On the diaphony and the star-diaphony of the Roth sequences and the Zaremba sequences

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Abstract

In this paper, we estimate the star-diaphony and the diaphony of the Roth sequence and the Zaremba sequence using their L^2-discrepancy formula given by Halton and Zaremba (see [3]), and White (see [17]). The optimal estimates and the exact asymptotic behaviours of the star-diaphony and the diaphony of both sequences are given. Moreover, the exact asymptotic behaviours of the star-diaphony are the same for both sequences, and the same is true for the diaphony.

Key words: diaphony, star-diaphony, L^2-discrepancy, Roth sequence, Zaremba sequence

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1 Introduction

Let $\sigma = (\xi_n)_{n\geq 0}$ be a finite or infinite sequence of points in the unit cube $I^s = [0,1]^s$ and let it contain at least N terms. For each $\mathbf{x} = (x_1, \dots, x_s)$ in I^s, $A_N(\mathbf{x}, \sigma)$ denotes the number of index n such that $0 \leq n \leq N-1$ and $\xi_n \in \prod_{i=1}^{s}[0, x_i)$ and $E_N(\mathbf{x}, \sigma)$ the remainder to ideal distributon:

$$E_N(\mathbf{x}, \sigma) = A_N(\mathbf{x}, \sigma) - x_1 \dots x_s N.$$

An infinite sequence σ is called uniformly distributed in I^s if for each $\mathbf{x} \in I^s$, we have

$$\lim_{N\to\infty} \frac{E_N(\mathbf{x}, \sigma)}{N} = 0.$$

There are various quantitative measures for the irregularity of the distribution based on different point of view of uniform distribution (see, for example [8]). A classical measure for the irregularity of the distribution of a sequence σ in I^s is the L^2-discrepancy $T_N(\sigma)$ which is defined for every positive integer N by,

$$T_N(\sigma) = \frac{1}{N}\left(\int_{I^s} \mid E_N(\mathbf{x},\sigma) \mid^2 d\mathbf{x}\right)^{\frac{1}{2}}.$$

Another measure for the irregularity of the distribution of a sequence σ in I^s is the diaphony $F_N(\sigma)$ which is defined for every positive integer N by,

$$F_N(\sigma) = \left(\sum_{\mathbf{h}\in\mathbf{Z}^s-\{\mathbf{0}\}} \frac{1}{r(\mathbf{h})^2} \mid \frac{1}{N}\sum_{n=0}^{N-1} e^{2\pi i n \mathbf{h}\cdot\xi_n} \mid^2\right)^{\frac{1}{2}},$$

where $\mathbf{h} = (h_1,\ldots,h_s)$, $\mathbf{h}\cdot\mathbf{x} = \sum_{i=1}^s h_i x_i$ and

$$r(\mathbf{h}) = \Pi_{i=1}^s \max\{1,|h_i|\}.$$

The diaphony $F_N(\sigma)$ was originaly introduced by Zinterhof (see [20]) for numerical integration of regular periodic functions and was defined in the following equivalent form. For $\mathbf{x} = (x_1,\ldots,x_s) \in \mathbf{R}^s$, denote $H(\mathbf{x}) = h(x_1)\ldots h(x_s) - 1$ where $h(t) = 1 - \frac{\pi^2}{6} + \frac{\pi^2}{2}(1-2\{t\})^2$ with $\{t\}$ the fraction part of a real t. Then

$$F_N(\sigma) = (\frac{1}{N^2}\sum_{k,n=0}^{N-1} H(\xi_k - \xi_n))^{\frac{1}{2}}.$$

It is well known that the infinite sequence $\sigma = (\xi_n)_{n\geq 0}$ is uniformly distributed is equivalent to both $\lim_{N\to\infty} T_N(\sigma) = 0$ and $\lim_{N\to\infty} F_N(\sigma) = 0$.

In 1954, Roth [13] established a general lower bound on the L^2-discrepancy T_N, that is, for any N points in I^s we have

$$T_N > C_s\frac{(\log N)^{\frac{s-1}{2}}}{N} \tag{1}$$

with a positive constant C_s only depending on s. According to [6] we may take $C_s = 2^{-4s}((s-1)\log 2)^{(1-s)/2}$ for $s \geq 2$.

The lower bound (1) for T_N is best possible as far as the order of magnitude is concerned since there exist finite sequences with $T_N = O(\frac{(\log N)^{\frac{s-1}{2}}}{N})$

and we can see these from the Zaremba sequences for $s = 2$ in the following and the results of Roth (see [14] and [15]) for $s \geq 3$.

For the diaphony F_N, Proinov [11] proved that for any N points in I^s with $s \geq 2$ we have

$$F_N > \alpha_s \frac{(\log N)^{\frac{s-1}{2}}}{N} \tag{2}$$

with a positive constant $\alpha_s = \pi^s 3^{s/2} 2^{-2s-3}((3^s - 1)((\pi^2 + 6)^s - \pi^{2s})((s - 1)\log 2)^{(s-1)})^{-1/2}$ only depending on s. For $s = 1$, we have from the result of Stegbuchner [16]

$$F_N \geq \frac{\pi}{\sqrt{3}N},$$

it is best possible since the lower bounded is reached by the van der Corput finite sequence, see below.

For $s = 2$, the exactness of the lower bound (2) was shown in [5], in the sense that a net of N points S in $[0, 1]^2$ exists for which $F_N = O(\frac{(\log N)^{\frac{1}{2}}}{N})$. In this paper we will show that the lower bound (2) is also reached by the Roth sequence and the Zaremba sequence. Moreover, both sequences have the same asymptotic behaviou.

