Weak solutions for the exterior Stokes problem in weighted Sobolev spaces.

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Résumé - On donne ici des résultats d'existence et d'unicité pour le problème de Stokes extérieur en controlant le comportement à l'infini des solutions. On pose pour cela le problème dans des espaces de Sobolev avec poids. On obtient aussi un développement asymptotique des solutions qui décroissent suffisament.

Abstract - *We establish here some existence and uniqueness properties for the exterior Stokes problem with prescribed growth or decay at infinity for the solutions. For this purpose, the problem is set in some suitable weighted Sobolev spaces. We also obtain an asymptotic expansion for some well behaved solutions.*

Consider an open region Ω of $\mathbb{R}^n$. In the sequel, we shall call this set an exterior domain if there exists a non-empty bounded open set Ω' with a finite number of connected components having Lipschitz-continuous boundaries such that $\Omega = \mathbb{R}^n - \overline{\Omega'}$. We shall also suppose for the sake of simplicity that each connected component of Ω' has a connected boundary (*i.e.* Ω' has no "holes"), which implies in particular that Ω is connected.

This paper is devoted to the following Stokes problem in an exterior connected domain :

$$(S) \qquad \begin{aligned} -\nu\Delta \boldsymbol{u} + \nabla\pi &= \boldsymbol{f} && \text{in } \Omega,\\ -\operatorname{div} \boldsymbol{u} &= g && \text{in } \Omega,\\ \boldsymbol{u} &= \boldsymbol{\varphi} && \text{on } \partial\Omega, \end{aligned}$$

where ν is a positive coefficient.

Existence, uniqueness and regularity properties of this problem are well-known when the domain Ω is bounded. In that case, the classical Sobolev spaces $W^{m,p}(\Omega)$ provide a suitable functional framework (see [5, 2] for instance), in particular thanks to Poincaré's inequalities. Nevertheless, when Ω is an exterior domain, these inequalities are not satisfied any more and it is necessary to introduce a specific functional framework which takes into account the behaviour of the functions at infinity.

Among the many works devoted to the exterior problem, some of them introduce the homogeneous Sobolev spaces :

$$\hat{H}_0^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{\| \nabla \ \|_{L^p(\Omega)}},$$

which fit the case of homogeneous Dirichlet boundary conditions. In particular, when $n/(n-1) < p < n$ with $n \geq 3$, H. Kozono and H. Sohr consider in [12] data $\boldsymbol{f}$ that belong to $\hat{H}^{-1,p}(\Omega)$ (which space is the dual space of $\hat{H}_0^{1,p'}(\Omega)$ with $1/p' = 1 - 1/p$), $g \in L^p(\Omega)$ and $\boldsymbol{\varphi} = 0$. Under such assumptions, they prove existence of a unique solution $(\boldsymbol{u}, \pi)$ to problem (S) with $\boldsymbol{u} \in \hat{H}_0^{1,p}(\Omega)$ and $\pi \in L^p(\Omega)$. The same authors have later removed the restrictions on p in [13] and also treated the case $n = 2$. In particular, they conclude that when $p \leq n'$ if $n \geq 3$ or $p < 2$ if $n = 2$, a solution exists if and only if the data satisfy some compatibility conditions. On the other hand, when $p \geq n \geq 3$ or $p > 2$ if $n = 2$, they show that the solution is no longer unique and they give a characterisation of the null-space of the problem. G. P. Galdi and C. G. Simader obtain similar results in [7]. The case of more regular data is investigated in [18] by W. Varnhorn who gives conditions for the velocity field $\boldsymbol{u}$ and the pressure π to vanish at infinity.

In a more recent work, G. P. Galdi also considers the case where $\boldsymbol{f} = \operatorname{div} F$ with $(1 + |\boldsymbol{x}|^2)F \in L^\infty(\Omega)$ where $\Omega \subset \mathbb{R}^3$ and proves the existence of a unique solution $(\boldsymbol{u}, \pi)$ satifying $(1 + |\boldsymbol{x}|)\boldsymbol{u} \in \mathbf{L}^\infty(\Omega)$ and $\pi \in L^p(\Omega)$ with $p > 3/2$. This result has in particular an interesting application to the steady-state Navier-Stokes equations.

In this article, our approach consists in looking for solutions in weighted Sobolev spaces of the type :

$$W_\alpha^{1,p}(\Omega) = \{v/(1 + |\boldsymbol{x}|)^{\alpha-1} v \in L^p(\Omega), (1 + |\boldsymbol{x}|)^\alpha \nabla v \in L^p(\Omega)\}, \text{ if } n/p + \alpha \neq 1,$$

for $p \in]1, +\infty[$, and $\alpha \in \mathbb{R}$, and with an additional logarithmic weight when $n/p + \alpha = 1$ (see definition 1.1 below). These spaces are well adapted to the Laplace and Stokes equations because they satisfy optimal Poincaré-type weighted inequalities. Moreover, they provide an explicit description of the behaviour of the functions at infinity, which is not obvious from the definition of $\hat{H}_0^{1,p}(\Omega)$. Even more important, they are much more general for, thanks to the parameter α, one may consider a much larger variety of behaviours at infinity than it is possible to do with the spaces $\hat{H}_0^{1,p}(\Omega)$.

The investigation of the exterior problem (S) in such spaces has partially been made for $n = 2$ or 3, $p = 2$, $\alpha = 0$ by A. Sequeira and V. Girault in [8]. M. Specovius-Neugebauer also gives in [16] more general results when $n \geq 3$ using integral equations techniques for strong solutions of (S). She has later extended these results to the bidimensional case in [17] but never used the logarithmic weight, therefore leaving the problem unsolved in several critical cases.

In the present article, we prove the existence, uniqueness and regularity of the solutions of the problem (S) for very general data (See Theorem 3.16). In particular, we give a complete characterisation of the null space of the problem (S) in weighted spaces. Our approach is based on the ideas of J. Giroire developed in [9] and makes use of the principle that exterior linear problems can be solved combining their properties on the whole space $\mathbb{R}^n$ and on bounded domains.

On the other hand, we take a special interest in the case of solutions going to zero at infinity and prove a new explicit asymptotic expansion for such solutions within the framework of weighted spaces.

Our paper is organised as follows : weighted Sobolev spaces, their fundamental properties, and some preliminary results are described in section 1. Sections 2 and 3 are devoted to the existence and uniqueness properties of the exterior Stokes problem (S). Regularity results are developed in section 4 as well as an application to the control of the second derivatives of $\boldsymbol{u}$ in the L^p norm which improves the results established by H. Kozono and T. Ogawa in [14] (see Theorem 4.3). At last, in section 5, we provide a precise description of the asymptotic behaviour of the solutions in some interesting cases (see Theorems 5.4 and 5.5).

1 Function spaces and preliminary results

In the sequel, n denotes an integer greater than or equal to 2 and p a real number in the interval $]1, +\infty[$. The dual exponent of p, denoted by p', is defined by the relation $1/p + 1/p' = 1$. When $p < n$, we set the Sobolev exponent $p*$ to be the real number defined by $1/p* = 1/p - 1/n$. We denote by B_R the open ball of radius $R > 0$ centered at the origin. Finally, if X is a Banach space, with dual space X', and Y is a closed subspace of X, we denote by $X' \perp Y$ the subspace of X' orthogonal to Y, that is :

$$X' \perp Y = \{x \in X', \quad \forall y \in Y, < x, y >= 0\} = (X/Y)'.$$

Without loss of generality, we consider exterior domains Ω such that the origin of $\mathbb{R}^n$ belongs to Ω'. We introduce the weight function :

$$\rho(\boldsymbol{x}) = 2 + |\boldsymbol{x}|,$$

and the following weighted Sobolev spaces.

Definition 1.1 For any real number α, we define the spaces,

$$W_\alpha^{0,p}(\Omega) = \{u \in \mathcal{D}'(\Omega),\ \rho^\alpha u \in L^p(\Omega)\},$$

$$W_\alpha^{1,p}(\Omega) = \{u \in \mathcal{D}'(\Omega),\ \rho^{\alpha-1} u \in L^p(\Omega),\ \rho^\alpha \nabla u \in L^p(\Omega)\}, \quad \text{if } n/p+\alpha \neq 1,$$

$$W_\alpha^{1,p}(\Omega) = \{u \in \mathcal{D}'(\Omega),\ \rho^{\alpha-1}(\ln\rho)^{-1} u \in L^p(\Omega),\ \rho^\alpha \nabla u \in L^p(\Omega)\},\ \text{if } n/p+\alpha = 1.$$

They are reflexive Banach spaces with respect to the norms :

$$\| u \|_{W_\alpha^{0,p}(\Omega)} = \| \rho^\alpha u \|_{L^p(\Omega)},$$

$$\| u \|_{W_\alpha^{1,p}(\Omega)} = (\| \rho^{\alpha-1} u \|^p_{L^p(\Omega)} + \| \rho^\alpha \nabla u \|^p_{L^p(\Omega)})^{1/p} \ \text{ if } n/p+\alpha \neq 1,$$

$$\| u \|_{W_\alpha^{1,p}(\Omega)} = (\| \frac{\rho^{\alpha-1}}{\ln\rho} u \|^p_{L^p(\Omega)} + \| \rho^\alpha \nabla u \|^p_{L^p(\Omega)})^{1/p} \ \text{ if } n/p+\alpha = 1.$$

We also define the semi-norm : $| u |_{W_\alpha^{1,p}} = \| \rho^\alpha \nabla u \|_{L^p(\Omega)}$.

Let us point out that the logarithmic weight only appears for the so-called critical exponents (see also [15]) and they are an essential ingredient of those spaces, which otherwise would have poor interest (see theorem 1.2 below).

We first recall some elementary properties of these spaces. The space $\mathcal{D}(\overline{\Omega})$ is dense in $W_\alpha^{1,p}(\Omega)$ whereas, like in bounded domains, this is not true for $\mathcal{D}(\Omega)$. Moreover, the functions of $W_\alpha^{1,p}(\Omega)$ belong to $W^{1,p}(\mathcal{O})$, for all bounded domain $\mathcal{O}$ contained in Ω, and they satisfy the usual trace theorems on the boundary $\partial\Omega$ which, we recall, is Lipschitz-continuous. Let us now introduce the space $\overset{\circ}{W}{}_\alpha^{1,p}(\Omega) = \overline{\mathcal{D}(\Omega)}^{\|\cdot\|_{W_\alpha^{1,p}(\Omega)}}$. It is easy to check that :

$$\overset{\circ}{W}{}_\alpha^{1,p}(\Omega) = \{v \in W_\alpha^{1,p}(\Omega), \gamma v = 0\},$$

where γv denotes the trace of v on the boundary $\partial\Omega$. The dual space denoted by $W_{-\alpha}^{-1,p'}(\Omega)$ is a subspace of $\mathcal{D}'(\Omega)$. Recall that in the whole space, we also have $W_{-\alpha}^{-1,p'}(\mathbb{R}^n) = (W_\alpha^{1,p}(\mathbb{R}^n))'$.

Let $\mathcal{P}_l$ denote the space of polynomials whose degree is not greater than l with the convention that $\mathcal{P}_l = \{0\}$ in the case $l < 0$. One easily sees that the larger space of polynomials contained in $W_\alpha^{1,p}(\Omega)$ is $\mathcal{P}_j$ with

$$\begin{array}{ll} j = [1-(n/p+\alpha)], & \text{if } \ n/p+\alpha \notin \{i \in \mathbb{Z}, i \leq 0\} \\ j = -(n/p+\alpha), & \text{otherwise,} \end{array}$$

and where $[s]$ denotes the integer part of the real number s.

A fundamental property of the weighted Sobolev spaces $W^{1,p}_\alpha(\Omega)$ is that their elements satisfy Poincaré-type inequalities (see [3]). This property strongly depends on the introduction of the logarithmic weight for critical exponents and is not satisfied in the cases $p = 1$ and $p = +\infty$.

Theorem 1.2 (Amrouche-Girault-Giroire,[3]) *Let Ω be an exterior domain, and α a real number.*

i) The semi-norm $|\,.\,|_{W^{1,p}_\alpha(\Omega)}$ defines on $W^{1,p}_\alpha(\Omega)/\mathcal{P}_{j'}$ a norm which is equivalent to the quotient norm, where $j' = \min(j, 0)$.

ii) The semi-norm $|\,.\,|_{W^{1,p}_\alpha(\Omega)}$ defines on $\overset{\circ}{W}{}^{1,p}_{\alpha}(\Omega)$ a norm which is equivalent to the full norm $\|\,.\,\|_{W^{1,p}_\alpha(\Omega)}$.

Remark 1.3 Thanks to the latter property, it is straightforward to prove that :

$$\overset{\circ}{W}{}^{1,p}_{0}(\Omega) = \hat{H}^{1,p}_0(\Omega), \quad \forall\, p \in \,]1, +\infty[,$$

so that weighted Sobolev spaces are a generalisation of the spaces $\hat{H}^{1,p}_0$.

We conclude this short review of weighted Sobolev spaces with more detailed asymptotic properties (see the proofs in [1]).

Proposition 1.4 (Alliot-Amrouche,[1]) *Let α be a real number and p such that $n/p + \alpha \neq 1$ and $R > 0$ such that $\Omega' \subset B_R$. Then, every function u in the space $W^{1,p}_\alpha(\Omega)$ satisfies :*
i) For all $\boldsymbol{x}$ with $|\boldsymbol{x}| > R$,

$$\|u(|\boldsymbol{x}|,\,.\,)\|_{L^p(\Sigma)} \leq C|\boldsymbol{x}|^{1-n/p-\alpha}\|\,u\,\|_{W^{1,p}_\alpha} \quad \text{and } |\boldsymbol{x}|^{\alpha+n/p-1}\|u(|\boldsymbol{x}|,\,.\,)\|_{L^p(\Sigma)} \overset{|\boldsymbol{x}|\to\infty}{\longrightarrow} 0,$$

where Σ denotes the unit sphere $\{|\boldsymbol{x}| = 1\}$ in $\mathbb{R}^n$.
ii) If $p > n$: for all $\boldsymbol{x}$ such that $|\boldsymbol{x}| > R$, $\quad |u(\boldsymbol{x})| \leq C|\boldsymbol{x}|^{1-n/p-\alpha}\|u\|_{W^{1,p}_\alpha}$.
Moreover, $|\boldsymbol{x}|^{\alpha+n/p-1}|\,u(\boldsymbol{x})| \overset{|\boldsymbol{x}|\to\infty}{\longrightarrow} 0$.
Finally, if $n/p + \alpha = 1$, the same properties hold if one replaces the function $|\boldsymbol{x}|^{1-n/p-\alpha}$ by $\ln(2 + |\boldsymbol{x}|)$.

2 First results : the case $p = 2$

Let us first briefly recall the result by V. Girault and A. Sequeira which states the existence of a solution to problem (S) with $\boldsymbol{u} \in \mathbf{W}^{1,2}_0(\Omega)$ and $\pi \in L^2(\Omega)$, and its uniqueness in this space. The proof of this result is detailed in [8] for $n = 2$ or 3, but remains valid for all higher dimensions.

