Algebraic Analysis of Linear Multidimensional Control Systems

J.F. Pommaret and A. Quadrat
CERMICS
Ecole Nationale des Ponts et Chaussées
6 et 8 avenue Blaise Pascal,
77455 Marne-La-Vallée Cedex 02, France
e-mail: {pommaret, quadrat}@cermics.enpc.fr

Abstract

The purpose of this paper is to show how to use the modern methods of algebraic analysis in partial differential control theory, when the input/output relations are defined by systems of partial differential equations in the continuous case or by multi-shift difference equations in the discrete case. The essential tool is the duality existing between the theory of differential modules or D-modules and the formal theory of systems of partial differential equations. We reformulate and generalize all the formal results that can be found in the extensive literature on multidimensional systems (controllability, observability, primeness concepts, poles and zeros, ...). All the results are presented through effective algorithms.

Keywords: Control theory, Primeness, Multidimensional Systems, Algebraic Analysis, Extension Functor, Janet Conjecture, Formal Theory of Partial Differential Equations, Duality, Homological Algebra.

1 Introduction

In 1963, R.E. Kalman related, in [10], the controllability of a linear ordinary differential control system, with constant coefficients, of the form $\dot{y} = Ay + Bu$, to the full row rank of the controllability matrix $(B, AB, ..., A^{m-1}B)$, where m is the number of outputs y. In 1969, this criterion was shown by Hautus [4] to be equivalent to the full row rank of the matrix $(A - \chi I, B)$ for all values of the indeterminate χ in an attempt to study the transfer matrix $(\chi I - A)^{-1}B$. Then, more general polynomial systems of the form $D(\chi)y = N(\chi)u$, with D a non degenerate square matrix, were considered in attempt to study the transfer matrix $D(\chi)^{-1}N(\chi)$. In particular, left-coprimeness conditions for the matrices D and N were given for multi-input/multi-output (MIMO) systems generalizing the case of single input/single output (SISO) systems where common factors of D (denominator) and N (numerator) could disappear in the transfer function [9]. One must notice that the Kalman criterion came from an explicit integration by means of exponential of matrices, which is not easily available in the general case. Little by little, the preceding condition for D and N separately, has been reformulated for the full matrix $(D, -N)$ in terms of Bezout identity, a result showing that controllability is a built-in property of the control system, not depending on the separation of the variables between inputs and outputs. Meanwhile, a few people tried to extend these results to matrices over polynomial ring $k[\chi] = k[\chi_1, ..., \chi_n]$ in n indeterminates over a field k of constants or to operator matrices with variable coefficients [12, 17, 18, 33, 34, 36, 38]. It was soon discovered that the case $n = 1$, where $k[\chi]$ is a principal ideal ring, should be distinguished with care from the case $n = 2$ and $n \geq 3$ [33, 38]. It is only recently that people paid attention to algebraic analysis, pionneered by V.P. Palamodov [19] for the constant coefficients case and by M. Kashiwara [11] for the general case. We quote in particular the very recent work of U. Oberst [17] showing, in first place, that a control system is controllable if and only if the corresponding differential module is torsion-free.

In this paper, the mathematical results are not new and we provide all corresponding references as their homological proofs are often awfully delicate. However, the applications to control are quite new. In particular, the main purpose of this paper is to combine the formal theory of differential operators with that of differential modules and a description by extension functors in order to avoid the introduction of signals spaces, while recovering and generalizing all the results previously quoted. All these results will be developped in a forthcoming book [25].

In view of the amount of mathematical tools needed in order to understand algebraic analysis, we suppose that the reader has a basic familiarity with differential sequences or resolutions and their use for defining the extension functor [5, 16, 31].

2 Algebraic Analysis

2.1 D-modules

Definition 1 A *differential ring* A with n commuting derivations $\partial_1, \ldots, \partial_n$ is a ring which satisfies $\forall\, a, b \in A,\ \forall\, i, j = 1, \ldots, n$:

- $\partial_i\, a \in A$,
- $\partial_i (a + b) = \partial_i\, a + \partial_i\, b$,
- $\partial_i (ab) = (\partial_i\, a)\, b + a\, \partial_i\, b$,
- $\partial_i\, \partial_j = \partial_j\, \partial_i$.

For applications, the differential ring A will either be a differential field K containing $\mathbb{Q}$ or its subfield of constants $k = \mathrm{cst}(K) = \{a \in K \mid \forall\, i = 1, ...n : \partial_i a = 0\}$. If $d_1, ..., d_n$ are n commuting formal derivative operators, we shall introduce the noetherian ring $D = A[d] = A[d_1, ..., d_n]$ of differential operators. Any element of D has the form $P = \sum_{\text{finite}} a^\mu\, d_\mu$, where $\mu = (\mu_1, \ldots, \mu_n)$ is a multi-index with length $\mid \mu \mid = \mu_1 + \ldots + \mu_n$, $a^\mu \in A$ and $d_\mu = (d_1)^{\mu_1} ... (d_n)^{\mu_n}$. D is a non-commutative integral domain which satisfies

$$\forall\, a, b \in A : a d_i\, (b\, d_j) = ab\, d_i\, d_j + a\, (\partial_i\, b)\, d_j,$$

and possesses the left (right) Ore property: $\forall\, (P, Q) \in D^2$, $\exists\, (U, V) \in (D \backslash 0)^2$ such that $U\, P = V\, Q$ $(P\, U = Q\, V)$.

Example 1 The field of rational functions $\mathbb{R}(t)$ is a differential field with derivative $\dfrac{d}{dt}$. Indeed, $\forall\, a(t), 0 \neq b(t) \in \mathbb{R}(t)$, we have:

$$\frac{d}{dt}(\frac{a(t)}{b(t)}) = \frac{\dot{a}(t)\, b(t) - a(t)\, \dot{b}(t)}{b^2(t)} \in \mathbb{R}(t).$$

Let $D = \mathbb{R}(t)[\frac{d}{dt}]$ be the non-commutative ring of linear operators with coefficients in $\mathbb{R}(t)$. Any element $P \in D$ has the form $P = \sum_{\text{finite}} a_i(t) (\frac{d}{dt})^i$, with $a_i \in \mathbb{R}(t)$.

In the general case, as D is a non-commutative ring, we define the notion of filtration and gradation in order to pass from the non-commutative ring D to the commutative ring gr(D) and thus to use all the results and techniques developped in the commutative case, for the non-commutative one [1, 14, 20]. Moreover, the ring of differential operators $D = A[d]$ looks like a polynomial ring and thus we may like to generalize the well-known notion of degree of a polynomial to a differential operator in D. This can be done by introducing the notion of graded ring. For more details, see [1, 14, 20].

Definition 2 A *filtration* of an A-algebra D is a sequence of A-modules $\{D_r\}_{k\in\mathbb{N}}$ satisfying:

1. $0 = D_{-1} \subseteq D_0 \subseteq D_1 \subseteq ... \subseteq D$,
2. $\cup_{r\geq 0} D_r = D$,
3. $D_r D_s \subseteq D_{r+s}$.

The *associated graded* A-algebra gr(D) of D is defined by:

1. $\mathrm{gr}\,(D) = \oplus_{r\in\mathbb{N}} D_r/D_{r-1}$,
2. $\forall \overline{P} \in D_r/D_{r-1}, \forall \overline{Q} \in D_s/D_{s-1} : \overline{P}.\overline{Q} = \overline{PQ} \in D_{r+s}/D_{r+s-1}$.

D_r/D_{r-1} is called the *homogeneous component of degree* r of D.

Example 2 The sequence of A-modules $D_r = \{\sum_{0\leq|\alpha|\leq r} a^\mu\, d_\mu, a^\mu \in A\}$ is a filtration of $D = A[d_1, ..., d_n]$. In particular, we have $D_0 = A \subset D$ and thus D_r is a free left A-module with basis $\{d_\mu, 0 \leq |\mu| \leq r\}$. In the next sections, D_r will always refered to this filtration and we shall endow $T = D_1/D_0$ with a bracket induced by the filtration of D, namely $[P, Q] = P \circ Q - Q \circ P,\ P, Q \in D$.

