Localization and Parametrization of Differential Operators in Control Theory *

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Abstract

We study the link existing between the parametrization of differential operators by potential-like arbitrary functions and the localization of differential modules, while applying these results to the parametrization of control systems. In particular, we insist on the fact that the localization of differential modules is the natural way to generalize some well-known results on transfer matrix, classically obtained by using Laplace transform, to time-varying ordinary differential control systems and to partial differential control systems with variable coefficients. Among the many results presented, we include the comparison between scalar observables, namely functions of the control system variables and their derivatives, satisfying at least one ordinary or partial differential equation, and first integrals, in the ordinary case.

Keywords: Parametrization of differential operators, localization, controller form, controllability, transfer function, minimal realization, formal integrability, differential module, commutative algebra, homological algebra.

1 Introduction

If we consider an operator $\mathcal{D}_0 : E \to F_0$ and a section η of the vector bundle F_0, then a necessary condition for the local existence of a section ξ of E satisfying the inhomogenous system $\mathcal{D}_0\,\xi = \eta$ is of the form $\mathcal{D}_1\,\eta = 0$. The operator $\mathcal{D}_1$ only depends on the operator $\mathcal{D}_0$ and it is called the *compatibility conditions* of $\mathcal{D}_0$. An historical problem was to construct effectively the operator $\mathcal{D}_1$. This problem was investigated by Riquier and Cartan at the beginning of the century [2, 26] but received a nice improvement with Janet's work in the twenties [7]. Let us recall that, following Hadamard's advice, his thesis advisor, Janet went for a few months to Göttingen to study syzygies with Hilbert. Janet showed that the operator $\mathcal{D}_1$ could be constructed by bringing the operator $\mathcal{D}_0$ to *involutiveness* and that $\mathcal{D}_1$ was, in general, of high order. Starting anew with the operator $\mathcal{D}_1$, Janet proved that the procedure

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had to stop after at most $n+1$ steps, where n is the number of partial derivatives. We obtain a sequence of differential operators $\mathcal{D}_i$ on the "right" of $\mathcal{D}_0$ and depending only on $\mathcal{D}_0$

$$0 \longrightarrow \Theta \longrightarrow E \xrightarrow{\mathcal{D}_0} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} \ldots \xrightarrow{\mathcal{D}_{n-1}} F_{n-1} \xrightarrow{\mathcal{D}_n} F_n \longrightarrow 0,$$

where Θ denotes the solutions of $\mathcal{D}_0$. More recently, in the seventies, the study of systems of partial differential equations has been developped again by Spencer and students (Quillen, Goldschmidt ...) [3, 15, 24, 25] with the differential geometric point of view. They formulated systems of partial differential equations in terms of the *jet theory* of Ehresmann. Moreover, Spencer introduced groups of δ-cohomology to study the involutiveness of partial differential operators. This theory is called the *Formal Theory of Partial Differential Equations* and it is centered around the concept of *formal integrability*. Quillen and Malgrange have reformulated the differential geometric Spencer's approach in terms of a differential algebraic one, within the framework of *differential modules* [14, 24].

Dually, if we have a system of partial differential equations $\mathcal{D}_0\,\xi = 0$, it is sometimes useful, in physics or in applied mathematics, to know when the solutions of the systems can be parametrized by certain arbitrary functions called "*potentials*" (for example, $\mathrm{curl}\,\eta = 0 \Rightarrow \exists\,\xi$ such that $\mathrm{grad}\,\xi = \eta$). The new problem is the following: starting with an operator $\mathcal{D}_0 : E \to F_0$, can we find a criterion to decide on the existence of an operator $\mathcal{D}_{-1} : E_{-1} \to E$ such that $\mathcal{D}_0\,\xi = 0$ represents exactly the compatibility conditions of the inhomogeneous system $\mathcal{D}_{-1}\,\theta = \xi$. We shall say that $\mathcal{D}_{-1}$ is a parametrization of the operator $\mathcal{D}_0$. Thus, we would like to construct a differential sequence, on the left of $\mathcal{D}_0$ and depending only on $\mathcal{D}_0$,

$$E_{-k-1} \xrightarrow{\mathcal{D}_{-k}} E_{-k} \xrightarrow{\mathcal{D}_{-k+1}} \ldots \xrightarrow{\mathcal{D}_{-2}} E_{-2} \xrightarrow{\mathcal{D}_{-1}} E_{-1} \xrightarrow{\mathcal{D}_0} E,$$

in which each operator exactly describes the compatibility conditions of the previous one. This criterion exists for sygyzies. It was first provided in the seventies by Palamodov for linear partial differential operators with constant coefficients, and was extended by Kashiwara for analytic ones [10, 17]. Their results are based on the differential module approach. They say that the obstructions of embedding a left finitely generated $D = A[d_1, ..., d_n]$-module M, where A is a *differential ring* and d_i are the derivatives, into a sequence of k free D-modules D^{s_i},

$$0 \longrightarrow M \longrightarrow D^{s_0} \longrightarrow D^{s_1} \longrightarrow ... \longrightarrow D^{s_{k-1}} \longrightarrow D^{s_k}, \tag{1}$$

are:

- for k=0, the injectivity of the canonical map $M \xrightarrow{\phi} \hom_D(\hom_D(M, D), D)$ defined by $\forall x \in M, \forall f \in \hom_D(M, D) : \phi(x)(f) = f(x)$,
- for k=1, the injectivity *and* the surjectivity of ϕ,
- for $k > 1$, the injectivity and the surjectivity of ϕ plus other conditions on the vanishing of certain extension modules [28] ($\forall\, i = 1, ..., k-1 : \mathrm{ext}^i_D(\mathrm{Hom}_D(M, D), D) = 0$), a beautiful but delicate powerful tool of homological algebra which is out of our scope now and will be presented elsewhere.

As far as we are concerned, the quite abstract previous results of Palamodov-Kashiwara seem to be unknown in applied mathematics or in physics. The purpose of this paper is first of all to show that these methods are useful in control theory. We shall show that the introduction of differential modules and homological algebra in control theory gives a characterization of the structural properties of the control systems in terms of intrinsic properties of differential modules. In particular, the *controllability*, which is one of the key-concepts of the control theory, can be reformulated by means of *torsion-free* differential module. Every torsion-free differential module can be embedded in a free differential module ($k = 0$ in the Palamodov-Kashiwara's classification), which means, in the operator language, that the operator $\mathcal{D}_0$ defining the control system can be parametrized by an operator $\mathcal{D}_{-1}$. We shall show that the parametrization $\mathcal{D}_{-1}$ is just the generalization of the *controller form* for ordinary control systems (OD control systems) with time-varying coefficients or for partial differential control systems (PD control systems) with variable coefficients [8]. We shall show how to use the *formal duality* and the *formal integrability theory* (see the first paragraph) to check whether a control system is controllable or not and to compute effectively the operator $\mathcal{D}_{-1}$. In the nonlinear case, we show how to look for constrained observables and prove that such a search in the general framework supersides the search for first integrals, in the ordinary differential case. Secondly, The aim of this paper is to show that there exists another way to compute a parametrization $\mathcal{D}_{-1}$ of the controllable system $\mathcal{D}_0$, by means of a *localization* of the differential modules. We shall show that the localization approach is *the natural way* to generalize the results given by the Laplace transform techniques to the time-varying OD control systems or PD control with variable coefficients [5, 13]. Finally, we will study the link between those two approaches and we shall see that even if the parametrization found by localization is sometimes "better" than the one computed by using the formal duality and the formal integrability theory, in the sense that it is much simpler, it is in general "worst" in the sense of the Palamodov-Kashiwara's classification. Many explicit examples will illustrate the main results.