Also for numerical integration purpose, we have introduced in [10] a new version of diaphony, the so-called star-diaphony, which is defined, for every positive integer N, by

$$F_N^*(\sigma) = \frac{(2\pi)^s}{N}(\sum_{\mathbf{h}\in \mathbf{Z}^s - \{\mathbf{0}\}} |\int_{I^s} E_N(\mathbf{x}, \sigma) e^{2\pi i \mathbf{h}\cdot\mathbf{x}} d\mathbf{x}|^2)^{\frac{1}{2}}.$$

For $s = 1$, $F_N^* = F_N$.

The F_N^* is related naturally with T_N by the following so-called Koksma formula (cf. [4]),

$$T_N^2(\sigma) = I_N^2(\sigma) + \frac{1}{(2\pi)^{2s}}(F_N^*(\sigma))^2$$

where

$$I_N(\sigma) = (\frac{1}{N}\sum_{k=1}^{N}\prod_{j=1}^{s}(1 - \xi_k^j) - \frac{1}{2^s}).$$

Thus $F_N^*(\sigma) \leq (2\pi)^s T_N(\sigma)$.

We can show that F_N^* is also a measure for the irregularity of the distribution of a sequence in I^s (see [19]).

In this paper, we will estimate the diaphony and the star-diaphony of the Roth sequences and the Zaremba sequences. The results show that they are nearly the same for these two measures. In §2, we will give the principal results and the proof of Theorem 2.3 is completed in §4 using some lemmas in §3. We end with some remarks in §5.

2 The Roth sequence and the Zaremba sequence and their diaphonies

We will first give the definition of the Roth sequence and the Zaremba sequence. Let $b \geq 2$ be a positive integer. Let $n = \sum_{i=0}^{\infty} a_i(n) b^i$ be the b-adic expansion of the nonnegative integer n and let $\phi_b(n) = \sum_{i=0}^{\infty} a_i(n) b^{-i-1}$ be the van der Corput sequence in base b. For $m \in \mathbf{N}^*$, denote $\psi_{b,m} = (\psi_{b,m}(n))_{0 \leq n \leq b^m - 1}$.

Definition 2.1 *For any positive integer m, the Roth sequence and Zaremba sequence in base b of b^m points are defined respectively by*

$$R_{b,m} = (\psi_{b,m}(n), \phi_b(n))_{0 \leq n \leq b^m - 1} \tag{3}$$

and

$$Z_{b,m} = (\psi_{b,m}(n), \phi_b'(n))_{0 \leq n \leq b^m - 1} \tag{4}$$

where $\phi_b'(n) = \sum_{i=0}^{\infty} (a_i(n) \oplus \mu_i) b^{-i-1}$ with $\mu_i = 0$ if $i \geq m$ else $0 \leq \mu_i \leq b - 1$ such that $\mu_i \equiv i \pmod b$ or $\mu_i \equiv i + 1 \pmod 2$ for the original Zaremba sequence in base 2. The $\oplus$ denotes $\pmod b$ addition component-by-component.

Then, if we put $\epsilon_m = 0$ when m is even and 1 when m is odd, we have the following results.

Theorem 2.2 *For the star-diaphony,*

$$F^*(R_{b,m}) = \frac{(2\pi)^2}{b^m} \left(\frac{(b^2 - 1)(3b^2 + 13)}{720 b^2} m + \frac{1}{8} + O\left(\frac{1}{m}\right) \right)^{\frac{1}{2}}.$$

and

$$F^*(Z_{b,m}) = \frac{(2\pi)^2}{b^m}\left(\frac{(b^2-1)(3b^2+13)}{720b^2}m + C + O(\frac{1}{m})\right)^{\frac{1}{2}}$$

where C is a constant. The best case is obtained for the original Zaremba sequence with $C = \frac{1}{8} - \frac{23\epsilon_m}{64}$.

Remark Indeed, the exact formulas for $F^*(R_{b,m})$ and $F^*(Z_{b,m})$ can be given. For reason of simplicity, we show the results in the present form.

For the asymptotic behavior of the star-diaphony, we have the following result.

Corollary 2.3

$$\lim_{m\to\infty}\frac{b^m F^*(R_{b,m})}{\sqrt{\log b^m}} = \lim_{m\to\infty}\frac{b^m F^*(Z_{b,m})}{\sqrt{\log b^m}} = \frac{\pi^2}{6b}\left(\frac{(b^2-1)(3b^2+13)}{5\log b}\right)^{\frac{1}{2}}.$$

Theorem 2.4 *For the diaphony,*

$$-\frac{2\pi^4(5b+18)}{3} \le b^{2m}\left(F^2(R_{b,m}) - \frac{\pi^4(b^2-1)(b^2+1)m}{45b^{2m+2}}\right) \le \frac{2\pi^4(b^2-30b+23)}{45}$$

and

$$C_1 \le b^{2m}\left(F^2(Z_{b,m}) - \frac{\pi^4(b^2-1)(b^2+1)m}{45b^{2m+2}}\right) \le C_2,$$

where C_1 and C_2 being two constants.

Thus, we also have the exact asymptotic behavior of the diaphony.

Corollary 2.5

$$\lim_{m\to\infty}\frac{b^m F(R_{b,m})}{\sqrt{\log b^m}} = \lim_{m\to\infty}\frac{b^m F(Z_{b,m})}{\sqrt{\log b^m}} = \frac{\pi^2}{3b}\left\{\frac{(b^2-1)(b^2+1)}{5\log b}\right\}^{\frac{1}{2}}$$

In order to prove these results, we first need recall the results on their L^2-discrepancies.