Proposition 1 *The natural morphism*

$$\begin{array}{rccc} & gr\,(D) & \longrightarrow & A[\chi_1, ..., \chi_n] \\ D_r/D_{r-1} \ni & \sum_{|\mu|=r} a^\mu d_\mu & \longmapsto & \sum_{|\mu|=r} a^\mu \chi_\mu, \end{array}$$

is an isomorphism of A-algebra.

Definition 3 Let M be a D-module where D admits the filtration $\{D_r\}_{r\in\mathbb{N}}$. A family $\{M_q\}_{q\in\mathbb{N}}$ of A-modules is a filtration of M if

1. $0 = M_{-1} \subseteq M_0 \subseteq M_1 \subseteq ... \subseteq M$,
2. $\cup_{q\in\mathbb{N}} M_q = M$,
3. $D_r M_q \subseteq M_{q+r}$.

The *associated* graded gr(D)-module $G = \mathrm{gr}\,(M)$ is then defined by:

1. $G = \oplus_{q\in\mathbb{N}} G_q$, with $G_q = M_q/M_{q-1}$,
2. $\forall \overline{P} \in D_r/D_{r-1}, \forall \overline{m} \in M_q/M_{q-1} : \overline{P}\,\overline{m} = \overline{Pm} \in M_{q+r}/M_{q+r-1}$.

We have the short exact sequence:

$$0 \longrightarrow M_{q-1} \longrightarrow M_q \longrightarrow G_q \longrightarrow 0. \tag{1}$$

Definition 4 A filtration $\{M_q\}_{q\in\mathbb{N}}$ of a D-module M is called a *good filtration* if it satisfies one of the following equivalent conditions:

1. $\forall\, q \in \mathbb{N}, M_q$ is finitely generated over A and there exists $p \in \mathbb{N}$ such that:

$$D_r M_p = M_{p+r},\ r \geq 0,$$

2. G is a finitely generated gr(D)-module.

Example 3 1. Let M be a finitely generated D-module with the set of generators $\{e_1, ..., e_m\}$. Then, the filtration $M_q = \sum_{i=1}^m D_q e_i$ is a good filtration as we have $G = \sum_{i=1}^m \mathrm{gr}(D)\, e_i$, and thus G is finitely generated over $\mathrm{gr}(D)$ by $\{e_1, ..., e_m\}$.

2. Let $M = D$ be the left D-module and let $(M_q = D_{2q})_{q \in \mathbb{N}}$ be a filtration of M. Then, we have $D_r M_q = D_r D_{2q} \subseteq D_{2q+r} \subsetneq D_{2(q+r)} = M_{r+q}$ and thus $(D_{2q})_{q\in\mathbb{N}}$ is not a good filtration of D.

Proposition 2 *Let M be a left D-module then M admits a good filtration if and only if M is finitely generated over D. Moreover, if M has a good filtration then*

1. *Any submodule M' of M has a good filtration, defined by $M'_q = M_q \cap M$ and M' is finitely generated over D.*
2. *Any quotient M'' of M has a good filtration, defined by the image of the filtration of M onto the projection $M \longrightarrow M'' \longrightarrow 0$.*
3. *If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is a short exact sequence of filtred modules then we have the short exact sequence $0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0$, where the associated graded modules $G' = gr(M')$, $G = gr(M)$ and $G'' = gr(M'')$ are defined with respect to the above induced filtrations.*

Now, using the graded module $\mathrm{gr}(M)$ over the commutative ring $\mathrm{gr}(D)$, instead of the D-module M, we can use the results of algebraic geometry to give an intrinsic definition of the dimension of a module M.

Definition 5 Let M be a finitely generated D-module with a good filtration and let $G = \mathrm{gr}(M)$ be its associated graded $\mathrm{gr}(D)$-module, then the ideal $I(M) = \sqrt{\mathrm{ann}(G)} = \{a \in \mathrm{gr}(D) \,|\, \exists\, n \in \mathbb{N} : a^n G = 0\}$ does not depend on the filtration of M and we introduce the *characteristic set* $\mathrm{char}(M) = V(I(M)) = \{\mathfrak{p} \in \mathrm{spec}(\mathrm{gr}(D)) \,|\, \sqrt{\mathrm{ann}(G)} \subseteq \mathfrak{p}\}$, where $\mathrm{spec}(\mathrm{gr}(D))$ is the set of proper prime ideals of $\mathrm{gr}(D)$.

The previous definition and proposition lead to the following one [1, 14, 20].

Proposition 3 *Let M be a finitely generated left D-module admitting a good filtration then*

1. *For q large enough, there exists a unique Hilbert polynomial H_M such that $dim\,(M_r) = H_M(r) = \frac{m}{r!}\, r^d + ...$, where d is the degree of the polynomial. The degree $d = d(M)$ is called the dimension of M, the coefficient $m = m(M)$ is the multiplicity of M and they do not depend on the good filtration.*
2. *If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence of finitely generated filtred modules then*
   (a) *$char(M) = char(M') \cup char(M'')$,*
   (b) *$H_M = H_{M'} + H_{M''}$ and thus $d(M) = max(d((M'), d(M''))$ and if $d(M') = d(M'')$ then $m(M) = m(M') + m(M'')$.*

We shall see, in the next section, how to compute effectively the Hilbert polynomial and thus the dimension and the multiplicity of a D-module M.

Now, let us give some basic definitions of properties of modules that will be at the core of this paper.

Definition 6 • A D-module M is *free* if there are elements of M which generate M and which are independent over D.

- A D-module M is *projective* if there exist a free D-module F and a D-module N such as: $F = M \oplus N$. Hence, the module N is also a projective D-module.

- A D-module M is *reflexive* if $M \cong \hom_D(\hom_D(M, D), D)$.
- A D-module M is *torsion-free* if $t(M) = \{m \in M \mid \exists\, p \neq 0,\, p\, m = 0\} = 0$. We call $t(M)$ the *torsion submodule* of M. In any case, $M/t(M)$ is a torsion-free D-module.

If D is a principal ideal ring then any torsion-free module is free and if $A = k$ then, using the Quillen-Suslin theorem [31, 34], any projective module is free. Moreover, it follows immediately from (1) that M is torsion-free (reflexive, projective, free) whenever G is torsion-free (reflexive, projective, free). The converse is not true.

Example 4 The SISO-system $\dot{y} - y - u = 0$ is torsion-free while the graded part $\dot{y} = 0$ is not torsion-free.

2.2 Differential Operators

Let X be a differential manifold of dimension n with local coordinates $x = (x^1, ..., x^n)$. We denote by $T = T(X)$ the tangent bundle of X and by $T^\star = T^\star(X)$ the cotangent bundle of X. By $S_q T^\star, \bigwedge^r T^\star$ we shall mean the q^{th} symetric product of $T^\star$, the r^{th} exterior product of $T^\star$. Let E be a vector bundle of fiber dimension m over X, with local coordinates (x, y), with $y = (y^1, ..., y^m)$. We shall use the same notation E for a vector bundle and for its sheaf of germs of sections. We consider the vector bundle $J_q(E)$ of $q-$jets of E. Its fiber at $x \in X$ is the quotient of the space of germs of sections of E at x by the subspace of germs of sections which vanishs up to order q at x ($f, g \in J_q(E)_x \Leftrightarrow \partial_\mu f(x) = \partial_\mu g(x), 0 \leq |\mu| \leq q$). We identify $J_0(E)$ with E and we denote the projection of $J_q(E)$ onto X by π and the projection of $J_q(E)$ onto $J_{q-1}(E)$ by π^q_{q-1}. If ξ is a section of E, we denote by $j_q(\xi)(x)$ the equivalence class of germs of ξ at x. We have the following exact sequence [3, 21, 29]:

$$0 \longrightarrow S_q T^\star \otimes E \longrightarrow J_q(E) \xrightarrow{\pi^q_{q-1}} J_{q-1}(E) \longrightarrow 0. \tag{2}$$

Let F be a vector-bundle over X, of fiber dimension l.