Theorem 2.1 (Girault-Sequeira,[8]) *Let Ω be an exterior domain having a Lipschitz-continuous boundary. For any distribution $\boldsymbol{f}$ in $\mathbf{W}_0^{-1,2}(\Omega)$, for any g in $L^2(\Omega)$ and $\boldsymbol{\varphi}$ in $\mathbf{H}^{1/2}(\partial\Omega)$, problem (S) has a solution $(\boldsymbol{u}, \pi) \in \mathbf{W}_0^{1,2}(\Omega) \times L^2(\Omega)$. In this space, the solution is unique and there exists a constant $C > 0$ such that :*

$$\| \boldsymbol{u} \|_{\mathbf{W}_0^{1,2}(\Omega)} + \| \pi \|_{L^2(\Omega)} \leq C(\| \boldsymbol{f} \|_{\mathbf{W}_0^{-1,2}(\Omega)} + \| g \|_{L^2(\Omega)} + \| \boldsymbol{\varphi} \|_{\mathbf{H}^{1/2,2}(\partial\Omega)}).$$

Remark 2.2 Note that in this case, existence of solutions does not require any compatibility conditions on the data whereas, in a bounded domain $\mathcal{O}$, the existence of $\boldsymbol{u} \in \mathbf{H}^1(\mathcal{O})$ requires that

$$\int_{\mathcal{O}} g(\boldsymbol{x}) d\boldsymbol{x} + \int_{\partial\mathcal{O}} \boldsymbol{\varphi}.\boldsymbol{n} ds = 0 \ .$$

Moreover, the pressure π is unique in $L^2(\Omega)$ which is not the case in a bounded domain.

We still consider the case $p = 2$ but we choose more general data :

$$\boldsymbol{f} \in \mathbf{W}_l^{-1,2}(\Omega), \quad g \in W_l^{0,2}(\Omega), \quad \boldsymbol{\varphi} \in \mathbf{H}^{1/2}(\partial\Omega),$$

where l is an integer. In this context, we wonder if we still have existence and uniqueness of a solution to problem (S) such that :

$$(\boldsymbol{u}, \pi) \in \mathbf{W}_l^{1,2}(\Omega) \times W_l^{0,2}(\Omega).$$

In order to address the uniqueness of the solutions, we introduce the spaces :

$$\mathcal{N}_l^p(\Omega) = \{(\boldsymbol{u}, \pi) \in \mathring{\mathbf{W}}_l^{1,p}(\Omega) \times W_l^{0,p}(\Omega), -\nu\Delta\boldsymbol{u} + \nabla\pi = \mathbf{0}, \text{ div } \boldsymbol{u} = 0, \text{ in } \Omega\}. \tag{2.1}$$

When $l \geq 0$, as a consequence of Theorem 2.1 and of the imbedding

$$\mathring{\mathbf{W}}_l^{1,2}(\Omega) \times W_l^{0,2}(\Omega) \ \subset \mathring{\mathbf{W}}_0^{1,2}(\Omega) \times L^2(\Omega),$$

we have $\mathcal{N}_l^2(\Omega) = \{(\mathbf{0}, 0)\}$. However, the situation is different when $l < 0$. Let us recall what occurs when the Stokes problem is set in $\mathbb{R}^n$. In that case, the elements of the space :

$$\mathcal{N}_l^p(\mathbb{R}^n) = \{(\boldsymbol{u}, \pi) \in \mathbf{W}_l^{1,p}(\mathbb{R}^n) \times W_l^{0,p}(\mathbb{R}^n), -\nu\Delta\boldsymbol{u} + \nabla\pi = \mathbf{0}, \text{ div } \boldsymbol{u} = 0, \text{ in } \mathbb{R}^n\},$$

are polyharmonic tempered distributions on $\mathbb{R}^n$ and therefore polynomials. Hence, the space $\mathcal{N}_l^p(\mathbb{R}^n)$ equals the space :

$$N_k = \{(\boldsymbol{\lambda}, \mu) \in \boldsymbol{\mathcal{P}}_k \times \mathcal{P}_{k-1},\ \operatorname{div} \boldsymbol{\lambda} = 0, -\nu\Delta\boldsymbol{\lambda} + \nabla\mu = \mathbf{0}\}, \tag{2.2}$$

with $k = [1 - n/p - l]$ if $n/p + l \notin \{i \in \mathbb{Z}, i \leq 0\}$ and $k = -(n/p + l)$ otherwise.

We can adapt this characterisation to $\mathcal{N}_l^2(\Omega)$ when $n \geq 3$.

Proposition 2.3 *If $n \geq 3$ and l is an integer such as $n/2 \notin \{1, \dots, |l|\}$, then :*

$$\mathcal{N}_l^2(\Omega) = \{(\boldsymbol{u}, \pi),\ \boldsymbol{u} = \boldsymbol{v}(\boldsymbol{\lambda}) - \boldsymbol{\lambda},\ \pi = \eta(\boldsymbol{\lambda}) - \mu,\ \ (\boldsymbol{\lambda}, \mu) \in N_k\},$$

with $k = [1 - n/2 - l]$ if $n/2 + l \notin \{i \in \mathbb{Z}, i \leq 0\}$ and $k = -(n/2 + l)$ otherwise, and where $(\boldsymbol{v}(\boldsymbol{\lambda}), \eta(\boldsymbol{\lambda}))$ is the unique solution in $\mathbf{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ of problem

$$(S_{\boldsymbol{\lambda}}) \qquad \begin{array}{ll} -\nu\Delta\boldsymbol{v} + \nabla\eta = 0 & \textit{in } \ \Omega, \\ -\operatorname{div} \boldsymbol{v} = 0 & \textit{in } \ \Omega, \\ \quad \boldsymbol{v} = \boldsymbol{\lambda} & \textit{on } \ \partial\Omega. \end{array}$$

Proof : We have already proved the case $l \geq 0$, so we now consider the case $l < 0$. Let us first note that each pair $(\boldsymbol{u}, \pi)$ in $\mathcal{N}_l^2(\Omega)$ satisfies the Green's formula : for all pairs $(\boldsymbol{\psi}, \xi)$ in $\boldsymbol{\mathcal{D}}(\overline{\Omega}) \times \mathcal{D}(\overline{\Omega})$,

$$\int_\Omega [(-\nu\Delta\boldsymbol{\psi} + \nabla\xi).\boldsymbol{u} - \pi \operatorname{div} \boldsymbol{\psi}]\, d\boldsymbol{x} \ = < \boldsymbol{\psi}, (\nu\nabla\boldsymbol{u} - \pi I).\boldsymbol{n} >_{\partial\Omega}, \tag{2.3}$$

where $\boldsymbol{n}$ denotes the unit normal vector to $\partial\Omega$ pointing outside Ω, where I is the second order identity tensor, and $< .,. >_{\partial\Omega}$ denotes the duality pairing between $\mathbf{W}^{1/p,p'}(\partial\Omega)$ and its dual space $\mathbf{W}^{-1/p,p}(\partial\Omega)$.

In particular, if we extend $\boldsymbol{u}$ and π by zero in Ω', the extended functions, still denoted by $\boldsymbol{u}$ and π, respectively belong to $\mathbf{W}_l^{1,2}(\mathbb{R}^n)$ and $W_l^{0,2}(\mathbb{R}^n)$. Moreover, thanks to (2.3), we have the equalities :

$$-\nu\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{h},\ \ \operatorname{div} \boldsymbol{u} = 0, \quad \text{in } \mathcal{D}'(\mathbb{R}^n), \tag{2.4}$$

$$\text{with } \ \forall\boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^n),\ \ < \boldsymbol{h}, \boldsymbol{\psi} > \ = \ < (\nu\nabla\boldsymbol{u} - \pi I).\boldsymbol{n}, \boldsymbol{\psi} >_{\partial\Omega} . \tag{2.5}$$

By construction, $\boldsymbol{h}$ belongs to $\mathbf{W}_0^{-1,2}(\mathbb{R}^n)$ so that, since $n \geq 3$, Stokes problem (2.4) has a unique solution $(\boldsymbol{v}, \eta) \in \mathbf{W}_0^{1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ (see [1], Theorem 3.3, or the more general Theorem 2.6 below). Then, the difference $(\boldsymbol{v} - \boldsymbol{u}, \eta - \pi)$ belongs to $\mathbf{W}_l^{1,2}(\mathbb{R}^n) \times W_l^{0,2}(\mathbb{R}^n)$ and to $\mathcal{N}_l^2(\mathbb{R}^n)$. Thus, in view of characterisation (2.2),

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\lambda} \quad \text{and} \quad \eta = \pi + \mu,$$

where $(\boldsymbol{\lambda}, \mu)$ belongs to N_k and $k = [1 - n/2 - l]$ if $n/2 + l \notin \{i \in \mathbb{Z}, i \leq 0\}$ and $k = -(n/2+l)$ otherwise. Hence, for $\boldsymbol{h}$ is supported in $\partial\Omega$, one immediately checks that $(\boldsymbol{v}, \eta)$, restricted to Ω, provides a solution in $\mathbf{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ to problem $(S_{\boldsymbol{\lambda}})$. But, since $\boldsymbol{\lambda}$ is a polynomial, it also belongs to $\mathbf{H}^{1/2}(\partial\Omega)$ and considering Theorem 2.2, for each $\boldsymbol{\lambda}$, the problem $(S_{\boldsymbol{\lambda}})$ has a solution in $\mathbf{W}_0^{1,2}(\Omega) \times L^2(\Omega)$ which is unique in this space. Hence, $(\boldsymbol{v}, \eta)$ actually equals $(\boldsymbol{v}(\boldsymbol{\lambda}), \eta(\boldsymbol{\lambda}))$. $\diamondsuit$

Remark 2.4 As a consequence of Theorem 2.1, the mapping :

$$(\boldsymbol{\lambda}, \mu) \mapsto (\boldsymbol{v}(\boldsymbol{\lambda}) - \boldsymbol{\lambda}, \eta(\boldsymbol{\lambda}) - \mu),$$

is linear and injective. Thereby, $\mathcal{N}_l^2(\Omega)$ is isomorphic to $\mathcal{N}_l^2(\mathbb{R}^n)$ when $n \geq 3$ and both spaces have the same finite dimension. However, when $n = 2$, this is not necessarily the case. For instance, Theorem 2.1 shows that $\mathcal{N}_0^2(\Omega) = \{(\mathbf{0}, 0)\}$, but $\mathcal{N}_0^2(\mathbb{R}^2) = N_0 = \boldsymbol{\mathcal{P}}_0 \times \{0\}$. More generally, we shall see in the sequel that if $n = 2$, then $\mathcal{N}_l^p(\Omega)$ has a different structure than the one it has if $n \geq 3$.

We now establish, in the case $\boldsymbol{\varphi} = \mathbf{0}$, and when $l < 0$, the existence of a solution $(\boldsymbol{u}, \pi) \in \mathbf{W}_l^{1,2}(\Omega) \times W_l^{0,2}(\Omega)$ to the problem (S). In the sequel, we agree that the set $\{1, \ldots, k\}$ is empty if the integer k is not positive.

Proposition 2.5 *Let $n \geq 3$, and $l < 0$ such that $n/2 \notin \{1, \ldots, |l|\}$. Then for any $(\boldsymbol{f}, g)$ in $\mathbf{W}_l^{-1,2}(\Omega) \times W_l^{0,2}(\Omega)$ and $\boldsymbol{\varphi} = \mathbf{0}$, problem (S) has a solution $(\boldsymbol{u}, \pi)$ with $\boldsymbol{u} \in \mathbf{W}_l^{1,2}(\Omega)$ and $\pi \in W_l^{0,2}(\Omega)$.*

In order to prove this property, we shall use the analogous existence result for the Stokes problem set in $\mathbb{R}^n$ which we quote here in the general case (see[1], Theorems 3.3, 3.5 and Corollary 3.6).

Theorem 2.6 (Alliot-Amrouche,[1]) *Let l be an integer, $n \geq 2$ and p satisfying :*

$$(H) \quad n/p' \notin \{1, \ldots, l\} \text{ and } n/p \notin \{1, \ldots, -l\}$$

If $\boldsymbol{f} \in \mathbf{W}_l^{-1,p}(\mathbb{R}^n)$ and $g \in W_l^{0,p}(\mathbb{R}^n)$, problem (S) has a solution $(\boldsymbol{u}, \pi)$ with $\boldsymbol{u} \in \mathbf{W}_l^{1,p}(\mathbb{R}^n)$ and $\pi \in W_l^{0,p}(\mathbb{R}^n)$ if and only if $\boldsymbol{f}$ and g satisfy :

$$\forall (\boldsymbol{\lambda}, \mu) \in N_{[l+1-n/p']}, \quad < \boldsymbol{f}, \boldsymbol{\lambda} >_{\mathbf{W}_l^{-1,p} \times \mathbf{W}_{-l}^{1,p'}} + < g, \mu >_{W_l^{0,p} \times W_{-l}^{0,p'}} = 0. \qquad (2.6)$$

In $\mathbf{W}_l^{1,p}(\mathbb{R}^n) \times W_l^{0,p}(\mathbb{R}^n)$, this solution is unique up to an element of $N_{[1-l-n/p]}$.

Remark 2.7 *i*) We have also proved in [1] the continuous dependence of the solution with respect to the the data but we shall not use this property here.
ii) In the sequel, we shall often use the assumption (H) which already appears with $p = 2$ in Proposition 2.5. In particular, let us point out that this condition is empty if $l = 0$ and reads otherwise

$$n/p \notin \{1, \ldots, -l\} \text{ if } l < 0 \quad \text{and} \quad n/p' \notin \{1, \ldots, l\} \text{ if } l > 0.$$

Moreover, when (H) is satisfied, one has $\mathcal{N}_l^p(\mathbb{R}^n) = N_{[1-l-n/p]}$.
iii) Problem (S) can be solved even if condition (H) is no longer satisfied. It is then necessary to work in slightly different weighted Sobolev spaces.

We now come to the

Proof of Proposition 2.5 : We first solve the problem in $\mathbb{R}^n$ by extending the data. Indeed, it results from Theorem 1.2-(ii) and from the Closed Range Theorem of Banach that there exists a second order tensor $F \in W_l^{0,2}(\Omega)$ such that $\operatorname{div} F = \boldsymbol{f}$. In particular, extending F by zero in Ω', we get a continuous extension of $\boldsymbol{f}$ in $\mathbf{W}_l^{-1,2}(\mathbb{R}^n)$. We also extend g by zero in Ω' and still denote by $(\boldsymbol{f}, g)$ the pair of extended distributions which obviously belongs to $\mathbf{W}_l^{-1,2}(\mathbb{R}^n) \times W_l^{0,2}(\mathbb{R}^n)$.

Now, since $l + 1 - n/2 < 0$ and considering Theorem 2.6, there exists a pair $(\boldsymbol{w}, \tau) \in \mathbf{W}_l^{1,2}(\mathbb{R}^n) \times W_l^{0,2}(\mathbb{R}^n)$ satisfying :

$$-\nu \Delta \boldsymbol{w} + \nabla \tau = \boldsymbol{f}, \quad - \operatorname{div} \boldsymbol{w} = g \quad \text{in } \mathbb{R}^n.$$

Moreover, in view of Theorem 2.2, the problem $(S_{\boldsymbol{\lambda}})$ introduced in Proposition 2.3, with $\boldsymbol{\lambda} = \boldsymbol{w}$, has, in $\mathbf{W}_0^{1,2}(\Omega) \times L^2(\Omega)$, one and only one solution $(\boldsymbol{v}, \eta)$. But $l < 0$, so that $(\boldsymbol{v}, \eta)$ also belongs to $\mathbf{W}_l^{1,2}(\Omega) \times W_l^{0,2}(\Omega)$. Thus, the pair $(\boldsymbol{w} - \boldsymbol{v}, \tau - \eta)$ restricted to Ω solves problem (S) with $\boldsymbol{\varphi} = \mathbf{0}$ and belongs to the desired space. $\Diamond$

We can now conclude with a full existence and uniqueness result in the case $p = 2$ with homogeneous Dirichlet boundary conditions.