2 Linear Control Systems

Recently, the language of the differential modules has been introduced in control theory to understand in a intrinsic way the structural properties of a control system. This introduction was guided by the wish to have a tool, which formally looked like the Laplace transform, but which could permit a better understanding of the notion of *controllability* and *observability*. We recall that the *Laplace transform* changes a system of ordinary differential equations with constant coefficients into a matrix of rational fraction in the Laplace variable s (the *transfer matrix* of the system). For example, the system $\ddot{y} - \dot{y} - 2y - \dot{u} + 2u = 0$ with $y(0) = 0$, becomes, by the Laplace transform, $\hat{y} = \frac{(s-2)}{(s+1)(s-2)}\,\hat{u}$, where we denote by $\hat{y}, \hat{u}$ the Laplace transform of y, u. The transfer matrix of the system is $H(s) = \frac{(s-2)}{(s+1)(s-2)} = \frac{1}{s+1}$, after the cancellation of the common factor $(s-2)$. In 1960, Kalman was the first to understand that such a cancellation led to a loose of the controllability of the system, that is to the loose of the possibility of passing in finite time T from any initial condition $y(0) = 0$ to any point $p \in \mathbb{R}$ by a appropriated command u [9] ($\forall T$, $\forall p \in \mathbb{R}$, $\exists\, u : [0, T] \to \mathbb{R}$ such that $y(T) = p$). Moreover, such a cancellation, by a factor having a root with a negative real part, leads to a loose of the stability of the system and it is called "unstable cancellation". The use of the differential module prevents any cancellation [1, 4, 5, 13, 18]. Indeed, we shall write the system as $(\frac{d}{dt}+1)(\frac{d}{dt}-2)y - (\frac{d}{dt}-2)u = 0$, i.e., $(\frac{d}{dt}-2)((\frac{d}{dt}+1)y - u) = 0$. The last equation leads to the existence of a *torsion element*, i.e. an element $z = (\frac{d}{dt}+1)y - u$ satisfying the non zero equation $(\frac{d}{dt}-2)\,z = 0$. In this way, the controllability of the control system can be seen as the lack of any torsion element, that is to say, in the algebraic language, the module generated by the control system is a $\mathbb{R}[\frac{d}{dt}]$ *torsion-free* differential module [28]. We see that the use of module permits to give an intrinsic definition of controllability and it provides a clear understanding of the previous cancellation, a fact that cannot be done with the Laplace transform techniques (vector space over $\mathbb{R}(\frac{d}{dt})$).

2.1 Module Theory

Let $\mathcal{D}_1 : F_0 \to F_1$ be a linear partial differential operator, where F_0 and F_1 are vector bundles over a manifold X of dimension n with local coordinates $x = (x^1, ..., x^n)$. The operator $\mathcal{D}_1$ sends a section η of F_0 to a section ζ of F_1. In order to generalize the previous results to variable coefficients case, we have to use more general fields than $\mathbb{R}$ as for example $\mathbb{R}(t)$ or $\mathbb{R}(x^1, ..., x^n)$. Thus, we have to define the notion of *differential field.*

Definition 1 1. A *differential field* K with n commutative derivatives $\partial_1, ..., \partial_n$ is a ring which satisfies: $\forall\, a, b \in K,\ \forall\, i = 1, \ldots, n$:

- $\partial_i a \in K$,
- $\partial_i(a + b) = \partial_i a + \partial_i b$,
- $\partial_i(ab) = (\partial_i a)b + a\partial_i b$,
- $\partial_i \partial_j = \partial_j \partial_i$.

2. Let K be a *differential field* containing $\mathbb{Q}$ with the set of derivatives $\{\partial_1, ..., \partial_n\}$ and let us denote by D the K-algebra $K\,[d_1, ..., d_n]$ of the differential operators, satisfying:

   (a) $d_i d_j = d_j d_i$,

   (b) $\forall\, a \in K : d_i a = a\, d_i + \partial_i a$.

   The ring of differential operator D is a non commutative integral domain but it is a *left Ore algebra*, i.e., $\forall\,(p, q) \in D^2, \exists\,(u, v) \in (D \setminus 0)^2$ such that $u\,p = v\,q$.

3. Introducing the differential indeterminates $y = (y^1, ..., y^m)$, we denote by Dy the left D-module $Dy^1 + Dy^2 + ... + Dy^m$ and by M the finitely generated left D-module determined by the operator $\mathcal{D}_1$, i.e. $M = Dy/(\mathcal{D}_1 y)$, where $(\mathcal{D}_1 y)$ is the D-submodule generated by the linear differential expressions of $\mathcal{D}_1 y$.

Remark 1 We shall use either the language of jet theory for systems of partial differential equations (PDE) or the language of sections for operators [18]. In the first case, we write $d_i y^k_\mu = y^k_{\mu+1_i}$, where $\mu = (\mu_1, ..., \mu_n)$ is a multi-index of length $|\mu| = \mu_1 + ... + \mu_n$, whereas, in the second case, d_i must be replaced by ∂_i on sections. We shall use the notation $\partial_{ij} = \partial_i\, \partial_j$.

Example 1 Let us take the operator $\mathcal{D}_1 : \eta \to \zeta$ defined by

$$\left\{ \begin{array}{l} \partial_2 \eta^1 - \partial_1 \eta^1 - 2\partial_1 \eta^3 + 2\eta^1 + 2\eta^2 = \zeta^1, \\ \partial_2 \eta^1 + \partial_2 \eta^2 - \partial_1 \eta^3 = \zeta^2, \end{array} \right. \tag{2}$$

and $D = \mathbb{R}[d_1, d_2]$. We use $y = (y^1, y^2)$ and we consider the D-module $M = Dy/(y^1_2 - y^1_1 - 2y^3_1 + 2y^1 + 2y^2, y^1_2 + y^2_2 - y^3_1)$.

We recall certain properties of the D-module M [28].

Definition 2 • The D-module is *free* if there is a set of elements of M which generates M and whose elements are independent on D.

- The D-module is *projective* if there exist a free D-module F and a D-module N such that: $F = M \oplus N$. Thus, the module N is also a projective D-module.
- We call the *torsion submodule* of M, the D-module defined by $t(M) = \{m \in M \mid \exists\, p \neq 0,\ p\,m = 0\}$. The D-module M is *torsion-free* if $t(M) = 0$, and in any case, $M/t(M)$ is a torsion-free D-module.

It is quite easy to note that every free D-module is projective and every projective D-module is a torsion-free, which can be summed up by the following module inclusions:

$$\text{free} \subseteq \text{projective} \subseteq \text{torsion-free}.$$

We have the following theorem. See [28] for more details.

Theorem 1 *1. If D is a principal ideal ring (for example $D = K[\frac{d}{dt}]$ with K a differential field), every torsion-free D-module is a free D-module.*

2. Every projective module over a polynomial ring $k[\chi_1, ..., \chi_n]$, where k is a field, is free (Quillen-Suslin). Thus, over $D = k[\partial_1, ..., \partial_n]$, with k a constant field (i.e., $\forall\, i = 1, ..., n, \forall\, a \in k : \partial_i\, a = 0$), any projective module is free.