Definition 7 1. A *differential operator* $\mathcal{D} = \Phi \circ j_q : E \to F$ is given by a bundle morphism $\Phi : J_q(E) \to F$, where we may suppose that Φ is surjective.

2. The r-prolongation of Φ is the unique bundle morphism $\rho_r(\Phi) : J_{q+r}(E) \to J_r(F)$ such that $\rho_r(\Phi) \circ j_{q+r} = j_r \circ \mathcal{D} = j_r \circ \Phi \circ j_q$.
3. The linear system of partial differential equations (PDE) R_q defined by $\mathcal{D}$ is the kernel of Φ and a solution of R_q is a local section ξ of E, over an open set $U \subset X$, such that $j_q(\xi)(x) \in R_q, \forall\, x \in U$.
4. The r^{th} prolongation R_{q+r} of R_q is the kernel of $\rho_r(\Phi)$.
5. We denote by $R^{(s)}_{q+r}$ the projection of R_{q+r+s} onto $J_{q+r}(E)$, i.e., $R^{(s)}_{q+r} = \pi^{q+r+s}_{q+r}(R_{q+r+s})$.

Example 5 The bundle morphism Φ defined by

$$\begin{array}{rlcl} \Phi : & J_q(E) & \longrightarrow & F \\ & (x, y^k_\mu) & \longmapsto & (x, \sum_{0 \leq |\mu| \leq q, 1 \leq k \leq m} a^{\tau\mu}_k(x)\, y^k_\mu), \end{array}$$

with $1 \leq \tau \leq l$ gives rise to the differential operator $\mathcal{D}$ defined by:

$$\begin{array}{rlcl} \mathcal{D} : & E & \longrightarrow & F \\ & (x, \xi^k(x)) & \longmapsto & (x, \eta(x)^\tau = \sum_{0 \leq |\mu| \leq q, 1 \leq k \leq m} a^{\tau\mu}_k(x)\, \partial_\mu\, \xi^k), \end{array}$$

with $\tau = 1, ..., l$. The system R_q, defined by the differential operator $\mathcal{D}$, is given by:

$$\sum_{0 \leq |\mu| \leq q, 1 \leq k \leq m} a^{\tau\mu}_k(x)\, y^k_\mu = 0, 1 \leq \tau \leq l.$$

Now, using the sequence (2) and the definition of the r-prolongation of Φ, we obtain an induced map $\sigma_{q+r}(\Phi) : S_{q+r}T^\star \otimes E \to S_rT^\star \otimes E$ and we denote by g_{q+r} the kernel of $\sigma_{q+r}(\Phi)$. We call g_{q+r} the *symbol* of R_{q+r}. We easily see that $g_{q+r} = R_{q+r} \cap S_{q+r}T^\star \otimes E$.

Example 6 The map $\sigma_{q+r}(\Phi)$, where Φ is defined as in the previous example, is defined by:

$$\begin{array}{rlcl} \sigma_{q+r}(\Phi): & S_{q+r}T^\star \otimes E & \longrightarrow & S_rT^\star \otimes F \\ & (x, y^k_\mu, |\mu| = q+r) & \longmapsto & (x, \sum_{|\mu|=q, |\nu|=r, 1\leq k\leq m} a_k^{\tau\mu}(x)\, y^k_{\mu+\nu}), \end{array}$$

The symbol g_{q+r} of the system R_{q+r} is the kernel of $\sigma_{q+r}(\Phi)$.

Let us define the Spencer δ-*sequence* by

$$\Lambda^s\, T^\star \otimes g_{q+r+1} \stackrel{\delta}{\longrightarrow} \Lambda^{s+1}\, T^\star \otimes g_{q+r},$$

with $(\delta(\omega))^k_\mu = dx^i \wedge \omega^k_{\mu+1_i}$ where $\omega = v^k_{\mu,I}\, dx^I \in \Lambda^s\, T^\star \otimes g_{q+r+1}$, $dx^I = dx^{i_1} \wedge ...$ $\wedge\, dx^{i_s}, i_1 < ... < i_s$ and $|\mu| = q + r$. We easily verify that $\delta \circ \delta = 0$. The resulting cohomology at $\Lambda^s\, T^\star \otimes g_{q+r}$ is denoted by $H^s_{q+r}(g_q)$.

Definition 8 The symbol g_q of R_q is said to be s-*acyclic* if $H^1_{q+r} = ... = H^s_{q+r} = 0, \forall\, r \geq 0$. The symbol g_q is *involutive* if it is n-acyclic. In particular, every system R_q of ordinary differential equations (ODE) has an involutive symbol. A symbol g_q is *of finite type* if $\exists\, r \geq 0$ such that $g_{q+r} = 0$.

We can prove that the symbol g_q of a system R_q is such that g_{p+r} becomes involutive for r large enough. If g_q is an involutive symbol, we may define integers α^i_q called *characters* of g_q such that

$$\dim g_{q+r} = \sum_{i=1}^{n} \frac{(r+i-1)!}{r!\,(i-1)!}\,\alpha^i_q, \forall\, r \geq 0,$$

and the following relations are statisfied:

1. $\dim g_q = \alpha^1_q + ... + \alpha^n_q$,
2. $\alpha^1_q \geq \alpha^2_q \geq ... \geq \alpha^n_q \geq 0$,
3. $0 \leq \alpha^n_q \leq m$.

Definition 9 A system R_q is said to be *formally integrable* if $\forall\, r, s \geq 0$, R_{q+r} is a vector bundles and the projection $\pi^{q+r+s}_{q+r} : R_{q+r+s} \to R_{q+r}$ is surjective. A system R_q is *involutive* if R_q is formally integrable and has an involutive symbol g_q.

If Φ is sufficiently regular, then R_{q+r} are vector bundles for any $r \geq 0$ and if R_q is formally integrable then we have the exact sequences:

$$0 \longrightarrow g_{q+r} \longrightarrow R_{q+r} \stackrel{\pi^{q+r}_{q+r-1}}{\longrightarrow} R_{q+r-1} \longrightarrow 0. \tag{3}$$

Corollary 1 *If the system R_q is involutive system then*

$$dim\,(R_{q+r}) = dim\,(R_{q-1}) + \sum_{i=1}^{n} \frac{(r+i)!}{r!\,i!}\alpha^i_q = \frac{\alpha^n_q}{n!} r^n + ...,$$

where R_{q-1} is the projection of R_q on $J_{q-1}(\mathcal{E})$.

The formal solutions of the system R_q depend on α_q^1 functions in x^1, α_q^2 functions in (x^1, x^2), ..., and α_q^n functions in $(x^1, ..., x^n)$.

If the system R_q is not formally integrable, then by adding sufficiently enough equations, we can bring the system R_q to be a formally integrable system $R_{q+r}^{(s)}$ with the same solutions, by means of a finite algorithm [3, 21]. The knowledge of the latter system, which is the finite substitute for R_{infty}, is essential for studying the formal properties of the given system and of the corresponding differential module.

Definition 10 Let $\mathcal{D} : E \to F$ be an involutive operator then there exists at most n new operators $\mathcal{D}_i : F_{i-1} \to F_i$, with $F_0 = F$, such that the following sequence

$$E \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} \quad ... \quad \xrightarrow{\mathcal{D}_n} F_n \longrightarrow 0,$$

is stricly exact, i.e., the operator $\mathcal{D}_i$ generates all the compatibility conditions of $\mathcal{D}_{i-1}$ and the sequence is exact at any order on the jets level. This sequence is called the *Janet sequence* of $\mathcal{D}$.