Theorem 2.6 *We have*

$$\begin{aligned} T^2(R_{b,m}) &= b^{-2m}\Big(\frac{(b^2-1)^2}{144b^2}m^2 + \frac{(b^2-1)(3b^2+60b+13)}{720b^2}m \\ &\quad + \frac{3}{8} - \frac{b^2-1}{24b^{m+1}}m + \frac{1}{4b^m} - \frac{1}{72b^{2m}}\Big) \end{aligned}$$

and

$$T^2(Z_{b,m}) = b^{-2m}(\frac{(b^2-1)(3b^2+13)}{720b^2}m + O(1)).$$

The best result is obtained in base 2 for the the original Zaremba sequence in base 2,

$$T^2(Z_{2,m}) = 2^{-m}(\frac{5m}{192} + \frac{3}{8} - \frac{7\epsilon_m}{64} + \frac{1}{4\times 2^m} + \frac{\epsilon_m}{16\times 2^m} - \frac{1}{72\times 2^{2m}}).$$

The results on the Roth sequence in base 2 and the original Zaremba sequence were given in [3], and the other results were given in [17].

We also need recall and generalize the results on their I_N estimates.

Proposition 2.7 *a) For the Roth sequence (see [1]),*

$$I(R_{b,m}) = b^{-m}(\frac{(b^2-1)}{12b}m + \frac{1}{2} + \frac{1}{4b^m}).$$

b) For the original Zaremba sequence (see [3]),

$$I(Z_{2,m}) = b^{-m}(-\frac{\epsilon_m}{8} + \frac{1}{2} + \frac{1}{42^m}).$$

c) For the general Zaremba sequence,

$$I(Z_{b,m}) = b^{-m}(-\frac{(b-1)[b(c_m+1)-c_m^2-1]}{4b} + \frac{1}{2} + \frac{1}{4b^m});$$

where $0 \le c_m \le b-1$ *and* $c_m \equiv m-1 \pmod b$.

Proof We will only prove c).

Let $m-1 = qb + c_m$ and write $a'_j(n) = (a_j(n) \oplus \mu_j)$, we have

$$\begin{aligned}
\sum_{n=0}^{b^m-1} \psi_{b,m}(n)\phi'_b(n) &= \sum_{i,j=0}^{m-1}\sum_{n=0}^{b^m-1} \frac{a_i(n)a'_j(n)}{b^{m-i+j+1}} \\
&= \sum_{i=0}^{m-1}\sum_{n=0}^{b^m-1} \frac{a_i(n)a'_i(n)}{b^{m+1}} + \sum_{i\neq j}\sum_{n=0}^{b^m-1} \frac{a_i(n)a'_j(n)}{b^{m-i+j+1}} \\
&= \frac{b^{m-1}}{b^{m+1}}\sum_{i=0}^{m-1}\sum_{a_i(n)=0}^{b-1} a_i(n)(a_i(n)\oplus\mu_i) + \frac{b^{m-2}}{b^{m+1}}\sum_{i\neq j}^{m-1}\frac{1}{b^{-i+j}}\sum_{a_i(n),a'_j(n)=0}^{b-1} a_i(n)a'_j(n)
\end{aligned}$$

with

$$\sum_{i\neq j}^{m-1}\frac{1}{b^{-i+j}}\sum_{a_i(n),a'_j(n)=0}^{b-1}a_i(n)a'_j(n)=\sum_{i\neq j}^{m-1}\frac{1}{b^{-i+j}}[\frac{b(b-1)}{2}]^2=b^3[-\frac{(b-1)^2}{4b}m+\frac{b^m}{4}-\frac{1}{2}+\frac{1}{4b^m}]$$

and

$$\begin{aligned}\sum_{i=0}^{m-1}\sum_{a_i(n)=0}^{b-1}a_i(n)(a_i(n)\oplus\mu_i) &= q\sum_{l=0}^{b-1}\sum_{i=1}^{b-1}i(i\oplus l)+\sum_{l=0}^{c_m-1}\sum_{i=1}^{b-1}i(i\oplus l)\\ &= (m-1-c_m)\frac{(b-1)^2b^2}{4}+\frac{(c_m-1)c_m(b-1)b}{4}.\end{aligned}$$

Thus, from

$$b^mI(Z_{b,m})=1+\sum_{n=0}^{b^m-1}\psi_{b,m}(n)\phi'_b(n)-\frac{b^m}{4},$$

the c) follows. ◇

Then, Theorem 2.2 is a direct application of Theorem 2.4, Proposition 2.5 and the Koksma formula $(F_N^*)^2=(2\pi)^4(T_N^2-I_N^2)$.

For the proof of Theorem 2.5, we need some notations and lemmas.

3 Some lemmas

We will first give some notations.

a) For a $[0,1]$-valued sequence, possibly finite, $\sigma=(x_n)_{n\geq 0}$, let

$$S_n(h)=\sum_{k=0}^{n-1}e^{2\pi ihx_k}.$$

b) For $h\in\mathbf{Z}^*$, $v_b(h)=\max\{k\geq 0 \text{ such that } b^k|h\}$ will denote the b-adic valuation of h.

c) For $n=\sum_{i=0}^{\infty}a_i(n)b^i\in\mathbf{N}$, denote $n_s=\sum_{i=0}^{s}a_i(n)b^i$ for $s\geq 0$.

We also recall the exponential sum formula of the Van Der Corput sequence (see [18] and [10]).

Lemma 3.1 *For the Van Der Corput sequence ϕ_b,*

$$S_n(h) = e^{2\pi i h\phi_b(n-n_{v_b(h)})}[b^{v_b(h)} \sum_{k=0}^{a_{v_b(h)}(n)-1} e^{2\pi i h\phi_b(kb^{v_b(h)})} + e^{2\pi i h\phi_b(a_{v_b(h)}(n)b^{v_b(h)})} n_{v_r(h)-1}].$$

In particular, we have

$$S_{b^m}(h) = \begin{cases} 0, & \text{if } v_b(h) \leq m-1, \\ b^m, & \text{if } v_b(h) \geq m. \end{cases}$$

and this is also valid for the sequence $\psi_{b,m}$. In addition, $S_n(h) = n$ if $v_b(h) > [\log_b n]$.