2.2 Controllability

Now, let a control system be defined by the operator $\mathcal{D}_1 : F_0 \to F_1$. Let M be the D-module determined by this operator, i.e., $M = Dy/(\mathcal{D}_1 y)$. The lack of controllability of the polynomial control system is due to the existence of torsion element. We find the following definition of controllability in [1, 4, 13, 18].

Definition 3 1. We call *observable* of the control system any scalar function of the inputs, outputs and their derivatives up to a certain order. In the linear case, an observable is an element of M.

2. A system is called *controllable* if every observable of the system does not satisfy by itself a PDE.

Theorem 2 *A control system defined by an operator $\mathcal{D}_1$ is controllable iff the D-module M determined by $\mathcal{D}_1$ is torsion-free.*

As a by-product, the control system defined by the D-module $M/t(M)$ is always controllable.

Example 2 1. Let the OD control system be defined by

$$\left\{ \begin{array}{l} \dot{\eta}^1 - \eta^2 - \eta^3 = 0, \\ \dot{\eta}^2 - \eta^1 + \alpha\, \eta^3 = 0, \end{array} \right. \tag{3}$$

where α is a constant coefficient in $\mathbb{R}$. We let the reader check that for $\alpha = -1$, the element $z = \eta^1 - \eta^2$ satisfies $\dot{z} + z = 0$ and thus z determines a torsion element in the corresponding D-module M. Moreover, if $\alpha = 1$, we have $z = \eta^1 + \eta^2$ satisfying $\dot{z} - z = 0$ and thus it determines a torsion element in M. At least for two values of the parameter α, the control system is not controllable.

2. Let us consider the system defined by:

$$\left\{ \begin{array}{l} \partial_2\eta^1 - \partial_1\eta^1 - 2\partial_1\eta^3 + 2\eta^1 + 2\eta^2 = 0, \\ \partial_{12}\eta^3 - \partial_2\eta^1 - \partial_2\eta^2 = 0. \end{array} \right.$$

This system is not controllable as the observable $z = \eta^1$ satisfies $\partial_{22}z - \partial_{12}z = 0$.

In the previous example, we have shown that for two disctinct values of the parametrer α, the system was uncontrollable. Now, we can wonder if for the others values of α, the system is controllable. Thus, we would like to have a formal test permitting us to know whenever a system is controllable, i.e., whether the D-module generated by the system is torsion-free. However, any submodule of a free module is torsion-free and reciproquely any torsion-free can be embedded in a free module. This is the first embedding in the Palamodov-Kashiwara classification and we have to know if $M \xrightarrow{\phi} \hom_D(\hom_D(M, D), D)$ is injective or not (see the introduction). This can be checked by using the formal duality and the formal theory of PDE as it has been independently discovered in [18].

3 Parametrization

Every torsion-free module can be embedded into a free module ($k = 0$ in the previous Palamodov-Kashiwara's classification), which means, in the operator language, that the operator $\mathcal{D}_0$ can be parametrized by a certain operator $\mathcal{D}_{-1}$. We shall show how to compute effectively the operator $\mathcal{D}_{-1}$ for controllable systems and we shall also show that it is the generalization of the well-known *controller form* for control systems in the Laplace domain, i.e. in polynomial form with respect to the Laplace variables $s = (s_1, ..., s_n)$ (see [8, 22]).

3.1 Torsion-free D-module

We denote by E a vector bundle over a manifold X, by $T^\star$ the cotangent bundle of X, by $E^\star$ the dual bundle of E and by $\tilde{E} = \bigwedge^n T^\star \otimes E^\star$ its adjoint bundle. The adjoint bundle $\tilde{E}$ is the right generalization of the concept of tensor density in physics [18].

Definition 4 If $\mathcal{D}_1 : F_0 \to F_1$ is a linear differential operator, its formal adjoint $\tilde{\mathcal{D}}_1 : \tilde{F}_1 \to \tilde{F}_0$ is defined by the following formal rules equivalent to the integration by parts:

- the adjoint of a matrix (zero order operator) is the transposed matrix,
- the adjoint of ∂_i is $-\partial_i$,
- for two linear PD operators P, Q that can be composed: $\widetilde{P \circ Q} = \tilde{Q} \circ \tilde{P}$.

We can easily verify that $\tilde{\tilde{\mathcal{D}}}_1 = \mathcal{D}_1$. It can be proved that, for any section λ of $\tilde{F}_1$, we have the relation

$$< \lambda, \mathcal{D}_1 \eta > - < \tilde{\mathcal{D}}_1 \lambda, \eta >= d(\cdot),$$

expressing a difference of n-forms ($\lambda \in \bigwedge^n T^\star \otimes F_1^\star \Rightarrow < \lambda, \mathcal{D}_1 \eta > \in \bigwedge^n T^\star$), where d is the standard exterior derivative. We can directly compute the adjoint of an operator by multiplying it by test functions on the left and integrating it by parts.

Example 3 Let us compute the adjoint of the operator $\mathcal{D}_1 : \eta \to \zeta$ defined by

$$\begin{cases} \dot{\eta}^1 - \eta^2 - \eta^3 = \zeta^1, \\ \dot{\eta}^2 - \eta^1 + \alpha\, \eta^3 = \zeta^2. \end{cases} \tag{4}$$

We multiply the system on the left by the row vector (λ_1, λ_2) and we integrate the result by parts, we find $\tilde{\mathcal{D}}_1 : (\lambda_1, \lambda_2) \to (\mu_1, \mu_2, \mu_3)$ defined by:

$$\begin{cases} -\dot{\lambda}_1 - \lambda_2 = \mu_1, \\ -\dot{\lambda}_2 - \lambda_1 = \mu_2, \\ -\lambda_1 + \alpha\, \lambda_2 = \mu_3. \end{cases} \tag{5}$$

As we have noticed in the introduction, it is sometimes useful to parametrize a system of PDE by some "potentials" considered as arbitrary functions. It leads to the following definition.

Definition 5 An operator $\mathcal{D}_0 : E \to F_0$ is said to be a *parametrization* of the operator $\mathcal{D}_1 : F_0 \to F_1$, if $\mathcal{D}_1$ represents exactly the compatibility conditions of $\mathcal{D}_0\,\xi = \eta$.

Let us describe the formal test checking if a D-module is torsion-free or not [18].

Torsion-free Test & Parametrization:

1. Start with $\mathcal{D}_1$.
2. Construct its adjoint $\tilde{\mathcal{D}}_1$.
3. Find the compatibility conditions of $\tilde{\mathcal{D}}_1\lambda = \mu$ and denote this operator by $\tilde{\mathcal{D}}_0$.
4. Construct its adjoint $\mathcal{D}_0\,(=\tilde{\tilde{\mathcal{D}}}_0)$.
5. Find the compatibility conditions of $\mathcal{D}_0\,\xi = \eta$ and call this operator $\mathcal{D}'_1$.

We are led to two different cases. If $\mathcal{D}_1$ is exactly the compatibility conditions $\mathcal{D}'_1$ of $\mathcal{D}_0$, then the system $\mathcal{D}_1$ determines a torsion-free D-module M and $\mathcal{D}_0$ is a parametrization of $\mathcal{D}_1$. Otherwise, the operator $\mathcal{D}_1$ is among, but not exactly, the compatibility conditions of $\mathcal{D}_0$. The torsion elements of M are all the new compatibility conditions modulo the equations $\mathcal{D}_1\eta = 0$.