2.3 Duality

The sequence (3) for $r = 0$ is the dual over A of the sequence (1) if $\mathcal{D}$ has coefficients in A and E, F are trivial bundles. Using coordinates (x, y) for E, we may identify $Dy = Dy^1 + ... + Dy^m$ with D^m. The duality between differential geometry and differential algebra is obtained by setting

$$J_q^\star = \hom_A(J_q, A) = D_q \Rightarrow J_q(E)^\star = D_q(E) = D_q \otimes_A E^\star,$$

whenever (X, A) is a ringed space [15]. Accordingly, we can define a differential module M by the cokernel in the exact sequence of modules:

$$D \otimes_A F^\star \longrightarrow D \otimes_A E^\star \longrightarrow M \longrightarrow 0,$$

or simply

$$D^l \longrightarrow D^m \longrightarrow M \longrightarrow 0,$$

in the trivial case if $\dim(E) = m$, $\dim(F) = l$. Hence, $R_\infty = \rho_\infty(R_q) = \hom_A(M, A)$, and the main difficulty is that certain properties of M, using injective limits, are not easily interpreted as properties of R_∞, using projective limits and vice-versa. When $\mathcal{D}$ is involutive and sufficiently regular, we notice that a canonical finite resolution of the sheaf Θ of solutions of $\mathcal{D}$, is of the form of the Janet sequence with $F = F_0$, $\dim(F_r) = l_r$ and $\dim(E) = m$,

$$0 \longrightarrow \Theta \longrightarrow E \xrightarrow{\mathcal{D}} F_0 \xrightarrow{\mathcal{D}_1} F_1 \xrightarrow{\mathcal{D}_2} ... \xrightarrow{\mathcal{D}_n} F_n \longrightarrow 0, \tag{4}$$

where $\mathcal{D}_i$ represents all the compatibility conditions of $\mathcal{D}_{i-1}$. The sequence (4) provides, by duality, a finite free resolution of M [5, 16, 31] :

$$0 \longrightarrow M \longleftarrow D^m \xleftarrow{.\mathcal{D}} D^{l_0} \xleftarrow{.\mathcal{D}_1} D^{l_1} \xleftarrow{.\mathcal{D}_2} ... \xleftarrow{.\mathcal{D}_n} D^{l_n} \longleftarrow 0, \tag{5}$$

The problem is to study the properties of operator $m \times l$-matrix, acting on column vectors on the right, in the operator sense or on row vectors on the left, in the module sense. Accordingly, a preliminary problem for being able to deal equivalently with $\mathcal{D}$ or with M is to bring effectively $\mathcal{D}$ or R_q to formal integrability or even to involutiveness, in such a way that $R_q = M_q^\star = \hom_A(M_q, A)$. Such a fact is rather hard to understand on the presentation of M when one asks equivalently for a *strict morphism* $D^l \longrightarrow D^m$ [11, 32]. It is important to notice that the dualities $\hom_A(\cdot, A)$ and $\hom_D(\cdot, D)$ that will be systematically used in this paper can lead to effective computations, contrary to the duality $\hom_D(\cdot, I)$, when I is an *injective* module, used by Oberst [17] and Willems [35] in their behavioural approach.

We have defined an algebraic set over k or K, namely the characteristic set $\mathrm{char}(M) = \mathrm{supp}(G)$ of M as the *support* of G, namely the set of prime ideals of $gr(D)$, containing the annihilator $\mathrm{ann}(G)$ of $G = \mathrm{gr}(M)$. Keeping the word variety for an irreductible algebraic set, we notice that the dimension $\dim(M) = \mathrm{d}(M)$ of the D-module M is the maximum dimension over an algebraic closure of k or K of the varieties corresponding to the minimum prime ideals in $\mathrm{char}(M)$, i.e., the degree d of the Hilbert polynomial H_M. Equivalently, to avoid dealing with many irreductible components, the Hilbert-Serre theorem says that $\mathrm{d}(M)$ is equal to the maximum number of non-zero characters α_q^i [32]. We denote by $\mathrm{cd}(M) = n - \mathrm{d}(M)$ the *codimension* of $\mathrm{char}(M)$.

We present the *extension functor* in the operator language (see [5, 16, 31] for a module approach). If $\mathcal{D} : E \longrightarrow F$ is a differential operator of order q, we denote by $\mathrm{ad}(\mathcal{D}){=}\tilde{\mathcal{D}}$: $\tilde{F} = \bigwedge^n T^\star \otimes F^\star \longrightarrow \tilde{E} = \bigwedge^n T^\star \otimes E^\star$, the *formal adjoint* of $\mathcal{D}$. The operator $\tilde{\mathcal{D}}$ is of the same order than $\mathcal{D}$, with coefficients in A. The formal adjoint $\tilde{\mathcal{D}} : \tilde{F} \to \tilde{E}$ can be easily computed by using the following three rules:

- The adjoint of a matrix (zero order operator) is the transposed matrix.
- The adjoint of ∂_i is $-\partial_i$.
- For two linear PD operators P, Q that can be composed: $\widetilde{P \circ Q} = \tilde{Q} \circ \tilde{P}$.

Moreover, we have the following relation

$$< \mu, \mathcal{D}\xi > = < \tilde{\mathcal{D}}\mu, \xi > + d(\cdot),$$

with d the exterior derivative. We compute the adjoint of an operator by multiplying it by test functions on the left and integrating the result by part, as we could do for distributions.

The two key problems are firstly that $\tilde{\mathcal{D}}$ may not be formally integrable when $\mathcal{D}$ is and secondly that $\tilde{\mathcal{D}}_r$ may not generate at all the compatibility conditions of $\tilde{\mathcal{D}}_{r+1}$ in the adjoint of the Janet sequence (4). Let us give an example.

Example 7 We take the operator $\mathcal{D} : \xi \to \eta$ defined on sections, by

$$\left\{ \begin{array}{l} \partial_{12}\xi = \eta^1, \\ \partial_{22}\xi = \eta^2, \end{array} \right.$$

and we easily see that the compatibility condition of $\mathcal{D}$ is the operator $\mathcal{D}_1 : \eta \to \zeta$, defined by $\partial_1\eta^2 - \partial_2\eta^1 = \zeta$. Then, the adjoint $\tilde{\mathcal{D}}_1 : \lambda \to \mu$ of $\mathcal{D}_1$ is then given by:

$$\left\{ \begin{array}{l} \partial_2\lambda = \mu_1, \\ -\partial_1\lambda = \mu_2. \end{array} \right.$$

The compatibility condition of $\tilde{\mathcal{D}}_1 : \mu \to \nu$ is defined by the operator $\partial_1\mu_1 + \partial_2\mu_2 = \nu$, which is not the adjoint $\tilde{\mathcal{D}}$ of the operator $\mathcal{D}$, defined by $\partial_{12}\,\mu_1 + \partial_{22}\,\mu_2 = \nu$.

One can roughly say that $\mathrm{ext}_D^r(M, D)$ measures the defect of exactness at $\tilde{F}_{r-1}$ in the adjoint sequence. However, as $\mathrm{ext}_D^r(M, D)$ does not depend on the presentation of M, the previous definition by means of the Janet sequence is, by far, the best one though one could use the second Spencer sequence too (another finite free resolution of the sheaf Θ of solutions of $\mathcal{D}$) [15, 21, 29], namely (do not confuse among standard notations)

$$0 \longrightarrow \Theta \xrightarrow{j_q} C_0 \xrightarrow{D_1} C_1 \xrightarrow{D_2} \ldots \xrightarrow{D_n} C_n \longrightarrow 0, \qquad (6)$$

and measure the defect of exactness at $\tilde{C}_r$ by dealing with first order operators D_r though with many more unknowns (take for example $E = T$, $F = \bigwedge^n T^\star$ and for $\mathcal{D}$ the divergence operator).

The first key result of algebraic analysis is the following theorem relating the vanishing of the extension functor to the codimension of the characteristic set [11, 19].

Theorem 1 $cd(M) \geq r \Leftrightarrow ext^i_D(M,D) = 0,\ \forall i < r.$

The second key result, instead of looking for the compatibility conditions $\mathcal{D}_1$ of a differential operator $\mathcal{D}$, deals with the converse problem of looking for a potential like expression of $\mathcal{D}$, namely to know whether one can find an operator $\mathcal{D}_{-1} : E_{-1} \longrightarrow E_0 = E$ such that $\mathcal{D}$ generates all the compatibility conditions of $\mathcal{D}_{-1}$. For example, one may keep in mind the Poincaré sequence for the exterior derivative. If there exists such an operator $\mathcal{D}_{-1}$, we say that the operator $\mathcal{D}$ is *parametrized* by $\mathcal{D}_{-1}$.