Remark For the generalized van der Corput sequences in base b, $\phi_b^{\Sigma}(n) = \sum_{i=0}^{\infty} \sigma_i(a_i(n))b^{-i-1}$ where $\Sigma = (\sigma_i)_{i\geq 0}$ being an infinite sequence of permutations of the set $\{0, 1, \ldots, b-1\}$, this exponential sum formula is still hold. Note that the second component $\phi_b'(n)$ of the Zaremba sequence is one of the generalized van der Corput sequences.

In the following, $S_n(h)$ concerns only the original van der Corput sequences even though all results are also valid for the generalized van der Corput sequences.

As an application of the above formula, the following result follows.

Lemma 3.2 *For the sequence ϕ_b and for any integer $n \in \mathbf{N}$,*

$$|S_n(h)| \leq \frac{1}{2} b^{v_b(h)+1}.$$

Applying Lemma 3.1 in the case of $n = b^m$, we get the following lemma.

Lemma 3.3 *a) For the diaphony of the sequence $(\phi_b(n))_{0\leq n\leq b^m-1}$ and the sequence $\psi_{b,m}$,*

$$F_{b^m}(\phi_b) = F(\psi_{b,m}) = \frac{\pi}{\sqrt{3}b^m}.$$

b)

$$\sum_{h=1}^{\infty} \frac{1}{h^2} |\sum_{n=0}^{b^m-1} \psi_{b,m}(n) e^{2\pi i h\phi_b(n)}|| \sum_{n=0}^{b^m-1} e^{2\pi i h\phi_b(n)}| = \frac{\pi^2(b^m-1)}{12b^m}.$$

The following two lemmas are crucial for the proof of Theorem 2.5.

Lemma 3.4 *a) Let* $h = b^{v_b(h)}(qb+j)$ *with* $1 \le j \le b-1$, *then*

$$\sum_{n=1}^{b^m-1} S_n(h) = \frac{b^{2v_b(h)+1}}{1 - e^{\frac{2\pi i(qb+j)}{b^{m-v_b(h)}}}}.$$

b)

$$\frac{b^2\pi^2}{360}(m\frac{(b^4-1)}{b^4} - 1) \le \frac{1}{b^{2m}} \sum_{h=1}^{b^m-1} \frac{1}{h^2} |\sum_{n=1}^{b^m-1} S_n(h)|^2 \le \frac{b^2\pi^2}{360}(m\frac{(b^4-1)}{b^4} + \frac{15(2b^2-1)}{b-1})$$

and thus

$$\lim_{m\to\infty} \frac{1}{mb^{2m}} \sum_{h=1}^{b^m-1} \frac{1}{h^2} |\sum_{n=1}^{b^m-1} S_n(h)|^2 = \frac{(b^4-1)\pi^2}{360b^2}.$$

Lemma 3.5 *We have a)*

$$\sum_{n=0}^{b^m-1} \psi_{b,m}(n) e^{2\pi i h\phi_b(n)} = \sum_{n=0}^{b^m-1} \phi_b(n) e^{2\pi i h\psi_{b,m}(n)},$$

b)

$$\sum_{h=1}^{\infty} \frac{1}{h^2} |\sum_{n=0}^{b^m-1} \psi_{b,m}(n) e^{2\pi i h\phi_b(n)}| \le \frac{\pi^2(b+2)(b^m-1)}{12b^m}.$$

and c)

$$\sum_{h=1}^{\infty} \frac{1}{h^2} |\sum_{n=0}^{b^m-1} \psi_{b,m}(n) e^{2\pi i h\phi_b(n)}|^2 \le \frac{1}{b^{2m}} \sum_{h=1}^{b^m-1} \frac{1}{h^2} |\sum_{n=1}^{b^m-1} S_n(h)|^2 + \pi^2(b+1)24$$

To prove Lemma 3.4 and Lemma 3.5, we will denote for each integer $l \in \mathbf{N}$

$$A_l = \{h \in \mathbf{N} \mid v_b(h) = l\}.$$

It is clear that $h \in A_l$ if and only if there is an integer $q \ge 0$ an integer $j \in \{1, \ldots, b-1\}$ such that $h = b^l(qb+j)$. In addition, $Card\{\ h \mid v_b(h) = l \le m-1 \ \text{ and } \ 1 \le h \le b^m - 1\} = (b-1)b^{m-l-1}$.

Proof of Lemma 3.4.

For a), applying Lemma 3.1,

$$\begin{aligned}\sum_{n=1}^{b^m-1} S_n(h) &= \sum_{n=1}^{b^m-1} e^{2\pi i h\phi_b(n-n_{v_b(h)})} b^{v_b(h)} \sum_{k=0}^{a_{v_b(h)}(n)-1} e^{2\pi i h\phi_b(kb^{v_b(h)})}\\ &\quad + \sum_{n=1}^{b^m-1} e^{2\pi i h[\phi_b(n-n_{v_b(h)})+\phi_b(a_{v_b(h)}(n)b^{v_b(h)})]} n_{v_b(h)-1}\end{aligned}$$

Because for $n = 1, \ldots, b^m-1$, $n-n_{v_b(h)-1} = kb^{v_b(h)}$ with $k = 0, \ldots, b^{m-v_b(h)}-1$, and $n_{v_b(h)-1} = 1, 2, \ldots, b^{v_b(h)} - 1$, so the second term is

$$\begin{aligned}&\sum_{n=1}^{b^m-1} e^{2\pi i h\phi_b(n-n_{v_b(h)-1})} n_{v_b(h)-1} = \sum_{n_{v_b(h)-1}=1}^{b^{v_b(h)}-1} n_{v_b(h)-1} \sum_{k=0}^{b^{m-v_b(h)}-1} e^{2\pi i h\phi_b(kb^{v_b(h)})}\\ =\ & \frac{(b^{v_b(h)}-1)b^{v_b(h)}}{2} S_{b^{m-v_b(h)}}(qb+j) = 0\end{aligned}$$

for $v_b(qb+j) = 0$ and $m - v_b(h) \geq 1$.