Theorem 2.8 *Let $n \geq 3$ and l an integer such that $n/2 \notin \{1, \ldots, |l|\}$. When $\boldsymbol{f} \in \mathbf{W}_l^{-1,2}(\Omega)$, $g \in W_l^{0,2}(\Omega)$ and $\boldsymbol{\varphi} = \mathbf{0}$, problem (S) has a solution satisfying $(\boldsymbol{u}, \pi) \in \mathbf{W}_l^{1,2}(\Omega) \times W_l^{0,2}(\Omega)$ if and only if $\boldsymbol{f}$ and g satisfy :*

$$\forall (\boldsymbol{v}, \eta) \in \mathcal{N}_{-l}^2(\Omega), \quad < \boldsymbol{f}, \boldsymbol{v} >_{\mathbf{W}_l^{-1,2} \times \mathring{\mathbf{W}}_{-l}^{1,2}} + < g, \eta >_{W_l^{0,2} \times W_{-l}^{0,2}} = 0. \tag{2.7}$$

This solution is unique in $\mathbf{W}_l^{1,2}(\Omega) \times W_l^{0,2}(\Omega)$*, up to an element of* $\mathcal{N}_l^2(\Omega)$ *and it satisfies the estimate :*

$$\inf_{(\boldsymbol{v},\eta)\in\mathcal{N}_l^2(\Omega)} (\| \boldsymbol{u} + \boldsymbol{v} \|_{\mathbf{W}_l^{1,2}} + \| \pi + \eta \|_{W_l^{0,2}}) \leq C(\| \boldsymbol{f} \|_{\mathbf{W}_l^{-1,2}} + \| g \|_{W_l^{0,2}}),$$

where $C > 0$ *only depends on* ν, n, l *and* Ω*.*

Proof : Recall that the case $l = 0$ is proved in Theorem 2.1. More generally, the result amounts to proving that the operator :

$$T \; : \; (\overset{\circ}{\mathbf{W}}{}_l^{1,2}(\Omega) \times W_l^{0,2}(\Omega))/\mathcal{N}_l^2(\Omega) \longrightarrow (\mathbf{W}_l^{-1,2}(\Omega) \times W_l^{0,2}(\Omega)) \perp \mathcal{N}_{-l}^2(\Omega), \tag{2.8}$$

$$(\boldsymbol{u}, \pi) \longmapsto (-\nu\Delta\boldsymbol{u} + \nabla\pi, \; - \operatorname{div} \boldsymbol{u}), \tag{2.9}$$

is an isomorphism. The operator T is obviously injective in the quotient space. Moreover, if $l < 0$, then $\mathcal{N}_{-l}^2(\Omega) = \{(\mathbf{0}, 0)\}$, and T is therefore surjective thanks to Proposition 2.6 . Since T is obviously continuous, it is an isomorphism. Now, set $k = -l > 0$, then the adjoint of T,

$$T^* \; : \; \overset{\circ}{\mathbf{W}}{}_k^{1,2}(\Omega) \times W_k^{0,2}(\Omega) \longrightarrow (\mathbf{W}_k^{-1,2}(\Omega) \times W_k^{0,2}(\Omega)) \perp \mathcal{N}_{-k}^2(\Omega),$$

is also an isomorphism and one can prove making use of a generalised Green's formula that $T^*(\boldsymbol{u}, \pi) = (-\nu\Delta\boldsymbol{u} + \nabla\pi, -\operatorname{div} u)$ which concludes the proof. $\diamondsuit$

Remark 2.9 The restrictions made so far on the value of the integer l have excluded to have existence and uniqueness properties when $n = 2$ unless $l = 0$. We shall treat this case in the next section when $p \neq 2$. We shall also extend Theorem 2.8 to non-homogeneous Dirichlet boundary conditions.

3 Existence and uniqueness in the general case

We now complete the latter Theorem 2.8 by treating the case $p \neq 2$. Moreover, we thoroughly investigate the specificities of the case $n = 2$. We first establish regularity properties for solutions of problem (S) with homogeneous Dirichlet boundary conditions given by Theorem 2.1, when data have compact support.

Lemma 3.1 *Let* Ω *be a* $C^{1,1}$ *exterior domain and let* l *be an integer. Assume that* $p > 2$ *satisfies both* (H) *and* $n/p' > l + 1$*. Then, for any pair* $(\boldsymbol{f}, g)$ *in*

$\mathbf{W}_0^{-1,p}(\Omega) \times L^p(\Omega)$ *with compact supports in* $\overline{\Omega}$ *and* $\boldsymbol{\varphi} = \mathbf{0}$, *problem* (S) *has a one and only one solution satisfying :*

$$\boldsymbol{u} \in \mathring{\mathbf{W}}_0^{1,2}(\Omega) \cap \mathring{\mathbf{W}}_l^{1,p}(\Omega), \quad \pi \in L^2(\Omega) \cap W_l^{0,p}(\Omega). \tag{3.1}$$

We are going to derive this result from regularity properties of the Stokes problem in bounded domains and in $\mathbb{R}^n$. In particular, we establish some similar regularity properties for the Stokes problem in $\mathbb{R}^n$ without assuming $p > 2$.

Proposition 3.2 *Let* l *and* p *satisfy* (H) *and* $n/p' > l+1$. *Then, for any pair* $(\boldsymbol{f}, g)$ *with* $\boldsymbol{f} \in \mathbf{W}_0^{-1,2}(\mathbb{R}^n) \cap \mathbf{W}_l^{-1,p}(\mathbb{R}^n)$, $\quad g \in L^2(\mathbb{R}^n) \cap W_l^{0,p}(\mathbb{R}^n)$, *problem* (S) *has a solution* $(\boldsymbol{u}, \pi)$ *with :*

$$\boldsymbol{u} \in \mathbf{W}_0^{1,2}(\mathbb{R}^n) \cap \mathbf{W}_l^{1,p}(\mathbb{R}^n), \quad \pi \in L^2(\mathbb{R}^n) \cap W_l^{0,p}(\mathbb{R}^n), \tag{3.2}$$

if and only if

$$< \boldsymbol{f}, \boldsymbol{\lambda} >_{\mathbf{W}_0^{-1,2} \times \mathbf{W}_0^{1,2}} = 0, \ \forall \boldsymbol{\lambda} \in \boldsymbol{\mathcal{P}}_0, \quad \textit{when} \quad n = 2, \tag{3.3}$$

This solution is the only one satisfying (3.2) *if* $n \geq 3$, *but up to an element of* N_0 *if* $n = 2$.

Proof : As a consequence of Theorem 2.6, problem (S) has solutions :

$$(\boldsymbol{u}^1, \pi^1) \in \mathbf{W}_0^{1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n) \ \text{ and } \ (\boldsymbol{u}^2, \pi^2) \in \mathbf{W}_l^{1,p}(\mathbb{R}^n) \times W_l^{0,p}(\mathbb{R}^n).$$

In particular, $(\boldsymbol{u}^1 - \boldsymbol{u}^2, \pi^1 - \pi^2) = (\boldsymbol{\lambda}, \mu)$ has to belong to N_k for some integer k. Moreover, considering Proposition 1.4-(i), and since

$$\| \boldsymbol{\lambda}(|\boldsymbol{x}|, .)\|_{L^p(\Sigma)} \leq \| \boldsymbol{u}^1(|\boldsymbol{x}|, .)\|_{L^p(\Sigma)} + \| \boldsymbol{u}^2(|\boldsymbol{x}|, .)\|_{L^p(\Sigma)},$$

a simple integration argument shows that the degree of $\boldsymbol{\lambda}$ cannot exceed the value $\max(1 - n/2, 1 - n/p - l)$. Hence, one can assign this value to k and therefore prove that $N_k \subset \mathbf{W}_0^{1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ or $N_k \subset \mathbf{W}_l^{1,p}(\mathbb{R}^n) \times W_l^{0,p}(\mathbb{R}^n)$. Both properties imply that at least one of the $(\boldsymbol{u}^i, \pi^i), i = 1, 2$ belongs to the desired intersection. Finally, the uniqueness properties are a straightforward consequence of the polynomial form of solutions of problem (S) with zero data in $\mathbb{R}^n$. $\diamondsuit$

Before giving the proof of Lemma 3.1, we also recall the result proved in [2] for bounded domains.

Theorem 3.3 (Amrouche,Girault,[2]) *Let $\mathcal{O}$ be a bounded domain in $\mathbb{R}^n$, with $n \geq 2$, and with a $C^{1,1}$ boundary. Then, when $\boldsymbol{f} \in \mathbf{W}^{-1,p}(\mathcal{O}), g \in L^p(\mathcal{O})$ and $\boldsymbol{\varphi} \in W^{1/p',p}(\partial\mathcal{O})$, problem (S) has a unique solution $(\boldsymbol{u}, \pi)$ in $\mathbf{W}^{1,p}(\mathcal{O}) \times L^p(\mathcal{O})$ with $\int_{\mathcal{O}} \pi d\boldsymbol{x} = 0$ if and only if :*

$$\int_{\mathcal{O}} g(\boldsymbol{x}) d\boldsymbol{x} + \int_{\partial\mathcal{O}} \boldsymbol{\varphi}.\boldsymbol{n} ds = 0 \ . \tag{3.4}$$

Moreover, there exists $C > 0$, depending only on $\mathcal{O}, p, n$ and ν, such that :

$$\| \boldsymbol{u} \|_{\mathbf{W}^{1,p}(\mathcal{O})} + \| \pi \|_{L^p(\mathcal{O})} \leq C(\| \boldsymbol{f} \|_{\mathbf{W}^{-1,p}(\mathcal{O})} + \| g \|_{L^p(\mathcal{O})} + \| \boldsymbol{\varphi} \|_{\mathbf{W}^{1/p',p}(\partial\mathcal{O})}).$$

Proof of Lemma 3.1 : As the pair $(\boldsymbol{f}, g)$ has bounded support, it also belongs to $\mathbf{W}_0^{-1,2}(\Omega) \times L^2(\Omega)$. In particular, problem (S) has a unique solution $(\boldsymbol{u}, \pi)$ in $\overset{\circ}{\mathbf{W}}{}_0^{1,2}(\Omega) \times L^2(\Omega)$. Once we have extended $\boldsymbol{u}$ and π by zero in Ω', we define the distributions over the whole space $\mathbb{R}^n$:

$$\tilde{\boldsymbol{f}} = -\Delta \boldsymbol{u} + \nabla \pi \quad \text{and} \quad \tilde{g} = -\operatorname{div} \boldsymbol{u},$$

and $(\tilde{\boldsymbol{f}}, \tilde{g})$ belongs by construction to $\mathbf{W}_0^{-1,2}(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and also satisfies (3.3) if $n = 2$ (see theorem 2.6 with $p = n = 2$ and $l = 0$).

We are going to prove that $\boldsymbol{u}$ and π are in fact more regular in $\Omega_{R_0} = \Omega \cap B_{R_0}$, that is :

$$(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}(\Omega_{R_0}) \times L^p(\Omega_{R_0}), \tag{3.5}$$

if R_0 is such that both $\overline{\Omega'}$ and the support of $(\boldsymbol{f}, g)$ are contained in B_{R_0}. Indeed, if (3.5) is fulfilled, then the pair $(\tilde{\boldsymbol{f}}, \tilde{g})$ belongs to $\mathbf{W}^{-1,p}(\Omega_{R_0}) \times L^p(\Omega_{R_0})$ but, considering its compact support, also belongs to $\mathbf{W}_l^{-1,p}(\mathbb{R}^n) \times W_l^{0,p}(\mathbb{R}^n)$. Hence, $(\boldsymbol{u}, \pi)$ is anything else but the solution satisfying (3.1) given by Proposition 3.2 and therefore satisfies the desired regularity on Ω.

We now conclude by proving (3.5). Let us define, for all $R > R_0$, the cut-off functions :

$$\psi_R \in \mathcal{D}(\overline{B_R}), \quad 0 < \psi_R \leq 1 \text{ in } B_R, \quad \text{and} \quad \psi_R = 1 \ \text{ in } B_{R_0}.$$

Then, the pair $(\boldsymbol{w}_R, \tau_R) = (\boldsymbol{u}\psi_R, \pi\psi_R)$ belongs to $\overset{\circ}{\mathbf{W}}{}_0^{1,2}(\Omega) \times L^2(\Omega)$ and has compact support in $\overline{B_R}$. Elementary calculations on distributions show that it satisfies for all $R > R_0$:

$$\begin{array}{ll} -\Delta \boldsymbol{w}_R + \nabla \tau_R = \boldsymbol{f}\psi_R + \mathbf{F}_R & \text{in } \ \Omega_R, \\ -\operatorname{div} \boldsymbol{w}_R = g\psi_R + G_R & \text{in } \ \Omega_R, \\ \ \boldsymbol{w}_R = \mathbf{0} & \text{on } \ \partial\Omega_R, \end{array}$$

with $\mathbf{F}_R = -2\nabla \boldsymbol{u} \nabla \psi_R + \pi \nabla \psi_R - \boldsymbol{u} \Delta \psi_R$ and $G_R = -\boldsymbol{u}.\nabla \psi_R$. In particular, one easily checks that $(\mathbf{F}_R, G_R)$ belongs to $\mathbf{L}^2(\Omega_R) \times H^1(\Omega_R)$.
i) We first assume that $n = 2$ and we consider a real number $R_1 > R_0$. Then, thanks to Sobolev imbeddings,

$$(\mathbf{F}_{R_1}, G_{R_1}) \in \mathbf{W}^{-1,q}(\Omega_{R_1}) \times L^q(\Omega_{R_1}), \quad \forall q > 2, \tag{3.6}$$

and in particular if $q = p$. Then, Theorem 1.7 shows that $(\boldsymbol{w}_{R_1}, \tau_{R_1})$ belongs to $\mathbf{W}^{1,p}(\Omega_{R_1}) \times L^p(\Omega_{R_1})$.

ii) Suppose now that $n \geq 3$, then the property (3.6) remains true for q in $[2, 2^*]$. Thus, if $p \leq 2^*$, we can apply the preceding argument and the result is straightforward. If $p > 2^*$ and $R_1 > R_0$, one can still prove that $(\boldsymbol{w}_{R_1}, \tau_{R_1})$ belongs for instance to $\mathbf{W}^{1,2^*}(\Omega_{R_1}) \times L^{2^*}(\Omega_{R_1})$ because the pair $(\boldsymbol{f}\psi_{R_1}, g\psi_{R_1})$ belongs to $\mathbf{W}^{-1,2^*}(\Omega_{R_1}) \times L^{2^*}(\Omega_{R_1})$. As a consequence, for any real number R_2 with $R_0 < R_2 < R_1$, the pair $(\boldsymbol{u}, \pi)$ belongs to $\mathbf{W}^{1,2^*}(\Omega_{R_2}) \times L^{2^*}(\Omega_{R_2})$.

Thus, $(\mathbf{F}_{R_2}, G_{R_2})$ belongs to $\mathbf{L}^{2^*}(\Omega_{R_2}) \times W^{1,2^*}(\Omega_{R_2})$ and we can apply the same argument with 2^* instead of 2 and R_2 instead of R_1. From then on, it is easy to see that repeating at most $[n/2]$ times this argument permits to reach any value of $p > 2$. $\diamondsuit$

Corollary 3.4 *Under the assumptions of Lemma 3.1, and if* $\boldsymbol{\lambda} \in \mathbf{W}^{1/p',p}(\partial\Omega)$, *problem* $(S_{\boldsymbol{\lambda}})$ *has one and only one solution* $(\boldsymbol{u}, \pi)$ *in the class defined by* (3.1).