Theorem 2 *There exists a sequence of differential operators*

$$E_{-r} \xrightarrow{\mathcal{D}_{-r}} E_{-r+1} \xrightarrow{\mathcal{D}_{-r+1}} \ldots \xrightarrow{\mathcal{D}_{-2}} E_{-1} \xrightarrow{\mathcal{D}_{-1}} E_0 \xrightarrow{\mathcal{D}} F,$$

where each operator generates all the compatibility conditions of the preceding one, if and only if $ext^i_D(N,D) = 0,\ \forall i = 1, \ldots, r$ *whenever* N *is the differential module determined by the operator* $\tilde{\mathcal{D}}$*, exactly as* M *was determined by* $\mathcal{D}$*.*

The above conditions can be checked effectively as we just need to construct the adjoint operator, find a sequence of compatibility conditions with length r, dualize it and check whether the adjoint sequence is formally exact, i.e., each operator generates exactly the compatibility conditions of the preceding one. The global dimension of D is n because, using the Spencer sequence (6), we obtain at once: $\mathrm{ext}^i_D(M,D) = 0,\ \forall i > n$.

Example 8 Let us take the divergence operator $\mathcal{D} : \xi \rightarrow \eta$, in $\mathbb{R}^3$, defined by

$$\partial_1 \xi^1 + \partial_2 \xi^2 + \partial_3 \xi^3 = \eta.$$

Dualizing the divergence operator, we obtain the operator $\tilde{\mathcal{D}} : \mu \rightarrow \nu$, defined by

$$\left\{ \begin{array}{l} -\partial_1 \mu = \nu_1, \\ -\partial_2 \mu = \nu_2, \\ -\partial_3 \mu = \nu_3, \end{array} \right.$$

which is nothing else than minus the gradient operator. We let the reader check by himself that the compatibility conditions $\tilde{\mathcal{D}}_{-1}$ of $\tilde{\mathcal{D}}$ is the curl operator and the adjoint of $\tilde{\mathcal{D}}_{-1}$ is still the curl operator, i.e., the curl is a *self-adjoint* operator. The compatibility conditions of the curl operator $\mathcal{D}_{-1}$ are the divergence and thus $\mathcal{D}$ is parametrized by the curl operator $\mathcal{D}_{-1}$. In other words, if M is the D-module defined by $\mathcal{D}$, we have $\mathrm{ext}^1_D(N,D) = 0$, where N is the D-module defined by $\tilde{\mathcal{D}}$. Moreover, we can check that the compatibilty conditions $\tilde{\mathcal{D}}_{-2}$ of $\tilde{\mathcal{D}}_{-1}$ is minus the divergence operator and thus its adjoint $\mathcal{D}_{-2}$ is the gradient which parametrizes the curl, i.e., $\mathrm{ext}^2_D(N,D) = 0$. We shall see in the next section that if $\mathcal{D}$ is a formally surjective operator, that is without any compatibility conditions, then N is a torsion D-module and thus $\hom_D(N,D) = \mathrm{ext}^0_D(N,D) = 0$. Using theorem 1, we obtain $\mathrm{cd}(N) > 2 \Leftrightarrow \mathrm{d}(N) = 0$, that is, $\alpha^i_1(\tilde{\mathcal{D}}) = 0,\ \forall i = 1, \ldots, 3$, and we find back that the solutions of the gradient operator only depend on constants.

It is essential to notice that the right D-module $N_r = \bigwedge^n T^\star \otimes_A N$, obtained from the left D-module $N = N_l$ by the side changing functor [1], must not be confused with $\tilde{M} = \hom_D(M,D)$ as we have the exact sequence

$$0 \longrightarrow \tilde{M} \longrightarrow E \otimes_A D \longrightarrow F \otimes_A D \longrightarrow N_r \longrightarrow 0,$$

and thus we have the relation: $\mathrm{ext}^i_D(N_r,D) = \mathrm{ext}^{i-2}_D(\hom_D(M,D),D),\ \forall i \geq 3$. Finally, the result 3.1.1 will prove that $\mathrm{ext}^i_D(N,D)$ depends in fact only on M, $i \geq 1$, while $\hom_D(N,D)$ is only determined up to a projective equivalence, according to the Schanuel's lemma [5, 16, 31].

3 Applications to Control Theory

We shall now divide the properties of control systems into two categories, depending on the fact that they do or do not depend on a separation of the variables of the control system between input and output.

3.1 Structural Properties

We first study the properties that do not depend on such a separation.

3.1.1 Primeness

The key idea, *not evident at all intuitively*, is to use $\tilde{\mathcal{D}}$ or N instead of $\mathcal{D}$ or M in order to achieve a classification of modules:

$$\text{free} \subseteq \text{projective} \subseteq ... \subseteq \text{reflexive} \subseteq \text{torsion-free}.$$

First of all, we recall that M is torsion-free (reflexive) if and only if the central morphisn in the long exact sequence of left D-modules

$$\begin{array}{cccccc} 0 \longrightarrow \mathrm{ext}^1_D(N_r, D) \longrightarrow & M & \xrightarrow{\epsilon} & \mathrm{hom}_D(\mathrm{hom}_D(M,D),D) \longrightarrow \mathrm{ext}^2_D(N_r,D) \longrightarrow 0, \\ & m & \longrightarrow & \epsilon(m), \end{array}$$

with $\forall\, f \in \mathrm{hom}_D(M,D) : \epsilon(m)(f) = f(m)$, is injective (bijective).

Corollary 2 *The following assertions are equivalent [11, 19, 21, 25]:*

1. *The control system defined by $\mathcal{D}$ is controllable.*
2. *The operator $\mathcal{D}$ is parametrizable by a $\mathcal{D}_{-1}$.*
3. *The D-module M is torsion-free.*
4. $ext^1_D(N_r, D) = \bigwedge^n T^\star \otimes_A ext^1_D(N, D) = 0$.

Remark 1 Moreover, if $\mathcal{D}$ is formally surjective, that is $\mathcal{D}_1 = 0$, then $\mathrm{hom}_D(N,D) = \mathrm{ext}^0_D(N,D) = 0 \overset{\mathrm{th.1}}{\Leftrightarrow} \mathrm{cd}(N) \geq 1 \Leftrightarrow \mathrm{d}(N) \leq n-1 \Leftrightarrow \alpha^n_q(N) = 0 \Leftrightarrow N$ is a torsion module. Then, M is torsion-free $\Rightarrow \mathrm{ext}^i_D(N,D) = 0,\ \forall i \leq 1 \Rightarrow \mathrm{cd}(N) \geq 2 \Rightarrow \mathrm{d}(N) \leq n-2$ and we find back the concept of *minor left-primess* (MLP) [33, 38, 36] for the operator matrix representing $\mathcal{D}$. One must care that, in the variable coefficient case, the matrix of $\tilde{\mathcal{D}}$ is not just the transposed of the matrix of $\mathcal{D}$. In the particular case $n = 1$, we find back the Hautus test [4] and the fact that the control system is controllable if and only if $\tilde{\mathcal{D}}$ is injective [23]. In that case, there is a lift-operator $\tilde{\mathcal{P}} : \tilde{E} \to \tilde{F}$ such that $\tilde{\mathcal{P}} \circ \tilde{\mathcal{D}} = id_{\tilde{F}}$ and thus $\mathcal{D} \circ \mathcal{P} = id_F$, a result amounting to the forward and reversed generalized Bezout identities [9, 27]. When $\mathcal{D}$ is not surjective, the above result amounts to *generalized factor left primeness* (see [33, 38] and p.12 of [36]).