The same, for $n = 1, \ldots, b^m - 1$, $n - n_{v_b(h)} = kb^{v_b(h)+1}$ with $k = 0, 1, \ldots, b^{m-v_b(h)-1} - 1$. In addition $\sum_{l=0}^{-1} = 0$, we have

$$\begin{aligned}&\sum_{n=1}^{b^m-1} S_n(h) = \sum_{n=1}^{b^m-1} e^{2\pi i h\phi_b(n-n_{v_b(h)})} b^{v_b(h)} \sum_{l=0}^{a_{v_b(h)}(n)-1} e^{2\pi i h\phi_b(lb^{v_b(h)})}\\ =\ & b^{v_b(h)} \sum_{n_{v_b(h)-1}=1}^{b^{v_b(h)}-1} \sum_{k=0}^{b^{m-v_b(h)-1}-1} e^{2\pi i h\phi_b(kb^{v_b(h)+1})} \sum_{a_{v_b(h)}(n)=1}^{b-1} \sum_{l=0}^{a_{v_b(h)}(n)-1} e^{2\pi i h\phi_b(lb^{v_b(h)})}\end{aligned}$$

with

$$\sum_{a_{v_b(h)}(n)=1}^{b-1} \sum_{l=0}^{a_{v_b(h)}(n)-1} e^{2\pi i h\phi_b(lb^{v_b(h)})} = \frac{b}{1-e^{\frac{2\pi i j}{b}}}$$

and

$$\begin{aligned}&\sum_{k=0}^{b^{m-v_b(h)-1}-1} e^{2\pi i h\phi_b(kb^{v_b(h)+1})} = \sum_{k=0}^{b^{m-v_b(h)-1}-1} e^{2\pi i b^{v_b(h)} \frac{(qb+j)\phi_b(k)}{b^{v_b(h)+1}}}\\ =\ & \sum_{\overline{k}=0}^{b^{m-v_b(h)-1}-1} e^{2\pi i \frac{(qb+j)}{b} \frac{\overline{k}}{b^{m-v_b(h)-1}}} = \frac{1-e^{\frac{2\pi i j}{b}}}{1-e^{\frac{2\pi i(qb+j)}{b^{m-v_b(h)}}}}\end{aligned}$$

where $\phi_b(k) = \frac{\overline{k}}{b^{m-v_b(h)-1}}$. Thus

$$\sum_{n=1}^{b^m-1} S_n(h) = b^{2v_b(h)} \frac{1-e^{\frac{2\pi ic}{b}}}{1-e^{\frac{2\pi i(qb+j)}{b^{m-v_b(h)}}}} \frac{b}{1-e^{\frac{2\pi ij}{b}}} = \frac{b^{2(v_b(h)+1)}}{1-e^{\frac{2\pi i(qb+j)}{b^{m-v_b(h)}}}}$$

For the proof of b), denote

$$T_m = \sum_{n=1}^{b^m-1} \frac{1}{h^2} | \sum_{n=1}^{b^m-1} S_n(h)|^2.$$

From a), we have

$$T_m = \sum_{n=1}^{b^m-1} \frac{1}{h^2} \frac{b^{4v_b(h)+2}}{4\sin^2 \frac{\pi(qb+j)}{b^{m-v_b(h)}}} = \sum_{l=0}^{m-1} \sum_{h\in A_l \bigcap [1,b^m-1]} \frac{b^{4l+2}}{4h^2 \sin^2 \frac{\pi(qb+j)}{b^{m-l}}}$$

$$= \frac{b^2}{4} \sum_{j=1}^{b-1} \sum_{l=0}^{m-1} b^{2l} \sum_{q=0}^{b^{m-l-1}-1} \frac{1}{(qb+j)^2 \sin^2 \frac{\pi(qb+j)}{b^{m-l}}}.$$

We will first estimate the lower bound of the T_m. Since $\sin x \leq x$ for $x > 0$, we have

$$\frac{4\pi^2 T_m}{b^{2m+2}} \geq \sum_{l=0}^{m-1} \sum_{q=0}^{b^{m-l-1}-1} \sum_{j=1}^{b-1} \frac{1}{(qb+j)^4}$$

$$= m(1-\frac{1}{b^4}) \sum_{q=1}^{\infty} \frac{1}{q^4} + \sum_{q=1}^{\infty} \frac{1}{q^4} + (1-\frac{1}{b^4}) \sum_{q=1}^{b-1} \frac{1}{q^4}$$

$$+\frac{1}{b^4} \sum_{q=b^m}^{\infty} \frac{1}{q^4} - (1-\frac{1}{b^4}) \sum_{l=1}^{m-1} \sum_{q=b^{l+1}}^{\infty} \frac{1}{q^4}$$

$$\geq m(1-\frac{1}{b^4}) \sum_{q=1}^{\infty} \frac{1}{q^4} + \sum_{q=1}^{\infty} \frac{1}{q^4}.$$

Thus, the lower bound of T_m follows.