Proof : One can lift $\boldsymbol{\varphi}$ by a field $\boldsymbol{w} \in \mathbf{W}_0^{1,p}(\Omega)$ with compact support in $\overline{\Omega}$ (the solution $\boldsymbol{w} \in \mathbf{W}^{1,p}(\Omega_R)$ of problem $-\Delta \boldsymbol{w} = \mathbf{0}$ in Ω_R, $\boldsymbol{w} = \boldsymbol{\varphi}$ on $\partial\Omega$, $\boldsymbol{w} = \mathbf{0}$ on ∂B_R, extended by zero out of B_R provides such a lifting if R is large enough). Then, problem $(S_{\boldsymbol{\lambda}})$ is equivalent to problem :

$$(S'_{\boldsymbol{\lambda}}) \qquad \begin{array}{ll} -\Delta \boldsymbol{v} + \nabla \eta = \Delta \boldsymbol{w} & \text{in } \Omega, \\ -\operatorname{div} \boldsymbol{v} = \operatorname{div} \boldsymbol{w} & \text{in } \Omega, \\ \boldsymbol{v} = \mathbf{0} & \text{on } \partial\Omega, \end{array}$$

where the pair $(\Delta \boldsymbol{w}, \operatorname{div} \boldsymbol{w})$ belongs to $\mathbf{W}_0^{-1,p}(\Omega) \times L^p(\Omega)$ and has compact support. Let $(\boldsymbol{v}, \eta)$ be the solution of problem $(S'_{\boldsymbol{\lambda}})$ given by Lemma 3.1 then the pair $(\boldsymbol{v} + \boldsymbol{w}, \eta)$ solves problem $(S_{\boldsymbol{\lambda}})$ and satisfies the desired regularity. Besides, its uniqueness follows from Theorem 2.2 . $\diamondsuit$

Remark 3.5 Note that in Lemma 3.1 and Corollary 3.4, we have assumed that $\partial\Omega$ is $C^{1,1}$ which was not the case in section 2. This assumption is made neccessary because we use Theorem 3.3 in the proof. Hence, if Theorem 3.3 remains true with weaker regularity of the boundary, so does Lemma 3.1.

The next result characterises the kernels $\mathcal{N}_l^p(\Omega)$ under the assumptions of Lemma 3.1. To this end, we shall use for the case $n = 2$ the bidimensional fundamental solution $(U, \mathbf{Q})$ of Stokes problem given by :

$$U_{ij}(\boldsymbol{x}) = \frac{1}{4\pi\nu}(-\delta_{ij}\ln|\boldsymbol{x}| + \frac{x_i x_j}{|\boldsymbol{x}|^2}), \quad Q_i(\boldsymbol{x}) = \frac{1}{2\pi}\frac{x_i}{|\boldsymbol{x}|^2}, \quad i,j = 1,\ldots,n. \tag{3.7}$$

Proposition 3.6 *Under the assumptions of Lemma 3.1, one has :*
i) If $n \geq 3$,

$$\mathcal{N}_l^p(\Omega) = \{(\boldsymbol{v}(\boldsymbol{\lambda}) - \boldsymbol{\lambda}, \eta(\boldsymbol{\lambda}) - \mu),\ (\boldsymbol{\lambda}, \mu) \in N_{[1-n/p-l]}\}, \tag{3.8}$$

where $(\boldsymbol{v}(\boldsymbol{\lambda}), \eta(\boldsymbol{\lambda}))$ *denotes the solution of problem* $(S_{\boldsymbol{\lambda}})$ *satisfying* (3.1) *given by Corollary 3.4.*
ii) If $n = 2$,

$$\mathcal{N}_l^p(\Omega) = \{\ (\boldsymbol{v}(\boldsymbol{\lambda}+U\boldsymbol{\lambda}(\mathbf{0}))-\boldsymbol{\lambda}-U\boldsymbol{\lambda}(\mathbf{0}),\ \eta(\boldsymbol{\lambda}+U\boldsymbol{\lambda}(\mathbf{0}))-\mu-\mathbf{Q}.\boldsymbol{\lambda}(\mathbf{0})),$$
$$(\boldsymbol{\lambda}, \mu) \in N_{[1-2/p-l]}\quad\}, \tag{3.9}$$

where $(\boldsymbol{v}(\boldsymbol{\lambda} + U\boldsymbol{\lambda}(\mathbf{0})), \eta(\boldsymbol{\lambda} + U\boldsymbol{\lambda}(\mathbf{0})))$ *denotes the solution of problem* $(S_{\boldsymbol{\lambda}+U\boldsymbol{\lambda}(\mathbf{0})})$ *satisfying* (3.1) *given by Corollary 3.4.*

Proof : When $n \geq 3$, the proof is very similar to that of Proposition 2.3 . Indeed, let us extend $(\boldsymbol{u}, \pi) \in \mathcal{N}_l^p(\Omega)$ by zero in Ω'. Then, the distribution $\boldsymbol{h}$ defined by (2.5) belongs to $\mathbf{W}_0^{-1,p}(\mathbb{R}^n)$ and has compact support. Hence, since $n \geq 3$ and $n/p' > l + 1$, problem (2.4) has a unique solution $(\boldsymbol{v}, \eta)$ with :

$$\boldsymbol{v} \in \mathbf{W}_0^{1,2}(\mathbb{R}^n) \cap \mathbf{W}_l^{1,p}(\mathbb{R}^n), \quad \eta \in L^2(\mathbb{R}^n) \cap W_l^{0,p}(\mathbb{R}^n),$$

as a consequence of Proposition 3.2. Thus, we obtain like above that

$$\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{\lambda} \quad \text{and} \quad \eta = \pi + \mu,$$

where $(\boldsymbol{\lambda}, \mu)$ belongs to $N_{[1-l-n/p]}$. Then, by construction, the restriction of $(\boldsymbol{v}, \eta)$ to Ω is the solution given by Corollary 3.2 of the problem $(S_{\boldsymbol{\lambda}})$.

We still have to treat the case $n = 2$ which is rather different. Indeed, to solve problem (2.4) with Proposition 3.2 requires the vector $\overline{\boldsymbol{h}}$ given by :

$$\overline{h}_i = < h_i, 1 >_{W_0^{-1,2}(\mathbb{R}^2)\times W_0^{1,2}(\mathbb{R}^2)}, \quad i = 1, 2 \tag{3.10}$$

to be the null-vector, which is not generally true. We can nevertheless solve a slightly modified equivalent problem. To this end, we introduce for $R > 0$, a function $\psi_R \in \mathcal{D}([0, +\infty[)$, with $\psi_R \geq 0$, that equals 1 on $[0, R/2]$ and 0 in $[R, +\infty[$. Setting $\chi_R = 1 - \psi_R$, we define the truncated fundamental solution :

$$U^R(\boldsymbol{x}) = \chi_R(|\boldsymbol{x}|)U(\boldsymbol{x}) \quad \text{and} \quad \mathbf{Q}^R(\boldsymbol{x}) = \chi_R(|\boldsymbol{x}|)\mathbf{Q}(\boldsymbol{x}),$$

which is no longer singular at the origin. On the one hand, one easily proves that $U^R \in W_k^{1,q}(\mathbb{R}^2)$ and $\mathbf{Q}^R \in \mathbf{W}_k^{0,q}(\mathbb{R}^2)$ if and only if $1 - 2/q - k > 0$, that is $2/q' > k + 1$. On the other hand, the functions

$$F = -\nu\Delta U^R + \nabla\mathbf{Q}^R \ \text{ and } \ \mathbf{G} = -\operatorname{div} U^R,$$

have compact supports and respectively belong to $W_0^{-1,2}(\mathbb{R}^2)$ and $\mathbf{L}^2(\mathbb{R}^2)$. Moreover, using symmetry properties of $(U, \mathbf{Q})$, one shows by some elementary tedious calculations that,

$$< F_{ij}, 1 >_{W_0^{-1,2}(\mathbb{R}^2)\times W_0^{1,2}(\mathbb{R}^2)} = \delta_{ij}, \quad i, j = 1, 2.$$

Therefore, thanks to Proposition 3.2, the problem :

$$-\Delta\boldsymbol{v} + \nabla\eta = \ \boldsymbol{h} - F\overline{\boldsymbol{h}}, \quad \operatorname{div}\boldsymbol{v} = \ -\mathbf{G}.\overline{\boldsymbol{h}} \ \text{ in } \mathbb{R}^n,$$

has a unique solution $(\boldsymbol{v}, \eta)$ satisfying (3.1). Then, we can show as above that :

$$\boldsymbol{v} - \boldsymbol{u} + U^R\overline{\boldsymbol{h}} = \boldsymbol{\lambda} \ \text{ and } \ \eta - \pi + \mathbf{Q}^R.\overline{\boldsymbol{h}} = \mu,$$

where $(\boldsymbol{\lambda}, \mu)$ belongs to $N_{[1-n/p-l]}$. In particular, if we choose R small enough, then $B_R \subset \Omega'$ and we have the equalities in Ω :

$$\boldsymbol{u} = \boldsymbol{v} - \boldsymbol{\lambda} + U\overline{\boldsymbol{h}} \ \text{ and } \ \pi = \eta - \mu - \mathbf{Q}.\overline{\boldsymbol{h}}, \tag{3.11}$$

where $(\boldsymbol{v}, \eta)$ is indeed the solution $(\boldsymbol{v}(\boldsymbol{\lambda} + U\overline{\boldsymbol{h}}), \eta(\boldsymbol{\lambda} + U\overline{\boldsymbol{h}}))$ of problem $(S_{\boldsymbol{\lambda}+U\overline{\boldsymbol{h}}})$ given by Corollary 3.4. Finally, note that if $\mathbf{a}$ is a constant vector, the solution of $(S_{\mathbf{a}})$ given by Corollary 3.4 is $(\mathbf{a}, 0)$. In particular, the functions $\boldsymbol{u}$ and π defined by (3.11) do not depend on the constant coefficient $\boldsymbol{\lambda}(\mathbf{0})$ of the polynomial $\boldsymbol{\lambda}$ so that we can choose $\boldsymbol{\lambda}(\mathbf{0}) = \overline{\boldsymbol{h}}$, which completes the proof. $\diamondsuit$

Remark 3.7 Proposition 3.3 implies again that $\mathcal{N}_l^p(\Omega)$ and $N_{[1-l-n/p]}$ are isomorphic (see Remark 2.4) when $n \geq 2, p > 2$ and $n/p' > l + 1$.

We now apply the preceding results in order to solve homogeneous Dirichlet problem (S).

Theorem 3.8 *Let Ω be an exterior domain with a $C^{1,1}$ boundary.*
i) Let l be an integer and let $p > 2$ satisfy both (H) and $n/p' > l+1$. If $\boldsymbol{f}$ in $\mathbf{W}_l^{-1,p}(\Omega)$, g in $W_l^{0,p}(\Omega)$ and $\boldsymbol{\varphi} = \mathbf{0}$, then problem (S) has a solution $(\boldsymbol{u}, \pi)$ in $\mathbf{W}_l^{1,p}(\Omega) \times W_l^{0,p}(\Omega)$. In this space, the solution is unique up to an element of $\mathcal{N}_l^p(\Omega)$ and satisfies the estimate :

$$\inf_{(\boldsymbol{v},\eta)\in\mathcal{N}_l^p(\Omega)} (\| \boldsymbol{u} + \boldsymbol{v} \|_{\mathbf{W}_l^{1,p}} + \| \pi + \eta \|_{W_l^{0,p}}) \leq C(\| \boldsymbol{f} \|_{\mathbf{W}_l^{-1,p}} + \| g \|_{W_l^{0,p}}),$$

where $C > 0$ only depends on ν, p, n, l and Ω.
ii) Let l be an integer and let $p < 2$ satisfy both (H) and $n/p > 1 - l$. For $\boldsymbol{f}$ in $\mathbf{W}_l^{-1,p}(\Omega)$, g in $W_l^{0,p}(\Omega)$ and $\boldsymbol{\varphi} = \mathbf{0}$, then problem (S) has a solution in $\mathbf{W}_l^{1,p}(\Omega) \times W_l^{0,p}(\Omega)$ if and only if :

$$\forall (\boldsymbol{v}, \eta) \in \mathcal{N}_{-l}^{p'}(\Omega), \quad < \boldsymbol{f}, \boldsymbol{v} >_{\mathbf{W}_l^{-1,p} \times \mathring{\mathbf{W}}_{-l}^{1,p'}} + < g, \eta >_{W_l^{0,p} \times W_{-l}^{0,p'}} = 0.$$

Such a solution $(\boldsymbol{u}, \pi)$ is unique and satisfies the estimate :

$$\| \boldsymbol{u} \|_{\mathbf{W}_l^{1,p}} + \| \pi \|_{W_l^{0,p}} \leq C(\| \boldsymbol{f} \|_{\mathbf{W}_l^{-1,p}} + \| g \|_{W_l^{0,p}}),$$

where $C > 0$ only depends on ν, p, n, l and Ω.

Proof : Statement $i)$ is equivalent to prove that the operator :

$$T \ : \ (\mathring{\mathbf{W}}_l^{1,p}(\Omega) \times W_l^{0,p}(\Omega))/\mathcal{N}_l^p(\Omega) \longrightarrow \mathbf{W}_l^{-1,p}(\Omega) \times W_l^{0,p}(\Omega),$$

defined by (2.9), is an isomorphism. This operator is obviously injective in the quotient space and continuous. Hence, we only have to prove that T is also surjective, that is, to prove existence of a solution in $\mathbf{W}_l^{1,p}(\Omega) \times W_l^{0,p}(\Omega)$ to problem (S) with $(\boldsymbol{f}, g)$ in $\mathbf{W}_l^{-1,p}(\Omega) \times W_l^{0,p}(\Omega)$.

We proceed as in Proposition 2.5. We extend by the same means the data and then the extended distributions $(\boldsymbol{f}, g)$ belong to $\mathbf{W}_l^{-1,p}(\mathbb{R}^n) \times W_l^{0,p}(\mathbb{R}^n)$. Since $n/p' > l + 1$, Theorem 2.6 provides a solution $(\boldsymbol{w}, \tau)$ in $\mathbf{W}_l^{1,p}(\mathbb{R}^n) \times W_l^{0,p}(\mathbb{R}^n)$ to problem :

$$-\nu\Delta\boldsymbol{w} + \nabla\tau = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{w} = g \quad \text{in } \mathbb{R}^n.$$

Moreover, problem $(S_{\boldsymbol{\lambda}})$ with $\boldsymbol{\lambda} = \boldsymbol{w}$ has a solution $(\boldsymbol{v}, \eta) \in \mathbf{W}_l^{1,p}(\Omega) \times W_l^{0,p}(\Omega)$, thanks to Corollary 3.4. Then, $(\boldsymbol{w} - \boldsymbol{v}, \tau - \eta)$ solves homogeneous Dirichlet problem (S) and belongs to $\mathbf{W}_l^{1,p}(\Omega) \times W_l^{0,p}(\Omega)$. Finally, statement $ii)$ follows from $i)$ by duality (see also the proof of Theorem 2.8) $\diamondsuit$

Remark 3.9 Theorem 3.8, *ii*) proves in particular that $\mathcal{N}_l^p(\Omega) = \{\,(\mathbf{0}, 0)\,\}$ whenever $p < 2$ and $l < 1 - n/p$. Hence, both characterisations established in Proposition 3.6 remain true with such assumptions

Let us now give a generalisation of Theorem 3.8 which is specific to the bidimensional case. Assume $n = 2$ and consider for any p the real number $\alpha(p) = 1 - 2/p$ (note that $\alpha(p)$ is an integer if and only if $p = 2$) then one can show that both Theorem 2.6 and Proposition 3.2 remain true if the integer l is replaced by $\alpha(p)$. As a consequence, we have the following result :

Theorem 3.10 *Let $\Omega \subset \mathbb{R}^2$ be an exterior domain with a $C^{1,1}$ boundary. For $\boldsymbol{f}$ in $\mathbf{W}^{-1,p}_{\alpha(p)}(\Omega)$, g in $W^{0,p}_{\alpha(p)}(\Omega)$ and $\boldsymbol{\varphi} = \mathbf{0}$, problem (S) has a solution $(\boldsymbol{u}, \pi)$ in $\mathbf{W}^{1,p}_{\alpha(p)}(\Omega) \times W^{0,p}_{\alpha(p)}(\Omega)$. In this space, the solution is unique and satisfies the estimate :*

$$\|\,\boldsymbol{u}\,\|_{\mathbf{W}^{1,p}_{\alpha(p)}} + \|\,\pi\,\|_{W^{0,p}_{\alpha(p)}} \leq C(\|\,\boldsymbol{f}\,\|_{\mathbf{W}^{-1,p}_{\alpha(p)}} + \|\,g\,\|_{W^{0,p}_{\alpha(p)}}),$$

where $C > 0$ only depends on ν, p and Ω.