Corollary 3 *M is reflexive $\Leftrightarrow$ $ext^i_D(N,D) = 0$, $\forall i = 1,2$.*

Remark 2 Moreover, if $\mathcal{D}$ is surjective, reasoning as before, we get $\mathrm{d}(N) \leq n-3$. The divergence operator provides a good example of a reflexive module which is nevertheless not projective.

Going on along theorem 2 with increasing r, we reach the case $\mathrm{ext}^i_D(N,D) = 0,\ \forall i = 1,..,n-1$, that is, $\mathrm{d}(N) = 0$ when $\mathcal{D}$ is surjective and this is the concept of *weakly zero left-primeness* [36, 38]. A particular example is provided by a system of finite type or *holonomic* module N such that $\mathrm{I}(N) = (\chi_1, ..., \chi_n) \Rightarrow$ the algebraic set $\mathrm{char}(N)$ is reduced to the origin and $\mathrm{ext}^i_D(N,D) = 0,\ \forall i \neq n$.

As N is a differential module too, we have $\mathrm{ext}^i_D(N,D)=0,\ \forall i>n$ and we are left with the only case $\mathrm{ext}^i_D(N,D), \forall i\geq 1$. In particular, when $\mathcal{D}$ is surjective, we obtain $\mathrm{ext}^i_D(N,D)=0,\ \forall i\geq 0$ and thus $\mathrm{d}(N)=-1$ that is $\mathrm{char}(N)=\emptyset$ and this is only possible if $N=0$. Hence, $\tilde{\mathcal{D}}$ admits a lift and M is a projective module [24, 27]. We find the generalization of *zero left-primeness* [33, 38, 36] as we are now dealing with variable coefficients. In the commutative case, one may use $\mathrm{ann}(M)$ instead of $\mathrm{ann}(G)$ [17, 18]. In general, when $\mathcal{D}$ is not surjective, as the $\mathrm{ext}^i_D(N,D)$ do not depend on the resolution of N, we may bring $\tilde{\mathcal{D}}$ to involutiveness and use the corresponding Spencer sequence to construct inductively lift operators P_r of the Spencer operator D_r in such a way that $D_r\circ P_r\circ D_r=D_r$, and reach the conclusion that N itself is projective, i.e., $\tilde{M}$ is projective and thus $M\cong\tilde{\tilde{M}}$ is projective as M is already reflexive. Finally, when $D=k[d]$ is commutative, it is known that $\mathrm{ann}(M)$ and ann(G) define algebraic sets with the same dimension, according to the Hilbert-Serre theorem [32]. Hence, such a generalization of all existing results explains the existence of a whole range of "possible types of primeness" conjectured in [36].

Example 9 When $n=1$, only one single type of primeness is left. Dealing with a formal integrable Kalman system $-\dot{y}+Ay+Bu=0$ and multiplying it on the left by a row vector of test functions λ, we find for the kernel of $\tilde{\mathcal{D}}$:

$$\left\{\begin{array}{l}\dot{\lambda}+\lambda A=0,\\ \lambda B=0,\end{array}\right. \Rightarrow \dot{\lambda}B=0\Rightarrow \lambda AB=0\Rightarrow \lambda A^2B=0...\Rightarrow \lambda A^{m-1}B=0,$$

and the Kalman test surprisingly amounts to the injectivity of the non-formally integrable operator $\tilde{\mathcal{D}}$, a result also equivalent to the lack of first integrals [23, 28].

Example 10 The system $\partial_1\xi^1+\partial_2\xi^2=0$ defines a torsion-free D-module with a first order parametrization, which is nevertheless not projective whereas $\partial_1\xi^1+\partial_2\xi^2-x^2\xi^1=0$ defines a projective (but not free) and thus reflexive D-module which is automatically torsion-free and admits a second order parametrization.

Example 11 The last operator $\mathcal{D}_n$ in a Janet sequence always provides a projective module.

Example 12 With $n=3$, let us consider the second order system

$$\left\{\begin{array}{l}\partial_{33}\xi-\partial_{13}\xi-\partial_3\xi=0,\\ \partial_{23}\xi-\partial_{12}\xi-\partial_2\xi=0,\\ \partial_{22}\xi-\partial_{12}\xi=0,\end{array}\right.$$

with characters $\alpha^1_2=3,\alpha^2_2=0,\alpha^3_2=0$. The algebraic sets defined by $\mathrm{ann}(M)$ and $\mathrm{ann}(G)$ are different though they are both unions of 3 varieties of dimension 1, and thus have the same dimension.

Example 13 The case of a surjective operator $\mathcal{D}:E\rightarrow F$ with $\dim(E)=\dim(F)$ is standard in physics (wave equations in elasticity, electromagnetism, ...). Indeed, M is a torsion module and thus $\mathrm{ext}^0_D(M,D)=\hom_D(M,D)=0\Rightarrow \mathrm{cd}(M)\geq 1$. If $\mathrm{cd}(M)=1$, then $\mathrm{ext}^1_D(M,D)\neq 0$ and $\mathcal{D}$ is a determined operator which is therefore always formally integrable. A good example is the Cauchy-Riemann system defining holomorphic transformations. However, $\mathrm{cd}(M)\geq 2\Rightarrow \mathrm{ext}^1_D(M,D)=0\Rightarrow N=0$. It follows that $\tilde{\mathcal{D}}$ is invertible, a fact showing that $\mathcal{D}$ is not formally integrable. This result implies that $M=0$. We have thus obtained a simple proof of the conjecture stated by M. Janet in 1920 [7] and first solved by J. Johnson in [8], saying that, for a system of this kind, there is a whole gap in the possible dimensions of the corresponding modules. The following second order system, with $n = 2$, dim(E)=dim(F)=2, provides a good example [6, 21]

$$\left\{ \begin{array}{l} \partial_{11}\xi^1 + \partial_{12}\xi^2 - \xi^2 = \eta^1, \\ \partial_{12}\xi^1 + \partial_{22}\xi^2 + \xi^1 = \eta^2, \end{array} \right.$$

and one can easily checked that the square matrix of $\mathcal{D}$ is unimodular with determinant equal to 1.

3.1.2 Pure Modules

We end this part with a generalization of the torsion-free property of a module and we follow [1]. In order to explain this extremely useful new direction for applications, we first provide a few examples.

Example 14 Starting with the system

$$\left\{ \begin{array}{l} y_{22} = 0, \\ y_{12} = 0, \end{array} \right.$$

we notice that $z = y_1$ only satisfies $z_2 = 0$ while $z = y_2$ does satisfy $z_1 = 0$, and $z_2 = 0$.

Hence, we may distinguish the torsion elements of a differential module according to the properties of the system of PDE they satisfy. Two examples from engineering science will particulary well illustrate the different behaviour of various torsion elements.

Example 15 In the linearized system for Euler equation for an incompressible fluid [22], namely

$$\left\{ \begin{array}{l} \vec{\nabla}.\vec{v} = 0, \\ \frac{\partial \vec{v}}{\partial t} + \vec{\nabla}p = 0, \end{array} \right.$$

where $\vec{v}$ is the speed and p the pressure of the fluid, one notices that we have the PDE

$$\triangle p = 0, \frac{\partial(\triangle \vec{v})}{\partial t} = 0.$$

Similarly, the Boussinesq stationary system for the Benard problem [21, 22], namely

$$\left\{ \begin{array}{l} \vec{\nabla}.\vec{v} = 0, \\ \triangle \vec{v} - \theta \vec{g} - \vec{\nabla}\pi = 0, \\ \triangle \theta - \vec{g}.\vec{v} = 0, \end{array} \right.$$

where $\vec{g} = (0, 0, -g)$ is the gravity while π and θ are perturbations of the pressure and temperature, we obtain from the vector analysis

$$\triangle\triangle\triangle\theta - g^2(\partial_{11} + \partial_{22})\theta = 0,$$

though, setting $w = \partial_1 v_2 - \partial_2 v_1$, we only get $\triangle w = 0$.