Proof The operator $\tilde{\mathcal{D}}_0$ describes exactly the compatibility conditions of the operator $\tilde{\mathcal{D}}_1$ and we have in particular $\tilde{\mathcal{D}}_0 \circ \tilde{\mathcal{D}}_1 = 0 \Rightarrow \mathcal{D}_1 \circ \mathcal{D}_0 = 0$. Thus, $\mathcal{D}_1$ is among the compatibility conditions of $\mathcal{D}_0$, which are described by the operator $\mathcal{D}'_1$. Now, computing the differential rank of the operators $\mathcal{D}'_1$ and $\mathcal{D}_1$, we find that diff rk $\mathcal{D}'_1$ = diff rk $\mathcal{D}_1$ (see [18] for more details). If $\mathcal{D}_1$ is strictly among the compatibility conditions of $\mathcal{D}_0$, then any new single compatibility condition ζ' in $\mathcal{D}'_1$ is a differential consequence of $\mathcal{D}_1$ (diff rk $\mathcal{D}'_1$ = diff rk $\mathcal{D}_1$), and we can find an operator $q \in D$ such that $q\,\zeta' = 0$ whenever $\mathcal{D}_1\,\eta = 0$. Thus, any new single compatibility condition of $\mathcal{D}_0$ (not in $\mathcal{D}_1$) determines a torsion element. If $\mathcal{D}_1$ describes exactly the compatibility conditions of $\mathcal{D}_0$, then the D-module M determined by $\mathcal{D}_1$ is torsion-free because $M \subseteq D\,\xi$ and $D\,\xi$ is a free D-module.

We can represent the test by the following differential sequences where the numbers indicate the different stages:

$$\begin{array}{cccccc}
 & & & \xrightarrow{\mathcal{D}'_1} & F'_1 & 5 \\
2 & E & \xrightarrow{\mathcal{D}_0} F_0 & \xrightarrow{\mathcal{D}_1} & F_1 & 1 \\
3 & \tilde{E} & \xleftarrow{\tilde{\mathcal{D}}_0} \tilde{F}_0 & \xleftarrow{\tilde{\mathcal{D}}_1} & \tilde{F}_1 & 2
\end{array}$$

If the D-module determined by the operator $\mathcal{D}_1$ is torsion-free, the test gives a parametrization $\mathcal{D}_0$ of $\mathcal{D}_1$. This operator is the generalization of the *controller form* [8, 22] for non-surjective time-varying OD control systems and PD control systems with variable coefficients (see [22] for more details). Let us illustrate it with example of ordinary differential time delay system presented in [16] but interpreted here as partial differential system.

Example 4 Let us try to know if the operator given by (2) is controllable or not. We multiply $\mathcal{D}_1\eta$ on the left by $\lambda = (\lambda_1, \lambda_2)$ and we integrate the result by parts in order to find $\tilde{\mathcal{D}}_1 : \lambda \to \mu$:

$$\begin{cases} -\partial_2\lambda_1 + \partial_1\lambda_1 - \partial_2\lambda_2 + 2\lambda_1 = \mu_1, \\ -\partial_2\lambda_2 + 2\lambda_1 = \mu_2, \\ 2\partial_1\lambda_1 + \partial_1\lambda_2 = \mu_3. \end{cases}$$

The system admits only one compatibility condition of second order (formal integrability theory) which defines the operator $\tilde{\mathcal{D}}_0 : \mu \to \nu$:

$$\partial_{22}\,\mu_3 - \partial_{12}\,\mu_3 - \partial_{12}\,\mu_2 + 2\partial_{12}\,\mu_1 - \partial_{11}\,\mu_2 + 2\partial_1\,\mu_1 - 2\partial_1\,\mu_2 = \nu.$$

Taking its adjoint, we finally find the operator $\mathcal{D}_0 : \xi \to \eta$ defined by:

$$\begin{cases} 2\partial_{12}\,\xi - 2\partial_1\,\xi = \eta^1, \\ -\partial_{12}\,\xi - \partial_{11}\,\xi + 2\partial_1\,\xi = \eta^2, \\ \partial_{22}\,\xi - \partial_{12}\,\xi = \eta^3. \end{cases} \tag{6}$$

We let the reader check that the operator $\mathcal{D}_1$ exactly generates the compatibility conditions of $\mathcal{D}_0$ and thus $\mathcal{D}_1$ determines a torsion-free D-module. A parametrization of (2) is (6).

We now describe how to compute the torsion elements.

Computation of torsion elements:

1. Compute $\mathcal{D}'_1$ and check that $\mathcal{D}_1$ is strictly among $\mathcal{D}'_1$.
2. For any new single compatibility condition of the form $\mathcal{D}'_1\eta = \zeta'$ of $\mathcal{D}'_1$, compute the compatibility conditions of the following system:

$$\begin{cases} \mathcal{D}_1\,\eta = 0, \\ \mathcal{D}'_1\,\eta = \zeta' \text{ (one equation only)}. \end{cases}$$

3. We find that ζ' is a torsion element of M satisfying $q\,\zeta' = 0$ with $0 \neq q \in D$.

We give an example of the search of torsion elements.

Example 5 Let us consider the operator $\mathcal{D}_1 : \eta \to \zeta$, adapted from a ordinary differential time delay system presented in [16], defined by:

$$\begin{cases} \partial_2\eta^1 - \partial_1\eta^1 - 2\partial_1\eta^3 + 2\eta^1 + 2\eta^2 = \zeta^1, \\ \partial_{12}\eta^3 - \partial_2\eta^1 - \partial_2\eta^2 = \zeta^2. \end{cases} \tag{7}$$

Its formal adjoint $\tilde{\mathcal{D}}_1 : \lambda \to \mu$ is defined by:

$$\begin{cases} -\partial_2\lambda_1 + \partial_2\lambda_2 + \partial_1\lambda_1 + 2\lambda_1 = \mu_1, \\ \partial_2\lambda_2 + 2\lambda_1 = \mu_2, \\ \partial_{12}\lambda_2 + 2\partial_1\lambda_1 = \mu_3. \end{cases}$$

There is one compatibility condition $-\partial_1\mu_2 + \mu_3 = 0$ and thus the operator $\tilde{\mathcal{D}}_0 : \mu \to \nu$ is given by:

$$-\partial_1\mu_2 + \mu_3 = \nu.$$

We find the operator $\mathcal{D}_0 : \xi \to \eta$ defined by

$$\left\{ \begin{array}{l} 0 = \eta^1, \\ \partial_1 \xi = \eta^2, \\ \xi = \eta^3, \end{array} \right.$$

and we find the following operator $\mathcal{D}'_1 : \eta \to \zeta'$

$$\left\{ \begin{array}{l} \eta^1 = \zeta'^1, \\ \partial_1 \eta^3 - \eta^2 = \zeta'^2 \end{array} \right. \tag{8}$$