For the upper bound of the T_m, by $\frac{1}{\sin x} \leq \frac{x}{3} + \frac{1}{x}$ if $x \in (0, \frac{\pi}{2}]$ and $\sin \pi x \geq 2x$ if $0 < x < \frac{1}{2}$, we have

$$T_m = \frac{b^2}{4} \sum_{j=1}^{b-1} \sum_{l=0}^{m-1} b^{2l} \sum_{g=0}^{b^{m-l-1}-1} \frac{1}{(qb+j)^2 \sin^2 \frac{\pi(qb+j)}{b^{m-l}}} = \frac{b^2}{4}(I_1 + I_2)$$

with

$$\begin{aligned} I_1 &= \sum_{j=1}^{b-1}\sum_{l=0}^{m-1} b^{2l} \sum_{\frac{qb+j}{b^{m-l}}\le\frac{1}{2}} \frac{1}{(qb+j)^2\sin^2\frac{\pi(qb+j)}{b^{m-l}}} \\ &\le \sum_{j=1}^{b-1}\sum_{l=0}^{m-1} b^{2l} \sum_{\frac{qb+j}{b^{m-l}}\le\frac{1}{2}} \frac{1}{(qb+j)^2}\left(\frac{b^{m-l}}{\pi(qb+j)}+\frac{\pi(qb+j)}{3b^{m-l}}\right)^2 \\ &\le \sum_{j=1}^{b-1}\sum_{l=0}^{m-1} b^{2l} \sum_{\frac{qb+j}{b^{m-l}}\le\frac{1}{2}} \frac{b^{2(m-l)}}{\pi^2(qb+j)^4} + \sum_{j=1}^{b-1}\sum_{l=0}^{m-1} b^{2l} \sum_{\frac{qb+j}{b^{m-l}}\le\frac{1}{2}} \frac{1}{(qb+j)^2} \\ &\le \frac{mb^{2m}}{\pi^2}\left(1-\frac{1}{b^4}\right)\sum_{q=1}^{+\infty}\frac{1}{q^4} + \frac{b^{2m}-1}{b-1}\left(1-\frac{1}{b^2}\right)\sum_{q=1}^{+\infty}\frac{1}{q^2} \end{aligned}$$

and

$$\begin{aligned} I_2 &= \sum_{j=1}^{b-1}\sum_{l=0}^{m-1} b^{2l} \sum_{\frac{qb+j}{b^{m-l}}>\frac{1}{2}} \frac{1}{(qb+j)^2\sin^2(\pi\frac{(qb+j)}{b^{m-l}})} \\ &= \sum_{j=1}^{b-1}\sum_{l=0}^{m-1} b^{2l} \sum_{\frac{qb+j}{b^{m-l}}>\frac{1}{2}} \frac{1}{[1-(1-\frac{(qb+j)}{b^{m-l}})]^2 b^{2(m-l)}\sin^2(\pi\frac{(qb+j)}{b^{m-l}})} \\ &\le \sum_{j=1}^{b-1}\sum_{l=0}^{m-1} b^{2l} \sum_{\frac{qb+j}{b^{m-l}}>\frac{1}{2}} \frac{2^2}{b^{2(m-l)}2^2(1-\frac{(qb+j)}{b^{m-l}})^2} \\ &= \sum_{j=1}^{b-1}\sum_{l=0}^{m-1} b^{2l} \sum_{\frac{qb+j}{b^{m-l}}>\frac{1}{2}} \frac{1}{(b^{m-l}-(qb+j))^2} \le \frac{b^{2m}-1}{b-1}\sum_{q=1}^{\infty}\frac{1}{q^2}. \end{aligned}$$

Thus, we have the upper bound of T_m.

Remark $\sum_{q=1}^{\infty}\frac{1}{q^4}=\frac{\pi^4}{90}$, the lemma follows. $\diamond$

Proof of Lemma 3.5. In the proof, Lemma 3.1 will be used repeatedly without mentioned.

The proof of a) can be obtained easily.

For c), write

$$b^{2m}\sum_{h=1}^{\infty}\frac{1}{h^2}\left|\sum_{n=0}^{b^m-1}\psi_{b,m}(n)e^{2\pi ih\phi_b(n)}\right|^2$$

$$= \sum_{v_b(h)\geq m} \frac{1}{h^2}|\sum_{n=0}^{b^m-1} ne^{2\pi i h\phi_b(n)}|^2 + \sum_{0\leq v_b(h)\leq m-1} \frac{1}{h^2}|\sum_{n=0}^{b^m-1} ne^{2\pi i h\phi_b(n)}|^2.$$

From the exchange of sum, we have

$$\sum_{n=0}^{b^m-1} ne^{2\pi i h\phi_b(n)} = \sum_{n=0}^{b^m-1} (S_{b^m}(h) - S_n(h)).$$

Thus, on the one hand,

$$\sum_{v_b(h)\geq m} \frac{1}{h^2}|\sum_{n=0}^{b^m-1} ne^{2\pi i h\phi_b(n)}|^2 = \sum_{k=1}^{\infty} \frac{1}{(kb^m)^2}|\sum_{n=0}^{b^m-1} (b^m - (l+1))|^2 = \frac{\pi^2}{24}(b^m-1)^2.$$