Proof : When $p > 2$, the proofs of both Lemma 3.1 an Corollary 3.4 can be adapted replacing the integer l by $\alpha(p)$. Moreover, one can prove, following the ideas of Proposition 3.6 that $\mathcal{N}^p_{\alpha(p)}(\Omega) = \{(\mathbf{0}, 0)\}$ because in this case the vector $\overline{\boldsymbol{h}}$ defined by (3.10) necessarily equals zero. Then, the existence of solutions is obtained as in Theorem 3.8, up to minor modifications. Finally, since one checks that $\alpha(p') = -\alpha(p)$ for any p, the case $p < 2$ follows from the case $p > 2$ by duality. $\diamondsuit$

Remark 3.11 This result covers all the cases when some logarithmic weight appears in the definition of $W^{1,p}_\alpha$ and $n = 2$. It also proves that $\mathcal{N}^p_{\alpha(p)}(\Omega)$ is reduced to $\{\,(\mathbf{0}, 0)\,\}$ and thus has not the same dimension as $\mathcal{N}^p_{\alpha(p)}(\mathbb{R}^2) = \boldsymbol{\mathcal{P}}_0 \times \{0\}$.

We now consider the cases left untreated by Theorem 3.8. The method is similar : it is also based on an adequate existence result for the problem $(S_{\boldsymbol{\lambda}})$, namely

Lemma 3.12 *i) Let $n \geq 3$, $p < 2$ and l_0 defined by :*

$$l_0 = -1 \quad \textit{if } p < n/2 \quad \textit{or} \quad l_0 = 0 \quad \textit{if} \quad n/2 \leq p < 2. \tag{3.12}$$

If $\boldsymbol{\lambda} \in \mathbf{W}^{1/p',p}(\partial\Omega)$, problem $(S_{\boldsymbol{\lambda}})$ has, in $\mathbf{W}^{1,p}_{l_0}(\Omega) \times W^{0,p}_{l_0}(\Omega)$, a unique solution.
ii) Let $n = 2$ and $\alpha(p) = 1 - 2/p$. If $\boldsymbol{\lambda}$ in $\mathbf{W}^{1/p',p}(\partial\Omega)$, problem $(S_{\boldsymbol{\lambda}})$ has in $\mathbf{W}^{1,p}_{\alpha(p)}(\Omega) \times W^{0,p}_{\alpha(p)}(\Omega)$, a unique solution.

Proof : In both cases, the proof looks like that of Corollary 3.4. We solve the equivalent problem $(S'_{\boldsymbol{\lambda}})$ where the field $\boldsymbol{w} \in \mathbf{W}_0^{1,p}(\Omega)$ has compact support, and equals $\boldsymbol{\varphi}$ on $\partial\Omega$. On the one hand, if $n \geq 3$, this problem has a solution $(\boldsymbol{u}, \pi) \in \mathbf{W}_{l_0}^{1,p}(\Omega) \times W_{l_0}^{0,p}(\Omega)$ in view of Theorem 3.8-(ii) (it is a simple matter of check to see that $n/p > 1 - l_0$) and when $n = 2$, problem $(S'_{\boldsymbol{\lambda}})$ is solved with Theorem 3.10. $\diamondsuit$

We now give a complete characterisation of the spaces $\mathcal{N}_l^p(\Omega)$.

Theorem 3.13 *Let l be an integer and p satisfy (H). Then, if $n \geq 3$ then* (3.8) *holds. If $n = 2$,* (3.9) *holds unless $p = 2$ and $l = 0$.*

Proof : We have already treated the case $p > 2$ and $n/p' > l + 1$ in Proposition 3.6 and the case $p = 2$ in Proposition 2.3. We now assume that $p < 2$ and $n/p' > l + 1$, and we proceed like in Proposition 3.6. Indeed, let $(\boldsymbol{u}, \pi) \in \mathcal{N}_l^p(\Omega)$ and $\boldsymbol{h}$ be the distribution defined by (2.5). Then, $\boldsymbol{h}$ has compact support in $\mathbb{R}^n$ and thus belongs to $\mathbf{W}_{l_0}^{-1,p}(\mathbb{R}^n)$ where l_0 is defined by (3.12). Therefore, if $n \geq 3$, one can show using theorem 2.6 that problem (2.4) has a solution $(\boldsymbol{v}, \eta)$ in $\mathbf{W}_{l_0}^{1,p}(\mathbb{R}^n) \times W_{l_0}^{0,p}(\mathbb{R}^n)$ that is unique in this space. Thus, since $l_0 \geq l$,

$$(\boldsymbol{v} - \boldsymbol{u}, \eta - \pi) = (\boldsymbol{\lambda}, \mu) \in N_{[1-n/p-l]},$$

so that the restriction of $(\boldsymbol{v}, \tau)$ to Ω is the only solution in $\mathbf{W}_{l_0}^{1,p}(\Omega) \times W_{l_0}^{0,p}(\Omega)$ of problem $(S_{\boldsymbol{\lambda}})$ given by Lemma 3.12. Now, since it is a polynomial, $\boldsymbol{\lambda}$ also belongs to $\mathbf{H}^{1/2}(\partial\Omega)$. Finally, a regularity argument similar to the one developed in the proof of Lemma 3.1, proves that $(\boldsymbol{v}, \eta)$ also belongs to $\mathbf{W}_0^{1,2}(\Omega) \times L^2(\Omega)$. When $n = 2$ and $2/p' > l + 1$, we can proceed similarly if we replace l_0 by the real exponent $\alpha(p) = 1 - 2/p$.

When $n/p' \leq l + 1$, elementary calculations show that $l \geq l_0$ if $n \geq 3$, and $l > \alpha(p)$ if $n = 2$. Therefore, $\mathcal{N}_l^p(\Omega) = \{(\mathbf{0}, 0)\}$ as a consequence of Remarks 3.9, 3.11 and of obvious imbeddings, which concludes the proof. $\diamondsuit$

The next result specifies the asymptotic behaviour of the elements of $\mathcal{N}_l^p(\Omega)$.

Corollary 3.14 *In characterisation* (3.8) *(resp.* (3.9) *), the vector field $\boldsymbol{v}(\boldsymbol{\lambda})$ (resp. $\boldsymbol{v}(\boldsymbol{\lambda} + U\boldsymbol{\lambda}(\mathbf{0}))$) is negligible compared to $\boldsymbol{\lambda}$ (resp. $\boldsymbol{\lambda} - \boldsymbol{\lambda}(\mathbf{0}) + U\boldsymbol{\lambda}(\mathbf{0})$) at infinity.*

Proof : *i)* Assume that $n \geq 3$. If $\boldsymbol{\lambda}$ is a polynomial function then it belongs to all spaces $\mathbf{W}^{1/q',q}(\partial\Omega)$ with $q \in]1, +\infty[$. In particular, Corollary 3.4 shows that $\boldsymbol{v}(\boldsymbol{\lambda})$ belongs to all spaces $\mathbf{W}_k^{1,q}(\Omega)$ with $q > n$ and k such that $n/q' > k + 1$ (*i.e.*

$1 - n/q - k > 2 - n$). If we choose $k = n - 2$ and make p vary in the interval $]n, +\infty[$, Proposition 1.4 then yields :

$$\boldsymbol{v}(\boldsymbol{\lambda})(\boldsymbol{x}) = o(|\boldsymbol{x}|^{\gamma}), \quad \forall \gamma > 2 - n \quad \text{as} \quad |\boldsymbol{x}| \longrightarrow +\infty.$$

ii) Similarly, when $n = 2$, Lemma 3.11-(ii) proves that $\boldsymbol{v}(\boldsymbol{\lambda} + U\boldsymbol{\lambda}(\mathbf{0})) \in \mathbf{W}^{1,q}_{\alpha_0}(\Omega)$ for all $q > 2$. In this case, Proposition 1.4 states that :

$$\boldsymbol{v}(\boldsymbol{\lambda} + U\boldsymbol{\lambda}(\mathbf{0}))(\boldsymbol{x}) = o(\ln |\boldsymbol{x}|), \quad \text{as} \quad |\boldsymbol{x}| \longrightarrow +\infty, \tag{3.13}$$

and is therefore negligible compared to $\boldsymbol{\lambda} - \boldsymbol{\lambda}(\mathbf{0}) + U\boldsymbol{\lambda}(\mathbf{0})$. $\diamondsuit$

Remark 3.15 These properties can be related to the well-known Stokes Paradox. This paradox arising from the modelling of viscous fluids flows originally states the following property : if Ω is the exterior of a disk of $\mathbb{R}^2$, problem (S) with zero data has no classical solution $(\boldsymbol{u}, \pi)$ such that $\boldsymbol{u}$ tends to a prescribed non-zero vector at infinity. This property may be generalised to arbitrary exterior domains of $\mathbb{R}^2$ with smooth boundary and to a wider class of solutions (see for instance [10]).

Corollary 3.14 provides another proof of this generalisation : let Ω be an exterior domain with a $C^{1,1}$ boundary, and let $(\boldsymbol{u}, \pi)$ solve problem (S) with null-data. Then, $\boldsymbol{u}$ and π are locally smooth functions. Let us now assume that $\boldsymbol{u}$ and π are tempered distributions. Then, it is not difficult to see that $(\boldsymbol{u}, \pi)$ necessarily belongs to $\mathcal{N}^p_l(\Omega)$ for some p and integer l . Therefore, the vector field $\boldsymbol{u}$ is of the form $\boldsymbol{\lambda} + U\boldsymbol{\lambda}(\mathbf{0}) - \boldsymbol{v}(\boldsymbol{\lambda} + U\boldsymbol{\lambda}(\mathbf{0}))$ for some polynomial $\boldsymbol{\lambda}$. Thus, because of (3.13), the field $\boldsymbol{u}$ cannot behave like $o(\ln |\boldsymbol{x}|)$ at infinity unless $\boldsymbol{\lambda} = \mathbf{0}$ which enlarges the original statement of the paradox since, in particular, $\boldsymbol{u}$ cannot tend to a prescribed non-zero constant vector (see [6], theorem V.3.5 for a similar result).

We conclude this section with a general existence and uniqueness result for problem (S) in weighted Sobolev spaces.

Theorem 3.16 *Let Ω be an exterior domain with a Lipschitz-continuous boundary if $p = 2$ and a $C^{1,1}$ boundary otherwise and let l be an integer satisfying (H). Then, for $\boldsymbol{f}$ in $\mathbf{W}^{-1,p}_l(\Omega)$, g in $W^{0,p}_l(\Omega)$ and $\boldsymbol{\varphi}$ in $\mathbf{W}^{1/p',p}(\partial\Omega)$, problem (S) has a solution in $\mathbf{W}^{1,p}_l(\Omega) \times W^{0,p}_l(\Omega)$ if and only if*

$$\forall (\boldsymbol{v}, \eta) \in \mathcal{N}^{p'}_{-l}(\Omega), \quad < \boldsymbol{f}, \boldsymbol{v} > \; + \; < g, \eta > \; + < \boldsymbol{\varphi}, (\nu \nabla \boldsymbol{v} - \eta I).\boldsymbol{n} >_{\partial\Omega} = 0. \tag{3.14}$$

In this space, a solution $(\boldsymbol{u}, \pi)$ is unique up to an element of $\mathcal{N}_l^p(\Omega)$ and satisfies the estimate :

$$\inf_{(\boldsymbol{v},\eta)\in\mathcal{N}_l^p(\Omega)}(\| \boldsymbol{u}+\boldsymbol{v} \|_{\mathbf{W}_l^{1,p}} + \| \pi+\eta \|_{W_l^{0,p}}) \leq C(\| \boldsymbol{f} \|_{\mathbf{W}_l^{-1,p}} + \| g \|_{W_l^{0,p}} + \| \boldsymbol{\varphi} \|_{\mathbf{W}^{1/p',p}}),$$

where $C > 0$ only depends on Ω, p, n, ν and l.

Proof : $i)$ We first complete the proof of the case $\boldsymbol{\varphi} = \mathbf{0}$. In the cases untreated so far, that is,

$$p < 2, n/p' > l+1, \quad \text{or} \quad p > 2, n/p < 1-l,$$

the proof is similar to that of Theorem 3.8, but uses Lemma 3.12 instead of Corollary 3.4.
$ii)$ Suppose now that $\boldsymbol{\varphi} \neq \mathbf{0}$, then we can lift it by a function $\boldsymbol{w}$ that belongs to $\mathbf{W}_l^{1,p}(\Omega)$, has compact support and satisfies the estimate :

$$\| \boldsymbol{w} \|_{\mathbf{W}_l^{1,p}(\Omega)} \leq C\|\boldsymbol{\varphi}\|_{\mathbf{W}^{1/p',p}(\partial\Omega)}. \tag{3.15}$$

In particular, setting $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \boldsymbol{w}$, problem (S) is equivalent to the problem :

$$(S') \qquad \begin{array}{ll} -\nu\Delta\tilde{\boldsymbol{u}} + \nabla\pi = \boldsymbol{f} + \nu\Delta\boldsymbol{w} & \text{in } \ \Omega, \\ -\operatorname{div}\tilde{\boldsymbol{u}} = g + \operatorname{div}\boldsymbol{w} & \text{in } \ \Omega, \\ \tilde{\boldsymbol{u}} = \mathbf{0} & \text{on } \ \partial\Omega, \end{array}$$

Now, problem (S') has a solution in $\mathring{\mathbf{W}}_l^{1,p}(\Omega) \times W_l^{0,p}(\Omega)$ if and only if :

$$\forall\,(\boldsymbol{v},\eta) \in \mathcal{N}_{-l}^{p'}(\Omega), \quad <\boldsymbol{f}+\nu\Delta\boldsymbol{w}, \boldsymbol{v}> \; + \; <g + \operatorname{div}\boldsymbol{w}, \eta> = 0.$$

But the latter condition is equivalent to (3.14). Indeed, for any $(\boldsymbol{v},\eta)$ belonging to $\mathcal{N}_{-l}^{p'}(\Omega)$, Green's formula (2.3) holds by density for any pair $(\boldsymbol{\psi},\theta)$ in $\mathbf{W}_l^{1,p}(\Omega) \times W_l^{0,p}(\Omega)$. In particular, we have for the pair $(\boldsymbol{w},0)$:

$$<\nu\Delta\boldsymbol{w}, \boldsymbol{v}> \; + \; <\operatorname{div}\boldsymbol{w}, \eta> = \; - <\boldsymbol{\varphi}, (\nu\nabla\boldsymbol{v} - \eta I).\boldsymbol{n}>_{\partial\Omega} .$$

Finally, estimate (3.15) and the one satisfied by $(\tilde{\boldsymbol{u}}, \pi)$ immediately provide the appropriate estimate. $\diamondsuit$

Remark 3.17 In view of Proposition 1.4, $\boldsymbol{u}$ vanishes at infinity when $p > n$ and $1 - n/p - l < 0$, which is the case for instance as soon as $l \geq 1$. When $n = 2$, such conditions imply that $\mathcal{N}_{-l}^{p'}(\Omega)$ is not reduced to zero and that the data must satisfy some compatibility conditions, which is not necessary when $n \geq 3$.