Thus, the D-module determined by $\mathcal{D}_1$ admits torsion elements which can be computed by finding the compatibility conditions of the systems

$$\left\{ \begin{array}{l} \eta^1 = \zeta'^1, \\ \partial_2 \eta^1 - \partial_1 \eta^1 - 2\partial_1 \eta^3 + 2\eta^1 + 2\eta^2 = 0, \\ \partial_{12} \eta^3 - \partial_2 \eta^1 - \partial_2 \eta^2 = 0, \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \partial_1 \eta^3 - \eta^2 = \zeta'^2, \\ \partial_2 \eta^1 - \partial_1 \eta^1 - 2\partial_1 \eta^3 + 2\eta^1 + 2\eta^2 = 0, \\ \partial_{12} \eta^3 - \partial_2 \eta^1 - \partial_2 \eta^2 = 0, \end{array} \right.$$

and we find the two torsion elements satisfying

$$\left\{ \begin{array}{l} \zeta'^1 = \eta^1, \\ \partial_{22} \zeta'^1 - \partial_{12} \zeta'^1 = 0, \end{array} \right.$$

and

$$\left\{ \begin{array}{l} \zeta'^2 = \partial_1 \eta^3 - \eta^2, \\ \partial_{22} \zeta'^2 - \partial_{12} \zeta'^2 = 0. \end{array} \right.$$

3.2 Projective & Free D-module

Let us turn to a projective D-module. We only give here a charactarization of a projective D-module determined by a surjective operator $\mathcal{D}_1$. We refer the reader to [22] for a general treatment of projective D-modules and for their applications in control theory to the generalized Bezout identity [8]. See also [6, 29, 30, 31].

Theorem 3 *A surjective operator $\mathcal{D}_1 : F_0 \to F_1$ determines a projective D-module M if its adjoint $\tilde{\mathcal{D}}_1$ is injective, i.e., if there exists an operator $\mathcal{P}_1 : F_1 \to F_0$ such that $\mathcal{D}_1 \circ \mathcal{P}_1 = id_{F_1}$, where id_{F_1} is the identity operator of F_1.*

Proof If the operator $\tilde{\mathcal{D}}_1$ is injective, then a differential consequence of the equations $\tilde{\mathcal{D}}_1 \lambda = 0$ is $\lambda = 0$. Using the formal integrability theory, we have $\tilde{\mathcal{D}}_1 \lambda = \mu \Rightarrow \lambda = \tilde{\mathcal{P}}_1 \mu$ and thus $\tilde{\mathcal{P}}_1 \circ \tilde{\mathcal{D}}_1 = id_{\tilde{F}_1} \Rightarrow \mathcal{D}_1 \circ \mathcal{P}_1 = id_{F_1}$. The operator $\mathcal{P}_1 : F_1 \to F_0$ is a right-inverse of $\mathcal{D}_1$ and $\mathcal{D}_1$ determines a projective D-module.

Finally, we have the obvious theorem.

Theorem 4 *An operator $\mathcal{D}_1$ determines a free D-module if it exists an injective parametrization $\mathcal{D}_0$, i.e., if it exists a left-inverse $\mathcal{P}_0$ of the operator $\mathcal{D}_0$.*

Let us give an example to illustrate both projective and free D-modules.

Example 6 Let us consider the following operator $\mathcal{D}_2 : \zeta \to \pi$, defined by:

$$\partial_2\zeta^2 - x^2\partial_1\zeta^1 + \zeta^1 = \pi. \tag{9}$$

Its formal adjoint $\tilde{\mathcal{D}}_2 : \kappa \to \lambda$ is given by

$$\left\{ \begin{array}{l} x^2\,\partial_1\kappa + \kappa = \lambda_1, \\ -\partial_2\kappa = \lambda_2, \end{array} \right.$$

and we easily see that $\tilde{\mathcal{D}}_1$ is an injective operator as we have

$$\kappa = -x^2\partial_2\lambda_1 - (x^2)^2\partial_1\lambda_2 - x^2\lambda_2 + \lambda_1. \tag{10}$$

Thus, the operator $\mathcal{D}_2$ generates a projective D-module, and taking the adjoint of (10), we obtain a right-inverse $\mathcal{P}_2 : \pi \to \zeta$ of $\mathcal{D}_2$:

$$\left\{ \begin{array}{l} x^2\partial_2\pi + 2\pi = \zeta^1, \\ (x^2)^2\partial_1\pi - x^2\pi = \zeta^2. \end{array} \right.$$

We let the reader check that $\mathcal{D}_2 \circ \mathcal{P}_2 = id_{F_2}$. We obtain the operator $\tilde{\mathcal{D}}_1 : \lambda \to \mu$, by substituting (10) in $\tilde{\mathcal{D}}_2$, and we find:

$$\left\{ \begin{array}{l} (x^2)^2\partial_{12}\lambda_1 + (x^2)^3\partial_{11}\lambda_2 + 2(x^2)^2\partial_1\lambda_2 - x^2\partial_1\lambda_1 + x^2\partial_2\lambda_1 + x^2\lambda_2 = \mu_1, \\ x^2\partial_{22}\lambda_1 + (x^2)^2\partial_{12}\lambda_2 + 2x^2\partial_1\lambda_2 + x^2\partial_2\lambda_2 = \mu_2. \end{array} \right. \tag{11}$$

Dualizing $\tilde{\mathcal{D}}_1$, we obtain a parametrization $\mathcal{D}_1 : \eta \to \zeta$ of $\mathcal{D}_2$:

$$\left\{ \begin{array}{l} x^2\,\partial_{22}\,\eta^2 + (x^2)^2\,\partial_{12}\,\eta^1 + 2\,\partial_2\,\eta^2 + 3x^2\,\partial_1\,\eta^1 - x^2\,\partial_2\,\eta^1 - \eta^1 = \zeta^1, \\ (x^2)^2\,\partial_{12}\,\eta^2 + (x^2)^3\,\partial_{11}\,\eta^1 - x^2\,\partial_2\,\eta^2 - 2(x^2)^2\,\partial_1\,\eta^1 + x^2\,\eta^1 - \eta^2 = \zeta^2. \end{array} \right. \tag{12}$$

We see that we can parametrize the operator $\mathcal{D}_2$ by two arbitrary functions η^1 and η^2.

Does there exist a parametrization of the new operator $\mathcal{D}_1$? To answer it, let us take the adjoint of $\mathcal{D}_1$ defined by (11) and let us see if it admits some compatibility conditions $\tilde{\mathcal{D}}_0$. We easily note that the two equations are not differentially independent (independent on D, see [12, 27]) as we have one compatibility condition between μ_1 and μ_2, i.e.

$$x^2\partial_2\mu_1 - (x^2)^2\partial_1\mu_2 - x^2\mu_2 - \mu_1 = 0.$$

Thus, the operator $\tilde{\mathcal{D}}_0 : \mu \to \nu$ is defined by

$$x^2\partial_2\mu_1 - (x^2)^2\partial_1\mu_2 - x^2\mu_2 - \mu_1 = \nu,$$

and dualizing it, we find the following operator $\mathcal{D}_0 : \xi \to \eta$:

$$\left\{ \begin{array}{l} -x^2\,\partial_2\,\xi - 2\,\xi = \eta^1, \\ (x^2)^2\,\partial_1\,\xi - x^2\,\xi = \eta^2. \end{array} \right. \tag{13}$$

The operator $\mathcal{D}_0$ is an injective operator as we have

$$\xi = \partial_2\,\eta^2 + x^2\,\partial_1\,\eta^1 - \eta^1,$$

and we can easily verify that $\mathcal{D}_0$ is a parametrization of $\mathcal{D}_1$. Thus, $\mathcal{D}_1$ determines a free D-module with ξ for basis and the operator $\mathcal{D}_2$ admits a parametrization $\mathcal{D}_1$ which admits itself a parametrization $\mathcal{D}_0$:

$$0 \longrightarrow E \xrightarrow{\mathcal{D}_0} F_1 \xrightarrow{\mathcal{D}_1} F_2 \xrightarrow{\mathcal{D}_2} F_2 \longrightarrow 0.$$

Let us give a useful corollary of theorem 4.