Accordingly, among the elements of a differential module, one can find the elements which are *free*, i.e., they do not satisfy any PDE, and the others (torsion elements) which are *constrained* by at least one PDE.

Definition 11 1. We introduce the D-submodules $t_r(M) = \{m \in M | \mathrm{cd}(Dm) > r\}$, with $t_0(M) = t(M)$, the torsion submodule of M.

2. A D-module is said to be *r-pure* if $t_r(M) = 0$ and $t_{r-1}(M) = M$.

The chain of inclusions

$$0 = t_n(M) \subseteq t_{n-1}(M) \subseteq .... \subseteq t_1(M) \subseteq t_0(M) = t(M) \subseteq M,$$

will be particularly useful for studying the specific properties of engineering quantities that can be observed experimentally by decoupling them from other quantities. Of course, $t_{r-1}(M)/t_r(M)$ is r-pure and one has the following delicate criterion for knowing whether a differential module is r-pure or not [1]

Theorem 3 M is r-pure $\Leftrightarrow M \subseteq \mathrm{ext}^r_D(\mathrm{ext}^r_D(M, D), D)$, with $\mathrm{cd}(M) = r$.

Corollary 4 *When M is r-pure then $char(M)$ is* r-equidimensional, *namely it can be decomposed into irreductible components of the same dimension r.*

We notice that the above criterion generalizes the situation of the torsion-free modules described in corollary 2 for the case $r = 0$.

Example 16 Without the previous criterion, it is not evident to prove that the differential module provided by example 12 is 2-pure and thus that the corresponding adjoint operator is torsion-free. More generally, any differential module defined by a finite type system is automatically n-pure. This is particulary clear in 2-dimensional elasticity, with $\mathcal{D} : (\xi_1, \xi_2) \longrightarrow (\partial_1\xi_1 = \epsilon_{11}, \frac{1}{2}(\partial_1\xi_2 + \partial_2\xi_1) = \epsilon_{12}, \partial_2\xi_2 = \epsilon_{22})$, $\mathcal{D}_1 : \epsilon \longrightarrow \partial_{11}\epsilon_{22} + \partial_{22}\epsilon_{11} - 2\partial_{12}\epsilon_{12} = 0$, defining the strain tensor and its compatibility condition, while the adjoint sequence allows to parametrize the stress equation by $\tilde{\mathcal{D}}_1$ acting on the Airy function.

To our knowledge, it does not seem that such a classification of systems/modules has ever been applied.

Another striking useful theorem is provided by the following non trivial theorem [11, 19].

Theorem 4 *We have the following relation:*

$$char(M) = \cup_{i=0}^{n} char\,(ext^i_D(M, D)).$$

Example 17 If $\mathcal{D}_1$ denotes the compatibility conditions of $\mathcal{D}$ and $\tilde{\mathcal{D}}$ generates the compatibility conditions of $\tilde{\mathcal{D}}_1$, in such a way that both the module M determined by $\mathcal{D}$ and the module N determined by $\tilde{\mathcal{D}}_1$ are torsion modules, then both $\mathcal{D}_1$ and $\tilde{\mathcal{D}}$ are surjective and $\mathrm{char}(M) = \mathrm{char}(N)$. This result generalizes the equality of the primeness degrees of left and right factor matrix descriptions of a given transfer matrix (see p. 74 of [36]). A typical example of this situation is provided by examples 12 and 16.

3.2 Input/Output Properties

We now turn to the properties involving inputs and outputs. First of all, contrary to the tradition, there is no reason at all for choosing the inputs as determining a maximum free differential submodule of M, though it is a possible choice. Accordingly, many concepts in control theory are based upon the two types of exact sequences that can be constructed from M

$$0 \longrightarrow t(M) \longrightarrow M \longrightarrow M/t(M) \longrightarrow 0, \tag{7}$$

$$0 \longrightarrow F \longrightarrow M \longrightarrow M/F \longrightarrow 0, \tag{8}$$

where $t(M)$ is the torsion submodule of M and F is a maximum free submodule of M. We notice that $M/t(M)$ is torsion-free while M/F is a torsion module. Setting $S = D\backslash\{0\}$, we may construct the field $Q(D) = S^{-1}D = DS^{-1}$ of quotients of D and tensor by $Q(D)$ the previous sequences in order to kill their torsion modules [11, 17, 28]. Such a construction, which is basic in algebraic analysis, gives the way to generalize the transfer matrix approach, even for variable coefficients, by considering the localization $S^{-1}M = Q(D)\otimes_D M$, without any reference to the Laplace transform [17, 28]. If we already know that M is torsion-free, it may provide a parametrization of $\mathcal{D}$ generalizing the *controller form* [9]. For more details, see [28]. We notice that $M/t(M)$ and M/F are two specializations of M giving rise to two subsystems R'_∞ and R''_∞ of R_∞. Taking into account that $t(M)\cap F = 0$, we obtain the following commutative and exact diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & F & = & F & \longrightarrow 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & t(M) & \longrightarrow & M & \longrightarrow & M/t(M) & \longrightarrow 0 & \\
 & & \| & & \downarrow & & \downarrow & & \\
0 \longrightarrow & & t(M) & \longrightarrow & M/F & \longrightarrow & M/(t(M)\oplus F) & \longrightarrow 0, & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

and, dualizing it, we obtain the following commutative and exact diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \uparrow & & \uparrow & & \\
 & & 0 & \longleftarrow & R_\infty/R''_\infty & = & R_\infty/R''_\infty & \longleftarrow 0 & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 \longleftarrow & & R_\infty/R'_\infty & \longleftarrow & R_\infty & \longleftarrow & R'_\infty & \longleftarrow 0 & \\
 & & \| & & \uparrow & & \uparrow & & \\
0 \longleftarrow & & R_\infty/R'_\infty & \longleftarrow & R''_\infty & \longleftarrow & R'_\infty\cap R''_\infty & \longleftarrow 0, & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

which provides at once the relation $R_\infty = R'_\infty + R''_\infty$. This very basic reason is hidden in [37, 38] where the underlying confusion concerning the choice of input and output comes from the fact that, when $n=1$, any torsion-free module is free and the first of the two preceding sequences splits. However, the resulting backward sequence must not be confused in general with the second sequence and the two sequences must be distinguished with care. In particular, only the first one entirely depends on M and provides the so-called minimum realization [28].

As input and output always play a reciprocal role and are made by elements of M, we shall consider two different differential submodules M_{in} and M_{out} of M such that $M_{in}+M_{out}$ may be a strict differential submodule of M if there are latent variables. There is no reason at all for supposing that M/M_{in} is a torsion module as M/M_{out} is not a torsion module in general. The main construction is to introduce $t(M)$ and set $M'_{in} = M_{in}+t(M)$, $M'_{out} = M_{out}+t(M)$ in M. Then, the idea of the minimal realization is to replace M_{in}, M_{out} and M by $M'_{in}/t(M) = M_{in}/(M_{in}\cap t(M))$, $M'_{out}/t(M) = M_{out}/(M_{out}\cap t(M))$ and $M/t(M)$ in order to deal only with torsion-free modules, always keeping in mind that the differential rank $\mathrm{rk}_D(M)$ of M, namely the last character, is intrinsically defined, does not depend on the presentation and is additive, that is to say, if $0\longrightarrow M'\longrightarrow M\longrightarrow M''\longrightarrow 0$ is a short exact sequence of differential modules then $\mathrm{rk}_D(M)$=$\mathrm{rk}_D(M')$+$\mathrm{rk}_D(M'')$. This is exactly the module analogue of the differential transcendence degree in differential algebra [13, 30] and one can prove that it is equal to the Euler characteristic of M. If one chooses $M_{in} = F$ as already defined, then $F\cap t(M) = 0$ and $M'_{in}/t(M)\cong F$ can always be considered as a submodule of $M/t(M)$.