4 Regularity results

As in the case of a bounded domain, one can establish that the local regularity of the solutions of the problem (S) increases with that of the data. The next result sets such a property in weighted Sobolev spaces and thus also takes into account the asymptotic behaviour of solutions and their derivatives. In particular, assume k is an integer, set $l = k+1$ and assume that p satisfy (H), we introduce the spaces :

$$W^{2,p}_{k+1}(\Omega) = \{\ u \in W^{1,p}_{k}(\Omega), \rho^{k+1}\nabla^2 u \in L^p(\Omega)\ \}, \quad \text{if } n/p + (k+1) \neq 1,$$

and if $n/p + (k+1) = 1$:

$$W^{2,p}_{k+1}(\Omega) = \{\ u/\ \rho^{k-1}(\ln\rho)^{-1}\nabla u \in L^p(\Omega), \rho^{k}(\ln\rho)^{-1} u \in L^p(\Omega), \rho^{k+1}\nabla^2 u \in L^p(\Omega)\ \}.$$

Let us recall then a regularity result established in [1] for problem (S) in $\mathbb{R}^n$.

Theorem 4.1 (Alliot-Amrouche,[1]) *Let k be an integer and p satisfy (H) with $l = k+1$. Then, for $\boldsymbol{f}$ in $\mathbf{W}^{0,p}_{k+1}(\mathbb{R}^n)$, g in $W^{1,p}_{k+1}(\mathbb{R}^n)$, problem (S) has a solution in $\mathbf{W}^{2,p}_{k+1}(\mathbb{R}^n) \times W^{1,p}_{k+1}(\mathbb{R}^n)$ if and only if $\boldsymbol{f}$ and g satisfy (2.6). Such a solution is unique up to an element of $N_{[1-k-n/p]}$ and satisfies the estimate :*

$$\inf_{(\boldsymbol{v},\eta)\in N_{[1-k-n/p]}} (\| \boldsymbol{u}+\boldsymbol{v} \|_{\mathbf{W}^{2,p}_{k+1}} + \| \pi+\eta \|_{W^{1,p}_{k+1}}) \leq C(\| \boldsymbol{f} \|_{\mathbf{W}^{0,p}_{k+1}} + \| g \|_{W^{1,p}_{k+1}}), \quad (4.1)$$

where $C > 0$ only depends on p, n, k and ν.

Owing to this result, it is not difficult to adapt the arguments developed above in order to prove Theorem 3.16. In particular, one can show that under the assumptions of Theorem 4.1, $\mathcal{N}^p_k(\Omega) \subset \mathbf{W}^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ and finally obtain the regularity result :

Theorem 4.2 *Let Ω be an exterior domain with $C^{1,1}$ boundary and k an integer such that p satisfies (H) with $l = k+1$. For $\boldsymbol{f}$ in $\mathbf{W}^{0,p}_{k+1}(\Omega)$, g in $W^{1,p}_{k+1}(\Omega)$ and $\boldsymbol{\varphi}$ in $\mathbf{W}^{1+1/p',p}(\partial\Omega)$, problem (S) has a solution $(\boldsymbol{u}, \pi)$ in $\mathbf{W}^{2,p}_{k+1}(\Omega) \times W^{1,p}_{k+1}(\Omega)$ if and only if $\boldsymbol{f}, g$ and $\boldsymbol{\varphi}$ satisfy (3.14). Such a solution is unique up to an element of $\mathcal{N}^p_k(\Omega)$ and satisfies the estimate :*

$$\inf_{(\boldsymbol{v},\eta)\in \mathcal{N}^p_k(\Omega)} (\| \boldsymbol{u}+\boldsymbol{v} \|_{\mathbf{W}^{2,p}_{k+1}} + \| \pi+\eta \|_{W^{1,p}_{k+1}}) \leq C(\| \boldsymbol{f} \|_{\mathbf{W}^{0,p}_{k+1}} + \| g \|_{W^{1,p}_{k+1}} + \| \boldsymbol{\varphi} \|_{\mathbf{W}^{1+1/p',p}}),$$

where $C > 0$ depends only on Ω, p, n, ν and k.

Let us recall that if we take $k = -1$ in Theorem 4.1, then :

$$1 - k - n/p = 2 - n/p < 2,$$

and $N_{[2-n/p]}$ only contains affine functions. In particular, estimate (4.1) implies that the second derivatives of a solution $\boldsymbol{u}$ are controlled in $L^p(\mathbb{R}^n)$ by the data, that is :

$$\| \nabla^2 \boldsymbol{u} \|_{L^p(\mathbb{R}^n)} + \| \nabla \pi \|_{L^p(\mathbb{R}^n)} \leq C(\| \boldsymbol{f} \|_{\mathbf{L}^p(\mathbb{R}^n)} + \| g \|_{W_0^{1,p}(\mathbb{R}^n)}).$$

More generally, this estimate holds true for any solution $(\boldsymbol{v}, \eta)$ of problem (S) with $D^2\boldsymbol{v}$ and $\nabla\eta$ in $L^p(\Omega)$. Indeed, such a solution differs from the solution $(\boldsymbol{u}, \pi)$ given by Theorem 4.2 by $(\boldsymbol{\lambda}, \mu) \in N_1$, so that in particular $\nabla^2\boldsymbol{u} = \nabla^2\boldsymbol{v}$ and $\nabla\pi = \nabla\eta$.

However, it is a well-known fact that the situation is different in exterior domains. For instance, when $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$, and $g, \boldsymbol{\varphi}$ vanish, W. Borchers and T. Miyakawa prove in [4] that every solution of problem (S) with $\boldsymbol{u} \in \mathbf{W}^{2,p}(\Omega)$ and $\pi \in W^{1,p}(\Omega)$ satisfies :

$$\| \nabla^2 \boldsymbol{u} \|_{L^p(\Omega)} + \| \nabla \pi \|_{L^p(\Omega)} \leq C \| \boldsymbol{f} \|_{\mathbf{L}^p(\Omega)} \quad \text{iff } \ n \geq 3 \text{ and } p < n/2. \tag{4.2}$$

In the other cases, H. Kozono and T. Ogawa prove in [14](Theorem 1.1) the following general *a priori* estimate (see also [11] for the case $p = 2$) :

$$\| \nabla^2 \boldsymbol{u} \|_{L^p(\Omega)} \leq C(\| \boldsymbol{f} \|_{\mathbf{L}^p(\Omega)} + \| \nabla \boldsymbol{u} \|_{L^r(\Omega)}) \quad \text{if } r \geq p \geq n/2. \tag{4.3}$$

Note that if $n/2 \leq p < n$, then any solution given by Theorem 4.2 satisfies $\nabla\boldsymbol{u} \in L^{p*}(\Omega)$ thanks to Sobolev imbeddings. On the contrary, if $p \geq n$, $\nabla\boldsymbol{u}$ does not necessarily belong to some L^r space so that the right-hand side of the estimate may not be defined.

The next result improves estimates (4.3) and (4.6) and naturally links the control of $\nabla^2\boldsymbol{u}$ in L^p with the uniqueness of the solution of problem (S) in $\mathbf{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$. Let us beforehand introduce the finite dimensional space :

$$\mathbf{E} = \{\boldsymbol{v} \in \mathbf{W}_0^{2,p}(\Omega), \ \exists\eta, (\boldsymbol{v}, \eta) \in \mathcal{N}_{-1}^p(\Omega)\}, \tag{4.4}$$

and denote by $\| \, . \, \|_{\mathbf{E}}$ a fixed norm on $\mathbf{E}$. We also introduce a continuous linear projection operator :

$$P \ : \ \mathbf{W}_0^{2,p}(\Omega) \longrightarrow \mathbf{E}, \tag{4.5}$$

whose existence follows from Hahn-Banach Theorem. Then, we have

Theorem 4.3 *Let $\boldsymbol{f} \in \mathbf{L}^p(\Omega)$ and assume that g and $\boldsymbol{\varphi}$ vanish. Then, any solution $(\boldsymbol{u}, \pi)$ of the problem (S) in $\mathbf{W}_0^{2,p}(\Omega) \times W_0^{1,p}(\Omega)$ satisfies the estimate :*

$$\| \nabla^2 \boldsymbol{u} \|_{L^p(\Omega)} \leq C(\| \boldsymbol{f} \|_{\mathbf{L}^p(\Omega)} + \| P\boldsymbol{u} \|_{\mathbf{E}}), \tag{4.6}$$

where $C > 0$ is independent of $\boldsymbol{f}$ and $\boldsymbol{u}$.

Proof : The estimate provided by Theorem 4.2 with $k = -1$ easily implies that :

$$\inf_{\boldsymbol{v} \in \mathbf{E}} \| \boldsymbol{u} + \boldsymbol{v} \|_{\mathbf{W}_0^{2,p}(\Omega)} \leq C \| \boldsymbol{f} \|_{\mathbf{L}^p(\Omega)}. \tag{4.7}$$

Moreover, since $\mathbf{E}$ is a finite dimensional subspace of $\mathbf{W}_0^{2,p}(\Omega)$, one can easily show the following equivalence of norms on $\mathbf{W}_0^{2,p}(\Omega)/\mathbf{E}$:

$$\inf_{\boldsymbol{v} \in \mathbf{E}} \| \boldsymbol{u} + \boldsymbol{v} \|_{\mathbf{W}_0^{2,p}(\Omega)} \sim \| \boldsymbol{u} - P\boldsymbol{u} \|_{\mathbf{W}_0^{2,p}(\Omega)}. \tag{4.8}$$

Thus, we can deduce from (4.7) and (4.8) the estimate :

$$\| D^2(\boldsymbol{u} - P\boldsymbol{u}) \|_{L^p(\Omega)} \leq C \| \boldsymbol{f} \|_{\mathbf{L}^p(\Omega)},$$

and therefore :

$$\| D^2 \boldsymbol{u} \|_{L^p(\Omega)} \leq C(\| \boldsymbol{f} \|_{\mathbf{L}^p} + \| D^2(P\boldsymbol{u}) \|_{L^p(\Omega)}).$$

This concludes the proof since the quantity $\| D^2 \,.\, \|_{L^p}$ defines a norm on $\mathbf{E}$ which is necessarily equivalent to $\| \,.\, \|_{\mathbf{E}}$. $\diamondsuit$

Remark 4.4 This estimate is more general than (4.2) and (4.3) in many aspects.
i) When $p < n/2$, the class of solutions to which the estimate (4.6) applies is much larger since $\mathbf{W}^{2,p}(\Omega)$ is strictly imbedded in $\mathbf{W}_0^{2,p}(\Omega)$.
ii) When $n/2 \leq p < n$, we recall that if $\boldsymbol{u} \in \mathbf{W}_0^{2,p}(\Omega)$ then $\nabla \boldsymbol{u}$ belongs to $L^{p^*}(\Omega)$. Anyway (4.6) is still sharper than (4.3) since one can prove that if $r \geq p$:

$$\| P\boldsymbol{u} \|_{\mathbf{E}} \leq C \| \nabla \boldsymbol{u} \|_{L^r(\Omega)},$$

(assume this is false and use the fact that P is a compact operator) whereas the reverse inequality is not satisfied. Indeed, if it was, then the kernel of P would be reduced to zero and therefore one would have $\mathbf{W}_0^{2,p}(\Omega) = \mathbf{E}$, which is obviously impossible.
iii) When $p \geq n$, estimate (4.6) applies to a larger class of solutions than (4.3), since we need not to assume that $\nabla \boldsymbol{u}$ belongs to some L^r space. Moreover, any solution of problem (S) with $\nabla^2 \boldsymbol{u} \in L^p(\Omega)$ in fact belongs to $\mathbf{W}_0^{2,p}(\Omega)$ (we omit here the proof of this property). Hence, in this case, all solutions $(\boldsymbol{v}, \eta)$ to problem (S) with $D^2 \boldsymbol{v} \in L^p(\Omega)$ satisfies (4.6).

Let us finally point out that estimate (4.6) readily extends when g and $\boldsymbol{\varphi}$ do not vanish any more. Moreover, one can also generalise such estimates in weighted spaces $\mathbf{W}_l^{0,p}(\Omega)$ provided (H) is satisfied.

5 Asymptotic properties

Let us investigate the asymptotic behaviour of some of the solutions we have constructed above. In most of the existing literature, the authors give sufficient conditions for the velocity field to behave like a given polynomial at infinity or to tend to $\mathbf{0}$. A few other works give more accurate results consisting in asymptotic representation formulae such as :

$$\boldsymbol{u}(\boldsymbol{x}) = \mathbf{P}(\boldsymbol{x}) + U(\boldsymbol{x})\mathbf{c} + \boldsymbol{\sigma}(\boldsymbol{x}),$$

where $\mathbf{P}$ is a polynomial, $\mathbf{c}$ is a constant vector and the remainder $\boldsymbol{\sigma}$ is negligible compared to each of the other terms. We recall that $(U, \mathbf{Q})$ denotes the fundamental solution of Stokes problem and is defined by (3.7) if $n = 2$, but by

$$U_{ij}(\boldsymbol{x}) = \delta_{ij}\frac{c_1(n)}{|\boldsymbol{x}|^{n-2}} - c_2(n)\frac{x_i x_j}{|\boldsymbol{x}|^n}, \quad Q_i(\boldsymbol{x}) = -2c_2(n)\frac{x_i}{|\boldsymbol{x}|^n}, \quad \text{if } n \geq 3.$$

C.G. Galdi proves such a property in the case where $\boldsymbol{f} \in L^p(\Omega)$ has compact support, $g = 0$, and $\boldsymbol{\varphi} = \mathbf{0}$ (see [6],theorem V.3.2) and obtains that :

$$\mathbf{c} = \int_\Omega \boldsymbol{f}\, d\boldsymbol{x} - \int_{\partial\Omega} (\nabla \boldsymbol{u} - \pi I).\boldsymbol{n} ds.$$

Note that the functional frame we used in Theorem 3.11 seems not to suit *a priori* the preceding formula. For instance, in general, if $\boldsymbol{f} \in \mathbf{W}_l^{-1,p}(\Omega)$, one cannot define its integral. Nevertheless, we prove that $\boldsymbol{u}$ has a similar expansion with much weaker assumptions. Moreover, we show that the constant vector $\mathbf{c}$ only depends on $\boldsymbol{f}, g, \boldsymbol{\varphi}$ and Ω, so that the representation formula becomes a real asymptotic expansion. On the other hand, we do not assume that $(\boldsymbol{f}, g)$ has compact support but that it belongs to appropriate weighted Sobolev spaces.

We recall that the vector $\overline{\boldsymbol{f}}$ is defined by :

$$\overline{f}_i =< f_i, 1 >, \quad i = 1, \ldots, n,$$

where the duality is understood between $W_l^{-1,p}(\mathbb{R}^n)$ and $W_{-l}^{1,p'}(\mathbb{R}^n)$ with l such that $\mathcal{P}_0 \subset W_{-l}^{1,p'}(\mathbb{R}^n)$. Then, in the whole space, we have the following expansion result.