Corollary 1 *A surjective OD operator $\mathcal{D}_1$ is controllable iff its adjoint is injective, i.e., iff it exists a right-inverse $\mathcal{P}_1$ of $\mathcal{D}_1$.*

Example 7 Let us test the controllability of the OD control system (4). The operator $\mathcal{D}_1$ defined by (4) is surjective and its formal adjoint is given by (5), i.e.

$$\begin{cases} -\dot{\lambda}_1 - \lambda_2 = \mu_1, \\ -\dot{\lambda}_2 - \lambda_1 = \mu_2, \\ -\lambda_1 + \alpha\, \lambda_2 = \mu_3 \end{cases}$$

Let us investigate the injectivity of $\tilde{\mathcal{D}}_1$. Differentiating the zero order equation and substituting it in the others, we find the new zero order equation

$$(1-\alpha)(1+\alpha)\lambda_2 = \dot{\mu}_3 - \mu_1 + \alpha\, \mu_2 - \alpha\, \mu_3.$$

We can easily verify that $\tilde{\mathcal{D}}_1$ is injective and thus controllable iff $(1-\alpha)(1+\alpha) \neq 0$. Finally, we obtain the following tree of integrability conditions:

$(1-\alpha)(1+\alpha) = 0$	$(1-\alpha)(1+\alpha) \neq 0$
not controllable	controllable

See [21, 23] for more general trees of integrability conditions. In this way, the controllability of a surjective OD control system with unknown time-varying coefficients depends on a single tree of integrability conditions. In the general situation, it depends on two problems of formal integrability ($\tilde{\mathcal{D}}_0$ and $\mathcal{D}_0$) and thus it depends on two trees of integrability conditions. We let the reader check that for $\alpha = -1$ and $\alpha = 1$, we find the torsion elements defined in example 2. Let us show the link between *torsion elements* and *first integrals of motion*. If $\alpha = 1$, the operator $\tilde{\mathcal{D}}_1$ is not injective and the solution of $\tilde{\mathcal{D}}_1\, \lambda = 0$ is, after one integration,

$$\lambda_1(t) = \lambda_2(t) = e^{-(t-t_0)}\lambda_1(t_0).$$

Moreover, we have $< \lambda, \mathcal{D}_1\eta > = < \tilde{\mathcal{D}}_1\lambda, \eta > + \frac{d}{dt}(\lambda_1\eta^1 + \lambda_2\eta^2)$ and if we take (η, λ) satisfying $\mathcal{D}_1\eta = 0$ and $\tilde{\mathcal{D}}_1\lambda = 0$, we obtain $\frac{d}{dt}(\lambda_1\eta^1 + \lambda_2\eta^2) = 0$ and thus we obtain a first integral of motion

$$\begin{cases} Z(t) = e^{-(t-t_0)}\lambda_1(t_0)(\eta^1(t) + \eta^2(t)), \\ \dot{Z}(t) = 0. \end{cases}$$

We can do the same for $\alpha = -1$.

We have the following theorem.

Theorem 5 *If $\mathcal{D}_1$ is a surjective and non controllable system, then the following numbers are equal:*

1. *The number of solutions of the adjoint operator $\tilde{\mathcal{D}}_1$ that are linearly independent over the constants of K.*

2. *The dimension over K of the jet space of order zero of the corresponding adjoint system.*

3. *The number of torsion elements which are linearly independent over the constants of K.*

4. *The number of first integrals that are linearly independent over the constants of K.*

Hence, we would like to stress the importance of torsion elements compared to first integrals in view of the following two comments:

1. The search for torsion elements is purely algebraic while the search for first integrals is purely analytic (integration needed).

2. The concept of torsion elements can be extended to the PD case as the concept of first integrals is only restricted to the OD case.

Remark 2 In the nonlinear framework, a similary comment is still valid but is out of the scope of this paper devoted to linear systems. Shortly, we study here the case of an affine OD system $\dot{y} = a(y) + \sum_{i=1}^{s} b_i(y)u_i$. We have already indicated in [18] that the number r of functionaly independent constrained observables, that is, observables satisfying at least one OD equation is equal to the corank of the strong controllability matrix generated by $b_i, [b_i, b_j], [a, b_i]...$, because each such observable must be killed by this distribution. If we denote by $z_1, ..., z_r$ such a functionaly independent set, we notice that the derivatives $\dot{z}_1, ..., \dot{z}_r$ are still constrained observables and we have, according to the implicit function theorem, $\dot{z}_i = \phi_i(z_1, ..., z_r)$. Hence, if $Z = f(t, z_1, ..., z_r)$ is a first integral, we have $\dot{Z} = \frac{\partial f}{\partial t} + \sum_{i=1}^{s} \frac{\partial f}{\partial z_i}\phi_i(z_1, ..., z_r) = 0$. Hence, we find by integration (as in the linear case indeed), exactly r functionaly independent first integrals $Z_1, ..., Z_r$ killed by the vector field $\frac{\partial}{\partial t} + \sum_{i=1}^{s} \phi_i(z)\frac{\partial}{\partial z_i}$. Once more, we notice that $z_i = g_i(y)$ only depend on y while $Z_i = h_i(t, y)$ explicitly depends on t in general. Of course, we notice that first integrals are trivially constrained observables.

We have shown how the theory of differential modules allows us to give a more intrinsic formulation of certain properties of the control systems. In particular, this theory makes clear the fact that the simplifications of the transfer matrix correspond to the existence of torsion elements in the module generated by the control system.

4 Localization

We would like to have a formal generalization, in the differential module language, of the Laplace transform but extended to time-varying ordinary differential equations or to partial differential equations with variable coefficients. We shall recall that it can be done by *localization*. This method leads to the well-known concept, in control theory, of the *minimal-realization* which is just equivalent to find a realization of the torsion-free part of M. Thus, by extension of the coefficients of M from D to $Q(D)$, we can copy much of the Laplace methods. This fact was first observed by Oberst [11, 13] in case of constant coefficients and by Fliess [5] for time-varying control systems. Let us recall those results and develop them for PD control systems with variable coefficients.

5 Definition & Properties

Let us recall some results about the localization. See [10] for more details.

Definition 6 Let S be a *multiplicative* subset of D, i.e. a subset of D satisfying the following properties:

1. $1 \in S$,
2. $\forall\, s, t \in S \Rightarrow st \in S$
3. $\forall\, a \in D, s \in S \Rightarrow \exists\, b \in D, t \in S$ such that $t\,a = b\,s$.
4. $\forall\, a \in D, s \in S$ such that $a\,s = 0 \Rightarrow \exists\, t \in S$ such that $ta = 0$.