The final idea is to define poles and zeros for multidimensional systems [17, 18, 36, 37]. First of all, we have seen (proposition 2) that if

$$0 \longrightarrow M' \xrightarrow{f} M \xrightarrow{g} M'' \longrightarrow 0,$$

is a short exact sequence of modules and if M is filtred, we can endow M' and M'' with the induced filtration $M'_q = M' \cap M_q$, $M''_q = g(M_q)$ and obtain, for these filtrations, the short exact sequence

$$0 \longrightarrow G' \longrightarrow G \longrightarrow G'' \longrightarrow 0,$$

of associated graded modules. Taking the radicals of the respective annhililators, we get

$$\sqrt{\mathrm{ann}(G)} = \sqrt{\mathrm{ann}(G')} \cap \sqrt{\mathrm{ann}(G'')},$$

and thus

$$\mathrm{char}(M) = \mathrm{char}(M') \cup \mathrm{char}(M''),$$

as the characteristic set does not depend on the filtration (see proposition 3). As, we are dealing with finitely generated modules, we also recall that, in the commutative case, the support $\mathrm{supp}(M)$ of a module M is the set of proper prime ideals of the corresponding ring, that contain the annihilator of M over the ring. The key point, in order to generalize the concept of transfer matrix approach, is to localize the graded sequence with respect to a prime ideal and get the short exact sequence

$$0 \longrightarrow G'_{\mathfrak{p}} \longrightarrow G_{\mathfrak{p}} \longrightarrow G''_{\mathfrak{p}} \longrightarrow 0,$$

with $\mathfrak{p} \in \mathrm{spec}(A[\chi])$, but we can also localize the filtred sequence when D is commutative. In the case of the SISO-system defined in example 4, we get $(\chi - 1)y = u$, and we can divide by $\chi - 1$ provided $\chi \neq 1$. Hence, the trick is to notice that $G'_{\mathfrak{p}} \cong G_{\mathfrak{p}}$ if and only if $G''_{\mathfrak{p}} = 0$, that is $\mathfrak{p} \notin \mathrm{supp}(G'')$, the true reason for looking at $\mathrm{char}(M'')$.

If N is any submodule of M, setting $N' = N + t(M)$, we have the following commutative and exact diagram,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & t(M) \cap N & \longrightarrow & N & \longrightarrow & N'/t(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & t(M) & \longrightarrow & M & \longrightarrow & M/t(M) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & t(M)/(t(M) \cap N) & \longrightarrow & M/N & \longrightarrow & M/N' & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

both with the isomorphisms:

$$t(M)/(t(M) \cap N) \cong N'/N, \tag{9}$$

$$N/(t(M) \cap N) \cong N'/t(M). \tag{10}$$

Setting M_{in}, M_{out} and $M_{in} + M_{out}$ in place of N, we get similar commutative and exact diagrams, both with short exact sequences of the type

$$0 \longrightarrow M_{in} \longrightarrow M_{in} + M_{out} \longrightarrow (M_{in} + M_{out})/M_{in} \longrightarrow 0, \tag{11}$$

$$0 \longrightarrow M'_{in} \longrightarrow M'_{in} + M'_{out} \longrightarrow (M'_{in} + M'_{out})/M'_{in} \longrightarrow 0, \tag{12}$$

and similar sequences with *in* and *out* interchanged.

Now, we have in general an exact sequence of the form

$$0 \longrightarrow N \longrightarrow N' \longrightarrow t(M)/(t(M) \cap N) \longrightarrow 0, \tag{13}$$

and similar sequences with M_{in}, M_{out} and $M_{in}+M_{out}$ in place of N. Combining the two preceding sequences starting respectively with M_{in} and M'_{in}, we obtain the short exact sequence:

$$0 \longrightarrow (t(M)\cap(M_{in}+M_{out}))/(t(M)\cap M_{in}) \longrightarrow (M_{in}+M_{out})/M_{in} \longrightarrow (M'_{in}+M'_{out})/M'_{in} \longrightarrow 0, \tag{14}$$

which is not evident at first sight and where many of the previous modules do appear.

We claim that all poles and zeros considered in classical control theory are only examples of the characteristic sets of the modules introduced above and all the relations among poles and zeros come from the preceding exact diagrams/sequences, by using the additive property of char$(\cdot)$ (see proposition 3). Of course, it is essential to notice the fact that the identification of char(M) with supp(G) when $G = \mathrm{gr}(M)$ only allows to use proper prime ideals of $A[\chi]$, a reason for setting char$(0)=\emptyset$.

For example, if $A = k$ and we use supp(M) instead of supp(G), there is nothing to change and we have (see p. 40 of [37]):

- { observables poles }=supp$((M_{in}+M_{out})/M_{in})$,
- { transmission poles }=supp$((M'_{in}+M'_{out})/M'_{in})$,
- { input decoupling zeros}=supp$(t(M))$,
- { input-output decoupling zeros }=supp$(t(M)/(t(M)\cap(M_{in}+M_{out})))$,

we obtain from the last exact sequence with evident notations:

$$\{\text{ob. p.}\} = \{\text{tr. p.}\} + \{\text{i.d.z.}\} - \{\text{i.o.d.z.}\} - \mathrm{supp}(t(M)\cap M_{in}).$$

If M_{in} is identifed with F, we obtain therefore $t(M)\cap M_{in} = 0$ and we recover the formula (23) of [2].

We may recapitulate the various modules involved on the following picture, explaining all the situations that can be met in the range of applications.

$$\begin{array}{ccccc}
 & & M & & \\
 & & \uparrow & & \\
 & & M'_{in}+M'_{out} & & \\
 & \nearrow & & \nwarrow & \\
M'_{in} & & \uparrow & & M'_{out} \\
 & \nwarrow & t(M) & \nearrow & \\
\uparrow & & & & \uparrow \\
M_{in} & & \uparrow & & M_{out} \\
 & \nwarrow & & \nearrow & \\
 & & 0 & &
\end{array}$$

Introducing also the sets:

- {system poles} = supp(M/M_{in}),
- {output decoupling zeros} = supp$(M/(M_{in}+M_{out}))$,

and using the short exact sequence

$$0 \longrightarrow (M_{in}+M_{out})/M_{in} \longrightarrow M/M_{in} \longrightarrow M/(M_{in}+M_{out}) \longrightarrow 0,$$

we obtain, with evident notations:

$$\{\text{sys. p.}\} = \{\text{ob.p.}\} + \{\text{o.d.z.}\}.$$

However, in pratice, there is no loss of generality in supposing $M = M_{in}+M_{out}$. In such a simple situation, combining the preceding results, we get:

$$\{\text{syst. p.}\} = \{\text{ob. p}\} = \{\text{tr. p.}\} + \{\text{i.d.z.}\} - \mathrm{supp}(t(M)\cap M_{in}),$$

and we may thus introduce the set:

$$\{\text{hidden modes}\} = \{\text{i.d.z.}\} - \text{supp}(t(M) \cap M_{in}).$$

The preceding results prove that input and output play a similar role and that it is thus better to use the only word "zero" or "supp" for the corresponding modules and not the word "pole".

Example 18 If we have a SISO system $\dot{y} - y = u$ with input u satisfying $\dot{u} + u = 0$, we obtain $\ddot{y} - y = 0$ and thus supp(M)=$\{(\chi-1),(\chi+1)\}$ while supp$(t(M) \cap M_{in}) = \{(\chi+1)\}$ and we find the hidden mode $(\chi + 1)$. Such a situation can happen in an electrical LCR circuit if we suppose conditions on a voltage input.

4 Conclusion

We hope to have convinced the reader that, despite the difficulty of the underlying mathematical tools, the formal methods of algebraic analysis allow to clarify and unify all the existing results on multidimensional control systems. In most cases, the corresponding algorithms are effective and can easily be checked. Finally, this approach is the only one which can separate the intrinsic/built-in properties of a control system such as torsion-freeness or pureness, from the other properties that depend on the choice of input and output. Meanwhile, another essential aspect is the possibility to bring the study of modules over non-commutative rings to the simpler study of modules over commutative rings. We do not believe that none of the results presented here could be obtained without the use of the extension functor and duality, a fact explaining why it took such a long time to estabish a link between algebraic analysis and control theory.

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