $[t_0,+\infty)$.

Thus, we have two different solutions of the MHD problem with homogeneous boundary conditions.

6 Conclusion

We have proved that the MHD system (2.1)-(2.8) has a unique solution on an interval $[0,T^*]$ as soon as the physical data are regular and small enough, with $T^*>0$ depending on the data. Note that the proof may probably be extended to the case of multifluid equations in two dimensions with constant viscosity and conductivity.

Moreover, we have shown that a solution is *not* unique if the operator $B\to \mathrm{curl}\,(\mathrm{curl}\,B) - \mathrm{curl}\,(u\times B)$ is not invertible. This may occur as soon as the velocity becomes too large, but it is an open question to show that the operator do *indeed* become not invertible.

The practical conclusion of this study is the following : even if the model presented here seems well-suited in some physical situations and even if it is mathematicaly well-posed under restrictive assumptions, it should be very carefuly used in numerical simulations since it could be ill-posed as soon as the velocity becomes too large.

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