On the other hand, because $0 \leq v_b(h) \leq m-1$ if and only if there are $u, w \in \mathbf{N}$ with $1 \leq w \leq b^m - 1$ and $h = ub^m + w$, we have

$$\sum_{0\leq v_b(h)\leq m-1} \frac{1}{h^2}|\sum_{n=0}^{b^m-1} ne^{2\pi i h\phi_b(n)}|^2 = \sum_{h=1}^{b^m-1} \frac{1}{h^2}|\sum_{n=1}^{b^m-1} S_n(h)|^2$$
$$+\sum_{u=1}^{\infty}\sum_{h=1}^{b^m-1} \frac{1}{(ub^m+h)^2}|\sum_{n=1}^{b^m-1} S_n(ub^m+h)|^2,$$

and so we will only need to estimate the last term. By Lemma 3.2,

$$\frac{4}{b^2(b^m-1)^2}\sum_{u=1}^{\infty}\sum_{h=1}^{b^m-1} \frac{1}{(ub^m+h)^2}|\sum_{n=1}^{b^m-1} S_n(ub^m+h)|^2$$
$$\leq \sum_{u=1}^{\infty}\sum_{h=1}^{b^m-1} \frac{b^{2v_b(h)}}{u^2b^{2m}} = \frac{\pi^2}{6b^{2m}}\sum_{l=0}^{m-1}\sum_{h\in A_l\bigcap[1,b^m-1]} b^{2l}$$
$$= \frac{\pi^2}{6b^{2m}}\sum_{l=0}^{m-1}(b-1)b^{m-l-1}b^{2l} = \frac{\pi^2}{6b^{2m}}b^{m-1}(b^m-1),$$

so the estimate follows.

For the proof of b), remark that

$$\sum_{l=0}^{m-1}\sum_{h\in A_l\bigcap[1,b^m-1]} \frac{b^l}{h^2} = \sum_{l=0}^{m-1}\frac{1}{b^l}\sum_{q=0}^{b^{m-1-l}-1}\sum_{j=1}^{b-1}\frac{1}{(qb+j)^2} \leq \frac{\pi^2(b+1)}{6b},$$

we can procede in a similar way as in the proof of c). ⋄

4 Proof of Theorem 2.5

We only show it for the Roth sequence. In the case of Zaremba sequence, the result can be shown similarly for Lemma 3.1 to Lemma 3.5 are also valid for $\phi'_b(n)$. The only differences are the constants C_1 and C_2.

Proof Denote

$$\Sigma_{12} = \sum_{n=0}^{b^m-1} e^{2\pi i(h_1\psi_{b,m}(n)+h_2\phi_b(n))},$$

$$\Sigma_1 = \sum_{n=0}^{b^m-1} e^{2\pi i h_1 \psi_{b,m}(n)} \quad \text{and} \quad \Sigma_2 = \sum_{n=0}^{b^m-1} e^{2\pi i h_2 \phi_b(n)}.$$

We have

$$b^{2m}F^2(R_{b,m}) = \sum_{h_1,h_2\neq 0} \frac{|\Sigma_{12}|^2}{h_1^2h_2^2} + b^{2m}F^2(\phi_{b,m}) + b^{2m}F^2(\psi_{b,m}). \tag{5}$$

In the following, we will use the $F^*(R_{b,m})$ to estimate $\sum_{h_1,h_2\neq 0} \frac{|\Sigma_{12}|^2}{h_1^2h_2^2}$. Indeed,

$$b^{2m}(F^*(R_{b,m}))^2 = \sum_{h_1,h_2\neq 0} \frac{1}{h_1^2h_2^2}|\Sigma_{12} - \Sigma_1 - \Sigma_2|^2 + S_1 + S_2 \tag{6}$$

with

$$S_1 = 8\pi^2 \sum_{h_1=1}^{\infty} \frac{1}{h_1^2} |\frac{1}{2} + \sum_{n=0}^{b^m-1} \psi_{b,m}(n) e^{2\pi i h_1 \phi_b(n)} - \sum_{n=0}^{b^m-1} e^{2\pi i h_1 \phi_b(n)}|^2$$

and

$$S_2 = 8\pi^2 \sum_{h_2=1}^{\infty} \frac{1}{h_2^2} |\frac{1}{2} + \sum_{n=0}^{b^m-1} \phi_b(n) e^{2\pi i h_2 \psi_{b,m}(n)} - \sum_{n=0}^{b^m-1} e^{2\pi i h_2 \psi_{b,m}(n)}|^2 = S_1,$$

as follows from Lemma 3.3 a) and Lemma 3.5 a). From Lemma 3.3, Lemma 3.4 and Lemma 3.5, we have

$$S_1 \leq 8\pi^2 \sum_{h=1}^{\infty} \frac{1}{h^2} [\frac{1}{4} + |\sum_{n=0}^{b^m-1} \psi_{b,m}(n) e^{2\pi i h \phi_b(n)}|^2 + |\sum_{n=0}^{b^m-1} e^{2\pi i h \phi_b(n)}|^2 + |\sum_{n=0}^{b^m-1} e^{2\pi i h \phi_b(n)}|$$

$$+|\sum_{n=0}^{b^m-1}\psi_{b,m}(n)e^{2\pi ih\phi_b(n)}|+2|\sum_{n=0}^{b^m-1}\psi_{b,m}(n)e^{2\pi ih\phi_b(n)}||\sum_{n=0}^{b^m-1}e^{2\pi ih\phi_b(n)}|]$$
$$\leq \frac{\pi^4}{3}(\frac{(b^4-1)}{15b^2}m+5b+15) \quad (7)$$

and

$$S_1 \geq 8\pi^2\sum_{h=1}^{\infty}\frac{1}{h^2}[\frac{1}{4}+|\sum_{n=0}^{b^m-1}\psi_{b,m}(n)e^{2\pi ih\phi_b(n)}|^2+|\sum_{n=0}^{b^m-1}e^{2\pi ih\phi_b(n)}|^2-|\sum_{n=0}^{b^m-1}e^{2\pi ih\phi_b(n)}|$$
$$-|\sum_{n=0}^{b^m-1}\psi_{b,m}(n)e^{2\pi ih\phi_b(n)}|-2|\sum_{n=0}^{b^m-1}\psi_{b,m}(n)e^{2\pi ih\phi_b(n)}||\sum_{n=0}^{b^m-1}e^{2\pi ih\phi_b(n)}|]$$
$$\geq \frac{\pi^4}{3}(\frac{(b^4-1)}{15b^2}m-\frac{b^2+30b-15}{15}). \quad (8)$$