Let M be a D-module, then we define the $S^{-1}D$-module $S^{-1}M$ as the quotient of the sets $(s, x) \in S \times M$ by the equivalence relation defined by:

$$(s_1, x_1) \sim (s_2, x_2) \Leftrightarrow \exists\, s_1', s_2', \in S \text{ such that } s_1' s_1 = s_2' s_2 \text{ and } s_1' x_1 = s_2' x_2.$$

We denote by $s^{-1}x$ the equivalence class of the pair (s, x) and such a procedure is called "left localization". We have $S^{-1}M = S^{-1}D \otimes_D M$.

In particular, if we take $S = D \backslash 0$, we obtain $S^{-1}D = Q(D)$ the left field of fraction of D and we have the following exact sequence

$$0 \longrightarrow t(M) \longrightarrow M \xrightarrow{i_S} S^{-1}M = Q(D) \otimes_D M,$$

where $t(M)$ is the torsion D-submodule of M. So, if the D-module M is a torsion-free D-module, then the homomorphism i_S is injective and M is embedded into the $Q(D)$-vector space $Q(D) \otimes_D M$. The torsion elements vanish in $Q(D) \otimes M$, a fact which is similar, in the constant coefficients case, to the cancellation in the transfer matrix. The following theorem extends the passage from left-coprime to right-coprime used in classical control theory ($D = \mathbb{R}[s]$) [8].

Theorem 6 *Let $S = D \backslash 0$ then we have:*

$$S^{-1}D = DS^{-1}. \tag{14}$$

Let us give an effective proof of this theorem, which makes clear the link between localization techniques and the use of duality through the formal test for checking whether a module is torsion-free or not.

Proof Let $a \in S$ and $b \in D$, we have to show that $\exists\, p \in D,\ q \in S$ such that $a^{-1}b = p\,q^{-1}$. If $b = 0$, then the result is obvious. Let us suppose that $b \neq 0$. We denote by $\mathcal{D}_1 : \eta \to \zeta$ the operator defined by:

$$a\,\eta^1 - b\,\eta^2 = \zeta.$$

The adjoint $\tilde{\mathcal{D}}_1 : \lambda \to \mu$ is defined by:

$$\left\{ \begin{array}{l} \tilde{a}\,\lambda = \mu_1, \\ \tilde{b}\,\lambda = \mu_2. \end{array} \right.$$

Now, using the fact that D is a left Ore algebra, we can find one compatibility condition $\tilde{\mathcal{D}}_0 : \mu \to \nu$ defined by

$$\tilde{p}\,\mu_1 - \tilde{q}\,\mu_2 = \nu, \tag{15}$$

with $\tilde{q} \neq 0$ and thus $\tilde{p} \neq 0$. Dualizing, we obtain the operator $\mathcal{D}_0 : \xi \to \eta$ given by :

$$\left\{ \begin{array}{l} p\,\xi = \eta^1, \\ q\,\xi = \eta^2. \end{array} \right.$$

Hence, $\xi = q^{-1}\eta^2 \Rightarrow \eta^1 = p\,q^{-1}\eta^2$. Finally, the kernel of the operator $\mathcal{D}_1$ is defined by $a\,\eta^1 - b\,\eta^2 = 0,\ a \neq 0 \Rightarrow \eta^1 = a^{-1}\,b\,\eta^2$ and thus $a^{-1}\,b = p\,q^{-1}$, which concludes the proof.

We have the following corollary.

Corollary 2 *D is a right Ore algebra, i.e., $\forall\,(a,b) \in D^2, \exists\,(p,q) \in (D\backslash 0)^2$ such that $ap = bq$.*

When we start with an operator with constant coefficients which determines a torsion-free $D = \mathbb{R}[d_1, ..., d_n]$-module, then we easily obtain a parametrization by localization. Indeed, we have the useful relation:

$$\partial_i^{-1}\partial_j = \partial_j\partial_i^{-1}.$$

Let us give an example.

Example 8 Let us try to find by localization a parametrization of the divergence operator in $\mathbb{R}^3$, defined by:

$$\partial_1\eta^1 + \partial_2\eta^2 + \partial_3\eta^3 = \zeta.$$

We have $\eta^3 = -\partial_3^{-1}(\partial_1\eta^1) - \partial_3^{-1}(\partial_2\eta^2) \Rightarrow \eta^3 = -\partial_1(\partial_3^{-1}\eta^1) - \partial_2(\partial_3^{-1}\eta^2)$. Finally, if we denote

$$\left\{ \begin{array}{l} \xi^1 = \partial_3^{-1}\eta^1, \\ \xi^2 = \partial_3^{-1}\eta^2, \end{array} \right.$$

we have the following parametrization of the divergence operator:

$$\left\{ \begin{array}{l} \partial_3\,\xi^1 = \eta^1, \\ \partial_3\,\xi^2 = \eta^2, \\ -\partial_1\,\xi^1 - \partial_2\,\xi^2 = \eta^3. \end{array} \right. \tag{16}$$

We remark that we do not find the usual parametrization of the divergence operator by the curl. Moreover, this new parametrization (16) cannot be parametrized in its turn whereas the curl is parametized by the gradient ($k = 1$ in the Palamodov-Kashiwara's classification). We shall see that it is a general fact that the parametrizations found by localization are "simpler" than those obtained using the formal test, but they are "worst" in the sense of Palamodov-Kashiwara's classification of differential modules (see the introduction).

The situation of operators with variable coefficients is more complicated. However, the proof of theorem 7 shows how to use the formal duality and the formal integrability to find a parametrization of an operator, when it determines a torsion-free D-module.

Example 9 Let us consider the following system of PDE:

$$\partial_2\,\zeta^2 - x^2\,\partial_1\,\zeta^1 + \zeta^1 = 0.$$

We can solve the system with respect to ζ^2: $\zeta^2 = \partial_2^{-1}(x^2\,\partial_1 - 1)\zeta^1$. We pose $a = \partial_2$ and $b = x^2\,\partial_2 - 1$, and let us search two elements p and $q \in D$ such that $ap = bq$. This is equivalent to search $\tilde{p}, \tilde{q} \in D$ such that $\tilde{p}\tilde{a} = \tilde{q}\tilde{b}$, i.e. to find one compatibility condition of the following operator:

$$\left\{ \begin{array}{l} -\partial_2\kappa = \lambda_1, \\ -x^2\partial_1\kappa - \kappa = \lambda_2. \end{array} \right. \tag{17}$$

This operator is injective as we have $\kappa = x^2\partial_2\lambda_2 - (x^2)^2\partial_1\lambda_1 - x^2\lambda_1 - \lambda_2$ and we find two different compatibility conditions of (17): the first is defined by

$$-\partial_{22}\lambda_2 + x^2\partial_{12}\lambda_1 + 2\partial_1\lambda_1 + \partial_2\lambda_1 = 0.,$$

whereas the second is given by:

$$-x^2\partial_{12}\lambda_2 + (x^2)^2\partial_{11}\lambda_1 + 2x^2\partial_1\lambda_1 + \partial_1\lambda_2 - \partial_2\lambda_2 + \lambda_1 = 0.$$

So, in the first case, we have $\tilde{p} = x^2\partial_{12} + 2\partial_1 + \partial_2$ and $\tilde{q} = \partial_{22}$, which give $p = x^2\partial_{12} - \partial_1 - \partial_2$ and $q = \partial_{22}$. Finally, we have

$$\partial_2(x^2\partial_{12} - \partial_1 - \partial_2) = (x^2\partial_1 - 1)\partial_{22},$$

and thus

$$\zeta^2 = \partial_2^{-1}(x^2\partial_1 - 1)\zeta^1 = (x^2\partial_{12} - \partial_1 - \partial_2)\partial_{22}^{-1}\zeta^1.$$