Proposition 5.1 *Assume* $(\boldsymbol{f}, g) \in \mathbf{W}_0^{-1,p}(\mathbb{R}^n) \times L^p(\mathbb{R}^n)$ *has compact support* K. *Then, problem* (S) *has a solution* $(\boldsymbol{u}, \pi)$ *such that for some* $R > 0$, *depending only on* K, $\boldsymbol{u}$ *and* π *are indefinitely differentiable out of the ball* B_R. *Moreover, for all* $\boldsymbol{x}$ *such that* $|\boldsymbol{x}| > R$ *:*

$$\boldsymbol{u}(\boldsymbol{x}) = U(\boldsymbol{x})\overline{\boldsymbol{f}} + \boldsymbol{v}(\boldsymbol{x}), \quad \pi(\boldsymbol{x}) = \mathbf{Q}(\boldsymbol{x}).\overline{\boldsymbol{f}} + \eta(\boldsymbol{x}), \tag{5.1}$$

with forall $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$,

$$|\partial^{\boldsymbol{\alpha}} \boldsymbol{v}(\boldsymbol{x})| + |\boldsymbol{x}||\partial^{\boldsymbol{\alpha}} \eta(\boldsymbol{x})| \leq C_{K,\boldsymbol{\alpha}}(\|\boldsymbol{f}\|_{\mathbf{W}_0^{-1,p}} + \|g\|_{L^p})|\boldsymbol{x}|^{1-n-|\boldsymbol{\alpha}|}. \tag{5.2}$$

i) When $n \geq 3$, *this solution belongs to* $\mathbf{W}_l^{1,p}(\mathbb{R}^n) \times W_l^{0,p}(\mathbb{R}^n)$ *if* $l < n/p' - 1$ *and is unique in this space if moreover* $l > 1 - n/p$.
ii) When $n = 2$, $(\boldsymbol{u}, \pi)$ *belongs to* $\mathbf{W}_{\alpha(p)}^{1,p}(\mathbb{R}^2) \times W_{\alpha(p)}^{0,p}(\mathbb{R}^2)$ *and is unique in this space up to an element of* $N_0 = \boldsymbol{\mathcal{P}}_0 \times \{0\}$.

Proof : The existence of a solution $(\boldsymbol{u}, \pi)$ satisfying (5.1) and (5.2) is proved in [1](Proposition 4.10). Such a solution necessarily belongs to $\mathbf{W}_{loc}^{1,p}(\mathbb{R}^n) \times L_{loc}^p(\mathbb{R}^n)$. Moreover, the decay properties (5.2) readily implies that $\boldsymbol{u}$ and π belong to the desired spaces. Finally, uniqueness follows from the polynomial characterisation of $\mathcal{N}_\alpha^p(\mathbb{R}^n)$ given in section 2. $\diamondsuit$

We now prove a similar expansion for an exterior domain when $n \geq 3$. For this purpose, we introduce the canonical basis $(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)$ of $\boldsymbol{\mathcal{P}}_0$ and, with the notations of (3.8), the family :

$$(\mathbf{V}_i, \Pi_i) = (\boldsymbol{e}_i - \boldsymbol{v}(\boldsymbol{e}_i), -\eta(\boldsymbol{e}_i)), \quad i = 1, \ldots, n,$$

which is a basis of $\mathcal{N}_k^q(\Omega)$ provided $0 \leq 1 - k - n/q < 1$ (see Remarks 2.4 and 3.7). We also introduce the constant vector $\overline{\mathbf{F}}$ defined by :

$$\overline{F}_i = < \boldsymbol{f}, \mathbf{V}_i > + < g, \Pi_i >, \quad i = 1, \ldots, n. \tag{5.3}$$

Theorem 5.2 *Let* Ω *be a* $C^{1,1}$ *exterior domain of* $\mathbb{R}^n, n \geq 3$, l *be an integer and* p *such that*

$$1 - n/p < l < n/p' - 1. \tag{5.4}$$

For $(\boldsymbol{f}, g) \in \mathbf{W}_0^{-1,p}(\Omega) \times L^p(\Omega)$ *with compact support* K *in* $\overline{\Omega}$ *and* $\boldsymbol{\varphi} = \mathbf{0}$, *the only solution* $(\boldsymbol{u}, \pi)$ *in* $\mathbf{W}_l^{1,p}(\Omega) \times W_l^{0,p}(\Omega)$ *of problem* (S) *is indefinitely differentiable out of a ball* B_R, *for some* $R > 0$ *depending only on* K. *Moreover, for all* $\boldsymbol{x}$ *such that* $|\boldsymbol{x}| > R$ *:*

$$\boldsymbol{u}(\boldsymbol{x}) = U(\boldsymbol{x})\overline{\mathbf{F}} + \boldsymbol{v}(\boldsymbol{x}), \quad \pi(\boldsymbol{x}) = \mathbf{Q}(\boldsymbol{x}).\overline{\mathbf{F}} + \eta(\boldsymbol{x}),$$

where $\boldsymbol{v}$ *and* η *satisfy* (5.2).

Proof : Recall that the existence and uniqueness in $\overset{\circ}{\mathbf{W}}{}_l^{1,p}(\Omega) \times W_l^{0,p}(\Omega)$ of $(\boldsymbol{u}, \pi)$ follows from Theorem 3.16 since (5.4) implies that (H) is satisfied and since the data have compact supports. Moreover, in this case, both spaces $\mathcal{N}_l^p(\Omega)$ and $\mathcal{N}_{-l}^{p'}(\Omega)$ are reduced to $\{(\mathbf{0}, 0)\}$.

i) We extend once again $\boldsymbol{u}, \pi, \boldsymbol{f}$ and g by zero in Ω'. Then, with the same notations, the extended distributions satisfy the relations in $\mathcal{D}'(\mathbb{R}^n)$:

$$-\nu\Delta\boldsymbol{u} + \nabla\pi = \boldsymbol{f} + \boldsymbol{h}, \quad -\operatorname{div} \boldsymbol{u} = g,$$

where $\boldsymbol{h} \in \mathbf{W}_0^{-1,p}(\mathbb{R}^n)$ has compact support and satisfies

$$\| \boldsymbol{h} \|_{\mathbf{W}_0^{-1,p}(\mathbb{R}^n)} \leq C_K(\| \boldsymbol{f} \|_{\mathbf{W}_0^{-1,p}(\Omega)} + \| g \|_{L^p(\Omega)}). \tag{5.5}$$

Hence, as a consequence of Proposition 5.1, $(\boldsymbol{u}, \pi)$ has the asymptotic expansion :

$$\boldsymbol{u}(\boldsymbol{x}) = U(\boldsymbol{x})(\overline{\boldsymbol{f}+\boldsymbol{h}}) + \boldsymbol{v}(\boldsymbol{x}), \quad \pi(\boldsymbol{x}) = \mathbf{Q}(\boldsymbol{x}).(\overline{\boldsymbol{f}+\boldsymbol{h}}) + \eta(\boldsymbol{x}),$$

where $\boldsymbol{v}$ and η satisfy (5.2).

ii) We now prove that $\overline{\boldsymbol{f}+\boldsymbol{h}} = \overline{\mathbf{F}}$ and we first assume that $(\boldsymbol{f}, g)$ belongs to $\boldsymbol{\mathcal{D}}(\Omega) \times \mathcal{D}(\Omega)$. In this case, since $\mathbf{V}_j$ and Π_j, $j = 1, \ldots, n$, are smooth, the components of $\overline{\mathbf{F}}$ can be written as integrals :

$$\begin{aligned} \overline{F}_j = <\boldsymbol{f}, \mathbf{V}_j> + <g, \Pi_j> = \int_\Omega f_j d\boldsymbol{x} + \int_\Omega \eta(\boldsymbol{e}_j) \operatorname{div} \boldsymbol{u} d\boldsymbol{x} \\ - \int_\Omega (-\nu\Delta\boldsymbol{u} + \nabla\pi).\boldsymbol{v}(\boldsymbol{e}_j) d\boldsymbol{x}. \end{aligned}$$

But, since $\boldsymbol{f}$ and g are smooth, $(\boldsymbol{u}, \pi)$ also belongs $\mathbf{W}_{l+1}^{2,p}(\Omega) \times W_{l+1}^{1,p}(\Omega)$ in view of Theorem 4.2. In particular, the traces of $\nabla\boldsymbol{u}$ and π exist in $W^{1/p',p}(\partial\Omega)$. Then, extending a classical Green's formula by density, it is not difficult to prove that :

$$\int_\Omega (-\nu\Delta\boldsymbol{u} + \nabla\pi).\boldsymbol{v}(\boldsymbol{e}_j) d\boldsymbol{x} - \int_\Omega \eta(\boldsymbol{e}_j) \operatorname{div} \boldsymbol{u} d\boldsymbol{x} = - <(\nu\nabla\boldsymbol{u} - \pi I)\boldsymbol{n}, \boldsymbol{e}_j>_{\partial\Omega},$$

and therefore that :

$$\overline{F}_j = <f_j, 1> + <(\nu\nabla\boldsymbol{u} - \pi I)\boldsymbol{n}, \boldsymbol{e}_j>_{\partial\Omega} .$$

On the other hand, $\boldsymbol{h}$ is defined by :

$$\forall \boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^n), < \boldsymbol{h}, \boldsymbol{\psi} > = -\nu \int_\Omega \boldsymbol{u}.\Delta\boldsymbol{\psi} d\boldsymbol{x} - \int_\Omega \pi \operatorname{div} \boldsymbol{\psi} d\boldsymbol{x} - < \boldsymbol{f}, \boldsymbol{\psi} > .$$

If we operate two integrations by parts on the first integral and only one on the second, we finally obtain :

$$\forall \boldsymbol{\psi} \in \boldsymbol{\mathcal{D}}(\mathbb{R}^n), < \boldsymbol{h}, \boldsymbol{\psi} > = < (\nu \nabla \boldsymbol{u} - \pi I)\boldsymbol{n}, \boldsymbol{\psi} >_{\partial\Omega} .$$

Hence, since $\boldsymbol{h}$ has compact support, one has :

$$< \boldsymbol{h}, \boldsymbol{e}_j > = < (\nu \nabla \boldsymbol{u} - \pi I)\boldsymbol{n}, \boldsymbol{e}_j >_{\partial\Omega},$$

and so we prove the desired equality for smooth $\boldsymbol{f}$ and g. Finally, the general case follows from the latter by density. Indeed, consider $(\boldsymbol{f}_m, g_m) \in \boldsymbol{\mathcal{D}}(\Omega) \times \mathcal{D}(\Omega)$ tends to $(\boldsymbol{f}, g)$ in $\mathbf{W}_0^{-1,p} \times L^p(\Omega)$. Assume without loss of generality that the support of $(\boldsymbol{f}_m, g_m)$ is contained in a compact set which is independent of m, then $\overline{\mathbf{F_m}} = \overline{\boldsymbol{f}_m + \boldsymbol{h}_m}$ for all m. Thus, by continuity of the duality pairing and owing to (5.5) :

$$\overline{\mathbf{F_m}} \overset{m\to+\infty}{\longrightarrow} \overline{\mathbf{F}}, \qquad \overline{\boldsymbol{f}_m + \boldsymbol{h}_m} \overset{m\to+\infty}{\longrightarrow} \overline{\boldsymbol{f} + \boldsymbol{h}},$$

so that $\overline{\mathbf{F}} = \overline{\boldsymbol{f} + \boldsymbol{h}}$ $\diamondsuit$

When $n = 2$, the situation is slightly different, and solutions in exterior domains have faster decay than those in the whole space. We also define the family,

$$(\mathbf{V}_i, \Pi_i) = (U\boldsymbol{e}_i - \boldsymbol{v}(U\boldsymbol{e}_i), \mathbf{Q}.\boldsymbol{e}_i - \eta(U\boldsymbol{e}_i)),\ i = 1, 2$$

which is still a basis of the space $\mathcal{N}_l^p(\Omega)$ provided $0 < 1 - l - 2/p < 1$.

Theorem 5.3 *Let Ω be a $C^{1,1}$ exterior domain of $\mathbb{R}^2$, $(\boldsymbol{f}, g) \in \mathbf{W}_0^{-1,p}(\Omega) \times L^p(\Omega)$ with compact support K in $\overline{\Omega}$ and $\boldsymbol{\varphi} = \mathbf{0}$. Then, the only solution $(\boldsymbol{u}, \pi) \in \mathbf{W}^{1,p}_{\alpha(p)}(\Omega) \times W^{0,p}_{\alpha(p)}(\Omega)$ of problem (S) is indefinitely differentiable out of a ball B_R for some $R > 0$ depending only on K. Moreover, for all $\boldsymbol{x}$ such that $|\boldsymbol{x}| > R$:*

$$\boldsymbol{u}(\boldsymbol{x}) = A.\overline{\mathbf{F}} + \boldsymbol{v}(\boldsymbol{x}), \quad \pi(\boldsymbol{x}) = \eta(\boldsymbol{x}),$$

where $\boldsymbol{v}$ and η satisfy (5.2) and $A = (A_{ij})$ is an inversible 2×2 matrix with constant coefficients depending only on Ω.

Proof : The existence and uniqueness of $(\boldsymbol{u}, \pi)$ follows from Theorem 3.10 because data have compact support. We extend again the solution $(\boldsymbol{u}, \pi)$ and the data by zero in Ω'. Thus, the extended distributions still denoted by $\boldsymbol{u}, \pi, \boldsymbol{f}$ and g satisfy the relations in $\mathcal{D}'(\mathbb{R}^2)$:

$$-\Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f} + \boldsymbol{h}, \quad - \operatorname{div} \boldsymbol{u} = g,$$

where $\boldsymbol{h}$ has compact support. Since $(\boldsymbol{u}, \pi)$ belongs to $\mathbf{W}^{1,p}_{\alpha_0}(\mathbb{R}^2) \times W^{0,p}_{\alpha_0}(\mathbb{R}^2)$ then, $(\boldsymbol{f} + \boldsymbol{h}, g) \in \mathbf{W}^{-1,p}_{\alpha_0}(\mathbb{R}^2) \times W^{0,p}_{\alpha_0}(\mathbb{R}^2)$ but also satisfies :

$$< f_j + h_j, 1 >= 0, \; j = 1, 2.$$

Therefore, the expansion given by Proposition 5.1 reads in this case :

$$u(\boldsymbol{x}) = \mathbf{a} + v(\boldsymbol{x}), \quad \pi(\boldsymbol{x}) = \eta(\boldsymbol{x}),$$

where $\boldsymbol{v}$ and η satisfy (5.2) and $\mathbf{a}$ belongs to $\boldsymbol{\mathcal{P}}_0$. Let us explicit the link between vector $\mathbf{a}$ and the data.
i) We first establish that the decomposition $\mathbf{a} + \boldsymbol{v}$ is unique for each pair $(\boldsymbol{f}, g)$. Indeed, if $\boldsymbol{u} = \mathbf{a}_1 + \boldsymbol{v}_1 = \mathbf{a}_2 + \boldsymbol{v}_2$ are two decompositions of such type, then $\boldsymbol{w} = \boldsymbol{v}_1 - \boldsymbol{v}_2$ satisfies :

$$-\nu \Delta \boldsymbol{w} = \mathbf{0}, \; \operatorname{div} \boldsymbol{w} = 0 \text{ in } \Omega, \quad \text{and} \quad \boldsymbol{w} = \mathbf{a}_2 - \mathbf{a}_1 \text{ on } \partial\Omega.$$