On the one hand, we have

$$\sum_{h_1,h_2\neq 0}\frac{1}{h_1^2h_2^2}|\Sigma_{12}|^2 \leq b^{2m}(F^*(R_{b,m}))^2-2S_1+\sum_{h_1,h_2\neq 0}\frac{1}{h_1^2h_2^2}(2|\Sigma_1||\Sigma_2|$$
$$+2|\Sigma_{12}||\Sigma_1|+2|\Sigma_{12}||\Sigma_2|-|\Sigma_1|^2-|\Sigma_2|^2), \quad (9)$$

and on the other hand,

$$\sum_{h_1,h_2\neq 0}\frac{1}{h_1^2h_2^2}|\Sigma_{12}|^2 \geq b^{2m}(F^*(R_{b,m}))^2-2S_1-\sum_{h_1,h_2\neq 0}\frac{1}{h_1^2h_2^2}(2|\Sigma_1||\Sigma_2|$$
$$+2|\Sigma_{12}||\Sigma_1|+2|\Sigma_{12}||\Sigma_2|+|\Sigma_1|^2+|\Sigma_2|^2). \quad (10)$$

From Lemma 3.1, we have

$$\sum_{h_1,h_2\neq 0}\frac{|\Sigma_1|^2}{h_1^2h_2^2}=\sum_{h_1,h_2\neq 0}\frac{|\Sigma_2|^2}{h_1^2h_2^2}=\sum_{h_1\neq 0}\frac{1}{h_1^2}\sum_{h_2\neq 0}\frac{|\Sigma_2|^2}{h_2^2}=4\sum_{k=1}^{\infty}\frac{1}{k^2}\sum_{k=1}^{\infty}\frac{(b^m)^2}{(kb^m)^2}=\frac{\pi^4}{9},$$

$$\sum_{h_1,h_2\neq 0}\frac{|\Sigma_1||\Sigma_2|}{h_1^2h_2^2}=\sum_{h_1\neq 0}\frac{|\Sigma_1|}{h_1^2}\sum_{h_2\neq 0}\frac{|\Sigma_2|}{h_2^2}=4(\sum_{k=1}^{\infty}\frac{b^m}{(kb^m)^2})^2=\frac{\pi^4}{9b^{2m}},$$

and since $|\Sigma_{12}|\leq b^m$

$$\sum_{h_1,h_2\neq 0}\frac{|\Sigma_{12}||\Sigma_1|}{h_1^2h_2^2}=\sum_{h_1,h_2\neq 0}\frac{|\Sigma_{12}||\Sigma_2|}{h_1^2h_2^2}=2\sum_{h_1\neq 0}\frac{1}{h_1^2}\sum_{k=1}\frac{|\Sigma_{12}|b^m}{(kb^m)^2}\leq 4(\sum_{k=1}^{\infty}\frac{1}{k^2})^2=\frac{\pi^4}{9}.$$

It follows from (7), (8), (9) and (10) that

$$b^{2m}F^2(R_{b,m}) \leq b^{2m}(F^*(R_{b,m}))^2 - \frac{2\pi^4(b^4-1)}{45b^2}m + \frac{2\pi^4(b^2-30b)}{45}$$

and

$$b^{2m}F^2(R_{b,m}) \geq b^{2m}(F^*(R_{b,m}))^2 - \frac{2\pi^4(b^4-1)}{45b^2}m - \pi^4(\frac{10}{3}b + 11).$$

From the estimate of $F^*(R_{b,m})$ in Theorem 2.2, the result follows. $\diamond$

5 Some remarks

The most popular measure of the irregularity of the distribution of a I^s-valued sequence $\sigma = (\xi_n)_{n\geq 0}$ in I^s is the star-discrepancy which is defined for every positive integer N by

$$D_N^*(\sigma) = \sup_{\mathbf{x}\in I^s} \frac{1}{N}|C_N(\mathbf{x}, \sigma)|.$$

We have $T_N(\sigma) \leq D_N^*(\sigma)$, thus the general lower bound of $T_N(\sigma)$ is also hold for the star-discrepancy.

For the star-discrepancy of the Roth sequence, an exact formula was obtained in [3], in which the leading term for $ND_N^*(\sigma)$ is $(1/3)\log_2 N$. The Zaremba sequence was introduced to improve the Roth sequence by permuting suitably the digits of the Roth sequence, and an exact formula was obtained in [17] for $ND_N^*(\sigma)$ with the leading term $(1/5)\log_2 N$. Curiously enough, we have seen that for the Zaremba sequence it is best possible apart from the value of the constant and it is not the case for the Roth sequence. This has been mentioned by Niederreiter in [6].

With the results on their star-diaphony and diaphony, we may explain this fact in the following way. The star-diaphony (or diaphony) is one part of the L^2 dicrepancy, the Zaremba sequence is obtained by improving the other part of the L^2 dicrepancy, I_N, and without changing the star-diaphony as it is already optimal.

Remark in [2], Grozdanov introduced a sequence by changing the digits of the Roth sequence as follows: $\Sigma_m = \{(\psi_{2,m}(n)+\frac{1}{2^{m+1}}, \phi_2(n)+\frac{1}{2^{m+1}})\}_{0\leq n\leq 2^m-1}$. By the definition of the diaphony, it has the same diaphony as the Roth sequence. With the result in this paper, we find that his estimate about the lower bound of the diaphony of the Roth sequence is wrong.

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