Let us pose $\eta = \partial_{22}^{-1}\zeta^1$, we obtain the parametrization:

$$\left\{ \begin{array}{l} \partial_{22}\eta = \zeta^1, \\ x^2\partial_{12}\eta - \partial_1\eta - \partial_2\eta = \zeta^2. \end{array} \right. \tag{18}$$

Similary, with the second compatibility condition, we shall obtain another parametrization:

$$\left\{ \begin{array}{l} x^2\partial_{12}\eta + 2\partial_1\eta - \partial_2\eta = \zeta^1, \\ (x^2)^2\partial_{11}\eta - 2x^2\partial_1\eta + \eta = \zeta^2. \end{array} \right. \tag{19}$$

We remark that we are in the same situation as in the example 8: we have obtained two different parametrizations of (2) which are more simple compared to (12). However, those two parametrizations cannot be parametrized at their turn whereas (12) is parametrized by an injective operator.

Let us try to explain why the localization techniques give more simple parametrizations than the formal test. Firstly, we have to remark that the parametrization (16) of the divergence operator has two arbitrary functions whereas the parametrization by the curl has three arbitrary functions in $\mathbb{R}^3$. The same remark may be done for the previous example: the parametrizations (18) and (19) have just one arbitary function whereas (12) has two. The number of arbitrary functions in $\mathcal{D}_0$ is equal to the number of equations of its adjoint $\tilde{\mathcal{D}}_0$ and thus, there is in general less compatibility conditions in the $\tilde{\mathcal{D}}_0$ computed by localization than on the formal test. Indeed, when we use localization, we do not need to compute all the compatibility conditions of $\tilde{\mathcal{D}}_1$ but just a differential transcendence basis. The fact that we still have a parametrization of $\mathcal{D}_1$ is due to the following non trivial theorem.

Theorem 7 *Let $\mathcal{D}_1 : F_0 \to F_0$ be an operator determining a torsion-free D-module and let $\mathcal{D}_0 : E \to F_0$ be a parametrization of $\mathcal{D}_1$ with a kernel having a non zero differential transcendence degree. Then, there exists a parametrization $\mathcal{D}'_0 : E' \to F_0$ of $\mathcal{D}_1$ with a kernel having zero differential transcendence degree. We call such a parametrization $\mathcal{D}'_0$ a minimal parametrization of $\mathcal{D}_1$.*

The proof of this useful result, which seems to be new, is quite technical and will be given in an appendix at this end of this paper as it involves diagram chasing in a necessary way. Finally, we understand that the localization techniques are a particular case of the formal test for torsion-free D-module, crucially using the duality, in which we do not have to compute all the $\tilde{\mathcal{D}}_0$ but only a differential transcendence basis. We notice that it is much more difficult to treat the PD case than the OD case, which is already delicated by itself, and this is the reason for which this technique has never been used up to now.

5.1 Minimal Realization

Let us generalize the well-known concept, in control theory, of minimal realization to time-varying OD control systems and PD control systems with variable coefficients [1, 8].

Definition 7 Let the operator $\mathcal{D}_1 : F_0 \to F_1$ determine a D-module M then a minimal realization of M is an operator $\mathcal{D}'_1 : F_0 \to F'_1$ which determines the D-module $M \backslash t(M)$, i.e. $M \backslash t(M) = Dy/(\mathcal{D}'_1 y)$.

Theorem 8 *Let $\mathcal{D}_1$ be an operator determining the D-module M, then a minimal realization of M is the operator $\mathcal{D}'_1 : F_0 \to F'_1$, given by the last step of the formal torsion-free test.*

Example 10 A minimal realization of the control system (7) is given by (8).

6 Conclusion

We hope to have convinced the reader that the *localization technique* is the only tool which, at the same time, is coherent with the transfer matrix approach in the case of constant coefficients ordinary control systems and can be extended to the variable coefficients or to the partial differential case along a procedure which constitutes the core of commutative algebra. The only difficulty met is to adapt such a procedure to the non commutative case in order to use the Ore property of the ring of differential operators. In this framework, we hope to have proved that the corresponding duality technique, based on a systematic use of the adjoint operator and the concept of formal integrability, will play a major constructive and effective role in the study of control theory for partial differential operators, delay and n-Dimensional systems.

7 Appendix

Let give a proof of theorem 7: If the kernel of the parametrization $\mathcal{D}_0$ of $\mathcal{D}_1$ has a zero differential transcendence degree, then $\mathcal{D}'_0 = \mathcal{D}_0$. Let us suppose that the kernel of $\mathcal{D}_0$ has a non zero differential transcendance degree [12, 27]. Let us select a maximal set of differentially independent compatibility conditions among $\tilde{\mathcal{D}}_0$ (the image of this new operator $\tilde{\mathcal{D}}'_0$ must produce a differential transcendence basis of the kernel of $\tilde{\mathcal{D}}_{-1}$). We have the commutative diagram

$$
\begin{array}{ccccccc}
0 & & 0 & & & & \\
\downarrow & & \downarrow & & & & \\
\tilde{E}_{-1} & \longleftarrow & \tilde{E}'' & \longleftarrow \Omega \longleftarrow 0 & & & \\
\| & & \downarrow & & & & \\
\tilde{E}_{-1} & \overset{\tilde{\mathcal{D}}_{-1}}{\longleftarrow} & \tilde{E} & \overset{\tilde{\mathcal{D}}_0}{\longleftarrow} & \tilde{F}_0 & \overset{\tilde{\mathcal{D}}_1}{\longleftarrow} & \tilde{F}_1 \\
\downarrow & & \downarrow & & \| & & \| \\
0 & \longleftarrow & \tilde{E}' & \overset{\tilde{\mathcal{D}}'_0}{\longleftarrow} & \tilde{F}_0 & \overset{\tilde{\mathcal{D}}_1}{\longleftarrow} & \tilde{F}_1, \\
 & & \downarrow & & & & \\
 & & 0 & & & &
\end{array}
$$

where by construction, the transcendence degree of the space of solutions Ω is zero (the corresponding differential module is a torsion module). We notice that the low row may not be formally exact at $\tilde{F}_0$. Taking the adjoint of all these operators, we get the commutative diagram:

$$
\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
0 & & E' & \xrightarrow{\mathcal{D}'_0} & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 \\
\downarrow & & \downarrow & & \| & & \| \\
E_{-1} & \xrightarrow{\mathcal{D}_{-1}} & E & \xrightarrow{\mathcal{D}_0} & F_0 & \xrightarrow{\mathcal{D}_1} & F_1 \\
\| & & \downarrow & & \downarrow & & \\
E_{-1} & \longrightarrow & E'' & \longrightarrow & 0. & & \\
\downarrow & & \downarrow & & & & \\
0 & & 0 & & & &
\end{array}
$$

An easy chase proves that the full diagram is formally exact and then the upper row is formally exact at F_0 as desired, that is, $\mathcal{D}'_0$ is a parametrization of $\mathcal{D}_1$ such that its kernel has a zero differential transcendence degree.

A dual module version of this proof can be given by introducing a maximal free D-submodule of the D-module determined by $\tilde{\mathcal{D}}_{-1}$.

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