But, the only solution of this problem in $\mathbf{W}^{1,p}_{\alpha(p)}(\Omega)$ is $\boldsymbol{v}_1 - \boldsymbol{v}_2 = \mathbf{a}_1 - \mathbf{a}_2$. Since $\boldsymbol{v}_1 - \boldsymbol{v}_2$ vanishes at infinity, we finally get that $\mathbf{a}_1 = \mathbf{a}_2$ and $\boldsymbol{v}_1 = \boldsymbol{v}_2$.
ii) By extension, we can introduce the linear mapping :

$$(\boldsymbol{f}, g) \mapsto \mathbf{a}(\boldsymbol{f}, g),$$

and study its kernel. Thanks to (5.2), it is not difficult to prove that

$$(\boldsymbol{v}, \eta) \in \mathbf{W}^{1,p}_{l}(\Omega) \times W^{0,p}_{l}(\Omega), \text{ with } \alpha(p) < l < 2/p'.$$

On the contrary, $\mathbf{a}(\boldsymbol{f}, g) \notin \mathbf{W}^{1,p}_{l}(\Omega)$ unless $\mathbf{a}(\boldsymbol{f}, g) = \mathbf{0}$ and considering Theorem 3.10, we have

$$(\boldsymbol{u}, \pi) \in \mathring{\mathbf{W}}^{1,p}_{l}(\Omega) \times W^{0,p}_{l}(\Omega) \Longleftrightarrow (\boldsymbol{f}, g) \in (\mathbf{W}^{-1,p}_{l}(\Omega) \times W^{0,p}_{l}(\Omega)) \perp \mathcal{N}^{p'}_{-l}(\Omega).$$

So we obtain the equivalence :

$$\mathbf{a}(\boldsymbol{f}, g) = \mathbf{0} \quad \Longleftrightarrow \quad \overline{F}_j =< \boldsymbol{f}, \mathbf{V}_j > + < g, \Pi_j >= 0, \quad j = 1, 2. \tag{5.6}$$

iii) We now conclude thanks to an appropriate decomposition of the data. Consider a non-empty compact set $K \subset \overline{\Omega}$, and the closed subspace $E(K)$ of $\mathbf{W}_0^{-1,p}(\Omega) \times L^p(\Omega)$ of pairs $(\boldsymbol{f}, g)$ with support in K. Then, $E(K)$ is a Banach space and thanks to Hahn-Banach Theorem, we introduce the decomposition :

$$E(K) = E(K)^{\perp} \oplus \mathbb{R}(\mathbf{H}^1, h^1) \oplus \mathbb{R}(\mathbf{H}^2, h^2),$$

where $E(K)^{\perp} = \{(\boldsymbol{f}, g) \in E(K), < \boldsymbol{f}, \mathbf{V}_j > + < g, \Pi_j >= 0, \quad j = 1, 2\}$ and $(\mathbf{H}^i, h^i) \in E(K)$ satisfies forall $i, j = 1, 2$:

$$< \mathbf{H}^i, \mathbf{V}_j > + < h^i, \Pi_j >= \delta_{ij}. \tag{5.7}$$

Hence, any pair $(\boldsymbol{f}, g) \in E(K)$ has a unique decomposition of the form :

$$(\boldsymbol{f}, g) = (\boldsymbol{f}, g)^{\perp} + c_1(\mathbf{H}^1, h^1) + c_2(\mathbf{H}^2, h^2),$$

where $(\boldsymbol{f}, g)^{\perp}$ belongs to $E(K)^{\perp}$. In particular, the vector $\mathbf{c}$ necessarily equals $\overline{\mathbf{F}}$. Finally, the kernel of the linear mapping from $E(K)$ into $\mathbb{R}^2$:

$$(\boldsymbol{f}, g) \longmapsto \mathbf{a}(\boldsymbol{f}, g),$$

is $E(K)^{\perp}$ thanks to (5.6). Therefore, its restriction to $\mathbb{R}(\mathbf{H}^1, h^1) \oplus \mathbb{R}(\mathbf{H}^2, h^2)$ is injective. Moreover, both spaces $\mathbb{R}^2$ and $\mathbb{R}(\mathbf{H}^1, h^1) \oplus \mathbb{R}(\mathbf{H}^2, h^2)$ have the same finite dimension, so that the restriction is in fact bijective. Thus, it can be represented by an invertible 2×2 matrix transforming the base $(\mathbf{H}^1, h^1), (\mathbf{H}^2, h^2)$ into the canonical base of $\mathbb{R}^2$. At last, it is not difficult to prove that the matrix A is independant of the compact K. $\diamondsuit$

We now investigate asymptotic properties of the solutions of the problem (S) when the data no longer have compact support. Even in this case, one still have some asymptotic expansion of solutions if the data have fast enough decay at infinity.

Theorem 5.4 *Let Ω be a $C^{1,1}$ exterior domain and $p > n \geq 3$. For $(\boldsymbol{f}, g)$ in $\mathbf{W}_{n-1}^{-1,p}(\Omega) \times W_{n-1}^{0,p}(\Omega)$ and $\boldsymbol{\varphi} = \mathbf{0}$, problem (S) has a unique solution $(\boldsymbol{u}, \pi)$ in $\mathbf{W}_{n-2}^{1,p}(\Omega) \times W_{n-2}^{0,p}(\Omega)$. Moreover, $\boldsymbol{u}$ has the asymptotic expansion :*

$$\boldsymbol{u}(\boldsymbol{x}) = U(\boldsymbol{x})\overline{\mathbf{F}} + o(|\boldsymbol{x}|^{\gamma}),$$

where $\gamma = 2 - n - n/p$ satisfies $1 - n < \gamma < 2 - n$.

Proof : The proof is once again based on an adequate decomposition of the data. Indeed, as in the previous theorem, Hahn-Banach Theorem implies that each pair $(\boldsymbol{f}, g) \in \mathbf{W}^{-1,p}_{n-1}(\Omega) \times W^{0,p}_{n-1}(\Omega)$ decomposes as follows :

$$(\boldsymbol{f}, g) = (\boldsymbol{f}^1, g^1) + (\boldsymbol{f}^2, g^2),$$

where $(\boldsymbol{f}^1, g^1)$ belongs to $(\mathbf{W}^{-1,p}_{n-1}(\Omega) \times W^{0,p}_{n-1}(\Omega)) \perp \mathcal{N}^{p'}_{1-n}(\Omega)$ and :

$$(\boldsymbol{f}^2, g^2) = \sum_{j=1}^{n} \overline{F}_j (\mathbf{H}^j, h^j),$$

where the distributions

$$(\mathbf{H}^i, h^i) \in \mathbf{W}_l^{-1,p}(\Omega) \times W_l^{0,p}(\Omega), \quad i = 1, \ldots, n,$$

have compact support and satisfy (5.7). Then, on the one hand, since $p > n$ and $l = n - 2$ satisfy (5.4), we can apply Theorem 5.2 to $(\boldsymbol{f}_2, g_2)$ that has compact support. Hence the associated problem (S) with $\boldsymbol{\varphi} = \mathbf{0}$ has a solution $(\boldsymbol{u}^2, \pi^2)$ in $\mathbf{W}^{1,p}_{n-2}(\Omega) \times W^{0,p}_{n-2}(\Omega)$ that satisfies :

$$\boldsymbol{u}^2(\boldsymbol{x}) = U(\boldsymbol{x})\overline{\mathbf{F}} + O(|\boldsymbol{x}|^{1-n}).$$

On the other hand, thanks to Theorem 3.16, problem (S) associated with $(\boldsymbol{f}^1, g^1)$ and $\boldsymbol{\varphi} = \mathbf{0}$ has a unique solution $(\boldsymbol{u}^1, \pi^1)$ in $\mathbf{W}^{1,p}_{n-1}(\Omega) \times W^{0,p}_{n-1}(\Omega)$. Moreover, since $p > n$, Proposition 1.4 yields

$$\boldsymbol{u}^1(\boldsymbol{x}) = o(|\boldsymbol{x}|^{2-n-n/p}).$$

Thus, since $1 - n < 2 - n - n/p$, the pair $(\boldsymbol{u}, \pi) = (\boldsymbol{u}^1 + \boldsymbol{u}^2, \pi^1 + \pi^2)$ obviously satisfies all the required properties. $\diamondsuit$

We now give the analogous result when the dimension equals 2. We shall not develop the proof of this result because it is very similar to the preceding one.

Theorem 5.5 *Let Let Ω be a $C^{1,1}$ exterior domain and $p > n = 2$. For $(\boldsymbol{f}, g)$ in $\mathbf{W}_1^{-1,p}(\Omega) \times W_1^{0,p}(\Omega)$ and $\boldsymbol{\varphi} = \mathbf{0}$, problem (S) has a unique solution $(\boldsymbol{u}, \pi)$ in $\mathbf{W}^{1,p}_{\alpha(p)}(\Omega) \times W^{0,p}_{\alpha(p)}(\Omega)$. Moreover, $\boldsymbol{u}$ has the following asymptotic expansion :*

$$u_i(\boldsymbol{x}) = A\overline{\mathbf{F}} + o(|\boldsymbol{x}|^{\gamma}),$$

where $\gamma = -2/p$ satisfies $-1 < \gamma < 0$ and A is given in theorem 5.2.

Remark 5.6 *i)* The last two theorems are optimal because the expansions does not always hold as soon as $p \leq n$ or $(\boldsymbol{f}, g) \in \mathbf{W}_l^{-1,p}(\Omega) \times W_l^{0,p}(\Omega)$ with $l < n-1$. They can nevertheless be refined considering data in $\mathbf{W}_\alpha^{-1,p}(\Omega) \times W_\alpha^{0,p}(\Omega)$ where $p > n$ and α is a real number such that $\alpha + 1 - n/p' > 0$.
ii) As we chose non-smooth data in Theorems 5.4 and 5.5, the pressure π does not admit any particular expansion. Nevertheless, if $(\boldsymbol{f}, g) \in \mathbf{W}_n^{0,p}(\Omega) \times W_n^{0,p}(\Omega)$ then, the pressure writes :

$$\begin{aligned} \pi(\boldsymbol{x}) = \ & \mathbf{Q}(\boldsymbol{x}).\overline{\mathbf{F}} + o(|\boldsymbol{x}|^{\gamma-1}), \quad \text{if } n \geq 3, \\ \pi(\boldsymbol{x}) = \ & o(|\boldsymbol{x}|^{\gamma-1}), \quad \text{if } n = 2. \end{aligned}$$

Under the same assumptions, $\nabla \boldsymbol{u}$ has the following expansion :

$$\begin{aligned} \nabla \boldsymbol{u}(\boldsymbol{x}) = \ & \nabla(U(\boldsymbol{x})\overline{\mathbf{F}}) + o(|\boldsymbol{x}|^{\gamma-1}), \quad \text{if } n \geq 3, \\ \nabla \boldsymbol{u}(\boldsymbol{x}) = \ & o(|\boldsymbol{x}|^{\gamma-1}), \quad \text{if } n = 2. \end{aligned}$$

iii) All the results in this section readily extends to non homogeneous boundary data $\boldsymbol{\varphi}$ if we replace the quantity $< \boldsymbol{f}, \mathbf{V}_j > + < g, \Pi_j >$ by

$$< \boldsymbol{f}, \mathbf{V}_j > + < g, \Pi_j > + < \boldsymbol{\varphi}, (\nabla \mathbf{V}_j - \Pi_j I)\boldsymbol{n} >_{\partial\Omega} .$$

In particular, one can improve the results of Corollary 3.14 to prove that the functions introduced in characterisations (3.8) and (3.9) satisfy the asymptotic representation formulae : forall $\beta > 1-n$,

$$\boldsymbol{v}(\boldsymbol{\lambda})(\boldsymbol{x}) = U(\boldsymbol{x})\overline{\boldsymbol{\mathcal{F}}} + o(|\boldsymbol{x}|^\beta), \quad \text{if } n \geq 3,$$

$$\boldsymbol{v}(\boldsymbol{\lambda} + U\boldsymbol{\lambda}(\mathbf{0}))(\boldsymbol{x}) = A\overline{\boldsymbol{\mathcal{F}}} + o(|\boldsymbol{x}|^\beta), \quad \text{if } n = 2,$$

where $\overline{\mathcal{F}}_j =< \boldsymbol{\lambda}, (\nabla \mathbf{V}_j - \Pi_j I)\boldsymbol{n} >_{\partial\Omega}$.

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References

[1] F. Alliot;C. Amrouche. The Stokes Problem in $\mathbb{R}^n$: an approach in weighted Sobolev spaces. to appear in Math. Meth. and Mod. in App. Sci.

[2] C. Amrouche;V.Girault. Decomposition of vector spaces and application to the Stokes problem in arbitrary dimensions. *Czechoslovak Mathematical Journal*, 44(119):109–140, 1994.

[3] C. Amrouche;V.Girault;J.Giroire. Dirichlet and Neumann exterior problems for the n-dimensional Laplace operator. An approach in weighted Sobolev spaces. *J. Math. Pures et Appliquées*, 76(1):55–81, 1997.

[4] W. Borchers;T.Miyakawa. On some coercive estimates for the stokes problem in unbounded domains. In Springer-Verlag, editor, *Navier-Stokes equations : Theory and numerical methods*, volume 1530, pages 71–84. Heywood J.G.; Masuda K.;Rautmann R.; Solonnikov V.A.; Lecture Notes in Mathematics, 1992.

[5] L. Cattabriga. Su un problema al contorno relativo al sistema di equazioni di Stokes. *Rend. Sem. Mat. Padova*, 31:308–340, 1961.

[6] G.P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations.*, volume I. Springer tracts in natural philosophy, 1994.

[7] G.P. Galdi;C.G.Simader. Existence uniqueness and L^q-estimates for the Stokes problem in an exterior domain. *Arch. Rat. Mech. Anal.*, 112:291–318, 1990.

[8] V. Girault;A.Sequeira. A well posed problem for the exterior Stokes equations in two and three dimensions. *Arch. Rat. Mech. Anal.*, 114:313–333, 1991.

[9] J. Giroire. Etude de quelques problèmes aux limites extérieurs et résolution par équations intégrales. Thèse de doctorat d'etat, Université Paris VI, 1987.

[10] J.G. Heywood. On some paradoxes concerning two dimensional Stokes flow past an obstacle. *Indiana University Math. Journal*, 24(5):443–450, 1974.

[11] J.G. Heywood. The Navier-Stokes equations : On the existence, Regularity and decay of Solutions. *Indiana Univ. Math. J.*, 29(5):639–681, 1980.

[12] H. Kozono;H.Sohr. New a priori estimates for the Stokes equations in exteriors domains. *Indiana Univ. Math. J.*, 40:1–25, 1991.

[13] H. Kozono;H.Sohr. On a new class of generalized solutions for the Stokes equations in exterior domains. *Ann.Scuola Norm. Sup. Pisa, Ser. IV*, 19:155–181, 1992.

[14] H. Kozono;T.Ogawa. Some L^p estimate for the exterior Stokes flow and an application to the non-stationary Navier-Stokes equations. *Indiana Univ. Math. J.*, 41(3), 1992.

[15] M.N. Leroux. Résolution numérique du problème du potentiel dans le plan par une méthode variationnelle d'éléments finis. Master's thesis, Thèse de troisième cycle, Université de Rennes, 1974.

[16] M. Specovius Neugebauer. Exterior Stokes problems and decay at infinity. *Math. Meth. in the Appl. Sci.*, (8):351–367, 1986.

[17] M. Specovius Neugebauer. The two-dimensional exterior Stokes problem, existence, regularity and decay properties. *Math. Methods Appl. Sci.*, 19(7):507–528, 1996.

[18] W. Varnhorn. On strong solutions of the Stokes Equations in exterior domains. *Acta Appl. Math.*, 37:205–214, 1994.

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